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Research article

Ekeland's variational principle in fuzzy quasi-normed spaces

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Abstract: Fuzzy quasi-normed space provides an ideal mathematical framework for studying asymmetric phenomena. In this paper, we prove a version of the Ekeland variational principle in fuzzy quasi-normed spaces and apply it to Caristi's fixed point theorem and Takahashi minimization theorem. Moreover, we prove the equivalence relations among these theorems.

Keywords: fuzzy quasi-norm; variational principle; fixed point theorem; minimization theorem
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1. Introduction

The research of Ekeland's variational principle (EVP) can be traced back to 1972. Ekeland [15] proposed a variational principle (see [16] for its proof), and Ekeland [17] described the wide application of this principle in 1979. Since then, EVP has attracted extensive attention in various fields and has been applied to optimization [7,9,21,28,33], control theory [6], nonlinear analysis [23,34], economics [5], biology [3,4], etc.

There are many generalized forms and equivalent forms of EVP. It is worth noting that Phelps [24] gave the EVP in a Banach space by using Bishop and Phelps cone lemma. Qiu [25–27] studied the EVP in locally convex spaces. In 1994, Chang and Luo [10] combined EVP with fuzzy mapping and established the equivalence between EVP and Caristi's fixed point theorem (CFPT).

A quasi-metric (a quasi-norm, resp.) is also called an asymmetric metric (an asymmetric norm, resp.), if it is a function satisfying all the axioms of a metric (a norm, resp.) with the exception of the symmetry. There are some interesting applications of quasi-metric and quasi-norm in the study of the complexity of algorithms and languages (see, e.g., [19,29,30]). In 2011, Cobzas [12] gave some versions of EVP and Takahashi minimization theorem (TMT) in T_1 quasi-metric spaces and proved the equivalence of the weak Ekeland variational principle (wEVP) and CFPT. The extension of wEVP to an arbitrary quasi-metric space was given in [22]. In 2012, Cobzas [13] proved two versions of EVP in asymmetric locally convex spaces. In 2019, Al-Homidan et al. [2] also presented an equilibrium version of EVP and extended TMT in the setting of quasi-metric spaces. At the same time, Cobzas [14] proved the versions of EVP, TMT and CFPT in the sequentially K-complete quasi pseudo-metric space. Recently, Wu and Tang [32] investigated EVP, TMT and CFPT in fuzzy quasi-metric spaces.

To the best of our knowledge, however, there are few research results on the variational principles in fuzzy quasi-normed spaces. Inspired by references [1,11,32], in this paper, we shall do some exploration in this field. The organization of the paper is as follows: In Section 2, we introduce some notations and results about fuzzy quasi-normed spaces. In Section 3, we extend EVP and wEVP to fuzzy quasi-normed spaces by means of the partial order induced in this paper. In Section 4, we give versions of TMT and CFPT in fuzzy quasi-normed spaces and prove the equivalence relation among these theorems. Finally, a brief conclusion is given in Section 5.

In this paper, X is a real vector space, θ is the zero vector, ϕ denotes the empty set, $\mathbb{N}$ and $\mathbb{R}$ mean the set of all natural numbers and the set of all real numbers, respectively.

2. Preliminaries

In this section, we recall some notations and results in fuzzy quasi-normed spaces.

Definition 2.1. ([31]) A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous $t-$norm if it satisfies the following conditions: for all $a, b, c, d \in [0,1]$,

(1) $a * b = b * a$ (commutativity),

(2) $(a * b) * c = a * (b * c)$ (associativity),

(3) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ (monotonicity),

(4) $a * 1 = a$ (boundary condition),

(5) $*$is continuous on $[0,1] \times [0,1]$ (continuity).

Three paradigmatic examples of continuous t-norms are $\wedge$, $\cdot$ and $*_L$ (the Lukasiewicz t-norm), which are defined by

$$a * b = a \wedge b, \ a * b = ab, \ a * b = max\{a + b - 1, 0\}.$$

Lemma 2.2. ([20]) Let $*$ be a continuous $t-$norm.

(1) If $1 > r_1 > r_2 > 0$, then there exists $r_3 \in (0,1)$ such that $r_1 * r_3 \ge r_2$;

(2) If $r_4 \in (0,1)$, then there exists $r_5 \in (0,1)$ such that $r_5 * r_5 \ge r_4$.

