AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 16102–16111.
DOI: 10.3934/math.2022881
Received: 17 March 2022
Revised: 19 June 2022
Accepted: 22 June 2022
Published: 01 July 2022

Research article

A note on the hybrid power mean involving the cubic Gauss sums and Kloosterman sums

Xiaoxue Li[1,*] **and Wenpeng Zhang**[2]

[1] School of Science, Xi'an Aeronautical Institute, Xi'an, Shaanxi, China
[2] School of Mathematics, Northwest University, Xi'an, Shaanxi, China

* **Correspondence:** Email: lxx20072012@163.com.

Abstract: The main purpose of this paper is to study the calculating problem of one kind hybrid power mean involving the cubic Gauss sums and Kloosterman sums, and using the elementary methods, analytic methods and the properties of the classical Gauss sums to give some interesting calculating formula for them. At the same time, the paper also provides an effective calculating method for the study of the hybrid power mean involving the k-th Gauss sums and Kloosterman sums.

Keywords: the cubic Gauss sums; Kloosterman sums; hybrid power mean; elementary method; recurrence formula
Mathematics Subject Classification: 11L03, 11L05

1. Introduction

Let q be a positive integer. For any integers m and n, The famous Kloosterman sums $K(m, n; q)$ is defined as follows:

$$K(m, n; q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{ma + n\overline{a}}{q}\right),$$

where $e(y) = e^{2\pi iy}$, $i^2 = -1$ and $\overline{a}$ is defined by the congruence $a \cdot \overline{a} \equiv 1 \bmod q$.

This sum plays a very important role in the study of analytic number theory, many number theory problems are closely related to it, so many scholars have studied the various properties about $K(m, n; q)$ and obtained a series of significant research results. For example, Kloosterman's pioneering work [3] proved the identity

$$\sum_{m=1}^{p} \left| \sum_{a=1}^{p-1} e\left(\frac{ma + \overline{a}}{p}\right) \right|^4 = p \cdot \left(2p^2 - 3p - 3\right),$$

where p is an odd prime.

Estermann [4] studied the upper bound estimation of $K(m, n; q)$, and obtained the best estimation

$$|K(m, n; q)| \le (m, n, q)^{\frac{1}{2}} \cdot d(q) \cdot q^{\frac{1}{2}},$$

where (m, n, q) denotes the greatest common divisor of m, n and q, $d(q)$ denotes the Dirichlet divisor function.

Zhang [5] used the elementary method to study the fourth power mean of $K(m, n; q)$ for general modulo q, and proved the identity

$$\sum_{m=1}^{q} \left| \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{ma + n\overline{a}}{q}\right) \right|^4 = 3^{\omega(q)} \cdot q^2 \cdot \phi(q) \cdot \prod_{p \| q} \left(\frac{2}{3} - \frac{1}{3p} - \frac{4}{3(p-1)}\right),$$

where $\omega(q)$ denotes the number of all distinct prime divisors of q, $\phi(q)$ denotes the Euler's function, and $\prod_{p\|q}$ denotes the product over all prime divisors of q such that $p \mid q$ and $p^2 \nmid q$.

In addition, Chen and Hu [6] studied the hybrid power mean involving the cubic Gauss sums $A(m)$ and Kloosterman sums $K(m, n; p)$, i.e.,

$$S_k(p) = \sum_{m=1}^{p-1} A^k(m) \cdot K^2(m, 1; p) = \sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) \right)^k \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma + \overline{a}}{p}\right) \right|^2,$$

and proved that for any positive integer $k \ge 1$, one has the third-order linear recurrence formula

$$S_{k+3}(p) = 3p \cdot S_{k+1}(p) + dp \cdot S_k(p),$$

with the first three values

$$S_1(p) = 2p^2 + dp \cdot A(1) - p \cdot A^2(1),$$

$$S_2(p) = 2p^3 + 2(d-1)p^2 + p(d^2 - p) \cdot A(1) - dp \cdot A^2(1),$$

$$S_3(p) = (d+6)p^3 + 3dp^2 \cdot A(1) - 3p^2 \cdot A^2(1) - dp(p+1),$$

where $4p = d^2 + 27 \cdot b^2$, and d is uniquely determined by $d \equiv 1 \bmod 3$.

Some related contents can also be found in [7–16], we would not list them all here.

