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Research article

Fractional modeling of COVID-19 pandemic model with real data from Pakistan under the ABC operator

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Abstract: In this study, the COVID-19 epidemic model is established by incorporating quarantine and isolation compartments with Mittag-Leffler kernel. The existence and uniqueness of the solutions for the proposed fractional model are obtained. The basic reproduction number, equilibrium points, and stability analysis of the COVID-19 model are derived. Sensitivity analysis is carried out to elaborate the influential parameters upon basic reproduction number. It is obtained that the disease transmission parameter is the most dominant parameter upon basic reproduction number. A convergent iterative scheme is taken into account to simulate the dynamical behavior of the system. We estimate the values of variables with the help of the least square curve fitting tool for the COVID-19 cases in Pakistan from 04 March to May 10, 2020, by using MATLAB.

Keywords: real data; pandemic model; stability analysis; sensitivity analysis; numerical simulations
Mathematics Subject Classification: 37G05, 37G15, 39A28, 92B05

1. Introduction

COVID-19 which is an infectious respiratory illness spreads in China in 2019 and affects the whole world in the shape of human and economic loss. It is known that 11 to 14 days are its latency period while old age people and comorbid patient are their favorite prey. Similar to the symptoms of common flu and cold it is advisable to do the test at a first glance. It is also reported that people have no symptoms even though they are infected. The basic reproduction number for COVID-19 stays between 2.2 and 3.58, which shows that the disease transmission rate is very high. Hence, it disperse throughout the world in the blink of an eye and disturbed 213 nations. Keeping this in mind, the World Health Organization (WHO) officially announced that COVID-19 is one of the global pandemics. At first, there were no proper vaccine/antiviral strategies to control the COVID-19 disease hence most the countries have adopted non-pharmaceutical interventions (NPIs), which are quarantining of exposed individuals, travel restrictions, closure of educational institutes, avoiding mass gatherings, and isolation of infected individuals.

Daily a lot of people die due to Infectious diseases and hence it is considered to be the 2nd biggest reason for death throughout the world. In Mathematics,modeling is considered to be the best tool to elaborate on the dynamic phenomena of infectious diseases. In this regard, researchers and mathematician try their best to explore infectious diseases by using mathematical modeling. The stability analysis of an SVEIR model with continuous age structure in the exposed and infection classes is discussed in [1]. The dynamics of covid-19 via the stochastic epidemic model are explained in [2]. The dynamics of dengue infection through fractal-fractional operator with real statistical data is shown in [3]. The stability analysis and optimal vaccination of a SIR epidemic model are proven in [4]. The global stability of the COVID-19 model involving the quarantine strategy and media coverage effects are depicted in [5]. Fractional COVID-19 transmission is formulated by using the Caputo fractional derivative in [6].

Fractional derivatives received much applause during the last few years as their involvement in science and engineering is much more than expected. Fractional derivatives split into Local and nonlocal theories in which the nonlocal definitions received much more attraction due to their interesting applications [7–14]. According to singularity, fractional derivatives have two kinds i.e singular and nonsingular kernel. The Caputo, the Grnwald, and the RiemannLiouville belong to the first kind while the AtanganaBaleanu and the CaputoFabrizio belong to the second kind. The Atangana-Baleanu which is based upon the Mittag-Leffler function answered the queries raised by the Caputo-Fabrizio definition. The classical queries of complex analysis have been easily addressed by the Mittag-Leffler function as can be seen from the power-law series analytic continuation which is convergent outside the disc. A dynamicalprocess based on fractional-order derivatives carries its past and present state information [15–19]. Thus, in epidemiological compartmental models, utilizing a fractional-order system is more reasonable to work the transmission dynamics of infectious diseases.

Section 2 is devoted to some basic definitions regarding fractal fractional operators. In Section 3 the model is reformulated with the help of the Atangana-Baleanu-Caputo fractional derivative. The real data comparison is shown in Section 4. Section 5 deals with the equilibrium points and basic reproduction number and local stability. Existence and uniqueness are carried out in Section 6. Section 7 deals with the Ulam-Hyers stability of the proposed model. The sensitivity of different parameters corresponding to basic reproduction is discussed in Section 8. Some plots are given to conclude the simplicity and reliability of the algorithm in Section 9. Section 10 deals with the conclusion of our work.

2. Preliminaries

The study presents basic definitions to set the tone for the mathematical derivation of the model.

Definition 2.1. *The Riemann-Liouville integral as [20]:*

$$J_{0,\vartheta}^{\alpha}[f(\vartheta)] = \frac{1}{\Gamma(\alpha)}\int_0^{\vartheta}(\vartheta-\delta)^{\alpha-1}f(\delta)d\delta, \quad \vartheta > 0. \tag{2.1}$$

Definition 2.2. *The definition of Caputo derivative as [21]:*

$${}^{c}D_{0,\vartheta}^{\alpha}[f(\vartheta)] = \frac{1}{\Gamma(n-\alpha)}\int_0^{\vartheta}(\vartheta-\delta)^{n-\alpha-1}f^{(n)}(\delta)d\delta, \quad \vartheta > 0, \tag{2.2}$$

where $n-1<\alpha\le n, n\in\mathbb{N}$. Note that if $\alpha\to 1$ then ${}^{C}D_{0,\vartheta}^{\alpha}f(\vartheta)$ approaches to $f'(\vartheta)$.

Definition 2.3. *The Caputo-Fabrizio operator of order $\zeta > 0$ is defined as follows [22]:*

$${}^{CF}D_{0,\vartheta}^{\zeta}[f(\vartheta)] = \frac{M(\zeta)}{1-\zeta}\int_0^{\vartheta}f'(\delta)\exp\left[-\zeta\frac{\vartheta-s}{1-\zeta}\right]d\delta, \quad \vartheta > 0. \tag{2.3}$$

Definition 2.4. *Suppose that $f(\vartheta)\in\mathbb{H}^1(0,T), T>0$, and $\lambda\in]0,1[$; then, the Atangana-Baleanu operator of order λ in the Caputo sense can be represented by [23]:*

$${}^{\mathrm{ABC}}D_{0,\vartheta}^{\lambda}[f(\vartheta)] = \frac{\mathfrak{B}(\lambda)}{1-\lambda}\int_0^{\vartheta}f'(\delta)\mathbf{E}_{\lambda}\left[-\frac{\lambda}{1-\lambda}(\vartheta-\delta)^{\lambda}\right]d\delta. \tag{2.4}$$

Definition 2.5. *The integral operator under the Atangana-Baleanu-Caputo sense is defined by the following expression [24]:*

$${}^{\mathrm{ABC}}I_{0,\vartheta}^{\lambda}[f(\vartheta)] = \frac{1-\lambda}{\mathfrak{B}(\lambda)}f(\vartheta) + \frac{\lambda}{\mathfrak{B}(\lambda)\Gamma(\lambda)}\int_0^{\vartheta}f(\delta)(\vartheta-\delta)^{\lambda-1}d\delta. \tag{2.5}$$

3. The model with the classical derivative

In this section, we consider the highly nonlinear epidemiological model for the transmission of the newly raised COVID-19 infection. In this model the total population is categorized into six time dependent classes namely: Recovered R; Hospitalized infected J, infected I; Quarantine Q, Exposed E, and Susceptible S. The model is expressed as [25]:

$$\begin{cases} \dfrac{dS}{d\varphi} = \Gamma - \beta(1-\kappa)W(S,E) - \mu S + r_1 Q, \\ \dfrac{dE}{d\varphi} = \beta(1-\kappa)W(S,E) - (\gamma_1+\gamma_2+\mu)E, \\ \dfrac{dQ}{d\varphi} = \gamma_1 E - (q_1+r_1+\mu)Q, \\ \dfrac{dI}{d\varphi} = \gamma_2 E + q_1 Q - (\alpha+r_2+d_1+\mu)I, \\ \dfrac{dJ}{d\varphi} = \alpha I - (r_3+d_2+\mu)J, \\ \dfrac{dR}{d\varphi} = r_2 I + r_3 J - \mu R. \end{cases} \tag{3.1}$$

with

$$S, E, Q, I, J, R \geq 0.$$

The description of the parameters are as stated in Table 1.

Table 1. Descriptions and values of the parameters.

Symbols	Descriptions	Values	References
β^λ	Coefficient of transmission	0.10700	[25]
α^λ	Rate at which infected individuals are being isolated	0.08910	Fitted
r_1^λ	Rate at which quarantined become again susceptible	0.04093	Fitted
r_2^λ	Recovery rate of infected individuals	0.00734	Fitted
r_3^λ	Recovery rate of isolated individuals	0.00974	Fitted
d_1^λ	Disease-induced mortality rate of infected individuals	0.00020	[25]
d_2^λ	Disease-induced mortality rate of isolated individuals	0.00326	Fitted
q_1^λ	Rate at which quarantined individuals become infected	0.00427	Fitted
γ_1^λ	Rate at which exposed are reduced due to being quarantined	0.06718	Fitted
γ_2^λ	Rate at which exposed are reduced due to being infected	0.08176	Fitted
κ^λ	Parameter related to lockdown	0.00026	Fitted

In addition, $W(S, E)$ is a feature known as the uptake function. The incidence of susceptible with exposed class is related to this feature. In this article, we provide the susceptible and exposed interaction, as in [26], which has the form:

$$W(S, E) := \frac{2SE}{S + E}. \tag{3.2}$$

In addition, the function described in (3.2) displaysthe susceptible and exposed harmonic mean. For several studies, the $W(S, E)$ uptake function is considered to be the product of susceptible and exposed, known as the [25] bi-linear incidence rate. The dynamics of the COVID-19 model were addressed by the writers of [25] when considering the uptake feature to be the product of S and E. Apart from these, it is very interesting to consider $W(S, E) := \frac{2SE}{S+E}$ as the uptake function. Since it is understood that if there aretwo quantities S and E in the sense that $S, E \geq 0$, then the following relation is true:

$$\frac{2SE}{S + E} \leq \sqrt{SE} \leq \frac{S + E}{2}.$$

In fact, the average of two values is a measure of the centrality of a data set. In addition, the geometric mean is mainly used to reach average data change ratios or rates. As far as harmonic means are concerned, as opposed to arithmetic or geometric means, it is less susceptible to a few broad values. It is used for highly skewed variables often [27]. The readers are referred to [28] for the biological interpretation to the harmonic mean.

