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Research article

Efficient one asset replacement scheme for an optimized portfolio

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Abstract: The traditional mean-variance portfolio optimization models in practice have suffered from complexity and heavy computation loads in the process of selecting the best assets for constructing a portfolio. If not, they are considerably departed from the theoretically optimized values. In this work, we develop the optimized portfolio investment strategy in which only one asset substitution occurs when re-balancing a portfolio. To do this, we briefly look into a quadratically constrained quadratic programming (QCQP), which has been well-studied for the non-negative solution. Based on the quadratic programming, an efficient scheme is presented for solving the large-scale inverse problem. We more precisely update the rank of an inverse matrix, so that the optimal solution can be easily and quickly obtained by our proposed scheme.

Various numerical and practical experiments are presented to demonstrate the validity and reliability of our scheme. Our empirical application to the U.S. and South Korea stock markets is tested and highlighted. Moreover, comparisons of a random allocation strategy and our proposed scheme reveal the better performance in the lower risks and higher expected returns obtained by our scheme.

Keywords: market portfolio; low-rank update; portfolio optimization; asset substitution problem; covariance matrix
Mathematics Subject Classification: 46N10, 91G10

1. Introduction

Modern portfolio theory is about determining how to distribute capital among available assets such that risk-averse investors can construct portfolios to relatively lower risk for a given expected return. In the work of Markowitz in 1952, variance was used as a measure of risk, which gave rise to the mean-variance portfolio optimization model. Although other mean-risk models have been proposed and studied in the finance literature, the mean-variance model continues to be the core of modern portfolio theory and it is still popular in finance [1–4].

Portfolio optimization has been widely used in the problem of portfolio selection. There is a large literature of numerical methods or algorithms for solving the portfolio selection problem to maximize profitability of a portfolio. The efficient model selection strategy to forecast stock returns is developed in [7], and using mean variance portfolio selection models in [8]. Still, portfolio optimization faces daunting challenges. The efficient portfolio where a fully diversified set of assets has the least possible risk for a given expected return can be hardly obtained since it often contains certain extreme positive or negative weights [9]. In many practical cases mixed integer linear programming is required to weight portfolio allocation [10], as well as tree-based genetic algorithm and optimized solution discussed in [11]. The stochastic volatility market models are often used for finding the optimal portfolio selection [12], and the authors in [13] argued on optimal portfolio choice.

Most existing investment strategies are far more sophisticated than simply choosing weights. In our model, we assume that short-selling is prohibited in the stock market, i.e., the market does not allow for unrestricted short-selling. This implies that some weights of assets cannot be negative. However, finding the optimal weights from the market assets is extremely hard with weights' non-negativity and unity constraints. Since our interesting in this paper is to develop a numerical scheme rather than numerical solvability, we simply normalize the investor's sum of portfolio weights to be 1 that involving the non-negativity constraint [5, 6] and reference therein.

The static Markowitz portfolio optimization that gives asset allocation under the desired expected return with minimum variance. Indeed, market data changes frequently as time marches on. However, the static Markowitz portfolio optimization is no longer to account for the dynamic market changes. Thus, the static Markowitz portfolio optimization is needed to maintain its property: low-risk and high-return by updating covariance matrix Σ. Our contribution of this work is to efficiently update covariance matrix Σ for finding the optimal portfolio weights daily subject to the nonnegative constraint, $\mathbf{x} \geq \mathbf{0}$. We assume that a portfolio consists of fixed assets, i.e. merely one asset replacement occurs to find the best portfolio under the given low-risk and high-return conditions.

An investment portfolio is a set of well-diversified assets owned by an institution or individual. Optimized portfolios can be constructed by applying mathematical programming to weight each portfolio asset. We assume that an institution or individual needs to know how to efficiently determine asset allocation.

The aim of this paper is to develop an efficient numerical scheme for finding a solution when one asset is replaced by new one among the rest of assets. Our main concern is to consider the symmetric rank-two update for solving the inverse problem. The rank-two update is based on the ShermanMorrison formulas [14] for rank-one modified matrix. The two steps of Sherman-Morrison formulas relate the inverse of a matrix after a small rank-two perturbation to the inverse of covariance matrix when one asset is substituted in the portfolio. Together with our proposed scheme, we can have the optimal portfolio weight where the optimized portfolios are constrained to have the non-negative weights and sum equals one.

In general, our proposed scheme for updating Σ is applicable to the all type of time series data. Our dynamic portfolio optimization allows for minutely, hourly, daily, weekly or monthly asset allocation maintenance, but we only consider daily stock closing prices in light of limits on available market data. As time goes by, we update the covariance matrix Σ, in other words, we replace one asset in a portfolio with one of the reference set, which induces the rank-two modification M to Σ, i.e.,

$\ddot{\Sigma} = \Sigma + M$. The main purpose of this work is not to consider the temporal derivative of Markowitz portfolio formulation, but to present the efficient way for updating covariance matrix Σ in more practical settings because financial dataset is dynamic.

The rest of the paper is organized as follows. We begin our discussion to consider the non-negative quadratic programming in Section 2. Section 3 is devoted to reformulating the inverse quadratic programming problem. We develop a symmetric rank-two update scheme for non-negative quadratic programming. We detail our proposed scheme in Section 5. Last, numerical tests are performed to validate the efficiency of the proposed algorithm in Section 6. Conclusion and discussion of this work are presented in Section 7.

2. Mathematical formulation

The main purpose in portfolio theory is to produce the lowest possible risk for any given level of expected return. More specifically, we look for a collection of the weights. Let us consider that one portfolio comprises n risky assets. Let x_i be the proportion of a portfolio in asset i such that $1 - \sum_i^n x_i = 0$. We denote the weight vector by $\mathbf{x} = (x_1 \cdots x_n)^T$. The return on the portfolio with no risk-free asset, r_p is then defined by $r_p = \mathbf{x}^T \mathbf{r}$, where $\mathbf{r} = (r_1 \cdots r_n)^T$ is an $n \times 1$ vector of returns on the n risky assets. Here, we define $\mu = \mathbb{E}(\mathbf{r})$, and $\Sigma := \text{var}(\mathbf{r})$ is an $n \times n$ non-singular variance-covariance matrix of $\mathbf{r}$. Since the return on the risky assets is uncertain, so is the return on the portfolio, and it could be evaluated by the expected value $\mathbb{E}(r_p) = \mathbf{x}^T \mathbb{E}(\mathbf{r}) = \mathbf{x}^T \mu$, and its variance is calculated by $\text{var}(r_p) = \sigma_p^2 = \mathbf{x}^T \Sigma \mathbf{x}$. The matrix Σ is assumed to be positive definite, and it has n positive eigenvalues $0 < \lambda_1 \leq \ldots \leq \lambda_n$. We use the notation $\text{Sym}_n(\mathbb{R})$ to denote the set of symmetric $n \times n$ and positive definite matrices. With given the return $\mathbf{r}$, let us find $\mathbf{x}$ to minimize the portfolio's risk.

We consider the inequality constrained quadratic programming problem. The objective of minimizing the variance of a portfolio is given as

$$\arg\min_{\mathbf{x}} \mathbf{x}^T \Sigma \mathbf{x} \quad \text{subject to } \mathbf{x}^T \mu = \mathbb{E}(r_p), \quad \|\mathbf{x}\|_1 = 1, \quad \text{and} \quad 0 \leq x_i, \text{ for } i = 1, \ldots, n.$$

The formation requires the weight $\mathbf{x}$ that minimizes variance subject to the inequality of expected return $\mathbb{E}(r_p)$ and non-negativity constraint. Each weight value $\hat{x}_i = x_i / \|\mathbf{x}\|_1$ is then normalized so as to comply with the budget constraint $\Sigma_{i=1}^n \hat{x}_i = 1$. Minimizing $\hat{\mathbf{x}}^T \Sigma \hat{\mathbf{x}}$ with inequality and equality constraints can be handled by

$$\arg\min_{\hat{\mathbf{x}}} \frac{1}{2} \hat{\mathbf{x}}^T \Sigma \hat{\mathbf{x}} \quad \text{subject to } \hat{\mathbf{x}} \geq 0 \quad \text{and} \quad \hat{\mathbf{x}}^T \mu = \mathbb{E}(r_p).$$

For sake of simplicity throughout the paper, we deal with a quadratically constrained quadratic problem (QCQP). We refer to [16, 17] for more details. The quadratic Lagrangian is given by

$$\mathcal{L}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{1}{2} \mathbf{x}^T \Sigma \mathbf{x} + \mathbf{y}^T (P\mathbf{x} - b) - \mathbf{z}^T \mathbf{x}, \tag{2.1}$$

where $b = (r_p \ 1)^T \in \mathbb{R}^{2 \times 1}$ and

$$P = \begin{pmatrix} \mathbf{r} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} r_1 & r_2 & \cdots & r_{n-1} & r_n \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \in \mathbb{R}^{2 \times n}.$$