Obviously, the result in [6] is meaningful, it gives a calculation method for all $S_k(p)$. However, this result does not look pretty, because it contains parameter $A(1)$. Therefore, we can not calculate the exact value of $S_k(p)$.

On the other hand, Kloosterman sums can be used to solve the Waring-Goldbach problem, to solve the problems of prime distribution over short intervals, mean estimation of the Riemann Zata function and Fourier coefficients in modular form, etc. Not only that, the properties of Kloosterman sums can also be used to determine the generalized Hamming weight or the weight distribution of some linear codes, so it has important applications in the field of communication, cryptography and coding theory. Therefore, it is necessary to further study the properties of Klooaterman sums.

Based on the above, in this paper, as a note of [6], we modify some parameters in [6] as follows:

$$\begin{aligned} C_k(h,p) &= \sum_{m=1}^{p-1} A^k(\overline{m}) \cdot K^h(m,p) \\ &= \sum_{m=1}^{p-1} \left(\sum_{b=0}^{p-1} e\left(\frac{\overline{m}b^3}{p} \right) \right)^k \cdot \left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p} \right) \right)^h, \end{aligned}$$

then we will get some more concise and prettier results. That is, we will prove the following two conclusions:

Theorem 1. Let p be a prime with $p \equiv 1 \bmod 3$. Then for any integer k, we have the third-order linear recurrence formula

$$C_k(1,p) = 3p \cdot C_{k-2}(1,p) + dp \cdot C_{k-3}(1,p), \quad k \geq 3,$$

with the first three values $C_0(1,p) = 1$, $C_1(1,p) = dp$ and $C_2(1,p) = 2p\cdot(p+1)$, where $4p = d^2+27\cdot b^2$, and d is uniquely determined by $d \equiv 1 \bmod 3$.

Theorem 2. Let p be a prime with $p \equiv 1 \bmod 3$. Then for any integer k, we have the third-order linear recurrence formula

$$C_k(2,p) = 3p \cdot C_{k-2}(2,p) + dp \cdot C_{k-3}(2,p), \quad k \geq 3,$$

with the first three values

$$C_0(2,p) = p^2 - p - 1, \quad C_1(2,p) = p \cdot \left(d^2 - 2p\right)$$

and

$$C_2(2,p) = p \cdot \left(2p^2 + dp - 2p - 2\right).$$

From these two theorems we may immediately deduce the following corollaries:

Corollary 1. For any prime p with $p \equiv 1 \bmod 3$, we have the identity

$$\sum_{m=1}^{p-1} \frac{\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)}{\sum_{b=0}^{p-1} e\left(\frac{\overline{m}b^3}{p}\right)} = \frac{2p-1}{d}.$$

Corollary 2. For any prime p with $p \equiv 1 \bmod 3$, we have the identity

$$\sum_{m=1}^{p-1} \left| \frac{\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)}{\sum_{b=0}^{p-1} e\left(\frac{\overline{m}b^3}{p}\right)} \right|^2 = \frac{3p^2 - 5dp + d^3 - 3p - 3}{d^2}.$$

Corollary 3. For any prime p with $p \equiv 1 \bmod 3$, we have the identity

$$\sum_{m=1}^{p-1}\left|\sum_{b=0}^{p-1} e\left(\frac{\overline{m}b^3}{p}\right)\right|^4 \cdot \left|\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right|^2 = p^2 \cdot \left(6p^2 + dp + d^3 - 6p - 6\right).$$

Some notes: In our theorems, we only consider the prime p with $p \equiv 1 \bmod 3$. In fact if $3 \nmid (p-1)$, then for any integer m with $(m, p) = 1$, we have $A(\overline{m}) = 0$ and $C_k(h, p) = 0$ for all $k \geq 1$. So in this case, the results are trivial.

Note that $K(m, n; p)$ is a real number, so if we replace $A(\overline{m})$ in Theorem 2 with $A(m)$, then the values of $C_k(h, p)$ are the same as in [6].

In addition, since $A(\overline{m}) \neq 0$ for all $(m, p) = 1$, so the third-order linear recurrence formulas in Theorem 1 and Theorem 2 are also hold for all integers $k < 3$.

For any integer $h \geq 3$, whether there is an exact calculating formula for $C_k(h, p)$ is an open problem. It remains to be further studied.