Remark 3.1. *The system (3.1) satisfies*

$$\frac{dN}{d\varphi} = \Gamma - dN, \tag{3.3}$$

where $N(\varphi) = S(\varphi) + E(\varphi) + Q(\varphi) + I(\varphi) + J(\varphi) + R(\varphi)$ at any time with t. Eq. (3.3) has the exact solution as:

$$N(\varphi) = \frac{A}{d} + \left(N_0 - \frac{A}{d}\right)e^{-d\varphi},$$

with

$$S(0) \geq 0, \quad E(0) \geq 0, \quad Q(0) \geq 0, \quad I(0) \geq 0, \quad J(0) \geq 0, \quad R(0) \geq 0.$$

We have

$$N(0) = S(0) + E(0) + Q(0) + I(0) + R(0).$$

Therefore, we get

$$S(\varphi) \geq 0, \quad E(\varphi) \geq 0, \quad Q(\varphi) \geq 0, \quad I(\varphi) \geq 0,, \quad J(\varphi) \geq 0, \quad R(\varphi) \geq 0.$$

Remark 3.2. *It should be noticed that $V(S) := \frac{S}{S+E}$ where $V : [0,\infty) \to [0,\infty)$ is referred as the uptake function and satisfies some of the features defined as [27]:*
1) $V(0) = 0, \quad V(S) > 0, \quad$ for $S > 0$,
2) $\lim_{P\to\infty} \frac{S(\varphi)}{S(\varphi)+E(\varphi)} = L_1, \quad 0 < L_1 < \infty$; where $L_1 = 1$,
3) $V(S)$ is continuously differentiable,
4) $\frac{dV}{dt} = \frac{E}{(S+E)^2} > 0$.

3.1. Model with the Atangana-Baleanu derivative

In this subsection, We reformulate the fractional variant of the model (3.1), as:

$$\begin{cases} {}^{\mathrm{ABC}}\mathbb{D}^{\lambda}_{0,\varphi}[S(\varphi)] = \Gamma^{\lambda} - \dfrac{2\beta^{\lambda}(1-\kappa)SE}{S+E} - \mu^{\lambda}S + r_1^{\lambda}Q, \\ {}^{\mathrm{ABC}}\mathbb{D}^{\lambda}_{0,\varphi}[E(\varphi)] = \dfrac{2\beta^{\lambda}(1-\kappa)SE}{S+E} - (\gamma_1^{\lambda} + \gamma_2^{\lambda} + \mu^{\lambda})E, \\ {}^{\mathrm{ABC}}\mathbb{D}^{\lambda}_{0,\varphi}[Q(\varphi)] = \gamma_1^{\lambda}E - (q_1^{\lambda} + r_1^{\lambda} + \mu^{\lambda})Q, \\ {}^{\mathrm{ABC}}\mathbb{D}^{\lambda}_{0,\varphi}[I(\varphi)] = \gamma_2^{\lambda}E + q_1^{\lambda}Q - (\alpha^{\lambda} + r_2^{\lambda} + d_1^{\lambda} + \mu^{\lambda})I, \\ {}^{\mathrm{ABC}}\mathbb{D}^{\lambda}_{0,\varphi}[J(\varphi)] = \alpha^{\lambda}I - (r_3^{\lambda} + d_2^{\lambda} + \mu^{\lambda})J, \\ {}^{\mathrm{ABC}}\mathbb{D}^{\lambda}_{0,\varphi}[R(\varphi)] = r_2^{\lambda}I + r_3^{\lambda}J - \mu^{\lambda}R. \end{cases} \tag{3.4}$$

with

$$S(\varphi), \quad E(\varphi), \quad Q(\varphi), \quad I(\varphi), \quad J(\varphi), \quad R(\varphi) \geq 0.$$

4. Real data comparison

The key and significant step for the problem validity is the comparison of real statistics which is helpful for achieving numerical values for the model validation. We consider here to obtain the nonlinear least square fitting method to fit the data to proposed model (3.4), we have thirteen parameters in which nine will be fitted, while the rest of the parameters must be assumed. Estimated parameters value given in Table 1. The parameters assumption techniques under the least-square fit via MATLAB software were used. We use the data of infected cases of COVID-19 in Pakistan from 04 March to May 10, 2020, the result is given in Figure 1. The associated mean relative error of the fitted data are obtained by formula $\frac{1}{6}\sum_{k=1}^{6}\left|\frac{x_k^{real}-x_k^{approximate}}{x_k^{real}}\right|$.

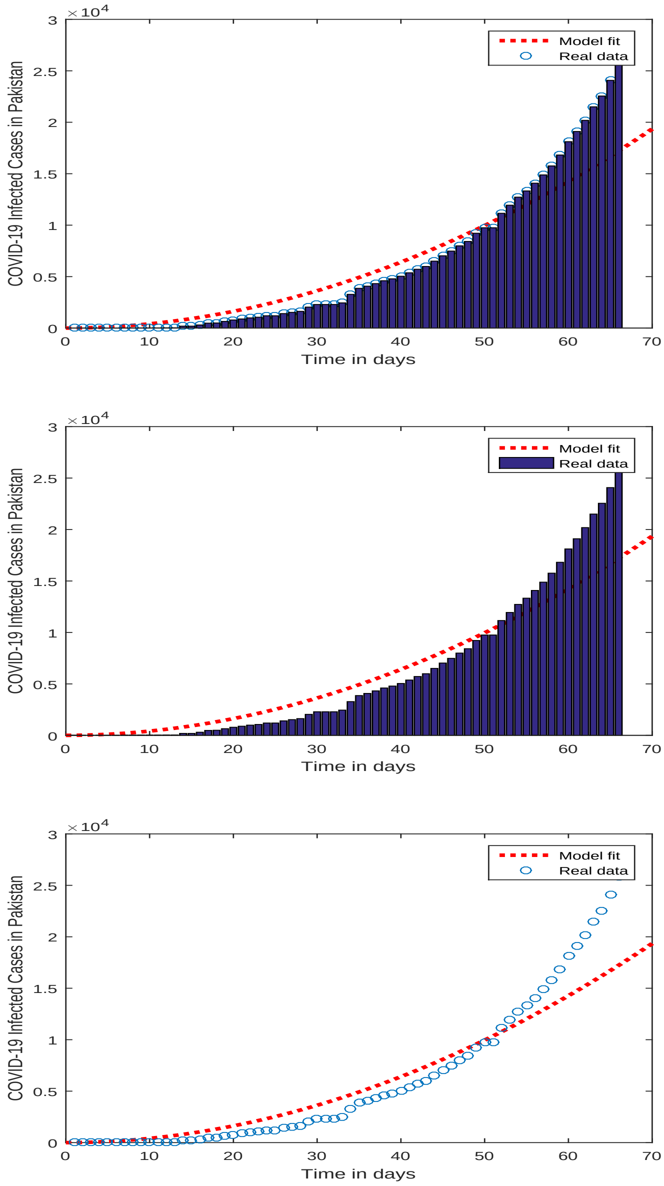

Figure 1. Conformed COVID-19 cumulative cases time series in Pakistan [29].

5. Equilibrium points and Basic reproductive number

5.1. Steady States

The disease free equilibrium (DFE) of the model system (3.4) can be computed by putting ${}^{ABC}\mathbb{D}^{\lambda}_{0,\varphi}[E(\varphi)] = 0, {}^{ABC}\mathbb{D}^{\lambda}_{0,\varphi}[Q(\varphi)] = 0, {}^{ABC}\mathbb{D}^{\lambda}_{0,\varphi}[I(\varphi)] = 0, {}^{ABC}\mathbb{D}^{\lambda}_{0,\varphi}[J(\varphi)] = 0$ and denoted by $E^0 = \left(S^0, 0, 0, 0, 0, R^0\right)$, where $S^0 = \frac{\Gamma^{\lambda}}{u^{\lambda}}$ and $R^0 = 0$.

5.2. Basic Reproductive Number R_0

By using the next generation method [28], we have

$$F = \begin{pmatrix} 2\beta^{\lambda}(1-\kappa^{\lambda})\mu^{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.1}$$

where the elements in matrix F constitute the new infection terms, and

$$V = \begin{pmatrix} \gamma_1^{\lambda}+\gamma_2^{\lambda}+\mu^{\lambda} & 0 & 0 & 0 \\ -\gamma_1^{\lambda} & q_1^{\lambda}+r_1^{\lambda}+\mu^{\lambda} & 0 & 0 \\ -\gamma_2^{\lambda} & -q_2^{\lambda} & d_1^{\lambda}+r_2^{\lambda}+\alpha^{\lambda}+\mu^{\lambda} & 0 \\ 0 & 0 & -\alpha^{\lambda} & d_2^{\lambda}+r_3^{\lambda}+\mu^{\lambda} \end{pmatrix}. \tag{5.2}$$

Also,

$$V^{-1} = \begin{pmatrix} \frac{1}{\gamma_1^{\lambda}+\gamma_2^{\lambda}+\mu^{\lambda}} & 0 & 0 & 0 \\ \frac{\gamma_1^{\lambda}}{(\gamma_1^{\lambda}+\gamma_2^{\lambda}+\mu^{\lambda})(q_1^{\lambda}+r_1^{\lambda}+\mu^{\lambda})} & \frac{1}{q_1^{\lambda}+r_1^{\lambda}+\mu^{\lambda}} & 0 & 0 \\ -\gamma_2^{\lambda} & -q_2^{\lambda} & \frac{1}{d_1^{\lambda}+r_2^{\lambda}+\alpha^{\lambda}+\mu^{\lambda}} & 0 \\ 0 & 0 & \frac{\alpha^{\lambda}}{(d_1^{\lambda}+r_2^{\lambda}+\alpha^{\lambda}+\mu^{\lambda})(d_2^{\lambda}+r_3^{\lambda}+\mu^{\lambda})} & \frac{1}{d_2^{\lambda}+r_3^{\lambda}+\mu^{\lambda}} \end{pmatrix}. \tag{5.3}$$

Now, $\mathcal{R}_0$ is dominant eigenvalue of $\rho(FV^{-1})$, and is given by;

$$\mathcal{R}_0 = \frac{2\beta^{\lambda}(1-\kappa^{\lambda})}{\left(\gamma_1^{\lambda}+\gamma_2^{\lambda}+\mu^{\lambda}\right)}. \tag{5.4}$$

5.2.1. Existence of endemic equilibrium

The existence of endemic equilibrium (EE) of (3.4) is established in terms of R_0 herein. $B_1 = \gamma_1^{\lambda}+\gamma_2^{\lambda}+\mu^{\lambda}, B_2 = q_1^{\lambda}+r_1^{\lambda}+\mu^{\lambda}, B_3 = d_1^{\lambda}+r_2^{\lambda}+\alpha^{\lambda}+\mu^{\lambda}$, and $B_4 = d_2^{\lambda}+r_3^{\lambda}+\mu^{\lambda}$.

