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Research article

The growth of entire solutions of certain nonlinear differential-difference equations

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Abstract: This paper is concerned with entire solutions of nonlinear differential-difference equations. We will characterize the growth of entire solutions for two classes of nonlinear differential-difference equations. Our results will contribute to the generalization and completion of some results obtained recently. The results are explained by various examples and remarks.

Keywords: nonlinear differential-difference equations; entire solutions; order; Nevanlinna theory
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1. Introduction

Throughout this paper, the word "meromorphic" means meromorphic in the complex plane $\mathbb{C}$. Assuming the reader is familiar with the elementary Nevanlinna theory, we will adopt the standard notations associated with the theory, such as the characteristic function $T(r, f)$, the counting function of the poles $N(r, f)$, and the proximity function $m(r, f)$. For standard terms and symbols of Nevanlinna theory, one can refer to [3, 12]. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as r tends to infinity outside a possible exceptional set of finite logarithmic measure. In addition, the order $\rho(f)$, the hyper order $\rho_2(f)$ of a meromorphic function f are defined in turn as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

In the past two decades, Nevanlinna theory has been used to study solvability and the existence of entire or meromorphic solutions of differential or difference equations in complex domains (see [8, 9, 16, 18]). In this paper, we consider general differential-difference equations, which can be traced back to 1962.

In [3], Clunie gave a proof for the result of $a_n(z)(f(z)+c)^n = b(z)e^{ng(z)}$ provided from

$$a_n(z)f^n(z) + a_{n-1}(z)f^{n-1}(z) + \cdots + a_0(z) = b(z)e^{ng(z)} \quad (a_n(z) \not\equiv 0), \tag{1.1}$$

where f and g are entire functions, and $a_i(i = 0, 1, ..., n)$ and b, c are small functions of f. In [5], Hayman considered the following nonlinear differential equation:

$$f^n(z) + Q_d(z, f) = g(z), \tag{1.2}$$

where f and g are nonconstant meromorphic functions, $Q_d(z, f)$ denotes a differential polynomial in f of degree d with coefficients being small functions and $d \leq n-1$ in (1.2). And he got that if $N(r, f) + N(r, \frac{1}{g}) = S(r, f)$, then $g(z) = (f(z) + \gamma)^n$, where γ is meromorphic and a small function of $f(z)$. From above we also have $N(r, g) + N(r, \frac{1}{g}) = S(r, f)$. Further, if f has finite order, then $g(z)$ is of the form $b(z)e^{\beta(z)}$, where β is a polynomial.

In [15], Yang and Laine investigated finite order entire solutions $f(z)$ of nonlinear differential-difference equations of the form

$$f^n(z) + L(z, f) = h(z), \tag{1.3}$$

where $L(z, f)$ is a linear differential-difference polynomial in f with coefficients being small functions, $h(z)$ is meromorphic, and $n \geq 2$ is an integer. In particular, it is known that the equation $f^2(z) + q(z)f(z+1) = P(z)$, where $P(z)$, $q(z)$ are polynomials, has no transcendental entire solutions of finite order. In [13], Wen et al. considered the finite order entire solutions f of the nonlinear difference equation

$$f^n(z) + q(z)e^{Q(z)}f(z+c) = P(z), \tag{1.4}$$

where $P(z)$, $q(z)$, and $Q(z)$ are polynomials, $Q(z)$ is not a constant, and $n \geq 2$ is an integer. In [11], Remark 1 (a), Liu showed that every meromorphic solution f of (1.4) is entire with the help of idea that appeared in Naftalevich [12]. Motivated by (1.4) and some results (see [10]), Chen et al. [2] discussed the following nonlinear differential-difference equation:

$$f^n(z) + \omega f^{n-1}(z)f'(z) + q(z)e^{Q(z)}f(z+c) = u(z)e^{v(z)}, \tag{1.5}$$

where n is a positive integer, $q \not\equiv 0$, Q, u, and v are nonconstant polynomials, and $c \neq 0$ and ω are constants. They proved the following result.

Theorem A. *Let n be an integer satisfying $n \geq 3$ for $\omega \neq 0$ and $n \geq 2$ for $\omega = 0$. Suppose f is a transcendental entire solution of finite order of (1.5). Then, every solution f satisfies one of the following:*

(1) $\rho(f) < \deg v = \deg Q$ *and* $f(z) = Ce^{-z/\omega}$*, where C is a constant.*

(2) $\rho(f) = \deg Q \geq \deg v$.

Remark 1.1. Li [6] proved that this result still holds for $n = 2$, $\omega \neq 0$.