Here, the two parameters $\mathbf{y} \in \mathbb{R}^{2\times 1},\ \mathbf{z} \in \mathbb{R}^{n\times 1}$ control the relation between the minimization $\mathbf{x}^T \Sigma \mathbf{x}$ and constraints $\mathbf{x} \geq 0$. Let us state the KKT conditions for (2.1) as follows

$$-\Sigma\mathbf{x} + P^T\mathbf{y} + \mathbf{z} = 0, \quad P\mathbf{x} = b, \quad \text{and} \quad XZ\mathbf{1} = (1/\rho)\cdot\mathbf{1},$$

where $\mathbf{1} = (1\cdots 1)^T \in \mathbb{R}^{n\times 1}$, $b = (r_p\ 1)^T \in \mathbb{R}^{2\times 1}$, $X = \text{diag}(\mathbf{x}) \in \mathbb{R}^{n\times n}$, $Z = \text{diag}(\mathbf{z}) \in \mathbb{R}^{n\times n}$,

$$P = \begin{pmatrix} \mathbf{r} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} r_1 & r_2 & \cdots & r_{n-1} & r_n \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \in \mathbb{R}^{2\times n},$$

and the parameter $1/\rho$ is set to a small positive number. As $\rho \to \infty$, i.e a small positive value $1/\rho$, $\mathbf{x}$ converges toward the optimal solution of the QCQP. A penalty parameter ρ approximately enforces complementary slackness, which prevents $\mathbf{x}$ from becoming negative. Linear equations for the Newton-Raphson method give that

$$\begin{pmatrix} \Sigma & P^T & -I_{n\times n} \\ P & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times n} \\ Z & \mathbf{0}_{n\times 2} & X \end{pmatrix} \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = \begin{pmatrix} -\Sigma\mathbf{x} - P^T\mathbf{y} + \mathbf{z} \\ b - P\mathbf{x} \\ (1/\rho)\cdot\mathbf{1} - XZ\mathbf{1} \end{pmatrix}, \tag{2.2}$$

where $I_{n\times n}$ is the $n \times n$ identity matrix whose elements along the diagonal are all 1s, and $\mathbf{0}_{m\times n} \in \mathbb{R}^{m\times n}$ is a null matrix all of whose entries are zero. Let us define the $(n+2)\times(n+2)$ matrix

$$X_0 := \begin{pmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times n} \\ \mathbf{0}_{n\times 2} & X \end{pmatrix},$$

and we set $L = (P \quad Z)^T$ and $U = (P^T \quad -I_{n\times n})$. With block matrices $X_0,\ L,\ U$, the matrix A can be expressed by

$$A = \begin{pmatrix} \Sigma & P^T & -I_{n\times n} \\ P & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times n} \\ Z & \mathbf{0}_{n\times 2} & X \end{pmatrix} = \begin{pmatrix} \Sigma & U \\ L & X_0 \end{pmatrix}.$$

Since the submatrix X_0 is singular when $\mathbf{x}$ has zero weighted assets, we use a non-singular submatrix Σ to find M^{-1}. Thus, the inversion of block matrices can be calculated by

$$A^{-1} = \begin{pmatrix} \Sigma & U \\ L & X_0 \end{pmatrix}^{-1} = \begin{pmatrix} \Sigma^{-1} + \Sigma^{-1}U(X_0 - L\Sigma^{-1}U)^{-1}L\Sigma^{-1} & -\Sigma^{-1}U(X_0 - L\Sigma^{-1}U)^{-1} \\ -(X_0 - L\Sigma^{-1}U)^{-1}L\Sigma^{-1} & (X_0 - L\Sigma^{-1}U)^{-1} \end{pmatrix}.$$

The Newton-Raphson method then proceeds iteratively from an initial point $x_s(0),\ y_s(0),\ z_s(0)$ through a sequence of points determined from the search directions described above the solution of linear equations (2.2):

$$\begin{aligned} \mathbf{x}^{k+1} &= \mathbf{x}^k + \alpha x_s, \\ \mathbf{y}^{k+1} &= \mathbf{y}^k + \alpha y_s, \\ \mathbf{z}^{k+1} &= \mathbf{z}^k + \alpha z_s. \end{aligned}$$

3. Two eigenvalues and eigenvectors by one asset substitution

We investigate the effect of one asset replacement for our portfolio, which is a set of n financial assets. For the asset index $1 \leq i \leq n$ and the number of total unit times l, e.g. n assets and l days, let us first define the following $l \times 1$ vectors $R_i = (r_{1,i}, \ldots, r_{k,i}, \ldots, r_{l,i})^T$ containing unit time's the asset returns, where $r_{k,i}$ represents the return of i-th asset on k-th day. The i-th asset's mean return in the portfolio is also defined as $\mu_{R_i} := \mathbb{E}(R_i)$. Next, we construct the variancecovariance (or covariance) matrix Σ. (We refer to Appendix for constructing the variancecovariance Σ.) Given Σ with n assets, we now assume that i-th asset is replaced with another asset. This induces that R_i is replaced with R_{i^*}. The variance-covariance matrix $\ddot{\Sigma}$ by the replacement of i-th asset in Σ can be expressed by matrix addition of the rank two matrix $M = \sigma_1 uu^T + \sigma_2 vv^T$, so we can write that

$$\ddot{\Sigma} = \Sigma + M = \Sigma + \begin{pmatrix} & m_{1,i} & \\ & \vdots & \\ m_{1,i} \cdots m_{i,i} & \cdots & m_{n,i} \\ & \vdots & \\ & m_{n,i} & \end{pmatrix} = \Sigma + \sigma_1 uu^T + \sigma_2 vv^T.$$

Here, σ_1, σ_2 are eigenvalues of M and u, v are corresponding eigenvectors of M. The elements of matrix M are computed by $m_{j,i} = cov(R_j, R_{i^*}) - cov(R_j, R_i)$. That is, we obtain $m_{j,i}$ by

$$m_{j,i} = \frac{1}{l-1} \sum_{k=1}^{l} (r_{k,j} - \mu_{R_j})[(r_{k,i^*} - r_{k,i}) - (\mu_{R_{i^*}} - \mu_{R_i})].$$

Indeed, the symmetric matrix M has a rank of 2, and it has one positive eigenvalue $\sigma_1 > 0$ and one negative eigenvalue $\sigma_2 < 0$. To see that, we solve the below system by elimination:

$$(M - \lambda I)w = \begin{pmatrix} -\lambda & & m_{1,i} & & \\ & -\lambda & \vdots & & \\ m_{1,i} & \cdots & m_{i,i} - \lambda & \cdots & m_{n,i} \\ & & \vdots & -\lambda & \\ & & m_{n,i} & & -\lambda \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{pmatrix} = \mathbf{0}.$$

This readily leads to the quadratic equation

$$\lambda^2 - m_{i,i}\lambda - \left(\sum_{j=1}^{n} m_{j,i}^2 - m_{i,i}^2 \right) = 0.$$

It implies that M has two different eigenvalues $\sigma_1 \neq \sigma_2$ and $\sigma_1 \sigma_2 < 0$. Furthermore, we can express the eigenvalues explicitly as

$$\sigma_{\{1,2\}} = \frac{1}{2} \left(m_{i,i} \pm \sqrt{4 \sum_{j=1}^{n} m_{j,i}^2 - 3m_{i,i}^2} \right) = \frac{1}{2} (m_{i,i}) \pm R_i, \tag{3.1}$$

where we define the distance R_i from the center $m_{i,i}/2$ to the eigenvalues $\sigma_{\{1,2\}}$ by

$$R_i = \frac{1}{2}\sqrt{4\sum_{j=1}^{n} m_{j,i}^2 - 3m_{i,i}^2}.$$

Since $m_{i,i} < 2R_i$, we know that $\sigma_1 > 0$, $\sigma_2 < 0$. Letting $w_i = \sigma_1$ or $w_i = \sigma_2$ and normalizing by

$$\beta_1 := (m_{1,i}^2 + \cdots + m_{i-1,i}^2 + \sigma_1^2 + m_{i+1,i}^2 \cdots + m_{n,i}^2)^{1/2},$$
$$\beta_2 := (m_{1,i}^2 + \cdots + m_{i-1,i}^2 + \sigma_2^2 + m_{i+1,i}^2 \cdots + m_{n,i}^2)^{1/2},$$

we have the corresponding eigenvectors for σ_1 and σ_2:

$$u_j = \begin{cases} \sigma_1/\beta_1 & \text{if } j = i \\ m_{j,i}/\beta_1 & \text{otherwise,} \end{cases} \quad \text{and} \quad v_j = \begin{cases} \sigma_2/\beta_2 & \text{if } j = i \\ m_{j,i}/\beta_2 & \text{otherwise.} \end{cases}$$