Let an arbitrary EE point of (3.4) be depicted by $E_1^* = (S^*, E^*, Q^*, I^*, J^*, R^*)$. Then, we get

$$\begin{aligned}
&\Gamma^{\lambda}-\tfrac{2\beta^{\lambda}(1-\kappa)S^{*}E^{*}}{S^{*}+E^{*}}-\mu^{\lambda}S^{*}+r_{1}^{\lambda}Q^{*}=0,\\
&\tfrac{2\beta^{\lambda}(1-\kappa)S^{*}E^{*}}{S^{*}+E^{*}}-B_{1}E^{*}=0,\\
&\gamma_{1}^{\lambda}E^{*}-B_{2}Q^{*}=0,\\
&\gamma_{2}^{\lambda}E^{*}+q_{1}^{\lambda}Q^{*}-B_{3}I^{*}=0,\\
&\alpha^{\lambda}I^{*}-B_{4}J^{*}=0,\\
&r_{2}^{\lambda}I^{*}+r_{3}^{\lambda}J^{*}-\mu R^{*}=0.
\end{aligned} \tag{5.5}$$

Solving (5.5), one reaches

$$\begin{aligned}
S^{*}&=\frac{B_{1}B_{2}^{2}B_{3}B_{4}\mu^{\lambda}\Gamma^{\lambda}+B_{1}B_{2}\Gamma^{\lambda}\left(B_{2}\gamma_{2}^{\lambda}+\gamma_{1}^{\lambda}q_{1}^{\lambda}\right)\left(B_{4}\left(\mu^{\lambda}+r_{2}^{\lambda}\right)+\alpha^{\lambda}r_{3}^{\lambda}\right)}{\mu^{\lambda}\left[B_{1}B_{2}B_{3}B_{4}\left(B_{1}B_{2}-\gamma_{1}^{\lambda}r_{1}^{\lambda}\right)(R_{0}-1)+B_{1}B_{2}^{2}B_{3}B_{4}\mu+B_{1}B_{2}\left(B_{2}\gamma_{2}^{\lambda}+\gamma_{1}^{\lambda}q_{1}\right)(B_{4}(\mu+r_{2})+\alpha r_{3})\right]},\\
Q^{*}&=\frac{\gamma_{1}^{\lambda}E^{*}}{B_{2}},\quad I^{*}=\frac{\left(B_{2}\gamma_{2}^{\lambda}+\gamma_{1}^{\lambda}q_{1}^{\lambda}\right)E^{*}}{B_{2}B_{3}},\quad J^{*}=\frac{\alpha\left(B_{2}\gamma_{2}+\gamma_{1}q_{1}\right)E^{*}}{B_{2}B_{3}B_{4}},R^{*}=\frac{\left(B_{4}\gamma_{2}^{\lambda}+r_{3}^{\lambda}\alpha^{\lambda}\right)(q_{1}\gamma_{1}+B_{2}\gamma_{2})E^{*}}{B_{2}B_{3}B_{4}\mu},\\
E^{*}&=\frac{S^{*}\mu^{\lambda}\left(\beta^{\lambda}(1-\kappa^{\lambda})\left(B_{4}+\alpha^{\lambda}r_{I}^{\lambda}\right)\left(B_{2}\gamma_{2}^{\lambda}+\gamma_{1}^{\lambda}q_{1}^{\lambda}\right)-B_{1}B_{2}B_{3}B_{4}\right)}{B_{1}B_{2}B_{3}B_{4}\mu+B_{1}\left(B_{2}\gamma_{2}+\gamma_{1}q_{1}^{\lambda}\right)\left(B_{4}\left(\mu^{\lambda}+r_{2}^{\lambda}\right)+\alpha^{\lambda}r_{3}^{\lambda}\right)}\\
&=\frac{S^{*}\mu B_{1}B_{2}B_{3}B_{4}(R_{0}-1)}{B_{1}B_{2}B_{3}B_{4}\mu+B_{1}(B_{2}\gamma_{2}+\gamma_{1}q_{1})\left(B_{4}(\mu+r_{2})+\alpha^{\lambda}r_{3}^{\lambda}\right)},
\end{aligned} \tag{5.6}$$

since $B_1B_2-\gamma_1^{\lambda}r_1^{\lambda}=\left(\gamma_2^{\lambda}+\mu^{\lambda}\right)\left(q_1^{\lambda}+r_1^{\lambda}+\mu^{\lambda}\right)+\gamma_1^{\lambda}\left(q_1^{\lambda}+\mu^{\lambda}\right)>0$, it is clear that the model system (3.4) has a unique endemic equilibrium when $R_0>1$ and no endemic equilibrium when $R_0<1$.

5.3. Local stability of DFE

Theorem 5.1. *Let $m>0$ and $n>0$ are the integers such that* $\gcd(m,n)=1$ *for $\lambda=\frac{m}{n}$ and $M=n$, then the DFE E^0 of the model in fractional derivative is locally asymptotically stable if* $|\arg(\vartheta)|>\frac{\pi}{2M}$ *for all roots ϑ of the associated characteristic equation,*

$$det(diag[\vartheta^{a}\vartheta^{a}\vartheta^{a}\vartheta^{a}\vartheta^{a}\vartheta^{a}\vartheta^{a}\vartheta^{a}]-J_{E^{0}})=0. \tag{5.7}$$

Proof. The Jacobian matrix of model (3.4) at E^0 is given by

$$J_{E^{0}}=\begin{pmatrix}
-(2\beta^{\lambda}(1-\kappa)+\mu^{\lambda}) & -2\beta^{\lambda}(1-\kappa) & \gamma_{1}^{\lambda} & 0 & 0 & 0\\
2\beta^{\lambda}(1-\kappa) & a_{22} & 0 & 0 & 0 & 0\\
0 & \gamma_{1}^{\lambda} & -\left(q_{1}^{\lambda}+r_{1}^{\lambda}+\mu^{\lambda}\right) & 0 & 0 & 0\\
0 & \gamma_{2}^{\lambda} & q_{1}^{\lambda} & -a_{44} & 0 & 0\\
0 & 0 & 0 & \alpha^{\lambda} & -\left(d_{2}^{\lambda}+r_{3}^{\lambda}+\mu^{\lambda}\right) & 0\\
0 & 0 & 0 & r_{2}^{\lambda} & r_{3}^{\lambda} & -\mu^{\lambda}
\end{pmatrix}, \tag{5.8}$$

where $a_{22} = 2\beta^\lambda(1-\kappa) - (\gamma_1^\lambda - \gamma_2^\lambda - \mu^\lambda)$ and $a_{44} = \left(d_1^\lambda + r_2^\lambda - \alpha^\lambda - \mu^\lambda\right)$. The three eigenvalues of the matrix (5.8) have negatine real part, i.e., $\vartheta_1^a = -\mu^\lambda$, $\vartheta_2^a = -\left(d_2^\lambda + r_3^\lambda + \mu^\lambda\right)$, $\vartheta_3^a = -\left(d_1^\lambda + r_2^\lambda + \alpha^\lambda + \mu^\lambda\right)$. Moreover, for last three eigenvalues we use the following reduce matrix.

$$RJ_{E_0} = \begin{pmatrix} -(2\beta^\lambda(1-\kappa)+\mu^\lambda) & -2\beta^\lambda(1-\kappa) & \gamma_1^\lambda \\ 2\beta^\lambda(1-\kappa) & 2\beta^\lambda(1-\kappa) - (\gamma_1^\lambda + \gamma_2^\lambda + \mu^\lambda) & 0 \\ 0 & \gamma_1^\lambda & -\left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right) \end{pmatrix}. \tag{5.9}$$

The characteristic equation of RJ_{E^0} (5.9) is given by

$$C(\vartheta) = \vartheta^3 + a_2\vartheta^2 + a_1\vartheta + a_0, \tag{5.10}$$

where

$$\begin{aligned} a_2 =& \left(2\beta^\lambda(1-\kappa)+\mu^\lambda\right) - \left(2\beta^\lambda(1-\kappa) - (\gamma_1^\lambda + \gamma_2^\lambda + \mu^\lambda)\right) + \left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right) \\ =& \left(2\beta^\lambda(1-\kappa)+\mu^\lambda\right) + \left(\gamma_1^\lambda + \gamma_2^\lambda + \mu^\lambda\right)(1-\mathcal{R}_0) + \left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right), \\ a_1 =& -\left(2\beta^\lambda(1-\kappa) - (\gamma_1^\lambda + \gamma_2^\lambda + \mu^\lambda)\right)\left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right) + 2\beta^\lambda(1-\kappa)2\beta^\lambda(1-\kappa), \\ & -\left(2\beta^\lambda(1-\kappa) - (\gamma_1^\lambda + \gamma_2^\lambda + \mu^\lambda)\right)(2\beta^\lambda(1-\kappa) + (2\beta^\lambda(1-\kappa)\left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right), \\ =& \left(\gamma_1^\lambda + \gamma_2^\lambda + \mu^\lambda\right)(1-\mathcal{R}_0)\left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right) + 2\beta^\lambda(1-\kappa)2\beta^\lambda(1-\kappa), \\ & (1-\mathcal{R}_0)\left(\gamma_1^\lambda + \gamma_2^\lambda + \mu^\lambda\right)(2\beta^\lambda(1-\kappa) + (2\beta^\lambda(1-\kappa)\left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right), \\ a_0 =& -\left(2\beta^\lambda(1-\kappa) - (\gamma_1^\lambda + \gamma_2^\lambda + \mu^\lambda)\right)(2\beta^\lambda(1-\kappa)\left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right) + \left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right), \\ & 2\beta^\lambda(1-\kappa)2\beta^\lambda(1-\kappa) + \gamma_1^2 2\beta^\lambda(1-\kappa), \\ =& (1-\mathcal{R}_0)\left(\gamma_1^\lambda + \gamma_2^\lambda + \mu^\lambda\right)(2\beta^\lambda(1-\kappa)\left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right) + \left(q_1^\lambda + r_1^\lambda + \mu^\lambda\right) \\ & 2\beta^\lambda(1-\kappa)2\beta^\lambda(1-\kappa) + \gamma_1^2 2\beta^\lambda(1-\kappa). \end{aligned} \tag{5.11}$$

Clearly, a_2, a_1 and a_0 are all positive, if $\mathcal{R}_0 < 1$. The argument of the root of equations

$$\left(\vartheta_1^a + \mu^\lambda\right) = 0, \left(\vartheta_2^a + \left(d_2^\lambda + r_3^\lambda + \mu^\lambda\right)\right) = 0,$$

and

$$\left(\vartheta_3^a + \left(d_1^\lambda + r_2^\lambda + \alpha^\lambda + \mu^\lambda\right)\right) = 0,$$

are similar, that is:

$$\left[|arg(\vartheta_k)| > \frac{\pi}{m} + k\frac{2\pi}{m} > \frac{\pi}{M} > \frac{\pi}{2M},\right]$$

where $k = 0, 1, 2, 3, ..., (m-1)$. Then, we get

$$\left(\vartheta^3 + a_2\vartheta^2 + a_1\vartheta + a_0\right) = 0, \tag{5.12}$$

are all greater than $\frac{\pi}{2M}$ if $\mathcal{R}_0 < 1$, having an argument less than $\frac{\pi}{2M}$ for $\mathcal{R}_0 > 1$. Thus the DFE is locally asymptotically stable for $\mathcal{R}_0 < 1$.