2. Several lemmas

To complete the proofs of our all results, we need three necessary lemmas. The proofs of these lemmas requires some knowledge of elementary or analytic number theory, all these can be found in references [1, 2], so we do not repeat them here. First we have the following:

Lemma 1. Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any third-order character λ modulo p, we have the identity

$$\tau^3(\lambda) + \tau^3\left(\overline{\lambda}\right) = dp,$$

where $\tau(\chi) = \sum_{a=1}^{p-1}\chi(a)e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums, $4p = d^2 + 27 \cdot b^2$, and d is uniquely determined by $d \equiv 1 \bmod 3$.

Proof. The proof of this Lemma see Zhang and Hu [17] or Berndt and Evans [18].

Lemma 2. Let p be a prime with $p \equiv 1 \bmod 3$, then for any three-order character λ modulo p, we have

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{\overline{m}a^3}{p}\right)\right) \cdot \left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right) = dp$$

and

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{\overline{m}a^3}{p}\right)\right)^2 \cdot \left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right) = 2p(p+1).$$

Proof. Let p be a prime with $p \equiv 1 \bmod 3$, let λ be any third-order character modulo p. Then for any integer m with $(m, p) = 1$, from the properties of the third-order character modulo p we have the identity

$$A(\overline{m}) = 1 + \sum_{a=1}^{p-1}\left(1 + \lambda(a) + \overline{\lambda}(a)\right)e\left(\frac{\overline{m}a}{p}\right) = \lambda(m)\tau(\lambda) + \overline{\lambda}(m)\tau\left(\overline{\lambda}\right). \tag{2.1}$$

From the properties of the classical Gauss sums we have

$$\sum_{m=1}^{p-1}\lambda(m)\left(\sum_{a=1}^{p-1}e\left(\frac{ma+\overline{a}}{p}\right)\right)=\sum_{a=1}^{p-1}e\left(\frac{\overline{a}}{p}\right)\sum_{m=1}^{p-1}\lambda(m)e\left(\frac{ma}{p}\right)=\tau(\lambda)\sum_{a=1}^{p-1}\overline{\lambda}(a)e\left(\frac{\overline{a}}{p}\right)=\tau^2(\lambda)\quad (2.2)$$

and

$$\sum_{m=1}^{p-1}\overline{\lambda}(m)\left(\sum_{a=1}^{p-1}e\left(\frac{ma+\overline{a}}{p}\right)\right)=\tau^2\left(\overline{\lambda}\right). \quad (2.3)$$

From (2.1)–(2.3) and Lemma 1, we have

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1}e\left(\frac{\overline{m}a^3}{p}\right)\right)\cdot\left(\sum_{a=1}^{p-1}e\left(\frac{ma+\overline{a}}{p}\right)\right)=\tau^3(\lambda)+\tau^3\left(\overline{\lambda}\right)=dp. \quad (2.4)$$

Note that $\tau(\lambda)\cdot\tau\left(\overline{\lambda}\right)=p$, from (2.4) and $\lambda^2=\overline{\lambda}$ we also have

$$A^2\left(\overline{m}\right)=\lambda(m)\tau^2\left(\overline{\lambda}\right)+\overline{\lambda}(m)\tau^2\left(\lambda\right)+2p. \quad (2.5)$$

$$\sum_{m=1}^{p-1}\left(\sum_{a=1}^{p-1}e\left(\frac{ma+\overline{a}}{p}\right)\right)=1. \quad (2.6)$$

From (2.2), (2.3), (2.5), (2.6) and Lemma 1, we have

$$\begin{aligned}&\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1}e\left(\frac{\overline{m}a^3}{p}\right)\right)^2\cdot\left(\sum_{a=1}^{p-1}e\left(\frac{ma+\overline{a}}{p}\right)\right)\\ =\ &\sum_{m=1}^{p-1}\left(\overline{\lambda}(m)\tau^2\left(\lambda\right)+\lambda(m)\tau^2\left(\overline{\lambda}\right)+2p\right)\cdot\left(\sum_{a=1}^{p-1}e\left(\frac{ma+\overline{a}}{p}\right)\right)\\ =\ &\tau^2(\lambda)\cdot\tau^2\left(\overline{\lambda}\right)+\tau^2(\lambda)\cdot\tau^2\left(\overline{\lambda}\right)+2p=2p\cdot(p+1). \quad (2.7)\end{aligned}$$

Now Lemma 2 follows from (2.4) and (2.7).