Moreover, we set the i-th column of matrix M by $m := (m_{1,i}, \ldots, m_{n,i})^T$ and $\alpha := \|m\|_\infty$. From (3.1), the two eigenvalues σ_1 and σ_2 follow that

$$\sigma_2 < \frac{1}{2}m_{i,i} < \sigma_1 \leq \frac{\alpha}{2}\left[1 + (4n-3)^{1/2}\right].$$

As for the two eigenvalues, the difference and sum of σ_1 and σ_2 are respectively given by

$$|\sigma_1 - \sigma_2| \leq (4n-3)^{1/2}\alpha \quad \text{and} \quad \sigma_1 + \sigma_2 = m_{i,i}.$$

In addition, we denote the eigenvector corresponding to the smallest eigenvalues of Σ by w. Then, the relation between two eigenvectors u, v and w has

$$\sigma_1(w^T u)^2 \geq -\sigma_2(w^T v)^2,$$

due to the condition of semi-positive definite matrix $\ddot{\Sigma}$. It also implies that $|w^T u| \geq |w^T v|$ and $|\sigma_2| > |\sigma_1|$ if $m_{i,i} < 0$.

4. Inverse update with a symmetric rank two matrix

Let us define $\dot{\Sigma} := \Sigma + \sigma_1 uu^T$, $\ddot{\Sigma} := \dot{\Sigma} + \sigma_2 vv^T$. Suppose that we have a linear system $\ddot{\Sigma}x = b$, where $\ddot{\Sigma} \in \mathbb{R}^{n\times n}$ is nonsingular and nonzero $b \in \mathbb{R}^n$. The solution $x \in \mathbb{R}^n$ of the linear system $\ddot{\Sigma}x = b$ is calculated as follows. We set $\xi = v^T x$, so that $\ddot{\Sigma}x = \dot{\Sigma}x + (\sigma_2\xi)v$. With $\dot{\Sigma}$, $n \times n$ identity matrix I, and ξ, we rewrite the linear system as lower and upper triangular submatrices:

$$\begin{pmatrix} \dot{\Sigma} & \sigma_2 v \\ v^T & -1 \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} I & \mathbf{0} \\ v^T\dot{\Sigma}^{-1} & 1 \end{pmatrix}\begin{pmatrix} \dot{\Sigma} & \sigma_2 v \\ \mathbf{0}^T & -1 - \sigma_2 v^T\dot{\Sigma}^{-1}v \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Since each triangular submatrix is nonsingular, this amounts to solve the linear system

$$x = \dot{\Sigma}^{-1}(b - \sigma_2\xi v) \quad \text{and} \quad \xi = \frac{v^T\dot{\Sigma}^{-1}b}{1 + \sigma_2 v^T\dot{\Sigma}^{-1}v}.$$

It implies that

$$\ddot{\Sigma}^{-1} b = x = \dot{\Sigma}^{-1}\left(I - \frac{\sigma_2 v v^T \dot{\Sigma}^{-1}}{1 + \sigma_2 v^T \dot{\Sigma}^{-1} v}\right) b.$$

Hence, the inverse of symmetric rank one and two updates respectively can be written by

$$\dot{\Sigma}^{-1} = \Sigma^{-1} - \frac{\sigma_1 \Sigma^{-1} u u^T \Sigma^{-1}}{1 + \sigma_1 u^T \Sigma^{-1} u}, \quad \text{and} \quad \ddot{\Sigma}^{-1} = \dot{\Sigma}^{-1} - \frac{\sigma_2 \dot{\Sigma}^{-1} v v^T \dot{\Sigma}^{-1}}{1 + \sigma_2 v^T \dot{\Sigma}^{-1} v}. \tag{4.1}$$

To find the solution x in the system $(\Sigma + \sigma_1 u u^T + \sigma_2 v v^T)x = b$, we multiply each of the above equations (4.1) by the vectors b and v:

$$\ddot{\Sigma}^{-1} b = \dot{\Sigma}^{-1} b - \frac{\sigma_2 \dot{\Sigma}^{-1} v v^T \dot{\Sigma}^{-1} b}{1 + \sigma_2 v^T \dot{\Sigma}^{-1} v}, \ \dot{\Sigma}^{-1} v = \Sigma^{-1} v - \frac{\sigma_1 \Sigma^{-1} u u^T \Sigma^{-1} v}{1 + \sigma_1 u^T \Sigma^{-1} u} \text{ and } \dot{\Sigma}^{-1} b = \Sigma^{-1} b - \frac{\sigma_1 \Sigma^{-1} u u^T \Sigma^{-1} b}{1 + \sigma_1 u^T \Sigma^{-1} u}.$$

If we respectively denote $\Sigma^{-1} u$ and $\dot{\Sigma}^{-1} v$ by p and q, it simply follows that

$$\ddot{\Sigma}^{-1} b = \Sigma^{-1} b - \frac{\sigma_1 p p^T b}{1 + \sigma_1 u^T p} - \frac{\sigma_2 q q^T b}{1 + \sigma_2 v^T q}. \tag{4.2}$$

Next, we observe the characterization of rank two perturbation for Σ. The distance of the two matrices are computed by the operator norm, which is defined by

$$\|A\|_{op} = \max_{\|x\|_2 = 1} \|Ax\|_2 \quad \text{for} \quad A \in \mathrm{Sym}_n(\mathbb{R}).$$

We write the operator norm of $\dot{\Sigma}$, and give the following estimation:

$$\|\dot{\Sigma}^{-1}\|_{op} = \left\| \left[I_{n\times n} - \frac{\sigma_1}{1 + \sigma_1 u^T \Sigma^{-1} u} \left(\Sigma^{-1} u u^T\right) \right] \Sigma^{-1} \right\|_{op} \leq \left\| \left[I_{n\times n} - \sigma_1 \left(\Sigma^{-1} u u^T\right) \right] \Sigma^{-1} \right\|_{op} \quad \leq \frac{1}{\lambda_1} C_{\sigma_1,\lambda_1},$$

where C_{σ_1,λ_1} is defined by

$$C_{\sigma_1,\lambda_1} = \begin{cases} 1 - \frac{\sigma_1}{\lambda_1} & \text{if } \sigma_1 \leq \lambda_1 \\ \frac{\sigma_1}{\lambda_1} - 1 & \text{otherwise.} \end{cases}$$

From $p = \Sigma^{-1} u$ and $q = \dot{\Sigma}^{-1} v$ and C_{σ_1,λ_1}, we also write the operator norm of two matrices, as follows

$$\|p p^T\|_{op} = \|\Sigma^{-1} u u^T \Sigma^{-1}\|_{op} \leq \lambda_1^{-2} \quad \text{and} \quad \|q q^T\|_{op} = \|\dot{\Sigma}^{-1} v v^T \dot{\Sigma}^{-1}\|_{op} \leq C^2_{\sigma_1,\lambda_1}.$$

Therefore, the difference between two matrices $\ddot{\Sigma}^{-1}$ and Σ^{-1} can be measured by

$$\begin{aligned} \left\|\ddot{\Sigma}^{-1} - \Sigma^{-1}\right\|_{op} &= \left\| \frac{\sigma_1 p p^T}{1 + \sigma_1 u^T p} + \frac{\sigma_2 q q^T}{1 + \sigma_2 v^T q} \right\|_{op} \leq \left(\frac{\sigma_1}{1 + \sigma_1 u^T p} \right) \left\| p p^T \right\|_{op} + \left| \frac{\sigma_2}{1 + \sigma_2 v^T q} \right| \left\| q q^T \right\|_{op} \\ &\leq \left(\frac{\sigma_1}{1 + \sigma_1/\lambda_n} \right) \left\| p p^T \right\|_{op} + \left| \frac{\sigma_2}{1 + \sigma_2/(\lambda_n + \sigma_1)} \right| \left\| q q^T \right\|_{op} \\ &\leq \left(\frac{\sigma_1}{1 + \sigma_1/\lambda_n} \right) \lambda_1^{-2} + \left| \frac{\sigma_2}{1 + \sigma_2/(\lambda_n + \sigma_1)} \right| C_{\sigma_1,\lambda_1}. \end{aligned}$$

Note that the second inequality follows from $\|\dot{\Sigma}\|_{op} \leq \lambda_n + \sigma_1$.