It is natural to ask what happens if the higher-order differential is included in dominant term on the left-hand side of (1.5)? In this paper, we consider this problem. We need some notations to state the following results. Suppose that p is a positive integer and $c \in \mathbb{C} \cup \{\infty\}$. We use $N_{p)}(r, 1/(f -$

$c)$) ($N_{(p}(r, 1/(f-c))$) to denote the counting function of zeros of $f-c$, whose multiplicities are less than or equal to p (greater than or equal to p). Define

$$\delta(c, f) = 1 - \limsup_{r\to\infty} \frac{N(r, 1/(f-c))}{T(r, f)},$$

$$\delta_{p)}(c, f) = 1 - \limsup_{r\to\infty} \frac{N_{p)}(r, 1/(f-c))}{T(r, f)}.$$

Specifically, if the dominant term $f^n(z)+\omega f^{n-1}(z)f'(z)$ is replaced by $bf^n(z)+af^{n-1}(z)f^{(k)}(z)$ in (1.5), we obtain the following Theorem 1.2.

Theorem 1.2. *Let $n \geq 3$ and $k \geq 1$ be integers, $c \neq 0$, a, and b be constants, and $(a, b) \neq (0, 0)$, q, Q, u, and v be nonconstant polynomials. Suppose that the nonlinear differential-difference equation*

$$bf^n(z) + af^{n-1}(z)f^{(k)}(z) + q(z)e^{Q(z)}f(z+c) = u(z)e^{v(z)} \tag{1.6}$$

satisfies $bf^n(z) + af^{n-1}(z)f^{(k)}(z) \not\equiv 0$ and admits a transcendental entire solution of finite order $f(z)$ with $\delta_{1)}(0, f) > 0$. Then $\rho(f) = \deg Q \geq \deg v$.

Remark 1.3. Obviously, the condition $bf^n(z) + af^{n-1}(z)f^{(k)}(z) \not\equiv 0$ in Theorem 1.2 is necessary. Otherwise, we can obtain the form of the solution directly. In particular, from the proof of Theorem A, it can be seen that the reason for conclusion (1) is $f^n(z) + \omega f^{n-1}(z)f'(z) \equiv 0$. Now, we provide an example to illustrate the necessity of the condition.

Example 1.4 ([2, Example 1.5])**.** The function $f(z) = 2e^{-z}$ is a transcendental entire solution of the differential-difference equation

$$f^3(z) + f^2(z)f'(z) + ze^{z^2+z+1}f(z+1) = 2ze^{z^2}.$$

By Example 1.4, we can observe that $f^3(z) + f^2(z)f'(z) \equiv 0$, and the solution $f(z)$ is not consistent with the conclusion of Theorem 1.2.

Remark 1.5. (a) If only $q(z) \equiv 0$ in Eq (1.6), then Eq (1.6) can be rewritten to the form $bf^n(z) + af^{n-1}(z)f^{(k)}(z) = u(z)e^{v(z)}$. Obviously, $\deg v \leq \rho(f)$. If $\deg v < \rho(f)$, Eq (1.6) can be also rewritten to $bf^n(z) + af^{n-1}(z)f^{(k)}(z) = u_1(z)$, where $T(r, u_1(z)) = S(r, f)$. Thus, $N\left(r, \frac{1}{f}\right) = S(r, f)$ for $n \geq 2$. By Lemma 2.1 in Section 2, we have $m(r, \frac{1}{f}) = S(r, f)$. Thus, $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1) = S(r, f)$, a contradiction. Therefore, $\rho(f) = \deg v$.

(b) If $Q(z) \equiv C$ with $q(z)$, $u(z)$, $v(z) \not\equiv 0$ in Eq (1.6), where C is a constant, then Eq (1.6) can be rewritten to

$$bf^n(z) + af^{n-1}(z)f^{(k)}(z) + q_1(z)f(z+c) = u(z)e^{v(z)},$$

where $T(r, q_1(z)) = S(r, f)$. Obviously, $\deg v \leq \rho(f)$. If $\deg v < \rho(f)$, Eq (1.6) can be rewritten to

$$bf^n(z) + af^{n-1}(z)f^{(k)}(z)f + q(z)f(z+c) = u_2(z),$$

where $T(r, u_2(z)) = S(r, f)$. From the proof of Lemma 2.7 in Section 2, we have $\deg v = \rho(f)$.

Remark 1.6. Let us recall the definition of an exponential polynomial of the form

$$f(z) = P_1(z)e^{Q_1(z)} + \cdots + P_k(z)e^{Q_k(z)}, \tag{1.7}$$

where $P_j(z)$ and $Q_j(z)(j = 1, ..., k)$ are polynomials. Denote

$$\Gamma = \{e^{\alpha(z)} : \alpha(z) \text{ is a noncontant polynomial}\}.$$

The condition "$\delta_{1)}(0, f) > 0$" due to an idea that appeared in [7]. Obviously, if a solution $f(z)$ belongs to Γ, then $\delta_{1)}(0, f) = 1 > 0$. Therefore, if solutions $f(z)$ of Eq (1.6) belong to Γ, we can eliminate the condition "$\delta_{1)}(0, f) > 0$" in Theorem 1.2.