6. Existence and uniqueness

Assume that a continuous real-valued function denoted by $B(J)$ containing the sup norm property is a Banach space on $J = [0, b]$ and $P = B(J) \times B(J) \times B(J) \times B(J) \times B(J) \times B(J)$ with norm $\|(S, E, Q, I, J, R)\| = \|S\| + \|E\| + \|Q\| + \|I\| + \|J\| + \|R\|$, where $\|S\| = \sup_{\varphi\in J} |S(\varphi)|, \|E\| = \sup_{t\in J} |E(\varphi)|, \|Q\| = \sup_{\varphi\in J} |Q(\varphi)|, \|I\| = \sup_{\varphi\in J} |I(\varphi)|, \|J\| = \sup_{\varphi\in J} |J(\varphi)|, \|R\| = \sup_{\varphi\in J} |R(\varphi)|$. We get

$$\begin{cases} S(\varphi) - S(0) = ^{\mathrm{ABC}}\, \mathbb{D}^{\lambda}_{0,\varphi}[S(\varphi)] \left\{ \Gamma^{\lambda} - \dfrac{2\beta^{\lambda}(1-\kappa)SE}{S+E} - \mu^{\lambda}S + r_1^{\lambda}Q \right\}, \\ E(\varphi) - E(0) = ^{\mathrm{ABC}}\, \mathbb{D}^{\lambda}_{0,\varphi}[E(\varphi)] \left\{ \dfrac{2\beta^{\lambda}(1-\kappa)SE}{S+E} - (\gamma_1^{\lambda} + \gamma_2^{\lambda} + \mu^{\lambda})E \right\}, \\ Q(\varphi) - Q(0) = ^{\mathrm{ABC}}\, \mathbb{D}^{\lambda}_{0,\varphi}[Q(\varphi)] \left\{ \gamma_1^{\lambda}E - (q_1^{\lambda} + r_1^{\lambda} + \mu^{\lambda})Q \right\}, \\ I(\varphi) - I(0) = ^{\mathrm{ABC}}\, \mathbb{D}^{\lambda}_{0,\varphi}[I(\varphi)] \left\{ \gamma_2^{\lambda}E + q_1^{\lambda}Q - (\alpha^{\lambda} + r_2^{\lambda} + d_1^{\lambda} + \mu^{\lambda})I \right\}, \\ J(\varphi) - J(0) = ^{\mathrm{ABC}}\, \mathbb{D}^{\lambda}_{0,\varphi}[J(\varphi)] \left\{ \alpha^{\lambda}I - (r_3^{\lambda} + d_2^{\lambda} + \mu^{\lambda})J \right\}, \\ R(\varphi) - R(0) = ^{\mathrm{ABC}}\, \mathbb{D}^{\lambda}_{0,\varphi}[R(\varphi)] \left\{ r_2^{\lambda}I + r_3^{\lambda}J - \mu^{\lambda}R. \right\}. \end{cases} \tag{6.1}$$

Now using the Definition 2.5 on each of the above equations, one obtains

$$\begin{aligned} S(\varphi) - S(0) &= \frac{1-\lambda}{B(\lambda)} K_1(\lambda, t, S(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi-\delta)^{\lambda-1} K_1(\lambda, \delta, S(\delta)) d\delta, \\ E(\varphi) - E(0) &= \frac{1-\lambda}{B(\lambda)} K_2(\lambda, \varphi, E(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi-\delta)^{\lambda-1} K_2(\lambda, \delta, E(\delta)) d\delta, \\ Q(\varphi) - Q(0) &= \frac{1-\lambda}{B(\lambda)} K_3(\lambda, \varphi, Q(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi-\delta)^{\lambda-1} K_3(\lambda, \delta, Q(\delta)) d\delta, \\ I(\varphi) - I(0) &= \frac{1-\lambda}{B(\lambda)} K_4(\lambda, \varphi, I(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi-\delta)^{\lambda-1} K_4(\lambda, \delta, I(\delta)) d\delta, \\ J(\varphi) - J(0) &= \frac{1-\lambda}{B(\lambda)} K_5(\lambda, \varphi, J(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi-\delta)^{\lambda-1} K_5(\lambda, \delta, J(\delta)) d\delta, \\ R(\varphi) - R(0) &= \frac{1-\lambda}{B(\lambda)} K_6(\lambda, \varphi, R(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi-\delta)^{\lambda-1} K_6(\lambda, \delta, R(\delta)) d\delta, \end{aligned} \tag{6.2}$$

where

$$
\begin{aligned}
&K_1(\lambda,\varphi,S(\varphi)) = \Gamma^{\lambda} - \frac{2\beta^{\lambda}(1-\kappa)SE}{S+E} - \mu^{\lambda}S + r_1^{\lambda}Q,\\
&K_2(\lambda,\varphi,E(\varphi)) = \frac{2\beta^{\lambda}(1-\kappa)SE}{S+E} - (\gamma_1^{\lambda}+\gamma_2^{\lambda}+\mu^{\lambda})E,\\
&K_3(\lambda,\varphi,Q(\varphi)) = \gamma_1^{\lambda}E - (q_1^{\lambda}+r_1^{\lambda}+\mu^{\lambda})Q,\\
&K_4(\lambda,\varphi,I(\varphi)) = \gamma_2^{\lambda}E + q_1^{\lambda}Q - (\alpha^{\lambda}+r_2^{\lambda}+d_1^{\lambda}+\mu^{\lambda})I,\\
&K_5(\lambda,\varphi,J(\varphi)) = \alpha^{\lambda}I - (r_3^{\lambda}+d_2^{\lambda}+\mu^{\lambda})J,\\
&K_6(\lambda,\varphi,R(\varphi)) = r_2^{\lambda}I + r_3^{\lambda}J - \mu^{\lambda}R.
\end{aligned}
$$

Thus, we reach

$$
\|K_1(\lambda,\varphi,S) - K_1(\lambda,\varphi,S^*)\| = \left\|-\left(\tfrac{2\beta^{\lambda}(1-\kappa)SE}{S+E} - \mu^{\lambda}S\right)(S-S^*)\right\|. \tag{6.3}
$$

Taking into account

$$
\lambda_1 := \left\|-\left(\frac{2\beta^{\lambda}(1-\kappa)SE}{S+E} + \mu^{\lambda}\right)\right\|,
$$

one reaches

$$
\|K_1(\lambda,\varphi,S) - K_1(\lambda,\varphi,S^*)\| \le \Omega_1 \|S - S^*\|. \tag{6.4}
$$

Then, we acquire

$$
\begin{aligned}
&\|K_2(\lambda,\varphi,E) - K_2(\lambda,\varphi,E^*)\| \le \Omega_2 \|E - E^*\|,\\
&\|K_3(\lambda,\varphi,Q) - K_3(\lambda,\varphi,Q^*)\| \le \Omega_3 \|Q - Q^*\|,\\
&\|K_4(\lambda,\varphi,I) - K_4(\lambda,\varphi,I^*)\| \le \Omega_4 \|I - I^*\|,\\
&\|K_5(\lambda,\varphi,J) - K_5(\lambda,\varphi,J^*)\| \le \Omega_5 \|J - J^*\|,\\
&\|K_6(\lambda,\varphi,R) - K_5(\lambda,\varphi,R^*)\| \le \Omega_6 \|R - R^*\|.
\end{aligned} \tag{6.5}
$$

Where

$$
\begin{aligned}
\lambda_2 &= \left\|-\left(\gamma_1^{\lambda}+\gamma_2^{\lambda}+\mu^{\lambda}\right)\right\|,\\
\lambda_3 &= \left\|-\left(q_1^{\lambda}+r_1^{\lambda}+\mu^{\lambda}\right)\right\|,\\
\lambda_4 &= \left\|-\left(\alpha^{\lambda}+r_2^{\lambda}+d_1^{\lambda}+\mu^{\lambda}\right)\right\|,\\
\lambda_5 &= \left\|-\left(r_3^{\lambda}+d_2^{\lambda}+\mu^{\lambda}\right)\right\|,\\
\lambda_6 &= \left\|-\left(\mu^{\lambda}\right)\right\|.
\end{aligned}
$$

Then, we reach

$$
\begin{aligned}
S_n(\varphi)-S(0)=&\frac{1-\lambda}{B(\lambda)}K_1(\lambda,\varphi,S_{n-1}(\varphi))+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\times\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}K_1(\lambda,\delta,S_{n-1}(\delta))d\delta,\\
E_n(\varphi)-E(0)=&\frac{1-\lambda}{B(\lambda)}K_2(\lambda,\varphi,E_{n-1}(\varphi))+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\times\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}K_2(\lambda,\delta,E_{n-1}(\delta))d\delta,\\
Q_n(\varphi)-Q(0)=&\frac{1-\lambda}{B(\lambda)}K_3(\lambda,\varphi,Q_{n-1}(\varphi))+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\times\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}K_3(\lambda,\delta,Q_{n-1}(\delta))d\delta,\\
I_n(\varphi)-I(0)=&\frac{1-\lambda}{B(\lambda)}K_4(\lambda,\varphi,I_{n-1}(\varphi))+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\times\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}K_4(\lambda,\delta,I_{n-1}(\delta))d\delta,\\
J_n(\varphi)-J(0)=&\frac{1-\lambda}{B(\lambda)}K_5(\lambda,\varphi,J_{n-1}(\varphi))+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\times\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}K_5(\lambda,\delta,J_{n-1}(\delta))d\delta,\\
R_n(\varphi)-R(0)=&\frac{1-\lambda}{B(\lambda)}K_6(\lambda,\varphi,R_{n-1}(\varphi))+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\times\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}K_6(\lambda,\delta,R_{n-1}(\delta))d\delta,
\end{aligned}
\tag{6.6}
$$

together with $S_0(\varphi)=S(0), E_0(\varphi)=E(0), Q_0(\varphi)=Q(0), I_0(\varphi)=I(0), J_0(\varphi)=J(0), R_0(\varphi)=R(0)$. Thus, we get