5. Covariance inversion scheme for portfolios' one asset substitution

We now assume that an investor daily renews portfolio proportions as weights by replacing the poorest performing one asset with a new asset. Mathematically, short selling helps the market to find the fair value of stocks, however we constrain all the weights in the portfolio to be non-negative as mentioned in Section 2. Our scheme allows only portfolio weight vector $\mathbf{x} \geq 0$ that minimizes variance subject to meeting the target return r_p. Note that even our portfolio weight vector $\mathbf{x}$ obtained from our scheme meets the target return r_p, it does not guarantee the optimal solution or it is possible to have another weight vector that achieves r_p due to the non-negativity constraint. In this work, we aim to focus on the efficient computation by using the rank two update. Meanwhile, note that the unique global solvability of optimal problem is beyond the scope of this paper.

Our work mainly concerns about how to develop an efficient scheme when replacing a low-performing asset to improve overall portfolio's performance. Based on the classical quadratic programming, we can create a non-negative portfolio weight vector $\mathbf{x} \geq 0$. Furthermore, our proposed scheme finds the lowest performance asset and then replaces it with the new one for improving portfolio's performance.

The data we use in this paper is stock prices of KOSPI and NASDAQ from 2020 to 2021. Let S_D denote the historical return data for D unit times. A unit time length can be a second, a minute, an hour, or a day. In these simulations, we select the total D days of historical return data to create a covariance matrix Σ. Also, we let P_S be a set of selected assets and a set of reference assets P_R. The framework for our proposed portfolio optimization is given specifically as follows:

Investment strategy by replacing an existing portfolio's asset for one period of performance evaluation.

1: **Input:** historical data S_D, selected assets P_S and reference assets P_R, target return r_p
2: **Define** : D:= total days, n_s:= number of selected assets, n_r:= number of reference assets
3: **For** $d = 1$ **to** D
    $R_p \leftarrow$ set of initial selected assets for D days
    $\Sigma \leftarrow$ covariance matrix of R_p
    $\mu_p \leftarrow$ expected values of R_p
4: **Do** Algorithm 2
    Input: covariance matrix Σ, target return r_p, expected values of initial selected assets μ_p
    Output: non-negative weight vector $\mathbf{x}_1 \geq 0$
    risk1 $\leftarrow \mathbf{x}_1^T \Sigma \mathbf{x}_1$
5: **For** $i = 1$ **to** n_r
    $R_i^* \leftarrow$ return of i-th reference asset
    $\mu_i^* \leftarrow \mathbb{E}(R_i^*)$
6: **For** $j = 1$ **to** n_s {j-th selected asset}
    $\mu_p^* \leftarrow \mu_p$
    $\mu_p^*(j) \leftarrow \mu_i^*$
7: **Do** Algorithm 3
    Input: inverse of covariance matrix Σ^{-1}, return of initial selected assets R_p, return of an asset to be replaced R_i^*, return of j-th selected asset R_j
    Output: inverse of rank two update matrix $\ddot{\Sigma}^{-1}$, rank two update matrix $\ddot{\Sigma}$

Do Algorithm 2

Input: rank two update matrix $\ddot{\Sigma}$, inverse of rank two update matrix $\ddot{\Sigma}^{-1}$, target return r_p, expected values of selected assets μ_p^*

Output: non-negative weight vector $\mathbf{x}_2 \geq 0$

risk2 $\leftarrow \mathbf{x}_2^T \ddot{\Sigma} \mathbf{x}_2$

If risk2 < risk1

risk1 $\leftarrow$ risk2

record $\leftarrow (i,\ j)$

End if

End for

End for

$R_p \leftarrow$ Portfolio with asset j replaced by asset i in R_p

End for

Output : optimal portfolio with non-negative weight vector $\mathbf{x}^* \geq 0$

Together with a vector $v \in \mathbb{R}^n$, the operator diag(v) gives us an $n \times n$ square diagonal matrix with the elements of vector v on the main diagonal. The optimal non-negative weight vector in the given assets

Input: Non-singular covariance matrix Σ, inverse of covariance matrix Σ^{-1}, target return r_p, expected values of n-selected assets $\mu = \{\mu_1, \mu_2, \ldots, \mu_n\}$

Define: n:= number of selected assets, n_t := number of iterations, η := 0.95

Initialize: $P \leftarrow \begin{pmatrix} \mu_1 & \mu_2 & \cdots \mu_n \\ 1 & 1 & \cdots 1 \end{pmatrix}$, $\quad b \leftarrow \begin{pmatrix} r_p & 1 \end{pmatrix}^T$, $\quad \mathbf{x} \leftarrow \mathbf{1}_{n\times 1}$, $\quad \mathbf{y} \leftarrow \begin{pmatrix} 1 & 1 \end{pmatrix}^T$, $\quad \mathbf{z} \leftarrow \mathbf{1}_{n\times 1}$,

$U \leftarrow \begin{pmatrix} P^T & -I_{n\times n} \end{pmatrix}$

For $i = 1$ **to** n_t **do**

$X \leftarrow \mathrm{diag}(\mathbf{x}), \quad Z \leftarrow \mathrm{diag}(\mathbf{z}), \quad L \leftarrow \begin{pmatrix} P \\ Z \end{pmatrix}$

$X_0 \leftarrow \begin{pmatrix} \mathbf{0_{2\times 2}} & \mathbf{0_{2\times n}} \\ \mathbf{0_{n\times 2}} & X \end{pmatrix}$

$M_{inv} \leftarrow (X_0 - L\Sigma^{-1}U)^{-1}$

$M_1 \leftarrow \Sigma^{-1} + \Sigma^{-1} U M_{inv} L \Sigma^{-1}, \quad M_2 \leftarrow -\Sigma^{-1} U M_{inv},$

$M_3 \leftarrow -M_{inv} L \Sigma^{-1}, \quad M_4 \leftarrow M_{inv}$

$A_{inv} \leftarrow \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$

$\begin{pmatrix} \mathbf{x}_s \\ \mathbf{y}_s \\ \mathbf{z}_s \end{pmatrix} \leftarrow A_{inv} \begin{pmatrix} -\Sigma\mathbf{x} - P^T\mathbf{y} + \mathbf{z} \\ b - P\mathbf{x} \\ -XZ\mathbf{1}_{n\times 1} \end{pmatrix}$

$\mathbf{x}_1 \leftarrow \mathbf{x}, \quad \mathbf{z}_1 \leftarrow \mathbf{z}$

For $i = 1$ **to** n **do**

If $\mathbf{x}(i, 1) < 0$

$\mathbf{x}(i, 1) \leftarrow 0$

End if

If $\mathbf{z}(i, 1) < 0$

$\mathbf{z}(i, 1) \leftarrow 0$

End if

11: **If** $\mathbf{x}_s(i,1) > 0)$

$\mathbf{x}_N(i,1) \leftarrow -10^3, \quad \mathbf{x}_1(i,1) \leftarrow 1$

12: **Else**

$\mathbf{x}_N(i,1) \leftarrow \mathbf{x}(i,1)$

13: **End if**

14: **If** $\mathbf{z}_s(i,1) > 0)$

$\mathbf{z}_N(i,1) \leftarrow -10^3, \quad \mathbf{z}_1(i,1) \leftarrow 1$

15: **Else**

$\mathbf{z}_N(i,1) \leftarrow \mathbf{z}$

16: **End if**

$\alpha_x \leftarrow \min\{\mathbf{x}_1(1,1)/\mathbf{x}_N(1,1),\ \ldots\ ,\ \mathbf{x}_1(n,1)/\mathbf{x}_N(n,1)\}$

$\alpha_z \leftarrow \min\{\mathbf{z}_1(1,1)/\mathbf{z}_N(1,1),\ \ldots\ ,\ \mathbf{z}_1(n,1)/\mathbf{z}_N(n,1)\}$

$\pi \leftarrow \mathbf{x}^T\mathbf{z}/n, \quad \pi_s \leftarrow (\mathbf{x}+\alpha_x\mathbf{x}_s)^T(\mathbf{z}+\alpha_z\mathbf{z}_s)/n, \quad \tau \leftarrow (\pi/\pi_s)^3$

$$\begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} \leftarrow A_{inv} \begin{pmatrix} -\Sigma\mathbf{x} - P^T\mathbf{y} + \mathbf{z} \\ b - P\mathbf{x} \\ -XZ\mathbf{1}_{n\times 1} + \mathrm{diag}(\mathbf{x}_s)\mathrm{diag}(\mathbf{z}_s)\mathbf{1}_{n\times 1} + \tau\pi\mathbf{1}_{n\times 1} \end{pmatrix}$$

$\alpha \leftarrow \min\{1, \eta\alpha_x, \eta\alpha_z\}, \quad \mathbf{x} \leftarrow \mathbf{x} + \alpha d_x,\ \mathbf{y} \leftarrow \mathbf{y} + \alpha d_y, \mathbf{z} \leftarrow \mathbf{z} + \alpha d_z$