Next, we give an example to illustrate the existence of the solution in Theorem 1.2.

Example 1.7. The function $f(z) = e^z$ is a transcendental entire solution of the differential-difference equation

$$f^3(z) + f^2(z)f''(z) + ze^{2z-1}f(z + 1) = (2 + z)e^{3z}.$$

By Example 1.7, we can observe that for

$$bf^n(z) + af^{n-1}(z)f^{(k)}(z) = f^3(z) + f^2(z)f'(z) \not\equiv 0,$$

where $n = 3$, $k = 2$, $\delta_{1)}(0, f) = 1$, $\rho(f) = \deg Q = \deg v$ is consistent with the conclusion of Theorem 1.2.

In [2], Chen et al. also considered the entire solutions with finite order to the following differential-difference equation:

$$f^n(z) + \omega f^{n-1}(z)f'(z) + q(z)e^{Q(z)}f(z + c) = p_1e^{\lambda z} + p_2e^{-\lambda z}, \tag{1.8}$$

where n is an integer, c, λ, p_1, and p_2 are nonzero constants and ω is a constant, and $q \not\equiv 0$, Q are polynomials such that Q is not a constant. They proved the following result.

Theorem B. *If $f(z)$ is a transcendental entire solution with finite order to* (1.8)*, then the following conclusions hold:*

(1) *If $n \geq 4$ for $\omega \neq 0$ and $n \geq 3$ for $\omega = 0$, then every solution $f(z)$ satisfies $\rho(f) = \deg Q = 1$.*

(2) *If $n \geq 1$ and $f(z)$ is a solution to* (1.8) *which belongs to Γ, then*

$$f(z) = e^{\lambda z/n+B}, \quad Q(z) = -\frac{n+1}{n}\lambda z + b$$

or

$$f(z) = e^{-\lambda z/n+B}, \quad Q(z) = \frac{n+1}{n}\lambda z + b,$$

where b, $B \in \mathbb{C}$.

Remark 1.8. We can find the prototype of (1.8) in many places (see [1, 14, 18]). If $n \geq 3$ and $\omega = 0$ in Theorem B, we also can obtain the conclusions (1) and (2) by Chen et al. [1].

If $f^n(z) + \omega f^{n-1}(z)f'(z)$ be substituted by $bf^n(z) + af^{n-1}(z)f^{(k)}(z)$ in Theorem B, we can obtain the following theorem.

Theorem 1.9. *Let $n \geq 3$, $k \geq 1$ be integers, $c \neq 0$, $p_1 \neq 0$, $p_2 \neq 0$, a, b, $\lambda \neq 0$ be constants with $(a, b) \neq (0, 0)$, q, Q be nonconstant polynomials. Suppose that the nonlinear differential-difference equation*

$$bf^n(z) + af^{n-1}(z)f^{(k)}(z) + q(z)e^{Q(z)}f(z+c) = p_1e^{\lambda z} + p_2e^{-\lambda z} \tag{1.9}$$

satisfies $bf^n(z) + af^{n-1}(z)f^{(k)}(z) \not\equiv 0$ and admits a transcendental entire solution of finite order $f(z)$ with $\delta_{1)}(0, f) > 0$. Then $\rho(f) = \deg Q = 1$.

Remark 1.10. In Theorem 1.9, by adding the condition $\delta_{1)}(0, f) > 0$, we prove that the conclusion (1) is still true when $n = 3$, $\omega \neq 0$ in Theorem B. However, if a solution f belongs to Γ, then $\delta_{1)}(0, f) = 1 > 0$. Therefore, if solutions $f(z)$ of Eq (1.9) belong to Γ, we can eliminate the condition "$\delta_{1)}(0, f) > 0$" in Theorem 1.9.

Remark 1.11. (a) If $p_1e^{\lambda z} + p_2e^{-\lambda z} \equiv C$ in Eq (1.9), where C is a constant, then we can obtain $\rho(f) = \deg Q$ by Lemma 2.7 in Section 2. Thus, we default to $p_1e^{\lambda z} + p_2e^{-\lambda z} \not\equiv 0$ in Theorem 1.9. If either p_1 or p_2 is equal to zero, then Theorem 1.9 is equivalent to Theorem 1.2.

(b) If $q(z) \equiv C_1$ or $Q(z) \equiv C_2$ in Eq (1.9), where C_1, C_2 are constants, then we can see that $\rho(f) = 1$ directly from Eq (1.9).