$$
\begin{aligned}
\Xi_{S,n}(\varphi)=&S_n(\varphi)-S_{n-1}(\varphi)=\frac{1-\lambda}{B(\lambda)}(K_1(\lambda,\varphi,S_{n-1}(\varphi))-K_1(\lambda,\varphi,S_{n-2}(\varphi)))\\
&+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left(K_1\left(\lambda,\delta,S_{n-1}(\delta)\right)-K_1\left(\lambda,\delta,S_{n-2}(\delta)\right)\right)d\delta,\\
\Xi_{E,n}(\varphi)=&E_n(\varphi)-E_{n-1}(\varphi)=\frac{1-\lambda}{B(\lambda)}(K_2(\lambda,\varphi,E_{n-1}(\varphi))-K_2(\lambda,\varphi,E_{n-2}(\varphi)))\\
&+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\int_0^{l}(\varphi-\delta)^{\lambda-1}\left(K_2\left(\lambda,\delta,E_{n-1}(\delta)\right)-K_2\left(\lambda,\delta,E_{n-2}(\delta)\right)\right)d\delta,\\
\Xi_{Q,n}(\varphi)=&Q_{1n}(\varphi)-Q_{n-1}(\varphi)=\frac{1-\lambda}{B(\lambda)}(K_3(\lambda,\varphi,Q_{n-1}(\varphi))-K_3(\lambda,\varphi,Q_{n-2}(\varphi)))\\
&+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left(K_3\left(\lambda,\delta,Q_{n-1}(\delta)\right)-K_3\left(\lambda,\delta,Q_{n-2}(\delta)\right)\right)d\delta,\\
\Xi_{I,n}(\varphi)=&I_n(\varphi)-I_{n-1}(\varphi)=\frac{1-\lambda}{B(\lambda)}(K_4(\lambda,\varphi,I_{n-1}(\varphi))-K_4(\lambda,\varphi,I_{n-2}(\varphi)))\\
&+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left(K_4\left(\lambda,\delta,I_{n-1}(\delta)\right)-K_4\left(\lambda,\delta,I_{n-2}(\delta)\right)\right)d\delta,\\
\Xi_{J,n}(\varphi)=&J_n(\varphi)-J_{n-1}(\varphi)=\frac{1-\lambda}{B(\lambda)}(K_5(\lambda,\varphi,J_{n-1}(\varphi))-K_5(\lambda,\varphi,J_{n-2}(\varphi)))\\
&+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left(K_5\left(\lambda,\delta,J_{n-1}(\delta)\right)-K_5\left(\lambda,\delta,J_{n-2}(\delta)\right)\right)d\delta,\\
\Xi_{R,n}(\varphi)=&R_n(\varphi)-R_{n-1}(\varphi)=\frac{1-\lambda}{B(\lambda)}(K_6(\lambda,\varphi,R_{n-1}(\varphi))-K_5(\lambda,\varphi,R_{n-2}(\varphi)))\\
&+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left(K_6\left(\lambda,\delta,R_{n-1}(\delta)\right)-K_5\left(\lambda,\delta,R_{n-2}(\delta)\right)\right)d\delta.
\end{aligned}
\tag{6.7}
$$

$$S_n(\varphi) = \sum_{i=0}^{n} \Xi_{S,i}(\varphi), \quad E_n(\varphi) = \sum_{i=0}^{n} \Xi_{E,i}(\varphi), \quad Q_n(\varphi) = \sum_{i=0}^{n} \Xi_{Q,i}(\varphi),$$
$$I_n(\varphi) = \sum_{i=0}^{n} \Xi_{I,i}(\varphi), \quad J_n(\varphi) = \sum_{i=0}^{n} \Xi_{J,i}(\varphi), \quad R_n(\varphi) = \sum_{i=0}^{n} \Xi_{R,i}(\varphi).$$

Additionally, by using Eqs (6.4) and (6.5) and considering that

$$\Xi_{S,n-1}(\varphi) = S_{n-1}(\varphi) - S_{n-2}(\varphi), \quad \Xi_{E,n-1}(\varphi) = E_{n-1}(\varphi) - E_{n-2}(\varphi), \quad \Xi_{Q,n-1}(\varphi) = Q_{n-1}(\varphi) - Q_{n-2}(\varphi),$$
$$\Xi_{I,n-1}(\varphi) = I_{n-1}(\varphi) - I_{n-2}(\varphi), \quad \Xi_{J,n-1}(\varphi) = J_{n-1}(\varphi) - J_{n-2}(\varphi), \quad \Xi_{R,n-1}(\varphi) = R_{n-1}(\varphi) - R_{n-2}(\varphi),$$

we reach

$$\begin{aligned}
\left\|\Xi_{S,n}(\varphi)\right\| &\leq \tfrac{1-\lambda}{B(\lambda)}\Omega_1 \left\|\Xi_{S,n-1}(\varphi)\right\| \tfrac{\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_1 \times \int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left\|\Xi_{S,n-1}(\delta)\right\| d\delta, \\
\left\|\Xi_{E,n}(\varphi)\right\| &\leq \tfrac{1-\lambda}{B(\lambda)}\Omega_2 \left\|\Xi_{E,n-1}(\varphi)\right\| \tfrac{\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_2 \times \int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left\|\Xi_{E,n-1}(\delta)\right\| d\delta, \\
\left\|\Xi_{Q,n}(\varphi)\right\| &\leq \tfrac{1-\lambda}{B(\lambda)}\Omega_3 \left\|\Xi_{Q,n-1}(\varphi)\right\| \tfrac{\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_3 \times \int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left\|\Xi_{Q,n-1}(\delta)\right\| d\delta, \\
\left\|\Xi_{I,n}(\varphi)\right\| &\leq \tfrac{1-\lambda}{B(\lambda)}\Omega_4 \left\|\Xi_{I,n-1}(\varphi)\right\| \tfrac{\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_4 \times \int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left\|\Xi_{I,n-1}(\delta)\right\| d\delta, \\
\left\|\Xi_{J,n}(\varphi)\right\| &\leq \tfrac{1-\lambda}{B(\lambda)}\Omega_5 \left\|\Xi_{J,n-1}(\varphi)\right\| \tfrac{\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_5 \times \int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left\|\Xi_{J,n-1}(\delta)\right\| d\delta, \\
\left\|\Xi_{R,n}(\varphi)\right\| &\leq \tfrac{1-\lambda}{B(\lambda)}\Omega_6 \left\|\Xi_{R,n-1}(\varphi)\right\| \tfrac{\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_6 \times \int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left\|\Xi_{R,n-1}(\delta)\right\| d\delta.
\end{aligned} \tag{6.8}$$

Now, we are in position to prove the following theorem.

Theorem 6.1. *If*

$$\frac{1-\lambda}{B(\lambda)}\Omega_i + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} b^{\lambda}\Omega_i < 1, i = 1, 2, \ldots, 5. \tag{6.9}$$

Then, (3.4) has a unique solution for $t \in [0, b]$.

Proof. It is observed S, E, Q, I, J, R are bounded functions. In Addition, as can be seen from Eqs (6.4) and (6.5), the symbols K_1, K_2, K_3, K_4, K_5, and K_6 hold for Lipchitz condition. Therefore, utilizing Eq (6.8) together with a recursive hypothesis, we arrive at

$$\begin{aligned}
\left\|\Xi_{S,n}(\varphi)\right\| &\leq \|S_0(\varphi)\| \left(\tfrac{1-\lambda}{B(\lambda)}\Omega_1 + \tfrac{\lambda b^\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_1\right)^n, \\
\left\|\Xi_{E,n}(\varphi)\right\| &\leq \|E_0(\varphi)\| \left(\tfrac{1-\lambda}{B(\lambda)}\Omega_3 + \tfrac{\lambda b^\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_2\right)^n, \\
\left\|\Xi_{Q,n}(\varphi)\right\| &\leq \|Q_0(\varphi)\| \left(\tfrac{1-\lambda}{B(\lambda)}\Omega_3 + \tfrac{\lambda b^\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_3\right)^n, \\
\left\|\Xi_{I,n}(\varphi)\right\| &\leq \|I_0(\varphi)\| \left(\tfrac{1-\lambda}{B(\lambda)}\Omega_4 + \tfrac{\lambda b^\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_4\right)^n, \\
\left\|\Xi_{J,n}(\varphi)\right\| &\leq \|J_0(\varphi)\| \left(\tfrac{1-\lambda}{B(\lambda)}\Omega_5 + \tfrac{\lambda b^\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_5\right)^n, \\
\left\|\Xi_{R,n}(\varphi)\right\| &\leq \|R_0(\varphi)\| \left(\tfrac{1-\lambda}{B(\lambda)}\Omega_6 + \tfrac{\lambda b^\lambda}{B(\lambda)\Gamma(\lambda)}\Omega_6\right)^n.
\end{aligned} \tag{6.10}$$

Then, we get
$\left\|\Xi_{S,n}(\varphi)\right\| \to 0, \quad \left\|\Xi_{E,n}(\varphi)\right\| \to 0, \quad \left\|\Xi_{Q,n}(\varphi)\right\| \to 0, \quad \left\|\Xi_{I,n}(\varphi)\right\| \to 0, \quad \left\|\Xi_{J,n}(\varphi)\right\| \to 0, \quad \left\|\Xi_{R,n}(\varphi)\right\| \to 0$
as $n \to \infty$. Thus, we reach

$$\begin{aligned}
\|S_{n+k}(\varphi) - S_n(\varphi)\| &\leq \textstyle\sum_{j=n+1}^{n+k} Z_1^j = \frac{Z_1^{n+1}-Z_1^{n+k+1}}{1-Z_1}, \\
\|E_{n+k}(\varphi) - E_n(\varphi)\| &\leq \textstyle\sum_{j=n+1}^{n+k} Z_2^j = \frac{Z_2^{n+1}-Z_2^{n+k+1}}{1-Z_2}, \\
\|Q_{n+k}(\varphi) - Q_n(t)\| &\leq \textstyle\sum_{j=n+1}^{n+k} Z_3^j = \frac{Z_3^{n+1}-Z_3^{n+k+1}}{1-Z_3}, \\
\|I_{n+k}(\varphi) - I_n(\varphi)\| &\leq \textstyle\sum_{j=n+1}^{n+k} Z_4^j = \frac{Z_4^{n+1}-Z_4^{n+k+1}}{1-Z_4}, \\
\|J_{n+k}(\varphi) - J_n(\varphi)\| &\leq \textstyle\sum_{j=n+1}^{n+k} Z_5^j = \frac{Z_5^{n+1}-Z_5^{n+k+1}}{1-Z_5}, \\
\|R_{n+k}(\varphi) - R_n(\varphi)\| &\leq \textstyle\sum_{i=n+1}^{n+k} Z_6^j = \frac{Z_6^{n+1}-Z_6^{n+k+1}}{1-Z_6},
\end{aligned} \tag{6.11}$$

with $Z_i = \frac{1-\lambda}{B(\lambda)}\Omega_i + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} b^\lambda \Omega_i < 1$ by hypothesis.