17: **End for**

18: **Output** : non-negative weight vector $\mathbf{x} \geq 0$

Rank two update for the inverse of covariance matrix

1: **Input:** inverse of covariance matrix Σ^{-1}, return of j-selected assets R_j, return of an asset to be replaced R_i^*, return of i-th selected asset R_i

2: **Define** : $n :=$ number of selected assets

3: **For** $j = 1$ **to** n **do**

$m_{j,i} \leftarrow cov(R_j, R_i^*) - cov(R_j, R_i)$

$\Sigma(j,i) \leftarrow cov(R_j, R_i^*), \quad \Sigma(i,j) \leftarrow cov(R_j, R_i^*)$

4: **End for**

$\ddot{\Sigma} \leftarrow \Sigma$

$\sigma_1 \leftarrow \frac{1}{2}\left(m_{i,i} + \sqrt{4\sum_{j=1}^n m_{j,i}^2 - 3m_{i,i}^2}\right), \quad \sigma_2 \leftarrow \frac{1}{2}\left(m_{i,i} - \sqrt{4\sum_{j=1}^n m_{j,i}^2 - 3m_{i,i}^2}\right)$

$\beta_1 \leftarrow (m_{1,i}^2 + \cdots + m_{i-1,i}^2 + \sigma_1^2 + m_{i+1,i}^2 \cdots + m_{n,i}^2)^{1/2}$

$\beta_2 \leftarrow (m_{1,i}^2 + \cdots + m_{i-1,i}^2 + \sigma_2^2 + m_{i+1,i}^2 \cdots + m_{n,i}^2)^{1/2}$

5: **For** $j = 1$ **to** n **do**

6: **If** $j = i$

$u(j,1) \leftarrow \sigma_1/\beta_1,\ v(j,1) \leftarrow \sigma_2/\beta_2$

7: **Else**

$u(j,1) \leftarrow m_{j,i}/\beta_1,\ v(j,1) \leftarrow m_{j,i}/\beta_2$

8: **End if**

9: **End for**

$\dot{\Sigma}^{-1} \leftarrow \Sigma^{-1} - (\sigma_1\Sigma^{-1}uu^T\Sigma^{-1})/(1 + \sigma_1 u^T\Sigma^{-1}u)$

$\ddot{\Sigma}^{-1} \leftarrow \dot{\Sigma}^{-1} - (\sigma_2\dot{\Sigma}^{-1}vv^T\dot{\Sigma}^{-1})/(1 + \sigma_2 v^T\dot{\Sigma}^{-1}v)$

10: **Output** : inverse of rank two update matrix $\ddot{\Sigma}^{-1}$, rank two update matrix $\ddot{\Sigma}$

6. Numerical tests

In this section, we present the validity of the proposed method by evaluating the two different data sets, which are Monte-Carlo sampling data and real data. First, the simulation generates the stock prices denoted by $S_i(t)$ for $1 \leq i \leq n$ and $0 \leq t$, where $S_i(t)$ is the i-th index at time t. The stock price process is being driven by the geometric Brownian motion. We compare with volatilities and returns measured by our proposed investment scheme and random investment scheme. Second, the numerical simulations are also performed with real data from the KOSPI and NASDAQ. To apply our proposed model, we have used data related to both bullish and bearish markets in KOSPI and NASDAQ.

The portfolio's return profits by the proposed method overall outperform the top 30% best performing results of portfolios which are randomly selected. In the both setting of Monte-Carlo simulations and real data tests, our proposed model shows that consistently low risk and tends to have higher return. Each test by Monte-Carlo sampling is different with varying levels of risks. We increase the number of samples and volatilities as well in these simulations. Next, we demonstrate our proposed model's volatilities and returns on real data to see the robustness of our model. Remark that we use the historical volatilities, but implied volatilities can be used as in [15].

6.1. *Monte-Carlo simulations*

We generate the random path of i-th stock price S_i using geometric Brownian motion $\mathrm{d}S_i = rS_i dt + \sigma_i S_i \,\mathrm{d}W_i$, so that Ito calculus gives us

$$S_i(t) = S_i(0)\exp\left(\left(r - \frac{1}{2}\sigma_i^2\right)t + \sigma_i W_i(t)\right)$$

in which each $W_i(t)$ is normally distributed with zero mean and variance at t [18]. In this subsection, we generate 200 random assets using the geometric Brownian motion process, and present the validity of the proposed method using these random assets.

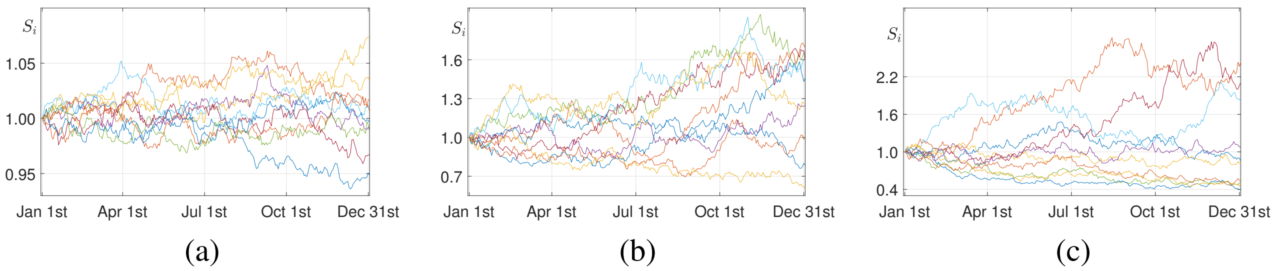


Figure 1. Ten random stock price paths are generated by geometric Brownian motion.

Figure 1 represents the 10 random paths generated by the standard Brownian motion process. Figures 1(a), (b), and (c) each shows its paths when σ_i is 0.05, 0.35 and 0.5 with $S_i(0) = 1$. In this way, we increase the number of random paths 200 and 500 for large scale simulations.

6.1.1. Large scale simulation with random paths

As we stated above, our proposed scheme can produce the optimal solution more efficiently and easier. Large scale empirical studies on the total 200 and 500 assets are performed to see the robustness of our scheme. We construct one portfolio, which consists of a set of 30, 50, and 75 assets, then the rest of assets is called the reference assets. Daily updating the portfolio by replacing one asset with one of the other assets, i.e., reference assets. Each asset class follows random walk generated by geometric Brownian motion. Also, the target return is set to 0.02 when the optimal volatility portfolio is constructed using QCQP method. Figure 2 represents the risk measured by ${\sigma_p}^2$. The risk ${\sigma_p}^2$ is