Lemma 3. Let p be a prime with $p\equiv 1 \bmod 3$. Then we have the identities

$$\sum_{m=1}^{p-1}A\left(\overline{m}\right)\cdot\left(\sum_{a=1}^{p-1}e\left(\frac{ma+\overline{a}}{p}\right)\right)^2=p\cdot\left(d^2-2p\right)$$

and

$$\sum_{m=1}^{p-1}A^2\left(\overline{m}\right)\cdot\left(\sum_{a=1}^{p-1}e\left(\frac{ma+\overline{a}}{p}\right)\right)^2=p\cdot\left(2p^2+dp-2p-2\right).$$

Proof. From the properties of the classical Gauss sums we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} \lambda(m) \cdot \left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right)^2 \\
= \;& \tau(\lambda) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \overline{\lambda}(a+b)\, e\left(\frac{\overline{a}+\overline{b}}{p}\right) \\
= \;& \tau(\lambda) \sum_{a=1}^{p-1} \overline{\lambda}(a+1) \sum_{b=1}^{p-1} \overline{\lambda}(b) e\left(\frac{\overline{b}\,(\overline{a}+1)}{p}\right) \\
= \;& \tau^2(\lambda) \sum_{a=1}^{p-1} \overline{\lambda}(a+1)\, \overline{\lambda}(\overline{a}+1) = \tau^2(\lambda) \sum_{a=1}^{p-1} \lambda(a) \overline{\lambda}^2(a+1) \\
= \;& \tau^2(\lambda) \sum_{a=1}^{p-1} \lambda(a)\lambda(a+1) = \frac{\tau^2(\lambda)}{\tau\left(\overline{\lambda}\right)} \sum_{b=1}^{p-1} \overline{\lambda}(b) \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{b(a+1)}{p}\right) \\
= \;& \frac{\tau^3(\lambda)}{\tau\left(\overline{\lambda}\right)} \sum_{b=1}^{p-1} \overline{\lambda}^2(b) e\left(\frac{b}{p}\right) = \frac{\tau^4(\lambda)}{\tau\left(\overline{\lambda}\right)} = \frac{\tau^5(\lambda)}{p}.
\end{aligned} \tag{2.8}$$

Similarly, we also have

$$\sum_{m=1}^{p-1} \overline{\lambda}(m) \cdot \left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right)^2 = \frac{\tau^5\left(\overline{\lambda}\right)}{p}. \tag{2.9}$$

From (2.1), (2.8), (2.9) and Lemma 1, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} A\left(\overline{m}\right) \cdot \left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right)^2 \\
= \;& \sum_{m=1}^{p-1} \left(\lambda(m)\tau(\lambda) + \overline{\lambda}(m)\tau\left(\overline{\lambda}\right)\right) \cdot \left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right)^2 \\
= \;& \frac{\tau^6(\lambda)}{p} + \frac{\tau^6\left(\overline{\lambda}\right)}{p} \\
= \;& \frac{1}{p} \cdot \left[\left(\tau^3(\lambda) + \tau^3\left(\overline{\lambda}\right)\right)^2 - 2\tau^3(\lambda) \cdot \tau^3\left(\overline{\lambda}\right)\right] \\
= \;& \frac{1}{p} \cdot \left(d^2 p^2 - 2p^3\right) = p \cdot \left(d^2 - 2p\right).
\end{aligned} \tag{2.10}$$

Note that the identity

$$\sum_{m=1}^{p-1} \left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right)^2 = p^2 - p - 1, \tag{2.11}$$

from (2.5), (2.8), (2.9), (2.11) and Lemma 1, we have the identity

$$\begin{aligned}
&\sum_{m=1}^{p-1} A^2\left(\overline{m}\right)\cdot\left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right)^2 \\
=\ &\sum_{m=1}^{p-1}\left(\overline{\lambda}(m)\tau^2(\lambda)+\lambda(m)\tau^2\left(\overline{\lambda}\right)+2p\right)\cdot\left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right)^2 \\
=\ &\frac{\tau^2(\lambda)\cdot\tau^5\left(\overline{\lambda}\right)}{p}+\frac{\tau^2\left(\overline{\lambda}\right)\cdot\tau^5\left(\lambda\right)}{p}+2p\left(p^2-p-1\right) \\
=\ &p\cdot\left(\tau^3(\lambda)+\tau^3\left(\overline{\lambda}\right)\right)+2p\left(p^2-p-1\right) \\
=\ &p\cdot\left(2p^2+dp-2p-2\right).
\end{aligned} \tag{2.12}$$

Now Lemma 3 follows from (2.10) and (2.12).