7. Hyers-Ulam stability

Definition 7.1. *[23] The model (3.4) is Hyers-Ulam stable if there exist a real number $\gamma \geq 0$, such that for every $\epsilon > 0$ and for any solution $\mathcal{Y} \in C^1(\mathcal{G}, \mathbb{R})$ of the following inequality:*

$$|{}^{\mathrm{ABC}}\mathbf{D}^\lambda \mathcal{Y}(\varphi) - \Psi(\varphi, \mathcal{Y}(\varphi))| \leq \epsilon, \quad \varphi \in \mathcal{G},$$

there is a unique solution $\mathfrak{W} \in C^1(\mathcal{G}, \mathbb{R})$ of the model (3.4), such that

$$|\mathcal{Y}(\varphi) - \mathfrak{W}(\varphi)| \leq \gamma\epsilon, \quad \varphi \in \mathcal{G}.$$

Definition 7.2. *The ABC fractional integral system given by Eq (6.2) is said to be Hyers-Ulam stable if exist constants $\Delta_i > 0, i \in \boldsymbol{N}^6$ fulfilling: For every $\gamma_i > 0, i \in \boldsymbol{N}^6$, for*

$$\begin{aligned}
&|S(\varphi) - \frac{1-\lambda}{B(\lambda)} K_1(\lambda, \varphi, S(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_1(\lambda, \delta, S(\delta)) d\delta| \leq \gamma_1, \\
&|E(\varphi) - \frac{1-\lambda}{B(\lambda)} K_2(\lambda, \varphi, E(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_2(\lambda, \delta, E(\delta)) d\delta| \leq \gamma_2, \\
&|Q(\varphi) - \frac{1-\lambda}{B(\lambda)} K_3(\lambda, \varphi, Q(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_3(\lambda, \delta, Q(\delta)) d\delta| \leq \gamma_3, \\
&|I(t) - \frac{1-\lambda}{B(\lambda)} K_4(\lambda, t, I(t)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{t} (t - \delta)^{\lambda-1} K_4(\lambda, \delta, I(\delta)) d\delta| \leq \gamma_4, \\
&|J(\varphi) - \frac{1-\lambda}{B(\lambda)} K_5(\lambda, \varphi, I(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_5(\lambda, \delta, I(\delta)) d\delta| \leq \gamma_5, \\
&|R(\varphi) - \frac{1-\lambda}{B(\lambda)} K_6(\lambda, \varphi, R(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_6(\lambda, \delta, R(\delta)) d\delta| \leq \gamma_6.
\end{aligned} \tag{7.1}$$

there exist $(\dot{S}(\varphi), \dot{E}(\varphi), \dot{Q}(\varphi), \dot{I}(\varphi), \dot{J}(\varphi), \dot{R}(\varphi))$ which are satisfying

$$\begin{aligned}
\dot{S}(\varphi) &= \frac{1-\lambda}{B(\lambda)} K_1(\lambda, \varphi, S(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_1(\lambda, \delta, \dot{S}(\delta)) d\delta, \\
\dot{E}(\varphi) &= \frac{1-\lambda}{B(\lambda)} K_2(\lambda, \varphi, E(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_2(\lambda, \delta, \dot{E}(\delta)) d\delta, \\
\dot{Q}(\varphi) &= \frac{1-\lambda}{B(\lambda)} K_3(\lambda, \varphi, Q(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_3(\lambda, \delta, \dot{Q}(\delta)) d\delta, \\
\dot{I}(\varphi) &= \frac{1-\lambda}{B(\lambda)} K_4(\lambda, \varphi, I(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_4(\lambda, \delta, \dot{I}(\delta)) d\delta, \\
\dot{J}(\varphi) &= \frac{1-\lambda}{B(\lambda)} K_5(\lambda, \varphi, J(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_5(\lambda, \delta, \dot{J}(\delta)) d\delta, \\
\dot{R}(t) &= \frac{1-\lambda}{B(\lambda)} K_6(\lambda, \varphi, R(\varphi)) + \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \times \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} K_6(\lambda, \delta, \dot{R}(\delta)) d\delta.
\end{aligned} \tag{7.2}$$

Such that

$$
\begin{aligned}
&|S(\varphi) - \dot{S}(\varphi)| \leq \zeta_1\gamma_1, |E(\varphi) - \dot{E}(\varphi)| \leq \zeta_2\gamma_2,\\
&|Q(\varphi) - \dot{Q}(\varphi)| \leq \zeta_3\gamma_3, |I(\varphi) - \dot{I}(\varphi)| \leq \zeta_4\gamma_4,\\
&|J(\varphi) - \dot{J}(\varphi)| \leq \zeta_5\gamma_5, |R(\varphi) - \dot{R}(\varphi)| \leq \zeta_6\gamma_6.
\end{aligned}
\tag{7.3}
$$

Theorem 7.1. *With assumption J, the suggested model of fractional order (3.4) is Hyers-Ulam stable.*

Proof. With the help of theorem (6.1), the given ABC fractional model (3.4) has a unique solution $(S(\varphi), E(\varphi), Q(\varphi), I(\varphi), J(\varphi), R(\varphi))$ satisfying equations of system (6.2). Then, we acquire

$$
\begin{aligned}
\|S(\varphi) - \dot{S}(\varphi)\| \leq{}& \frac{1-\lambda}{B(\lambda)} \left\|K_1(\lambda, \varphi, S(\varphi)) - K_1(\lambda, \varphi, \dot{S}(\varphi))\right\| \\
&+ \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} \left\|K_1(\lambda, \varphi, S(\varphi)) - K_1(\lambda, \varphi, \dot{S}(\varphi))\right\| d\delta \\
\leq{}& \left[\frac{1-\lambda}{B(\lambda)} + \frac{\lambda}{B(\lambda)\Gamma(\lambda)}\right] \Omega_1 \|S - \dot{S}\|,
\end{aligned}
\tag{7.4}
$$

$$
\begin{aligned}
\|E(\varphi) - \dot{E}(\varphi)\| \leq{}& \frac{1-\lambda}{B(\lambda)} \left\|K_2(\lambda, \varphi, E(\varphi)) - K_2(\lambda, \varphi, \dot{E}(\varphi))\right\| \\
&+ \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} \left\|K_2(\lambda, \varphi, E(\varphi)) - K_1(\lambda, \varphi, \dot{E}(\varphi))\right\| d\delta \\
\leq{}& \left[\frac{1-\lambda}{B(\lambda)} + \frac{\lambda}{B(\lambda)\Gamma(\lambda)}\right] \Omega_2 \|E - \dot{E}\|,
\end{aligned}
\tag{7.5}
$$

$$
\begin{aligned}
\|Q(\varphi) - \dot{Q}(\varphi)\| \leq{}& \frac{1-\lambda}{B(\lambda)} \left\|K_3(\lambda, \varphi, Q(\varphi)) - K_3(\lambda, \varphi, \dot{Q}(\varphi))\right\| \\
&+ \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} \left\|K_3(\lambda, \varphi, Q(\varphi)) - K_3(\lambda, \varphi, \dot{Q}(\varphi))\right\| d\delta \\
\leq{}& \left[\frac{1-\lambda}{B(\lambda)} + \frac{\lambda}{B(\lambda)\Gamma(\lambda)}\right] \Omega_3 \|Q - \dot{Q}\|,
\end{aligned}
\tag{7.6}
$$

$$
\begin{aligned}
\|I(\varphi) - \dot{I}(\varphi)\| \leq{}& \frac{1-\lambda}{B(\lambda)} \left\|K_4(\lambda, \varphi, I(\varphi)) - K_4(\lambda, \varphi, \dot{I}(\varphi))\right\| \\
&+ \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} \left\|K_4(\lambda, \varphi, I(\varphi)) - K_4(\lambda, \varphi, \dot{I}(\varphi))\right\| d\delta \\
\leq{}& \left[\frac{1-\lambda}{B(\lambda)} + \frac{\lambda}{B(\lambda)\Gamma(\lambda)}\right] \Omega_4 \|I - \dot{I}\|,
\end{aligned}
\tag{7.7}
$$

$$
\begin{aligned}
\|J(\varphi) - \dot{J}(\varphi)\| \leq{}& \frac{1-\lambda}{B(\lambda)} \left\|K_5(\lambda, \varphi, J(\varphi)) - K_5(\lambda, \varphi, \dot{J}(\varphi))\right\| \\
&+ \frac{\lambda}{B(\lambda)\Gamma(\lambda)} \int_0^{\varphi} (\varphi - \delta)^{\lambda-1} \left\|K_5(\lambda, \varphi, J(\varphi)) - K_5(\lambda, \varphi, \dot{J}(\varphi))\right\| d\delta \\
\leq{}& \left[\frac{1-\lambda}{B(\lambda)} + \frac{\lambda}{B(\lambda)\Gamma(\lambda)}\right] \Omega_5 \|J - \dot{J}\|,
\end{aligned}
\tag{7.8}
$$

$$\|R(\varphi)-\dot{R}(\varphi)\| \leq \frac{1-\lambda}{B(\lambda)}\left\|K_6(\lambda,\varphi,R(\varphi))-K_6(\lambda,\varphi,\dot{R}(\varphi))\right\| + \frac{\lambda}{B(\lambda)\Gamma(\lambda)}\int_0^{\varphi}(\varphi-\delta)^{\lambda-1}\left\|K_6(\lambda,\varphi,R(\varphi))-K_6(\lambda,\varphi,\dot{R}(\varphi))\right\|d\delta \leq \left[\frac{1-\lambda}{B(\lambda)}+\frac{\lambda}{B(\lambda)\Gamma(\lambda)}\right]\Omega_6\|R-\dot{R}\|. \tag{7.9}$$

Taking, $\gamma_i = \Omega_i, \Delta_i = \frac{1-\lambda}{B(\lambda)} + \frac{\lambda}{B(\lambda)\Gamma(\lambda)}$, this implies

$$\|S(\varphi)-\dot{S}(\varphi)\| \leq \gamma_1\Delta_1. \tag{7.10}$$

Thus, we get

$$\begin{cases} \|E(\varphi)-\dot{E}(\varphi)\| \leq \gamma_2\Delta_2 \\ \|Q(\varphi)-\dot{Q}(\varphi)\| \leq \gamma_3\Delta_3 \\ \|I(\varphi)-\dot{I}(\varphi)\| \leq \gamma_4\Delta_4 \\ \|J(\varphi)-\dot{J}(\varphi)\| \leq \gamma_5\Delta_5 \\ \|R(\varphi)-\dot{R}(\varphi)\| \leq \gamma_6\Delta_6 \end{cases} \tag{7.11}$$

With the help of Eqs (7.10) and (7.11), the ABC fractional integral system (6.2) is Hyers-Ulam and consequently the ABC-fractional order model (3.4) is Hyers-Ulam stable.

8. Sensitivity analysis

To know the influence of various parameters upon the basic reproduction number, we use sensitivity analysis. To calculate the sensitivity index we used the direct differentiation method. The sensitivity index $\Upsilon_{\tau}^{R_0}$ of a parameter τ is calculated by $\Upsilon_{\tau}^{R_0} = \frac{dR_0}{d\tau}\frac{\tau}{R_0}$. Sensitivity indices of different parameters for our proposed model are given in Figures 1–3. The effect of different parameters over basic reproduction number are shown grap hically in Figures 4–9.