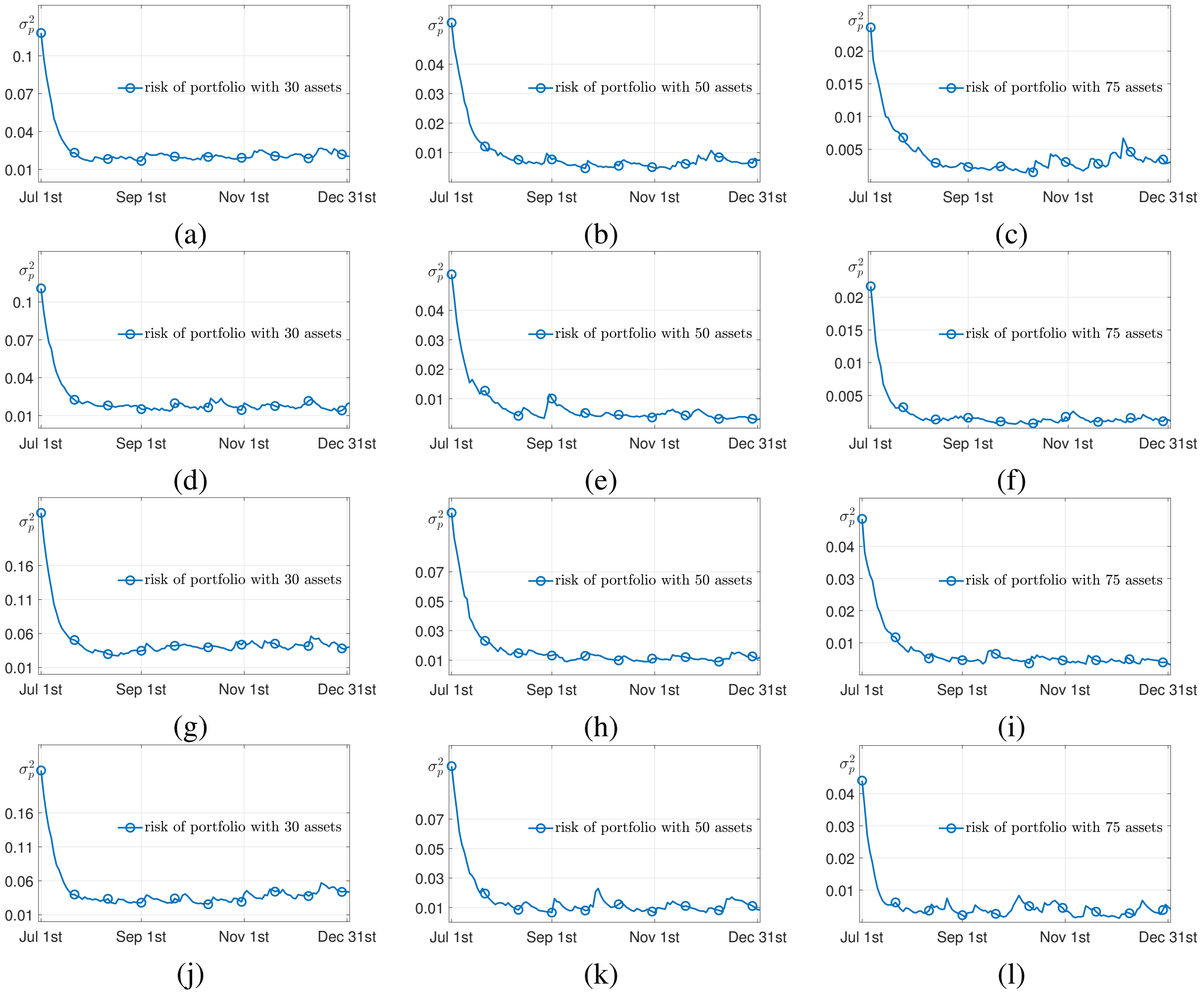


Figure 2. Daily risk by our daily-one-asset replacement scheme simulated with a random path.

calculated day by day, and the graph of ${\sigma_p}^2$ is plotted. Figures 2(a), (b), and (c), the first row, are the results of 200 paths simulations generated by the Brownian motion process when σ is given 0.35. Figures 2(d), (e), and (f), the second row, are the results of 500 paths simulations by the Brownian motion process when σ is given 0.35. The third row, (g), (h), and (i) are the results of 200 paths simulations by the Brownian motion process when σ is given 0.5, and the fourth row (j), (k), and (l) are the results of 500 paths simulations by the Brownian motion process when σ is given 0.5. Also, the first column of Figure 2 shows the results of simulations with 30 assets. Likewise the second column of Figure 2 is the results of one portfolio having 50 selected assets. The last column Figure 2 is the results of simulations having of 75 selected assets.

6.1.2. Portfolio comparison in random paths

We provide a comparison of ‘daily risk by our proposed daily-one asset replacement scheme and ‘daily plot by randomly selecting assets scheme. Each test is compared using 100 random stock prices artificially generated using the geometric Brownian motion. The number of selected assets is 30. In our daily-one asset replacement method, we randomly select 30 of 100 assets to form an initial portfolio allocation, and then each one asset is replaced by one of the reference 70 assets to improve the portfolio’s performance with a minimum risk. i.e., update the portfolio weight daily for finding optimal volatility. In the comparison, ‘portfolio by randomly selecting one asset scheme in which initially 30 assets are randomly selected from 100 assets, and then everyday the portfolio weight is updated to be minimum risk using QCQP method. The random simulation was repeatedly performed 10,000 times, and the results of ‘portfolio by randomly selecting one asset scheme in Figure 3 were the top 30% highest return portfolio, which is the $3,000$ portfolios of total $10,000$ portfolios.

The first row of Figures 3 (a), (b), and (c) are the comparisons when the target return r_p is 0.03, and the second and third rows are the comparisons when the target return r_p are 0.05 and 0.07, respectively. Also, the results of first column (a), (d), and (g) were simulated with 100 random paths with a given constant $\sigma = 0.5$, and the second and third columns’ results were simulated with 100 random paths and given $\sigma = 1.0,\ 1.5$.

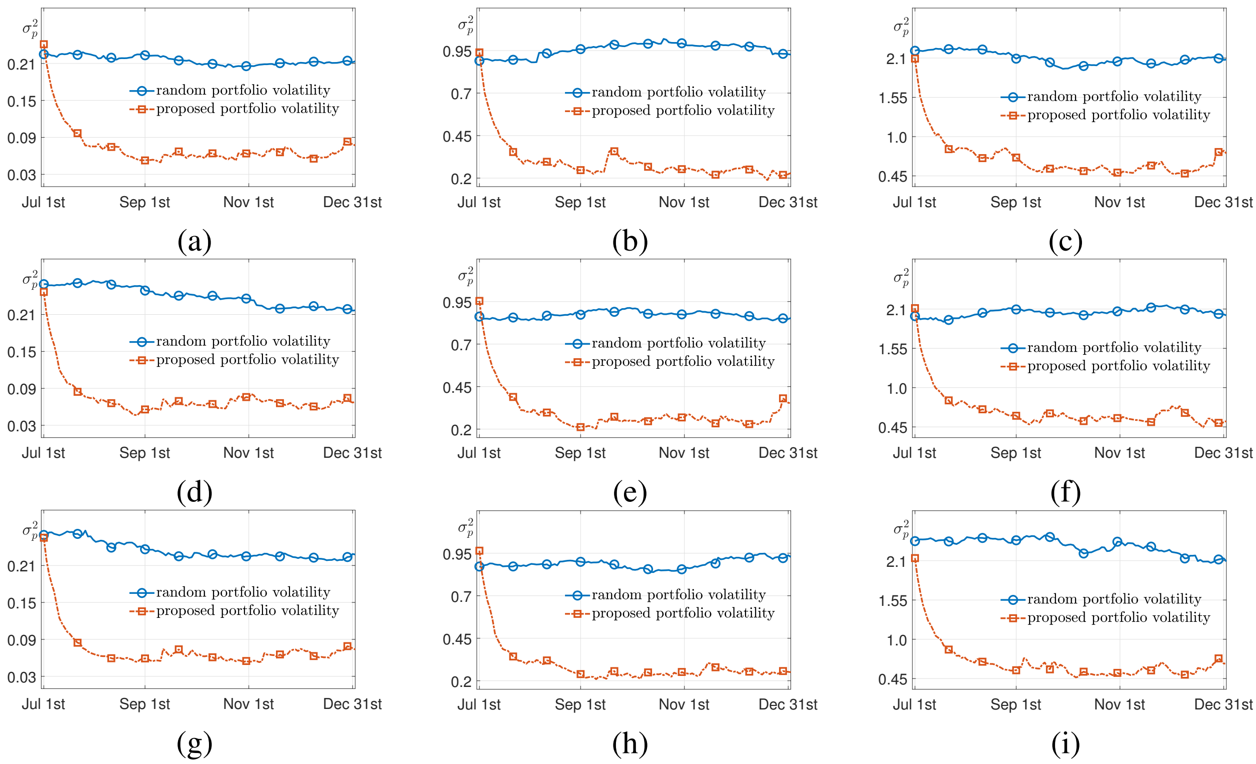


Figure 3. Risk comparison simulation of ‘daily risk by our daily-one asset replacement scheme and ‘daily plot by randomly selecting assets schemes using 100 random paths generated by the geometric Brownian motion.

6.2. *Practical examples*

We present a variety of practical examples for reporting proposed portfolio performance. The results imply that our proposed optimization scheme has the robustness and stability. In the following, we use real data which are KOSPI and NASDAQ in 2020 and 2021 to compute the optimal portfolio weights. Two different scenarios, bearish and bullish markets are illustrated to describe how the proposed portfolio better perform with respect to risk and return.

6.2.1. Risk evolutions in 2020

One of the major events in 2020 is the stock market crash caused by the COVID-19. It was a sudden global event that the corona-virus pandemic strongly affected South Korea in June, as well. The fear of the corona-virus diseases caused KOSPI to drop below 2000. Rapidly increased volatility in the stock market also influenced in our tests. However, the crash caused a short-lived bear market and does last only a short time.