3. Proofs of the theorems

Applying Lemma 1 and Lemma 2 we can easily prove our theorems. First we prove Theorem 1. For any prime p with $p \equiv 1 \bmod 3$ and integer m with $(m, p) = 1$, from (2.1) and Lemma 1 we have the identity

$$A^3\left(\overline{m}\right) = \tau^3(\lambda)+\tau^3\left(\overline{\lambda}\right)+3p\cdot A\left(\overline{m}\right) = dp+3p\cdot A\left(\overline{m}\right). \tag{3.1}$$

From (3.1) we can deduce that

$$\begin{aligned}
A^k\left(\overline{m}\right) &= A^{k-3}\left(\overline{m}\right)\cdot A^3\left(\overline{m}\right) = A^{k-3}\left(\overline{m}\right)\cdot\left(dp+3p\cdot A\left(\overline{m}\right)\right) \\
&= 3p\cdot A^{k-2}\left(\overline{m}\right)+dp\cdot A^{k-3}\left(\overline{m}\right).
\end{aligned} \tag{3.2}$$

From (3.2) we can deduce the third-order linear recurrence formula

$$C_k(h,p) = \sum_{m=1}^{p-1} A^k\left(\overline{m}\right)\cdot\left(\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right)^h = 3p\cdot C_{k-2}(h,p)+dp\cdot C_{k-3}(h,p),\quad k\geq 3. \tag{3.3}$$

Taking $h = 1$, from Lemma 2 we have $C_0(1, p) = 1$, $C_1(1, p) = dp$ and $C_2(1, p) = 2p\cdot(p + 1)$. This proves Theorem 1.

Now we prove Theorem 2. Taking $h = 2$ in (3.3), from (2.11) and Lemma 3 we have

$$C_0(2,p) = p^2-p-1,\ \ C_1(2,p) = p\cdot\left(d^2-2p\right)$$

and

$$C_2(2,p) = p\cdot\left(2p^2+dp-2p-2\right).$$

This proves Theorem 2.

From (3.1) we know that $A\left(\overline{m}\right) \neq 0$. So formula (3.3) also holds for all integers $k < 0$. Taking $k = 2$ in Theorem 1 we have

$$C_2(1,p) = 3p\cdot C_0(1,p)+dp\cdot C_{-1}(1,p)$$

or

$$\sum_{m=1}^{p-1} \frac{\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)}{\sum_{b=0}^{p-1} e\left(\frac{\overline{m}b^3}{p}\right)} = \frac{2p-1}{d}.$$

This proves Corollary 1.

Taking $k = 2$ in Theorem 2, note that

$$C_2(2, p) = 3p \cdot C_0(2, p) + dp \cdot C_{-1}(2, p)$$

and

$$C_1(2, p) = 3p \cdot C_{-1}(2, p) + dp \cdot C_{-2}(2, p).$$

Therefore, we have

$$\begin{aligned} C_{-1}(2, p) &= \frac{1}{dp} \cdot \left[2p^3 + dp^2 - 2p^2 - 2p - 3p \cdot \left(p^2 - p - 1\right)\right] \\ &= \frac{-p^2 + dp + p + 1}{d} \end{aligned}$$

and

$$\begin{aligned} C_{-2}(2, p) &= \frac{1}{dp} \cdot (C_1(2, p) - 3p \cdot C_{-1}(2, p)) \\ &= \frac{1}{dp} \cdot \left[d^2p - 2p^2 - 3p \cdot \frac{-p^2 + dp + p + 1}{d}\right] \\ &= \frac{3p^2 - 5dp + d^3 - 3p - 3}{d^2}. \end{aligned}$$

This completes the proofs of our all results.