(c) If $q(z)$ is not a constant, $Q(z) \equiv C$, $n = 3$, when $a = 0$ in Eq (1.9), where C is a constant, then we obtain that the equation does not have any transcendental entire solution of finite order by [14, 18].

Next, we give an example to illustrate the existence of the solution in Theorem 1.9.

Example 1.12. The function $f(z) = e^z$ is a solution of the differential-difference equation

$$f^3(z) + f^2(z)f''(z) + \frac{1}{2}e^{-4z}f(z + \log 2) = 2e^{3z} + e^{-3z}.$$

By Example 1.12, we can observe that $n = 3$ and $k = 2$, $a = b = 1$, $p_1 = 2$, $p_2 = 1$, $\lambda = 3$, $\delta_{1)}(0, f) = 1$. Thus, the conclusion $\rho(f) = \deg Q = 1$ is consistent with the conclusion of Theorem 1.9.

2. Preliminary lemmas

Lemma 2.1 ([17, Theorem 1.22])**.** *Let f be a meromorphic function and let $k \in N$. Then*

$$m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = S(r, f),$$

where $S(r, f) = O(\log T(r, f) + \log r)$ $(r \to \infty, r \notin E, mesE < \infty)$.

Lemma 2.2 ([4, Theorem 5.1])**.** *Let f be a nonconstant meromorphic function, $\varepsilon > 0$, $c \in \mathbb{C}$. If $\rho_2(f) < 1$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r, f)}{r^{1-\rho_2-\varepsilon}}\right),$$

$(r \to \infty, r \notin E, mesE < \infty)$.

Lemma 2.3 ([15, Lemma 2.2]). *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$H(z, f)P(z, f) = Q(z, f),$$

where $H(z, f)$, $P(z, f)$, and $Q(z, f)$ are difference polynomials in f such that the total degree of $Q(z, f)$ is less than or equal to that of $H(z, f)$. If $H(z, f)$ contains just one term of maximal total degree, then for any $\varepsilon > 0$,

$$m(r, P(z, f)) = O(r^{\rho-1+\varepsilon}) + S(r, f),$$

possibly outside of an exceptional set of finite logarithmic measure.

The following result is a Clunie-type lemma [3] for the differential-difference polynomials of a meromorphic function f. It can be proved by applying Lemma 2.3 with a similar reasoning as in [16]. It is stated as follows.

Proposition 2.4. *If in the above lemma $H(z, f) = f^n$, then a similar conclusion holds if $P(z, f)$, $Q(z, f)$ are differential-difference polynomials in f.*

Lemma 2.5 ([7, Lemma 2.4]). *Let $Q(z, f)$ be a differential polynomial in f of degree d with small functions of f as coefficients. Then, we have $m(r, Q) \leq dm(r, f) + S(r, f)$.*

Lemma 2.6 ([17, Theorem 1.51]). *Suppose that $f_1, f_2, \ldots, f_n$ $(n \geq 2)$ are meromorphic functions and $g_1, g_2, \ldots, g_n$ are entire functions satisfying the following conditions:*

(1) $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0$*;*

(2) $g_j - g_k$ *are not constant for* $1 \leq j < k \leq n$*;*

(3) $T(r, f_j) = o(T(r, e^{g_h - g_k}))$ $(r \to \infty, r \notin E, mesE < \infty)$, $1 \leq j \leq n, 1 \leq h < k \leq n$. *Then,* $f_j \equiv 0$, $j = 1, \ldots, n$.

Lemma 2.7. *Let $n \geq 3$, $k \geq 1$ be integers, $c \neq 0$, a, b be constants with $(a, b) \neq (0, 0)$, $q(z) \not\equiv 0$, $Q(z) \not\equiv 0$ be polynomials, and $u(z)$ be a small function of $f(z)$. Suppose that the nonlinear differential-difference equation*

$$bf^n(z) + af^{n-1}f^{(k)}(z) + q(z)e^{Q(z)}f(z + c) = u(z) \tag{2.1}$$

satisfies $bf^n(z) + af^{n-1}(z)f^{(k)}(z) \not\equiv 0$ and admits a transcendental entire solution of finite order $f(z)$. Then $\rho(f) = \deg Q$.