Definition 2.3. ([1]) A fuzzy quasi-norm on a real linear space X is a pair $(N,*)$ such that $*$ is a continuous $t-$norm and N is a fuzzy set in $X \times 0, +\infty)$ satisfying the following conditions: for any $x, y \in X$,

(FQN1) $N(x, 0) = 0$;

(FQN2) $N(x, t) = N(-x, t) = 1$, for all $t > 0 \Leftrightarrow x = \theta$;

(FQN3) $N(\lambda x, t) = N\left(x, \frac{t}{\lambda}\right)$, for all $\lambda, t > 0$;

(FQN4) $N(x + y, t + s) \ge N(x, t) * N(y, s)$, for all $t, s > 0$;

(FQN5) $N(x, \cdot): [0, \infty) \to [0,1]$ is left continuous;

(FQN6) $\lim_{t\to\infty} N(x,t) = 1$.

Obviously, the function $N(x,\cdot)$ is increasing for any $x \in X$.

A fuzzy quasi-norm $(N,*)$ is called a fuzzy norm if $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ for all $x \in X$ and $c \in \mathbb{R}\setminus\{0\}$. Let $(N,*)$ be a fuzzy quasi-norm, set

$$N^{-1}(x,t) = N(-x,t),\ \ N^{s}(x,t) = min\{N(x,t), N^{-1}(x,t)\},\ \forall x \in X,\ t > 0,$$

then $(N^{-1},*)$ and $(N^{s},*)$ are a fuzzy quasi-norm and a fuzzy norm on X, respectively.

If $(N,*)$ is a fuzzy quasi-norm (a fuzzy norm, resp.) on X, we call $(X,N,*)$ a fuzzy quasi-normed space (a fuzzy normed space, resp.).

Each fuzzy quasi-norm $(N,*)$ on X induces a topology τ_N which has a base given by the family of open balls at $x \in X$

$$\mathscr{B}(x) = \{B_N(x,r,t): r \in (0,1), t > 0\},$$

where

$$B_N(x,r,t) = \{y \in X: N(y-x,t) > 1-r\}.$$

It is easy to see that the topology τ_N is T_0 and first countable. Since $x + B_N(\theta,r,t) = B_N(x,r,t)$, the topology τ_N is translation invariant. A sequence $\{x_n\}$ of X converges to x with respect to (w.r.t.) τ_N (denoted by $x_n \overset{\tau_N}{\to} x$) if and only if $\lim_{n\to\infty} N(x_n - x, t) = 1,\ \forall t > 0$.

It is well known, a quasi-norm on a real vector space X is a function $p: X \to 0, +\infty)$ satisfying the following conditions: for all $x, y \in X$ and $\lambda \in [0,\infty)$,

(QN1) $p(x) = p(-x) = 0 \Rightarrow x = \theta$,

(QN2) $p(\lambda x) = \lambda p(x)$,

(QN3) $p(x+y) \le p(x) + p(y)$.

If p only satisfies the conditions (QN2) and (QN3), then it is called a quasi-seminorm.

Definition 2.4. ([18]) Let X be a linear space and $*$ be a continuous $t-$norm. $P = \{p_\alpha: X \to [0,+\infty), \alpha \in (0,1)\}$ is called a family of star quasi-seminorms if it satisfies the following conditions: for all $x, y \in X$, $\alpha, \beta \in (0,1)$ and $\lambda > 0$,

(*QN1) $p_\alpha(\lambda x) = \lambda p_\alpha(x)$,

(*QN2) $p_{\alpha*\beta}(x+y) \le p_\alpha(x) + p_\beta(y)$.

If P satisfies the following condition:

(*QN3) $p_\alpha(x) = p_\alpha(-x) = 0$ for every $\alpha \in (0,1)$ implies $x = \theta$, then P is said to be separating.

Proposition 2.5. ([18]) $(X,N,*)$ is a fuzzy quasi-normed space. For any $\alpha \in (0,1)$, define a function $\|\cdot\|_\alpha: X \to 0, +\infty)$ as:

$$\|x\|_\alpha = inf\{t > 0: N(x,t) \ge \alpha\},\ \forall x \in X. \tag{2-1}$$

$P_N = \{\|\cdot\|_\alpha: \alpha \in (0,1)\}$. Then,

(1) $\|x\|_\alpha = sup\{t > 0: N(x,t) < \alpha\}$ for all $x \in X$ and $\alpha \in (0,1)$;

(2) P_N is increasing, that is, $\|x\|_\alpha$ is increasing with respect to $\alpha \in (0,1)$ for the given $x \in X$;

(3) P_N is a separating family of star quasi-seminorms.

P_N is called the family of star quasi-seminorms induced by $(N,*)$.

Remark 2.6. If $* = \wedge$, then P_N is just a family of quasi-norms.