4. Conclusions

The main results of this paper is to give two theorems for the hybrid power mean involving the cubic Gauss sums and Kloosterman sums. In addition, we also obtained a third-order linear recurrence formula for $C_k(h, p)$ with $h = 1$ and 2. That is, for any integer k, we have the three-order linear recurrence formula

$$C_k(h, p) = 3p \cdot C_{k-2}(h, p) + dp \cdot C_{k-3}(h, p)$$

with the exact values $C_0(h, p)$, $C_1(h, p)$ and $C_2(h, p)$, where d is uniquely determined by $4p = d^2 + 27b^2$ and $d \equiv 1 \bmod 3$.

At the same time, our results also provides an effective method for the study of the hybrid power mean involving the k-th Gauss sums and Kloosterman sums.

Acknowledgments

This work is supported by the N. S. F. (12126357) of China, the Doctoral Scientific Research Project of XAAI. The authors would like to thank the referees for their very helpful and detailed comments, which have significantly improved the presentation of this paper.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. T. M. Apostol, *Introduction to analytic number theory*, New York: Springer-Verlag, 1976. https://doi.org/10.1007/978-1-4757-5579-4
2. K. Ireland, M. Rosen, *A classical introduction to modern number theory*, New York: Springer-Verlag, 1990. https://doi.org/10.1007/978-1-4757-2103-4
3. H. D. Kloosterman, On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$, *Acta Math.*, **49** (1927), 407–464. https://doi.org/10.1007/BF02564120
4. T. Estermann, On Kloosterman's sum, *Mathematica*, **8** (1961), 83–86.
5. W. P. Zhang, On the general Kloosterman sums and its fourth power mean, *J. Number Theory,* **104** (2004), 156–161. http://dx.doi.org/10.1016/S0022-314X(03)00154-9
6. L. Chen, J. Y. Hu, A linear recurrence formula involving cubic Gauss sums and Kloosterman sums, *Acta Math. Sin. Chin. Ser.*, **61** (2018), 67–72.
7. W. P. Zhang, On the fourth power mean of the general Kloosterman sums, *J. Number Theory*, **169** (2016), 315–326. https://doi.org/10.1016/j.jnt.2016.05.018
8. S. F. Cao, T. T. Wang, On the hybrid power mean of two-term exponential sums and cubic Gauss sums, *J. Math.*, **2021** (2021), 6638156. http://dx.doi.org/10.1155/2021/6638156
9. W. P. Zhang, D. Han, On the sixth power mean of the two-term exponential sums, *J. Number Theory*, **136** (2014), 403–413. http://dx.doi.org/10.1016/j.jnt.2013.10.022
10. W. P. Zhang, Y. Y. Meng, On the sixth power mean of the two-term exponential sums, *Acta. Math. Sin.-English Ser.*, **38** (2022), 510–518. http://dx.doi.org/10.1007/s10114-022-0541-8
11. X. X. Lv, On the hybrid power mean of three-th Gauss sums and two-term exponential sums, *Acta Math. Sin. Chin. Ser.*, **62** (2019), 225–231.
12. S. Chowla, J. Cowles, M. Cowles, On the number of zeros of diagonal cubic forms, *J. Number Theory,* **9** (1977), 502–506.
13. Z. Y. Chen, W. P. Zhang, On the fourth-order linear recurrence formula related to classical Gauss sums, *Open Math.*, **15** (2017), 1251–1255. https://doi.org/10.1515/math-2017-0104

14. L. Chen, On the classical Gauss sums and their some properties, *Symmetry*, **10** (2018), 625. https://doi.org/10.3390/sym10110625

15. T. T. Wang, G. H. Chen, A note on the classical Gauss sums, *Mathematics*, **6** (2018), 313. https://doi.org/10.3390/math6120313

16. W. P. Zhang, X. D. Yuan, On the classical Gauss sums and their some new identities, *AIMS Math.*, **7** (2022), 5860–5870. http://dx.doi.org/10.3934/math.2022325

17. W. P. Zhang, J. Y. Hu, The number of solutions of the diagonal cubic congruence equation mod p, *Math. Reports*, **20** (2018), 73–80.

18. B. C. Berndt, R. J. Evans, The determination of Gauss sums, *Bull. Amer. Math. Soc.*, **5** (1981), 107–128. http://dx.doi.org/10.1090/S0273-0979-1981-14930-2

AIMS Press