Proof. Set $f(z + c) = f_c$. From Lemmas 2.1 and 2.2, we see that

$$\begin{aligned} T(r, e^Q) = m(r, e^Q) &= m\left(r, \frac{u - bf^n - af^{n-1}f^{(k)}}{qf_c}\right) \\ &\leq m\left(r, \frac{1}{qf_c}\right) + m(r, u) + m(r, bf^n + af^{n-1}f^{(k)}) + O(1) \\ &\leq m\left(r, \frac{f}{qf_c}\right) + m\left(r, \frac{1}{f}\right) + nT(r, f) + S(r, f) \\ &\leq (n + 1)T(r, f) + S(r, f). \end{aligned} \tag{2.2}$$

Thus, we deduce that $\deg Q \leq \rho(f)$. If $\deg Q < \rho(f)$, then $T(r, e^Q) = S(r, f)$. Equation (2.1) can be written as

$$f^{n-1}(bf + af^{(k)}) = u(z) + q_1(z)f_c, \tag{2.3}$$

where $T(r, q_1) = S(r, f)$. Since $n \geq 3$, from Proposition 2.4, that $m(r, bf + af^{(k)}) = S(r, f)$, $m(r, bf^2 + aff^{(k)}) = S(r, f)$. Furthermore, we note that f is entire; thus, $T(r, bf + af^{(k)}) = S(r, f)$, and $T(r, bf^2 + aff^{(k)}) = S(r, f)$. By $bf^n + af^{n-1}f^{(k)} \not\equiv 0$, we conclude that

$$T(r, f) \leq T(r, bf^2 + aff^{(k)}) + T\left(r, \frac{1}{bf + af^{(k)}}\right) = S(r, f),$$

which is absurd. Hence, $\rho(f) = \deg Q$.

This completes the proof of Lemma 2.7.

3. Proof of Theorem 1.2

Suppose that f is a transcendental entire solution of finite order of Eq (1.6) with $\delta_{1)}(0, f) > 0$. Set $f(z + c) = f_c$. Lemmas 2.1 and 2.2 indicate that

$$\begin{aligned} T(r, e^Q) = m(r, e^Q) &= m\left(r, \frac{ue^v - bf^n - af^{n-1}f^{(k)}}{qf_c}\right) \\ &\leq m\left(r, \frac{1}{qf_c}\right) + m(r, ue^v) + m(r, bf^n + af^{n-1}f^{(k)}) + O(1) \\ &\leq m\left(r, \frac{f}{qf_c}\right) + m\left(r, \frac{1}{f}\right) + nT(r, f) + T(r, e^v) + S(r, e^v) \\ &\leq (n + 1)T(r, f) + S(r, f) + T(r, e^v) + S(r, e^v). \end{aligned} \tag{3.1}$$

We consider the following three cases.

Case 1. If $\rho(f) < \deg v$, then we obtain from (3.1) that $T(r, e^Q) \leq T(r, e^v) + S(r, e^v)$. Thus, $\deg Q \leq \deg v$. We shall show that $\deg Q = \deg v$. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned} T(r, e^v) = m(r, e^v) &= m\left(r, \frac{qe^Q f_c + bf^n + af^{n-1}f^{(k)}}{u}\right) \\ &\leq m(r, qf_c) + m(r, e^Q) + m(r, bf^n + af^{n-1}f^{(k)}) + S(r, f) \\ &\leq m\left(r, \frac{qf_c}{f}\right) + m(r, f) + T(r, e^Q) + nT(r, f) + S(r, f) \\ &\leq (n + 1)T(r, f) + S(r, f) + T(r, e^Q) + S(r, e^v) \\ &\leq T(r, e^Q) + S(r, e^v). \end{aligned}$$

Thus, we deduce that $\deg Q = \deg v$, and $\rho(f) < \deg Q$.

By differentiating both sides of Eq (1.6), we have

$$bnf^{n-1}f' + a(n - 1)f^{n-2}f'f^{(k)} + af^{n-1}f^{(k+1)} + (q'f_c + qf_c' + qf_cQ')e^Q = (u' + uv')e^v. \tag{3.2}$$

From (1.6) and (3.2), we have

$$A_2e^Q = A_1,$$

where

$$\begin{aligned}A_1 =&b(u'+uv')f^n + a(u'+uv')f^{n-1}f^{(k)} - bnuf^{n-1}f'\\&- a(n-1)uf^{n-2}f'f^{(k)} - auf^{n-1}f^{(k+1)},\\A_2 =&u(q'f_c + qf_c' + qf_cQ') - q(u'+uv')f_c.\end{aligned}$$

We discuss two subcases in the following:

Case 1 (i). If $A_2 \not\equiv 0$, then we obtain $e^Q = \frac{A_1}{A_2}$. Noting that $\rho(f) < \deg Q$, we obtain $T(r, f) = S(r, e^Q)$, and $T(r, A_j) = S(r, e^Q)$, $j = 1, 2$. Thus, $T(r, e^Q) \le S(r, e^Q)$, which yields a contradiction.

Case 1 (ii). If $A_2 \equiv 0$, then we have

$$\frac{q'}{q} + \frac{f_c'}{f_c} + Q' = \frac{u'}{u} + v'.$$

By integrating, we obtain $f_c = \frac{t_1}{q(z)}u(z)e^{v(z)-Q(z)}$, where t_1 is a nonzero constant.