Proposition 2.7. Let $(X, N,*)$ be a fuzzy quasi-normed space. The follows are equivalent:

(1) the topology τ_N is T_1,

(2) $N(x,t) = 1$ for all $t > 0 \Rightarrow x = \theta$,

(3) if $x \neq \theta$, then there exists an $\alpha \in (0,1)$ such that $\|x\|_\alpha > 0$.

Proof. (1)⇒(2): Suppose τ_N is T_1 and $N(x,t) = 1$ for all $t > 0$, meanwhile, $x \neq \theta$. Since τ_N is T_1, $\{x\}$ is a closed subset of X. So there exists $B_N(\theta, r_0, t_0)$, such that $B_N(\theta, r_0, t_0) \cap \{x\} = \phi$, which means that $x \notin B_N(\theta, r_0, t_0)$, that is, $N(x, t_0) \leq 1 - r_0 < 1$. Which is contradicted by $N(x,t) = 1$ for all $t > 0$.

(2)⇒(3): Suppose $x \neq \theta$. It follows from (2) that there exists $t_0 > 0$ such that $N(x, t_0) < 1$. Take $N(x, t_0) < \alpha < 1$, then $\alpha \in (0,1)$. From Proposition 2.5 (1), we know $\|x\|_\alpha \geq t_0 > 0$.

(3)⇒(1): Let $x, y \in X$ with $x \neq y$. It follows from (3) that there are $\alpha_1, \alpha_2 \in (0,1)$ such that $\|x - y\|_{\alpha_1} > 0$ and $\|y - x\|_{\alpha_2} > 0$. Taking $\|x - y\|_{\alpha_1} > s_1 > 0$ and $\|y - x\|_{\alpha_2} > s_2 > 0$. By using Proposition 2.5 (1), we know that there exist $t_1 > s_1$ and $t_2 > s_2$, such that $N(x - y, t_1) < \alpha_1$ and $N(y - x, t_2) < \alpha_2$. Therefore, $x \notin B_N(y, 1 - \alpha_1, t_1)$ and $y \notin B_N(x, 1 - \alpha_2, t_2)$. So τ_N is T_1. □

Remark 2.8. Let $(X, N,*)$ be a fuzzy quasi-normed space, and $P_N = \{\|\cdot\|_\alpha: \alpha \in (0,1)\}$ be the family of star quasi-seminorms induced by $(N,*)$. For each $x \in X$, set

$$U_{P_N}(x) = \{U(x; \alpha_1, \alpha_2, \dots, \alpha_n; \varepsilon): \varepsilon > 0; \alpha_1, \alpha_2, \dots, \alpha_n \in (0,1), n \in \mathbb{N}\},$$

where

$$\begin{aligned} U(x; \alpha_1, \alpha_2, \dots, \alpha_n; \varepsilon) &= \left\{y \in X: \|y - x\|_{\alpha_i} < \varepsilon, \alpha_i \in (0,1), i = 1,2,\dots,n\right\} \\ &= \cap_{i=1}^{n} \left\{y \in X: \|y - x\|_{\alpha_i} < \varepsilon, \alpha_i \in (0,1)\right\} \\ &= \left\{y \in X: \|y - x\|_{max\{\alpha_i: 1 \leq i \leq n\}} < \varepsilon\right\}. \end{aligned}$$

Then, from Proposition 5 and Theorem 1 in [18], we know that $U_{P_N}(x)$ is a basis of neighborhood of x w.r.t. τ_N. Therefore, for any sequence $\{x_n\}$ of X, $x_n \xrightarrow{\tau_N} x \in X$ if and only if $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$ for every $\alpha \in (0,1)$.

Definition 2.9. Suppose $(X, N,*)$ is a fuzzy quasi-normed space, P_N is the family of star quasi-seminorms induced by $(N,*)$, $\{x_n\}$ is a sequence of X.

(1) $\{x_n\}$ is called left N -Cauchy if for every $\varepsilon > 0$ and $\alpha \in (0,1)$, there exists$n_0 \in \mathbb{N}$ such that $\|x_m - x_n\|_\alpha < \varepsilon$, for any $m, n \in \mathbb{N}$ with $m > n > n_0$.

(2) $\{x_n\}$ is called right N -Cauchy if for every $\varepsilon > 0$ and $\alpha \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\|_\alpha < \varepsilon$, for any $m, n \in \mathbb{N}$ with $m > n > n_0$.

Definition 2.10. A fuzzy quasi-normed space $(X, N,*)$ is called left (right, resp.) $N - \tau_N$ complete if every left (right, resp.) N -Cauchy sequence is convergent w.r.t. τ_N.