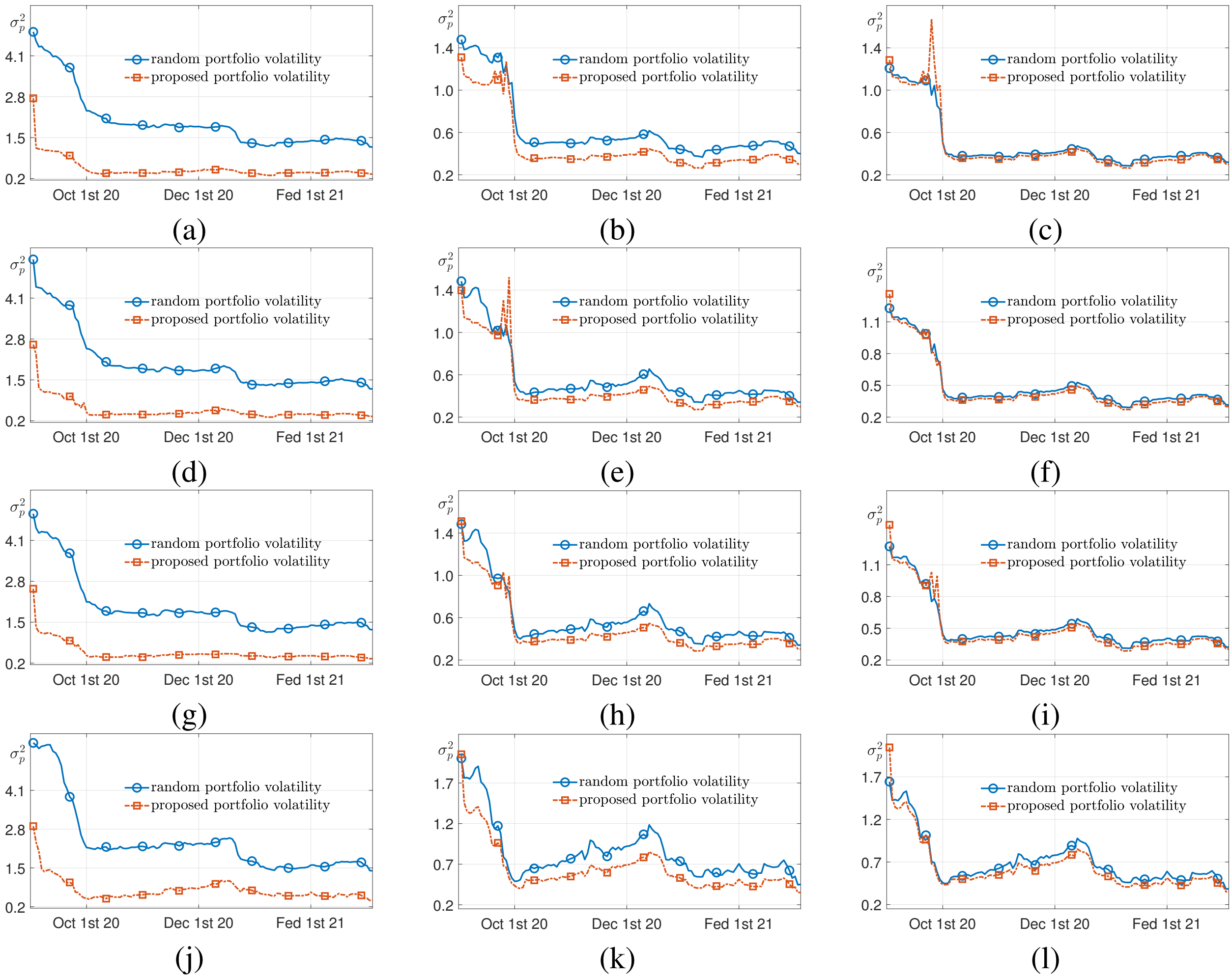


Figure 4. Risk comparison simulation of ‘daily risk by our daily-one asset replacement scheme and ‘daily plot by randomly selecting assets scheme using 100 KOSPI stock assets in the bearish market 2020.

Figure 4 represents the volatility or risk measured by ${\sigma_p}^2$. Data of σ_p^2 are based on the daily returns from February 28, 2020 to February 19th and 2021. The first 6 month data were evaluated, and proceed the evaluation on every previous 6 months. As a result, from August 24th, 2020 to February 19th, 2021, the risk was calculated by replacing one asset every day. The risk ${\sigma_p}^2$ of 'proposed portfolio volatility' is calculated and updated day by day. Meanwhile, the risk comparison of 'random portfolio volatility' is the average of the top 30% of 100 tests on random portfolios. In this simulation, 100 KOSPI stock assets in the bearish market 2020 was used. The first row of Figures 4, (a), (b), and (c) show a risk comparison when the target return is 0.03, and the second, third and fourth rows are risk comparisons for a given target returns 0.05, 0.07, 0.15, respectively. Also, the first column (a), (d), (g), and (j) are risk comparisons of simulations, corresponding to the portfolio consisted of 5 assets, and the second and third columns are risk comparisons of simulations when the number of portfolio assets are 30 and 50, respectively. Even though the risk ${\sigma_p}^2$ from random simulations is the result of the top 30% performance, the volatility of the proposed portfolio generated by our daily-one asset replacement method is much lower. Thus, our scheme gave more stable investments.

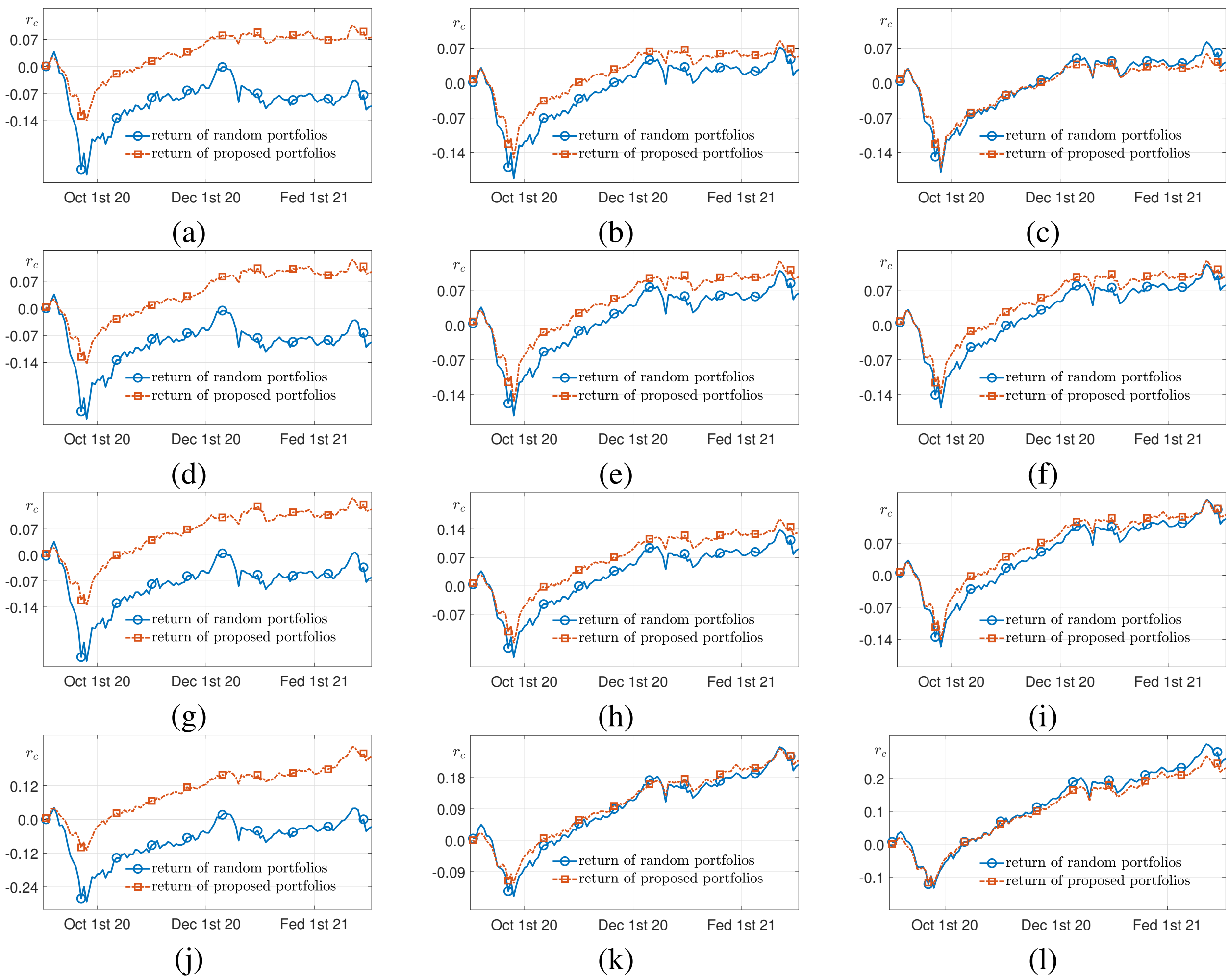


Figure 5. Cumulative returns are illustrated for comparison of proposed scheme and randomly selecting assets scheme. We use 100 KOSPI stock assets in the bearish market 2020.