If $t_1 = 1$, then $qf_c = ue^{v-Q}$. We substitute this expression back into (1.6), and we obtain $bf^n + af^{n-1}f^{(k)} \equiv 0$, a contradiction to the assumption that $bf^n + af^{n-1}f^{(k)} \not\equiv 0$.

If $t_1 \neq 1$, then we have $f(z) = g_1(z)e^{w_1(z)}$ with

$$g_1(z) = \frac{t_1}{q(z-c)}, \quad w_1(z) = v(z-c) - Q(z-c).$$

Then, we substitute this expression back into (1.6), it is not hard to see that

$$(bg_1^n + ag_1^{n-1}L)e^{nw_1} = (1-t_1)ue^v,$$

where L is a polynomial in $g_1, g_1', g_1'', ..., g_1^{(k)}, w_1', w_1'', ..., w_1^{(k)}$, so that $\deg w_1 = \deg v$, which is a contradiction to $\deg w_1 = \rho(f) < \deg v$.

Case 2. If $\rho(f) > \deg v$, then $T(r, e^v) = S(r, f)$. By Lemma 2.7, we conclude that $\deg v < \deg Q = \rho(f)$.

Case 3. If $\rho(f) = \deg v$, from (3.1), we have $\deg Q \le \rho(f)$. Now, we claim that $\deg Q = \rho(f)$. Suppose that $\deg Q < \rho(f)$. Denote $D = q(z)e^{Q(z)}$; then, $T(r, D) = S(r, f)$. By (1.6), we obtain

$$bf^n + af^{n-1}f^{(k)} + Df_c = ue^v. \tag{3.3}$$

Differentiating both sides of (3.3), we have

$$bnf^{n-1}f' + a(n-1)f^{n-2}f'f^{(k)} + af^{n-1}f^{(k+1)} + D'f_c + Df_c' = (u'+uv')e^v. \tag{3.4}$$

By eliminating e^v, from (1.6) and (3.4), we have

$$f^{n-2}\varphi = D'uf_c + Duf_c' - D(u'+uv')f_c, \tag{3.5}$$

where

$$\varphi = b(u'+uv')f^2 + a(u'+uv')ff^{(k)} - bnuff' - a(n-1)uf'f^{(k)} - auff^{(k+1)}. \tag{3.6}$$

Since $n \ge 3$, it follows from Proposition 2.4 that we have $m(r, \varphi) = S(r, f)$. By combining with the fact that f is entire, we obtain $T(r, \varphi) = S(r, f)$.

If $\varphi \not\equiv 0$, then from (3.6), we obtain

$$\frac{1}{f^2}=\frac{1}{\varphi}\left(b(u'+uv')+a(u'+uv')\frac{f^{(k)}}{f}-bnu\frac{f'}{f}\frac{f^{(k)}}{f}-a(n-1)u\frac{f'}{f}\frac{f^{(k)}}{f}-au\frac{f^{(k+1)}}{f}\right).$$

By Lemmas 2.1 and 2.5, we have

$$2m\left(r,\frac{1}{f}\right)\le T(r,\varphi)+S(r,f)=S(r,f).$$

On the other hand, if z_0 is a multiple zero of f which is not a zero or pole of u and v, then it follows from (3.6) that z_0 be a zero of φ. Thus,

$$N_{(2}\left(r,\frac{1}{f}\right)\le 2N\left(r,\frac{1}{\varphi}\right)+S(r,f)=S(r,f),$$

$$T(r,f)=T\left(r,\frac{1}{f}\right)+S(r,f)=m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+S(r,f)=N_{1)}\left(r,\frac{1}{f}\right)+S(r,f),$$

which contradicts the assumption that $\delta_{1)}(0,f)>0$.

If $\varphi\equiv 0$, then from (3.5), we have

$$\frac{q'}{q}+\frac{f_c'}{f_c}+Q'=\frac{u'}{u}+v'.$$

By integrating, we obtain $f_c=\frac{t_2}{q}u(z)e^{v-Q}$, where t_2 is a nonzero constant.

If $t_2=1$, then $qf_c=ue^{v-Q}$. We substitute this expression back into (1.6), and we obtain $bf^n+af^{n-1}f^{(k)}\equiv 0$, which is a contradiction to the assumption that $bf^n+af^{n-1}f^{(k)}\not\equiv 0$.

If $t_2\neq 1$, then we have $f(z)=g_2(z)e^{w_2(z)}$ with

$$g_2(z)=\frac{t_2}{q(z-c)},$$

$$w_2(z)=v(z-c)-Q(z-c).$$

Then, substituting this expression back into (1.6), we obtain

$$(bg_2^n+ag_2^{n-1}S)e^{nw_2}=(1-t_2)ue^{v},$$

where S is a polynomial in $g_2,g_2',g_2'',...,g_2^{(k)},w_2',w_2'',...,w_2^{(k)}$. By Lemma 2.6, we can obtain $t_2=1$, which yields a contradiction.