Let (X, τ) be a topological space. A function $f: X \to \mathbb{R}$ is called to be upper semi-continuous (u.s.c.), if for any $a \in \mathbb{R}$, $\{x: f(x) < a\} \in \tau$; $f: X \to \mathbb{R}$ is called to be lower semi-continuous (l.s.c.), if for any $a \in \mathbb{R}$, $\{x: f(x) > a\} \in \tau$.

Remark 2.11. If the topology τ is the first countable, it is well known that f is u.s.c. (l.s.c., resp.) if and only if $\lim_{n\to\infty} sup\, f(x_n) \le f(x)$ ($\lim_{n\to\infty} inf\, f(x_n) \ge f(x)$, resp.) for any sequence $\{x_n\}$ which converges to $x \in X$.

Let$(Z, \le)$ be a partial order set. For$x \in Z$, put$S_+(x) = \{z \in Z: x \le z\}$ and$S_-(x) = \{z \in Z: z \le x\}$, we shall use the notation $x < y$ to designate the situation$x \le y$ and $x \ne y$.

Lemma 2.12. ([7]) Let $(Z, \le)$ be a partial order set.

(1) Suppose that $\varphi: Z \to \mathbb{R}$ is a function satisfying the conditions

(a) φ is strictly increasing, i.e.,$x < y \Rightarrow \varphi(x) < \varphi(y)$;

(b) $\varphi(S_-(x))$ is bounded below for each $x \in Z$;

(c) for any decreasing sequence $\{x_n\}$ in Z there exists $y \in Z$ such that $y \le x_n$, $n \in \mathbb{N}$.

Then, for each $x \in Z$ there exists a minimal element z in Z such that $z \le x$.

(2) Let $\varphi: Z \to \mathbb{R}$ be a function satisfying the conditions:

(a') φ is strictly increasing, i.e.,$x < y \Rightarrow \varphi(x) < \varphi(y)$;

(b') $\varphi(S_+(x))$ is bounded above for each $x \in Z$;

(c') for any increasing sequence $\{x_n\}$ in Z there exists $y \in Z$ such that $x_n \le y$, $n \in \mathbb{N}$.

Then, for each $x \in Z$ there exists a maximal element z in Z such that $x \le z$.

3. Ekeland's variational principle

In this section, $(X, N, *)$ is always supposed to be a fuzzy quasi-normed space such that the topology τ_N is T_1, and $P_N = \{\|\cdot\|_\alpha: \alpha \in (0,1)\}$ is the family of star quasi-seminorms induced by $(N, *)$.

Theorem 3.1. Suppose $(X, N, *)$ is right $N - \tau_N$ complete, the function $\varphi: X \to \mathbb{R}$ is bounded below and l.s.c. with respect to τ_N, $\varepsilon > 0$. Define a relation "$\le_\varphi$" on X by

$$x \le_\varphi y \Leftrightarrow \varphi(x) + \varepsilon\|y - x\|_\alpha \le \varphi(y),\ \forall \alpha \in (0,1). \tag{3-1}$$

Then,

(1) the relation "$\le_\varphi$" is a partial order;

(2) every element of X is minored by a minimal element z in X.

Proof. (1) **Reflexivity.** It is obvious.

Transitivity. For any $\alpha \in (0,1)$, we take $\beta \in (0,1)$ such that $\beta * \beta \ge \alpha$. If $x \le_\varphi y$ and $y \le_\varphi z$, then $\varphi(x) + \varepsilon\|y - x\|_\beta \le \varphi(y)$ and $\varphi(y) + \varepsilon\|z - y\|_\beta \le \varphi(z)$, hence

$$\varphi(x) + \varepsilon\|z - x\|_\alpha \le \varphi(x) + \varepsilon\|z - x\|_{\beta*\beta} \le \varphi(x) + \varepsilon\|y - x\|_\beta + \varepsilon\|z - y\|_\beta \le \varphi(z).$$

Therefore $x \le_\varphi z$.

Anti-symmetry. Suppose $x \le_\varphi y$ and $y \le_\varphi x$. Then, for any $\alpha \in (0,1)$,

$$\varphi(x) + \varepsilon\|y - x\|_\alpha \le \varphi(y), \varphi(y) + \varepsilon\|x - y\|_\alpha \le \varphi(x).$$

So

$$\varphi(x) + \varepsilon\|y - x\|_\alpha + \varepsilon\|x - y\|_\alpha \le \varphi(x).$$

Thus $\|y - x\|_\alpha = \|y - x\|_\alpha = 0$. Since P_N is separating, we get $x = y$.