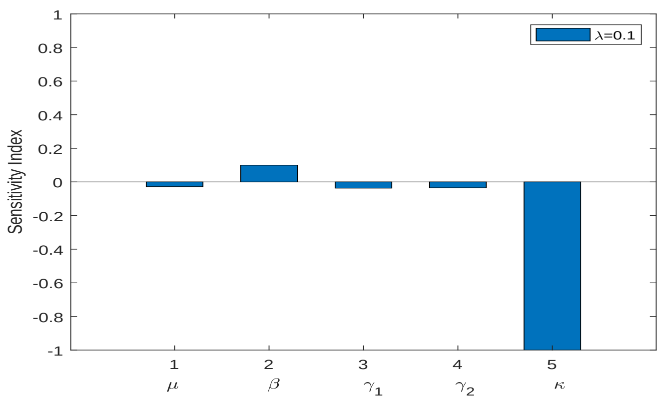

Figure 2. Sensitivity analysis of different parameters with fractional parameter $\lambda = 0.1$.

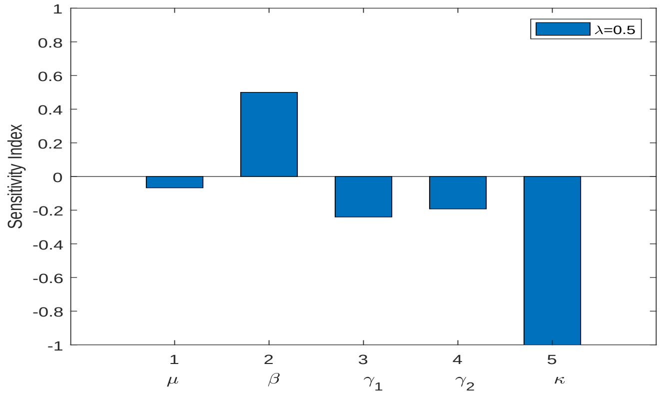

Figure 3. Sensitivity analysis of different parameters with fractional parameter $\lambda = 0.5$.

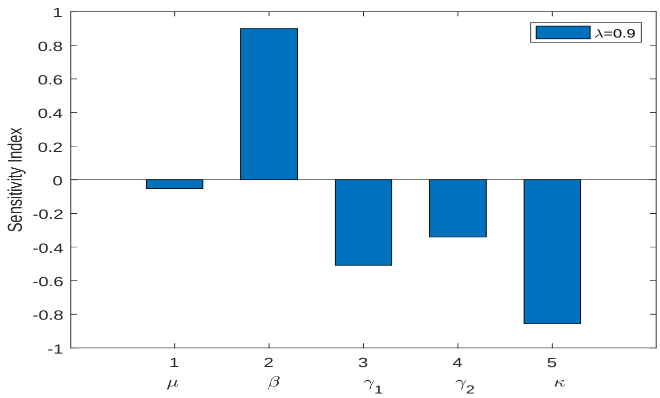

Figure 4. Sensitivity analysis of different parameters with fractional parameter $\lambda = 0.9$.

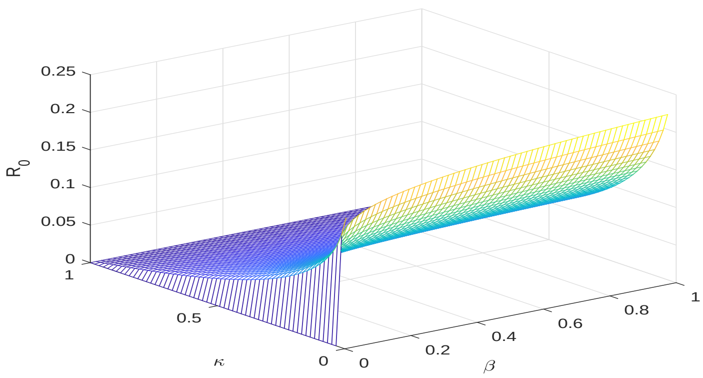

Figure 5. Sensitivity analysis of R_0 vs κ and β for $\lambda = 0.1$.

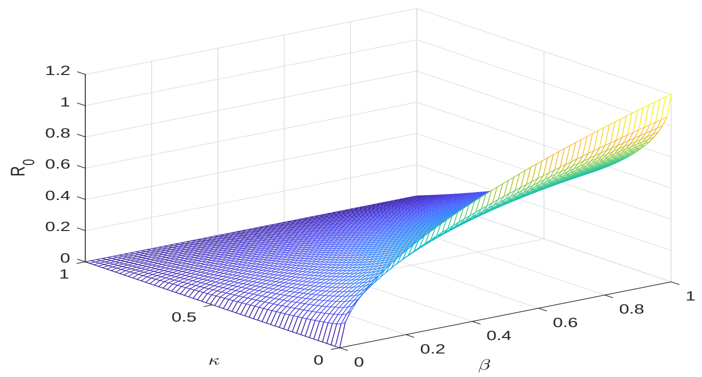

Figure 6. Sensitivity analysis of R_0 vs κ and β for $\lambda = 0.5$.

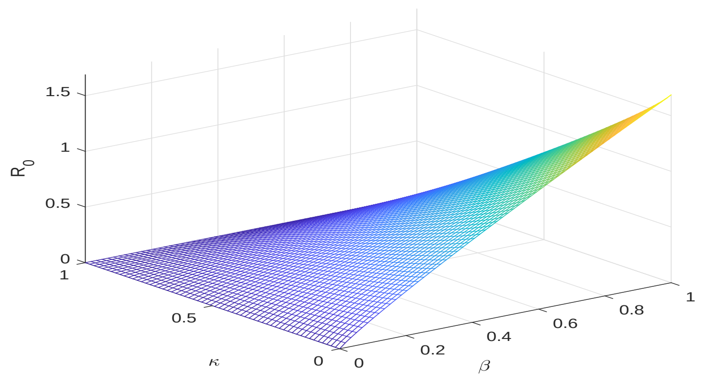

Figure 7. Sensitivity analysis of R_0 vs κ and β for $\lambda = 0.9$.

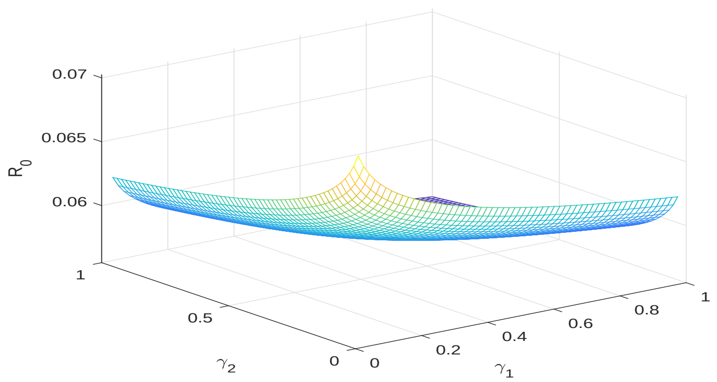

Figure 8. Sensitivity analysis of R_0 vs γ_1 and γ_2 for $\lambda = 0.1$.

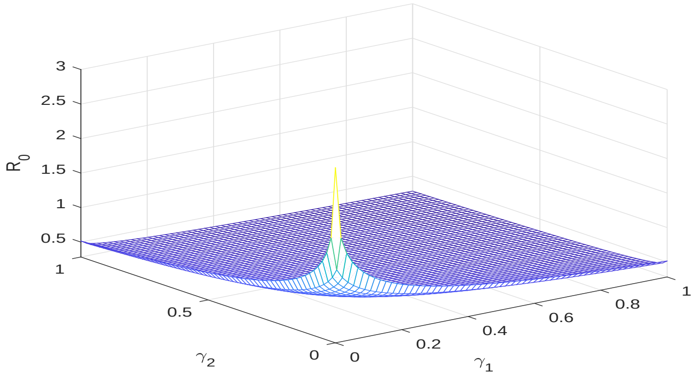

Figure 9. Sensitivity analysis of R_0 vs γ_1 and γ_2 for $\lambda = 0.5$.

9. Numerical dynamics

In the present section, we have simulated the proposed scholastic epidemiological system for the ABC. While simulating the system, convergent iterative scheme is employed. Fractional-order chaotic system via bifurcation is studied in [30] and through Caputo operator is discussed in [31]. Consider a general Cauchy problem of fractional order having autonomous nature

$$^{\star}D^{\lambda}_{0,\varphi}\Big(y(\varphi)\Big) = g\Big(y(\varphi)\Big), \lambda \in (0,1), \varphi \in [0,T], y(0) = y_0, \tag{9.1}$$

where $y = (a, b, c, w) \in \mathbb{R}^4_+$ being a continuous function (real-valued) fulfills the Lipchitz condition as:

$$||g(y_1(\varphi)) - g(y_2(\varphi))|| \leq M||y_1(\varphi) - y_2(\varphi)||, \tag{9.2}$$

where $M > 0$ is said to be Lipchitz constant.

$$y(\varphi) = y_0 + J^{\lambda}_{0,\varphi} g(y(\varphi)),\ \varphi \in [0,T], \tag{9.3}$$

where $J^{\lambda}_{0,\varphi}$ denotes the fractional integral operator. Consider an equispaced integration intervals over [0, T] with the fixed step size $h(= 10^{-2}$ for simulation$) = \frac{T}{n}, n \in N$. Suppose that y_q is the approximation of $y(\varphi)$ at $t = t_q$ for $q = 0, 1, n$. Consequently, our model becomes

$$
\begin{aligned}
{}^{ABC}S_{q+1} &= S_0 + \frac{1-\lambda}{AB(\lambda)}\Big(\Gamma - \frac{2\beta(1-\kappa)S_q E_q}{S_q + E_q} - \mu S_q + r_1 Q_q\Big) \\
&+ \frac{h^\lambda}{AB(\lambda)\Gamma(\lambda)} \sum_{k=0}^{q} \Big((q-k+1)^\lambda - (q-k)^\lambda\Big)\Big(\Gamma - \frac{2\beta(1-\kappa)S_k E_k}{S_k + E_k} - \mu S_k + r_1 Q_k\Big), \\
{}^{ABC}E_{q+1} &= E_0 + \frac{1-\lambda}{AB(\lambda)}\Big(\frac{2\beta(1-\kappa)S_q E_q}{S_q + E_q} - (\gamma_1 + \gamma_2 + \mu)E_q\Big) \\
&+ \frac{h^\lambda}{AB(\lambda)\Gamma(\lambda)} \sum_{k=0}^{q} \Big((q-k+1)^\lambda - (q-k)^\lambda\Big)\Big(\frac{2\beta(1-\kappa)S_k E_k}{S_k + E_k} - (\gamma_1 + \gamma_2 + \mu)E_k\Big), \\
{}^{ABC}Q_{q+1} &= Q_0 + \frac{1-\lambda}{AB(\lambda)}\Big(\gamma_1 E_q - (q_1 + r_1 + \mu)Q_q\Big) \\
&+ \frac{h^\lambda}{AB(\lambda)\Gamma(\lambda)} \sum_{k=0}^{q} \Big((q-k+1)^\lambda - (q-k)^\lambda\Big)\Big(\gamma_1 E_k - (q_1 + r_1 + \mu)Q_k\Big), \\
{}^{ABC}I_{q+1} &= I_0 + \frac{1-\lambda}{AB(\lambda)}\Big(\gamma_2 E_q + q_1 Q_q - (\alpha + r_2 + d_1 + \mu)\, I_q\Big) \\
&+ \frac{h^\lambda}{AB(\lambda)\Gamma(\lambda)} \sum_{k=0}^{q} \Big((q-k+1)^\lambda - (q-k)^\lambda\Big)\Big(\gamma_2 E_k + q_1 Q_k - (\alpha + r_2 + d_1 + \mu)\, I_k\Big), \\
{}^{ABC}J_{q+1} &= J_0 + \frac{1-\lambda}{AB(\lambda)}\Big(\alpha I_q - (r_3 + d_2 + \mu)\, J_q\Big) \\
&+ \frac{h^\lambda}{AB(\lambda)\Gamma(\lambda)} \sum_{k=0}^{q} \Big((q-k+1)^\lambda - (q-k)^\lambda\Big)\Big(\alpha I_k - (r_3 + d_2 + \mu)\, J_k\Big), \\
{}^{ABC}R_{q+1} &= R_0 + \frac{1-\lambda}{AB(\lambda)}\Big(r_2 I_q + r_3 J_q - \mu R_q\Big) \\
&+ \frac{h^\lambda}{AB(\lambda)\Gamma(\lambda)} \sum_{k=0}^{q} \Big((q-k+1)^\lambda - (q-k)^\lambda\Big)\Big(r_2 I_k + r_3 J_k - \mu R_k\Big).
\end{aligned} \tag{9.4}
$$