This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.9

Suppose that f is a transcendental entire solution of finite order of Eq (1.9) with $\delta_{1)}(0,f)>0$. Set

$$f(z+c)=f_c,\quad P(z)=p_1e^{\lambda z}+p_2e^{-\lambda z}\not\equiv C,$$

then $\rho(P) = 1$. From Lemmas 2.1 and 2.2, we can deduce that

$$\begin{aligned}
T(r, e^Q) = m(r, e^Q) &= m\left(r, \frac{P - bf^n - af^{n-1}f^{(k)}}{qf_c}\right) \\
&\leq m\left(r, \frac{1}{qf_c}\right) + m(r, P) + m(r, bf^n + af^{n-1}f^{(k)}) + O(1) \\
&\leq m\left(r, \frac{f}{qf_c}\right) + m\left(r, \frac{1}{f}\right) + nT(r, f) + T(r, p) + O(1) \\
&\leq (n+1)T(r, f) + S(r, f) + T(r, P). \qquad (4.1)
\end{aligned}$$

We consider the following three cases.

Case 1. If $\rho(f) < 1$, from (4.1), we obtain $T(r, e^Q) \leq T(r, P) + S(r, P)$, $\deg Q \leq 1$. Recall that $\deg Q \geq 1$. Thus, $\deg Q = 1$. Let $Q = mz + n$, $m \neq 0$, and n be constants. In this case, Eq (1.9) can be written as

$$bf^n + af^{n-1}f^{(k)} + q(z)e^{mz+n}f(z+c) = p_1e^{\lambda z} + p_2e^{-\lambda z}. \qquad (4.2)$$

By differentiating both sides of Eq (4.2), we have

$$\begin{aligned}
&nbf^{n-1}f' + (n-1)af^{n-2}f'f^{(k)} + af^{n-1}f^{(k+1)} \\
&\qquad + (q'f_c + qf_c' + mqf_c)e^{mz+n} = \lambda p_1e^{\lambda z} - \lambda p_2e^{-\lambda z}. \qquad (4.3)
\end{aligned}$$

Eliminating $e^{-\lambda z}$ from (4.2) and (4.3) yields

$$\begin{aligned}
&b\lambda f^n + a\lambda f^{n-1}f^{(k)} + nbf^{n-1}f' + (n-1)af^{n-2}f'f^{(k)} + af^{n-1}f^{(k+1)} \\
&\qquad + (q\lambda f_c + q'f_c + qf_c' + mqf_c)e^{mz+n} = 2\lambda p_1e^{\lambda z}. \qquad (4.4)
\end{aligned}$$

By differentiating both sides of Eq (4.4), we obtain

$$\begin{aligned}
&b\lambda nf^{n-1}f' + a\lambda(n-1)f^{n-2}f'f^{(k)} + a\lambda f^{n-1}f^{(k+1)} + n(n-1)bf^{n-2}(f')^2 + nbf^{n-1}f'' \\
&\qquad + (n-1)(n-2)af^{n-3}(f')^2f^{(k)} + (n-1)af^{n-2}f''f^{(k)} \\
&\qquad + 2(n-1)af^{n-2}f'f^{(k+1)} + af^{n-1}f^{(k+2)} + (A' + mA)e^{mz+n} = 2\lambda^2p_1e^{\lambda z}, \qquad (4.5)
\end{aligned}$$

where $A = q\lambda f_c + q'f_c + qf_c' + mqf_c$. Eliminating $e^{\lambda z}$ from (4.4) and (4.5) yields

$$\begin{aligned}
&b\lambda^2f^n + a\lambda^2f^{n-1}f^{(k)} + nb\lambda f^{n-1}f' + (n-1)a\lambda f^{n-2}f'f^{(k)} + a\lambda f^{n-1}f^{(k+1)} \\
&\qquad - b\lambda nf^{n-1}f' - a\lambda(n-1)f^{n-2}f'f^{(k)} - a\lambda f^{n-1}f^{(k+1)} - n(n-1)bf^{n-2}(f')^2 \\
&\qquad - nbf^{n-1}f'' - (n-1)(n-2)af^{n-3}(f')^2f^{(k)} - (n-1)af^{n-2}f''f^{(k)} \\
&\qquad - 2(n-1)af^{n-2}f'f^{(k+1)} - af^{n-1}f^{(k+2)} + (A\lambda - A' - mA)e^{mz+n} = 0.
\end{aligned}$$