(2) It is sufficient to verify that "$\le_\varphi$" satisfies the conditions (a)–(c) in Lemma 2.12. Let $x <_\varphi y$,

then

$$\varphi(x)+\varepsilon\|y-x\|_\alpha \le \varphi(y),\ \forall \alpha\in(0,1).$$

Therefore $\varphi(y)\ge\varphi(x)$. Since $x\neq y$, it follows from Proposition 2.7 that there exists $\alpha_0\in(0,1)$ such that $\|y-x\|_{\alpha_0}>0$. So $\varphi(y)\neq\varphi(x)$, i.e., $\varphi(y)>\varphi(x)$. Thus, φ is strictly increasing. The condition (a) holds.

Since φ is bounded below, (b) holds too.

To prove (c), we suppose that $\{y_n\}$ is a decreasing sequence in $(X,\le_\varphi)$. Then for any $n\in\mathbb{N}$, we have $y_m\le_\varphi y_n$ with $m>n$. By (3-1)

$$\varphi(y_m)+\varepsilon\|y_n-y_m\|_\alpha\le\varphi(y_n),\ \forall\alpha\in(0,1). \tag{3-2}$$

Firstly, we prove that $\{y_n\}$ is right N -Cauchy. Since $\{y_n\}$ is decreasing and φ is strictly increasing, $\{\varphi(y_n)\}$ is decreasing too. So $\{\varphi(y_n)\}$ is convergent. For any $\eta>0$, there exists $n_0\in\mathbb{N}$ such that

$$0\le\varphi(y_n)-\varphi(y_m)<\varepsilon\eta \text{ with } m>n>n_0.$$

Which together with (3-2) implies that

$$\|y_n-y_m\|_\alpha<\eta \text{ with } m>n>n_0,\ \forall\alpha\in(0,1).$$

Consequently, $\{y_n\}$ is right N -Cauchy.

Secondly, we prove that there exists $y\in X$ such that $y\le_\varphi y_n$ for all $n\in\mathbb{N}$. Recall that X is right $N-\tau_N$ complete, we know that $\{y_n\}$ converges to a point $y\in X$ w.r.t. τ_N. Hence, for any $\alpha\in(0,1)$, we have

$$\lim_{m\to\infty}\|y_m-y\|_\alpha=0. \tag{3-3}$$

Now, we take $\beta\in(0,1)$ such that $\beta*\beta\ge\alpha$. By (3-2), for every $n\in\mathbb{N}$, we have,

$$\begin{aligned}\varepsilon\|y_n-y\|_\alpha&\le\varepsilon\|y_n-y\|_{\beta*\beta}\le\varepsilon\left(\|y_n-y_m\|_\beta+\|y_m-y\|_\beta\right)\\&\le\varphi(y_n)-\varphi(y_m)+\varepsilon\|y_m-y\|_\beta \text{ with } m>n.\end{aligned} \tag{3-4}$$

Since φ is l.s.c. with respect to τ_N and $\{\varphi(y_n)\}$ is decreasing, we have $\varphi(y)\le\lim_{m\to\infty}\varphi(y_m)$. Let $m\to\infty$in (3-4), by using (3-3), we obtain

$$\varepsilon\|y_n-y\|_\alpha\le\varphi(y_n)-\varphi(y),\ \forall\alpha\in(0,1).$$

Thus, $y\le_\varphi y_n$ for all $n\in\mathbb{N}$. The condition (c) is verified. □

Theorem 3.2. (EVP) Suppose $(X,N,*)$ is right $N-\tau_N$ complete, the function $\varphi: X\to\mathbb{R}$ is bounded below and l.s.c. with respect τ_N, $\varepsilon>0$. Let $x_0\in X$ be such that

$$\varphi(x_0)\le\inf\varphi(X)+\varepsilon. \tag{3-5}$$

Then, there exists $z\in X$ such that

(i) $\varphi(z)+\varepsilon\|x_0-z\|_\alpha\le\varphi(x_0),\ \forall\alpha\in(0,1)$;

(ii) $\|x_0-z\|_\alpha\le 1,\ \forall\alpha\in(0,1)$;

(iii) for any $x\in X\backslash\{z\}$, there exists $\alpha_0\in(0,1)$ such that $\varphi(x)+\varepsilon\|z-x\|_{\alpha_0}>\varphi(z)$.