10. Discussions

Figures 2–4 shows the sensitivity indices of all the parameters corresponding to different values of fractional parameter λ. From Figures 2–4 it is noted that the largest positive parameter is β which means that by increasing the transmission rate β, the basic reproduction number will increase and hence β is the most sensitivity parameter upon R_0. While on the other hand, the smallest negative parameter is μ which means that by increasing the natural death rate μ, the basic reproduction number will decrease and hence μ is most senseless parameter upon R_0. From Figures 2–4, it is obtained that by increasing the fractional parameter λ the positive sensitive parameter is increasing while the negative sensitive parameters are decreasing. Figures 5–7 depicts the sensitivity analysis of R_0 vs κ and β for different values of $\lambda = 0.1, 0.5, 0.9$. It is depicted that transmission rate β is directly proportional to R_0 while the lockdown parameter κ is inversely proportional to R_0. Figures 8–10 shows the sensitivity analysis of R_0 vs γ_1 and γ_2 for different values of $\lambda = 0.1, 0.5, 0.9$. It is shown that the transmission rate from exposed class to quarantined class γ_1 and the transmission rate from exposed class to infected class γ_2 both are inversely proportional to R_0. Figure 11 shows the impact of fractional parameter λ on the first three classes S, E, and Q, while Figure 12 shows the impact of fractional parameter λ on the other three classes I, J, and R.

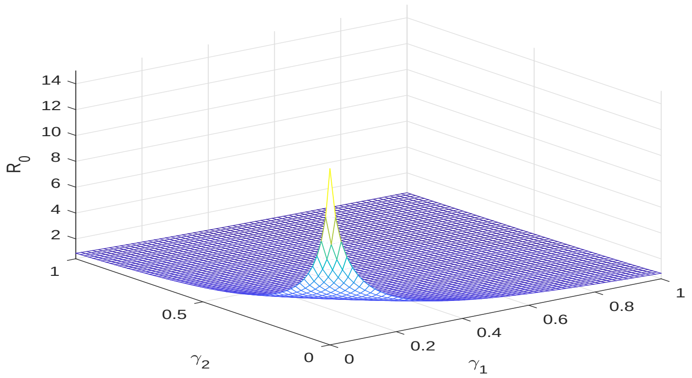

Figure 10. Sensitivity analysis of R_0 vs γ_1 and γ_2 for $\lambda = 0.9$.

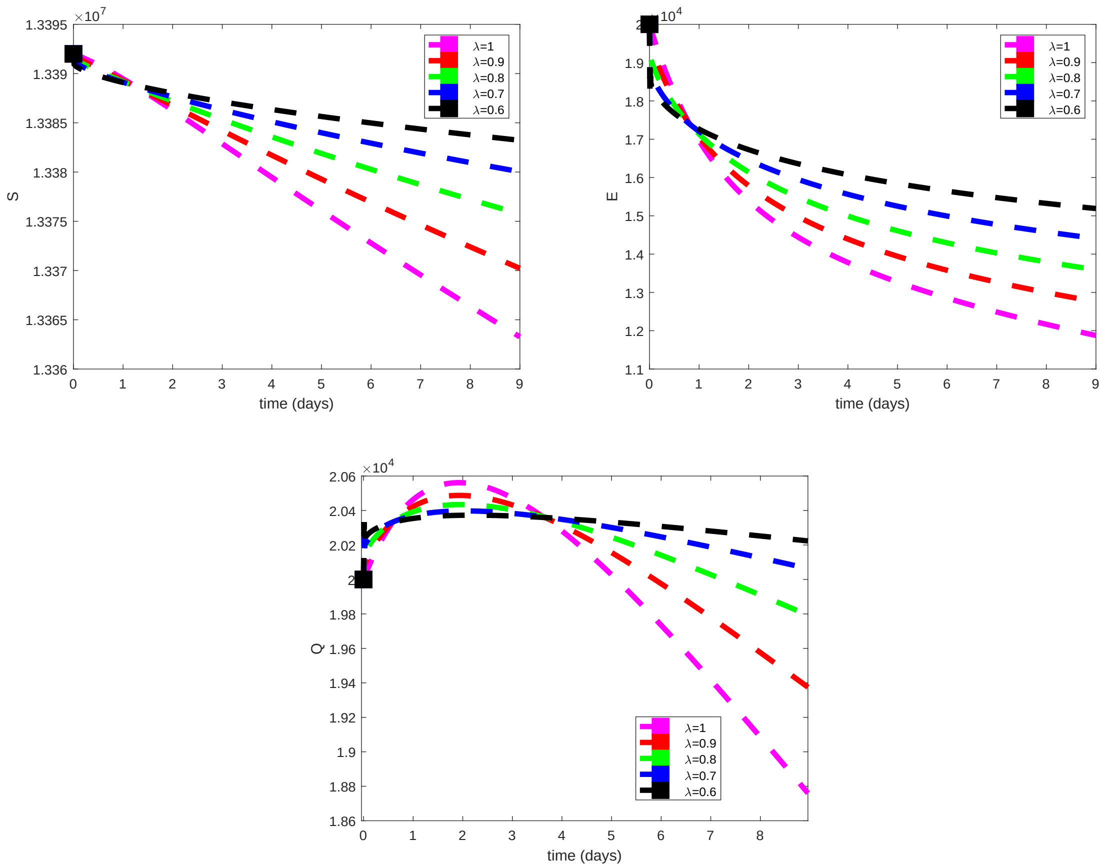

Figure 11. The behavior of the first three state variables for the COVID-19.

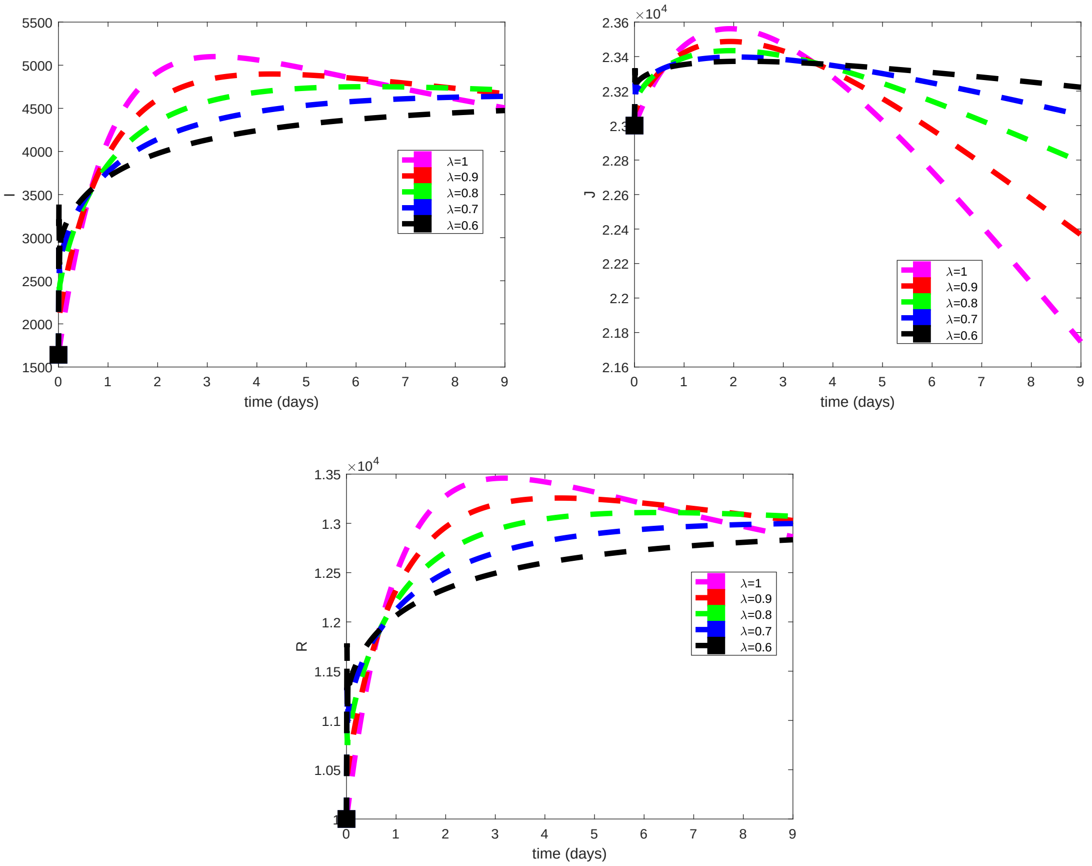

Figure 12. The behavior of the other three state variables for the COVID-19.

11. Conclusions

The fractional operator is convenient in the literature for working the transmission dynamics of COVID-19 disease. Many valuable properties of the given fractional version of the model have been presented, such as the model formation, the existence and uniqueness of the solution through the fixed point theorem, invariant region, stability analysis, the sensitivity analysis and, most importantly, the basic number of reproductions. It should be noted that the most sensitive variable to the basic reproduction number is β which is the disease transmission rate from infected to susceptible people. Also, the least sensitive parameters are μ and γ_2, which are the natural mortality rate and Recovery rate of isolated individuals. Also, the effect of fractional parameter on the sensitivity indices is very high, as the fractional parameter increasing the indices of all the parameters also increases.

Acknowledgments

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Conflict of interest

The authors declare no conflict of interest.

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