Note that when $\rho(f) < 1$ and $m \neq 0$, we have $A\lambda - A' - mA \equiv 0$. If $\lambda \neq m$, by integration, we see that there exists constant $C_1 \neq 0$ such that $A = C_1e^{(\lambda-m)z}$. Thus, $\rho(A) = 1$. Since $\rho(A) = \rho(f) < 1$, we obtain a contradiction. If $\lambda = m$, then $A' \equiv 0$. By integration, we see that there exists constant $C_2 \neq 0$ such that $q\lambda f_c + q'f_c + qf_c' + mqf_c = 2\lambda qf_c + (qf_c)' = C_2$. Solving this equation, we obtain $qf_c = -\frac{C_2}{2\lambda}e^{-4\lambda z} + C_3e^{-2\lambda z}$, where C_3 is a constant. Thus, we have $\rho(f_c) = \rho(f) = 1$, a contradiction.

Case 2. If $\rho(f) > 1$, then $T(r, P) = S(r, f)$. Rewrite (4.3) as

$$nbf^{n-1}f' + (n-1)af^{n-2}f'f^{(k)} + af^{n-1}f^{(k+1)} + He^{Q} = P', \tag{4.6}$$

where $H = q'f_c + qf_c' + Q'qf_c$. Eliminating e^{Q} from (1.9) and (4.6) yields

$$f^{n-2}\phi = PH - P'qf_c,$$

where

$$\phi = bHf^2 + aHff^{(k)} - nbqff'f_c - (n-1)aqf'f^{(k)}f_c - aqff^{(k+1)}f_c. \tag{4.7}$$

We discuss two subcases in the following:

Case 2 (i). If $\phi \equiv 0$, then $PH - P'qf_c = Pq'f_c + Pqf_c' + PQ'qf_c - P'qf_c \equiv 0$. This gives that

$$\frac{q'}{q} + \frac{f_c'}{f_c} + Q' = \frac{P'}{P}.$$

By integration, we see that there exists a constant $C_4 \neq 0$ such that $qf_c = C_4Pe^{Q}$. Thus,

$$f(z) = \frac{C_4P(z-c)e^{Q(z-c)}}{q(z-c)} = g(z)e^{w(z)},$$

where $g(z) = \frac{P(z-c)}{q(z-c)}$, $\quad w(z) = Q(z-c)$. Then, substituting this expression back into (1.9), we obtain

$$(bg^n + ag^{n-1}Y)e^{nw} + C_4Pe^{2Q} = P,$$

where Y is a polynomial in $g, g', g'', ..., g^{(k)}, w', w'', ..., w^{(k)}$. By Lemma 2.6, we can obtain $P \equiv 0$, which yields a contradiction.

Case 2 (ii). If $\phi \not\equiv 0$, it follows from Propositions 2.4 and $n \geq 3$ that $m(r, \phi) = S(r, f)$. By combining with the fact $N(r, f) = S(r.f)$, we have $T(r, \phi) = S(r, f)$. From (4.7), we have

$$\frac{1}{f^3} = \frac{1}{\phi}\left(\frac{bH}{f} + \frac{aH}{f}\frac{f^{(k)}}{f} - \frac{nbqf'}{f}\frac{f_c}{f} - \frac{(n-1)aqf'}{f}\frac{f_c}{f}\frac{f^{(k)}}{f} - \frac{aqf^{(k+1)}}{f}\frac{f_c}{f}\right),$$

where $\frac{H}{f} = q'\frac{f_c}{f} + q\frac{f_c'}{f} + qQ'\frac{f_c}{f}$. By Lemma 2.1 and Lemma 2.5, we have

$$3m\left(r, \frac{1}{f}\right) \leq T(r, \phi) + S(r, f) = S(r, f).$$

On the other hand, if z_0 is a multiple zero of f which is not a zero or pole of q and Q, then it follows from (4.7) that z_0 is a zero of ϕ. Hence,

$$N_{(2}\left(r, \frac{1}{f}\right) \leq 2N\left(r, \frac{1}{\phi}\right) + S(r, f) = S(r, f),$$

$$T(r, f) = T\left(r, \frac{1}{f}\right) + S(r, f) = m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + S(r, f) = N_{1)}\left(r, \frac{1}{f}\right) + S(r, f),$$

which contradicts the assumption that $\delta_{1)}(0, f) > 0$.

Case 3. If $\rho(f) = 1$, from (4.1), we obtain $\deg Q \leq 1$. Recall that $\deg Q \geq 1$. Thus, $\rho(f) = \deg Q = 1$.

This completes the proof of Theorem 1.9.

5. Conclusions

In this paper, we have described the growth of entire solutions for certain nonlinear differential-difference equations. Clunie lemma plays a key role in the proof. Our results generalize and complement some results obtained by Chen et al. and references therein. In addition, we have given specific examples and remarks to illustrate our results.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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