Proof. Define a relation " $\le_\varphi$ " on X by (3-1). From Theorem 3.1, there exists $z\in X$ such that $z\le_\varphi x_0$ and z is a minimal element of X, which means that (i) and (iii) hold. Moreover, by using

(3-5) and (i), we have

$$\varphi(z)+\varepsilon\|x_0-z\|_\alpha \le inf\,\varphi\,(X)+\varepsilon \le \varphi(z)+\varepsilon,\ \forall\alpha\in(0,1).$$

Which implies that (ii) holds. □

The following corollary is called the weak Ekeland's variational principle (wEVP).

Corollary 3.3. (wEVP) Suppose $(X,N,*)$ is right $N-\tau_N$ complete, the function $\varphi: X\to\mathbb{R}$ is bounded below and l.s.c. with respect to τ_N. Then, for any $\varepsilon>0$, there exists $z\in X$ such that: for any $x\in X\backslash\{z\}$, there exists $\alpha_0\in(0,1)$ such that $\varphi(x)+\varepsilon\|z-x\|_{\alpha_0}>\varphi(z)$.

Proof. For the given $\varepsilon>0$, there exists $x_0\in X$ such that $\varphi(x_0)<inf\,\varphi\,(X)+\varepsilon$. From Theorem 3.2, there exists $z\in X$ such that z satisfies the conclusions (i)–(iii) in Theorem 3.2. It can be seen that z is what is wanted. □

4. Caristi's fixed point theorem and Takahashi minimization theorem

In this section, we study CFPT and TMT in fuzzy quasi-normed spaces. Moreover, we prove the equivalence chain among wEVP, CFPT and TMT.

Theorem 4.1. (CFPT) Suppose $(X,N,*)$ is right N -τ_N complete, $\varphi: X\to\mathbb{R}$ is proper bounded below and l.s.c with respect to τ_N, $\varepsilon>0$. If $f: X\to X$ satisfies the condition: for any $x\in X$,

$$\varphi\big(f(x)\big)+\varepsilon\|x-f(x)\|_\alpha\le\varphi(x),\ \forall\alpha\in(0,1). \tag{4-1}$$

Then, f has a fixed point in X.

Proof. From wEVP, there exists $z\in X$ such that: for any $x\in X\backslash\{z\}$, there exists $\alpha_0\in(0,1)$ such that $\varphi(x)+\varepsilon\|z-x\|_{\alpha_0}>\varphi(z)$. If $f(z)\ne z$, then there exists $\alpha_1\in(0,1)$ such that $\varphi(f(z))+\varepsilon\|z-f(z)\|_{\alpha_1}>\varphi(z)$, which contradicts (4-1). Thus, $f(z)=z$, that is, f has a fixed point z. □

From the above proof, we see that CFPT is implied by wEVP. In fact, we have the following result.

Theorem 4.2. CFPT and wEVP are mutually equivalent.

Proof. It is sufficient to show that wEVP can be implied by CFPT. Suppose that wEVP does not hold. Then, there exists $\varepsilon>0$, for any $x\in X$, there is $y_x\in X\backslash\{x\}$ such that $\varphi(y_x)+\varepsilon\|x-y_x\|_\alpha\le\varphi(x)$, $\forall\alpha\in(0,1)$. Define $f: X\to X$ as $f(x)=y_x$, $\forall x\in X$. Then

$$\varphi\big(f(x)\big)+\varepsilon\|x-f(x)\|_\alpha\le\varphi(x),\ \forall\alpha\in(0,1).$$

From CFPT, there exists $z\in X$ such that $f(z)=z$. Which contradicts the fact that $f(x)=y_x\in X\backslash\{x\}$, $\forall x\in X$. Thus, wEVP holds. □

From wEVP, we can get Takahashi minimization theorem as following.

Theorem 4.3. (TMT) Suppose $(X,N,*)$ is right $N-\tau_N$ complete, the function $\varphi: X\to\mathbb{R}$ is bounded below and l.s.c. with respect to τ_N, $\varepsilon>0$. If the following condition **(A)** holds:

(A) for any $z\in X$ with $\varphi(z)>\inf\limits_{x\in X}\varphi(x)$, there is $x\in X\backslash\{z\}$ such that $\varphi(x)+\varepsilon\|z-x\|_\alpha\le\varphi(z)$ for all $\alpha\in(0,1)$,

then, there exists $u\in X$ such that $\varphi(u)=\inf\limits_{x\in X}\varphi(x)$.

Proof. It follows from wEVP, there exists $z\in X$ such that: for any $x\in X\backslash\{z\}$, there exists $\alpha_0\in(0,1)$ such that $\varphi(x)+\varepsilon\|z-x\|_{\alpha_0}>\varphi(z)$. It is easy to see that $\varphi(z)=\inf\limits_{x\in X}\varphi(x)$. In fact, if

$\varphi(z) \neq \inf_{x\in X} \varphi(x)$, then $\varphi(z) > \inf_{x\in X} \varphi(x)$, which is impossible by the condition (A). □

Moreover, we have the following theorem.

Theorem 4.4. TMT and wEVP are mutually equivalent.

Proof. From the proof of Theorem 4.3, it is sufficient to show that wEVP can be implied by TMT. To use the method of proof by contradiction, we assumed that wEVP does not hold, that is, there exists $\varepsilon > 0$, for any $z \in X$, there exists $x \in X\backslash\{z\}$ such that

$$\varphi(x) + \varepsilon\|z - x\|_\alpha \leq \varphi(z) \text{ for all } \alpha \in (0,1). \tag{4-2}$$

Therefore, the condition (A) is satisfied. Since τ_N is T_1, it follows from (4-2) that $\varphi(x) < \varphi(z)$. So, $\varphi(z) > \inf_{x\in X} \varphi(x)$. By the arbitrariness of $z \in X$, we know φ cannot get its minimum on X. This contradicts TMT. □

5. Conclusions

The Ekeland's variational principle, Caristi's fixed point theorem and Takahashi minimization theorem are extended to fuzzy quasi-normed spaces. The obtained results enrich and develop asymmetric analysis and provide important tools for studying asymmetry in mathematics and other fields. On the other hand, the main results in the paper are obtained under the condition that the topology τ_N is T_1, Therefore, the following question is also worthy of further study: Is it possible to give types of these theorems in the framework of the general fuzzy quasi-normed spaces?

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Conflict of interest

The authors declare that there is no conflict of interest in this paper.

References

1. C. Alegre, S. Romaguera, Characterizations of metrizable topological vector spaces and their asymmetric generalizations in terms of fuzzy (quasi-)norms, *Fuzzy Sets Syst.*, **161**(2010), 2181–2192. https://doi.org/10.1016/j.fss.2010.04.002
2. S. Al-Homidan, Q. H. Ansari, G. Kassay, Takahashi's minimization theorem and some related results in quasi-metric spaces, *J. Fixed Point Theory Appl.*, **21** (2019), 38. https://doi.org/10.1007/s11784-019-0676-0
3. T. Q. Bao, S. Cobzaş, A. Soubeyran, Variational principles, completeness and the existence of traps in behavioral sciences, *Ann. Oper. Res.*, **269** (2018), 53–79. https://doi.org/10.1007/s10479-016-2368-0
4. T. Q. Bao, B. S. Mordukhovich, A. Soubeyran, Variational analysis in psychological modeling, *J. Optim. Theory Appl.*, **164** (2015), 390–315. https://doi.org/10.1007/s10957-014-0569-8

5. T. Q. Bao, B. S. Mordukhovich, A. Soubeyran, Fixed points and variational principles with applications to capability theory of wellbeing via variational rationality, *Set-Valued Var. Anal.*, **23** (2015), 375–398. https://doi.org/10.1007/s11228-014-0313-4
6. T. Q. Bao, A. Soubeyran, Variational analysis in cone pseudo-quasimetric spaces and applications to group dynamics, *J. Optim. Theory Appl.*, **170** (2016), 458–475. https://doi.org/10.1007/s10957-016-0933-y
7. E. M. Bednarczuk, D. Zagrodny, A smooth vector variational principle, *SIAM J. Control Optim.*, **48** (2010), 3735–3745. https://doi.org/10.1137/090758271
8. H. Brézis, F. E. Browder, A general principle on ordered sets in nonlinear functional analysis, *Adv. Math.*, **21** (1976), 355–364. https://doi.org/10.1016/S0001-8708(76)80004-7
9. L. C. Ceng, G. Mastroeni, J. C. Yao, Existence of solutions and variational principles for generalized vector systems, *J. Optim. Theory Appl.*, **137** (2008), 485–495. https://doi.org/10.1007/s10957-007-9348-0
10. S. S. Chang, Q. Luo, Caristi's fixed point theorem for fuzzy mappings and Ekeland's variational principle, *Fuzzy Sets Syst.*, **64** (1994), 119–125. https://doi.org/10.1016/0165-0114(94)90014-0
11. S. Cobzas, *Functional analysis in asymmetric normed spaces*, Birkhäuser: Springer, 2013. https://doi.org/10.1007/978-3-0348-0478-3
12. S. Cobzas, Completeness in quasi-metric spaces and Ekeland variational principle, *Topol. Appl.*, **158** (2011), 1073–1084. https://doi.org/10.1016/j.topol.2011.03.003
13. S. Cobzaş, Ekeland variational principle in asymmetric locally convex spaces, *Topol. Appl.*, **159** (2012), 2558–2569. https://doi.org/10.1016/j.topol.2012.04.015
14. S. Cobzaş, Ekeland, Takahashi and Caristi principles in quasi-pseudo metric spaces, *Topol. Appl.*, **265** (2019), 106831. https://doi.org/10.1016/j.topol.2019.106831
15. I. Ekeland, Sur les problemes variationnels, *C. R. Acad. Sci. Paris*, **275** (1972), 1057–1059.
16. I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, **47** (1974), 324–353. https://doi.org/10.1016/0022-247X(74)90025-0
17. I. Ekeland, Nonconvex minimization problems, *Bull. Amer. Math. Soc.*, **1** (1979), 443–474. https://doi.org/10.1090/S0273-0979-1979-14595-6
18. R. Gao, X. X. Li, J. R. Wu, The decomposition theorem for a fuzzy quasi-norm, *J. Math.*, **2020** (2020), 8845283. https://doi.org/10.1155/2020/8845283
19. L. M. García-Raffi, S. Romaguera, E. A. Sánchez-Pérez, Weak topologies on asymmetric normed linear spaces and non-asymptotic criteria in the theory of complexity analysis of algorithms, *J. Anal. Appl.*, **2** (2004), 125–138.
20. A. George, P. Veeramani, On some results in fuzzy metric space, *Fuzz Sets Syst.*, **64** (1994), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7
21. C. Gutiérrez, B. Jiménez, V. Novo, L. Thibault, Strict approximate solutions in set-valued optimization with applications to the approximate Ekeland variational principle, *Nonlinear Anal.-Theor.*, **73** (2010), 3842–3855. https://doi.org/10.1016/j.na.2010.08.012
22. E. Karapinar, S. Romaguera, On the weak form of Ekeland's variational principle in quasi-metric spaces, *Topol. Appl.*, **184** (2015), 54–60. https://doi.org/10.1016/j.topol.2015.01.011
23. B. S. Mordukhovich, *Variational analysis and generalized differentiation, I: Basic Theory, II: Applications*, Berlin: Springer, 2006.
24. R Phelps, *Convex functions, monotone operators and differentiability*, Springer-Verlag, 1993. https://doi.org/10.1007/978-3-540-46077-0

25. J. H. Qiu, Ekeland's variational principle in locally complete spaces, *Math. Nachr.*, **257** (2003), 55–58. https://doi.org/10.1002/mana.200310077
26. J. H. Qiu, Local completeness, drop theorem and Ekeland's variational principle, *J. Math. Anal. Appl.*, **311** (2005), 23–39. https://doi.org/10.1016/j.jmaa.2004.12.045
27. J. H. Qiu, Ekeland's variational principle in locally convex spaces and the density of extremal points, *J. Math. Anal. Appl.*, **360** (2009), 317–327. https://doi.org/10.1016/j.jmaa.2009.06.054
28. J. Robinson, A survey of Ekeland's variational principle and related theorems and applications, UNLV Theses, 2014. http://doi.org/10.34917/7048610
29. S. Romaguera, M. Schellekens, Quasi-metric properties of complexity spaces, *Topol. Appl.*, **98** (1999), 311–322. https://doi.org/10.1016/S0166-8641(98)00102-3
30. S. Romaguera, M. Schellekens, Duality and quasi-normability for complexity spaces, *Appl. Gen. Topol.*, **3** (2002), 91–112. https://doi.org/10.4995/agt.2002.2116
31. B. Schweizer, A. Sklar, Statistical metric spaces, *Pac. J. Math.*, **10** (1960), 314–334. https://doi.org/10.2140/pjm.1960.10.313
32. J. R. Wu, X. Tang, Caristi's fixed point theorem, Ekeland's variational principle and Takahashi's maximization theorem in fuzzy quasi-metric spaces, *Topol.*, *Appl.*, **302** (2021), 107801. https://doi.org/10.1016/j.topol.2021.107801
33. A. Zaslavski, Symmetric optimization problems, In: *Turnpike phenomenon and symmetric optimization problems*, Springer, 2022. https://doi.org/10.1007/978-3-030-96973-8_2
34. S. D. Zeng, S. Migórski, A. A. Khan, Nonlinear quasi-hemivariational inequalities: existence and optimal control, *SIAM J. Control Optim.*, **59** (2021), 1246–1274. https://doi.org/10.1137/19M1282210

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