Remark that tests illustrated in (b), (c), and (e) contain a sudden surge of stock prices, and it happens during a pandemic due to the panic. The portfolios' profits by the proposed scheme overall outperform in risk and return, i.e., it produces the lower risks and higher returns.

6.2.2. Cumulative return in 2020

Previously, we have seen that the risk of our portfolios is always lower when we use the data of 2020. Likewise, we also verify that the risk of our portfolio is not only lower but also returns by our scheme are relatively higher. Throughout simulations, each test was conducted with daily return data in 2020, and the expected return was calculated over the past 6 months.

Figure 5 represents the cumulative returns measured by r_c. The cumulative return r_c of our proposed portfolio is calculated and updated day by day. For a comparative study, cumulative return of random portfolios is calculated as the average of top 30 percent of total 100 tests. The first to fourth rows show the evolutions of cumulative return over 6 months in the second half of 2020. The graph of target returns $r_p = \{3\%,\ 5\%,\ 7\%,\ 15\%\}$, are drawn in upper and lower rows. The first to third columns are the 5, 30, and 50 number of assets, respectively. The data of 100 KOSPI assets are daily returns from February 28th, 2020 to February 19th, 2021. Each the daily return from August 24th, 2020 to February 19th, 2021 was calculated, and then the cumulative return was plotted.

As the number of assets in the portfolio increased, the gap between 'return of random portfolio' and 'return of proposed portfolio' tended to decrease, and the gap in two graphs increases with 5 assets portfolio. When we applied our method to constructing one portfolio, which has a size of 5, i.e., it showed the greatest difference. Also, 'return of random portfolio' and 'return of proposed portfolio' followed the similar tendency to each other, but in general, it was verified that cumulative return of 'proposed portfolio' was superior.

6.2.3. Risk evolutions in 2021

In the simulation of Figure 4, we used stock market's return data in 2020. The risk of proposed portfolio was lower than the top 30% of risk of random portfolios. The same risk comparison but different market condition was considered. Stock market return data of 2021 is assumed to be bullish market for simulations. In the same way as in the previous simulations, the volatility was evaluated based on every past 6 months, and then the risk was estimated by replacing the poorest performing one asset on everyday from December 22nd, 2020 to June 25th, 2021.

Figure 6 represents the risk measured by ${\sigma_p}^2$. The risk ${\sigma_p}^2$ of 'proposed portfolio volatility' is calculated and updated day by day. The risk comparison of 'random portfolio volatility' is the average of the top 30% of 100 tests on the random portfolio's weights. In this simulation, 100 KOSPI stock assets in the bullish market 2021 were used. The first row of Figures 6, (a), (b), and (c) show a risk comparison when the target return is 0.03, and the second, third and fourth rows are risk comparisons for a given target returns 0.05, 0.07, 0.15. Also, the first column (a), (d), (g), and (j) are illustrated for the risk comparison. The simulations are conducted for the portfolio consisted of 5 assets, and the second and third columns are risk comparisons when the portfolio assets are 30 and 50, respectively. Even though random simulation is the result of the top 30%, the volatility of the proposed portfolio generated by our daily-one asset replacement method is lower.

The difference between 'random portfolio volatility' and 'proposed portfolio volatility' was

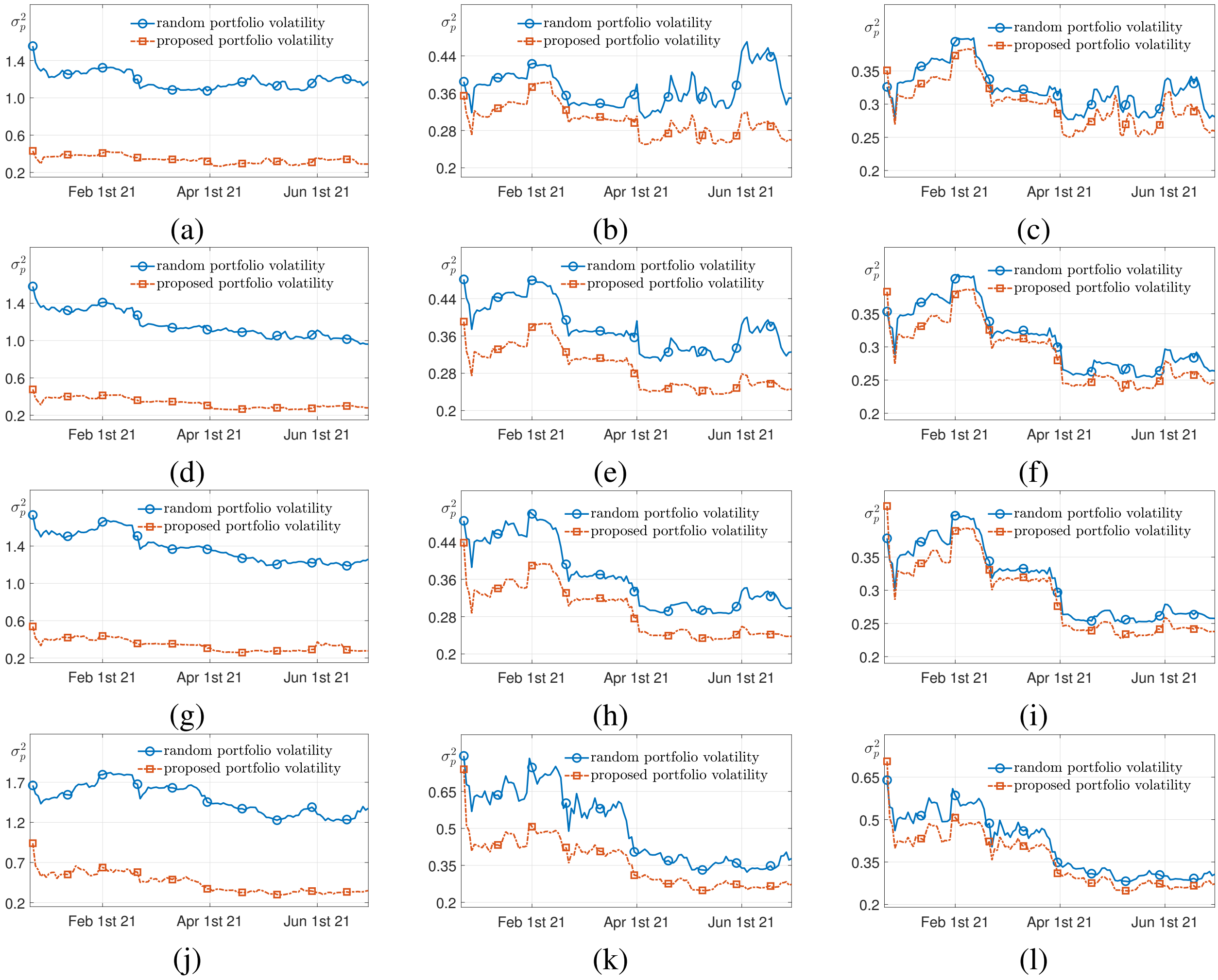


Figure 6. In the bullish market 2021, risks are measured for comparison between simulations of daily risk by our daily-one asset replacement scheme and daily plot by randomly selecting assets scheme. Both tests use 100 KOSPI stock assets. The 100 KOSPI assets in this simulation are used for calculating the daily returns from June 25th, 2020 to June 25th, 2021.

significantly different in first column than in third column. Thus, the smaller the number of portfolio's assets, the gap between proposed and 30% random portfolios is greater. In addition, although the tendency of two portfolio risks tends to be similar, the 'proposed portfolio volatility' is always low as seen in the risk simulation of Figure 4.

6.2.4. Cumulative return in 2021

In the previous Figure 5, we examined how the cumulative return was different when target return and portfolio size were fixed differently. We now observe the changes of 'return of random portfolios' and 'return of proposed portfolios' using 2021 KOSPI. KOSPI assets' return data for one year from June 25th, 2020 to June 25th, 2021 were used. The cumulative returns from December 22nd, 2020 to June 25th, 2021 were shown. They are evaluated on each day based on the previous 6 months of one year of data.

Figure 7 represents the cumulative return measured by r_c. The cumulative return of proposed portfolios r_c is updated by replacing one asset in portfolio with one of the reference asset set. For a comparison of proposed portfolio's return, the cumulative return is calculated by the average of the best of top 30% of 100 random portfolios cumulative returns. The first column are graphs of portfolios consisted of 5 assets, the second and third columns are graphs of portfolios of 30 and 50 assets. Also, the first row are graphs of the optimal portfolio when the target return is fixed at 0.03, and the second, third, and fourth rows are graphs of the optimal portfolio calculated by fixing the target return at 0.05, 0.07, and 0.15, respectively.

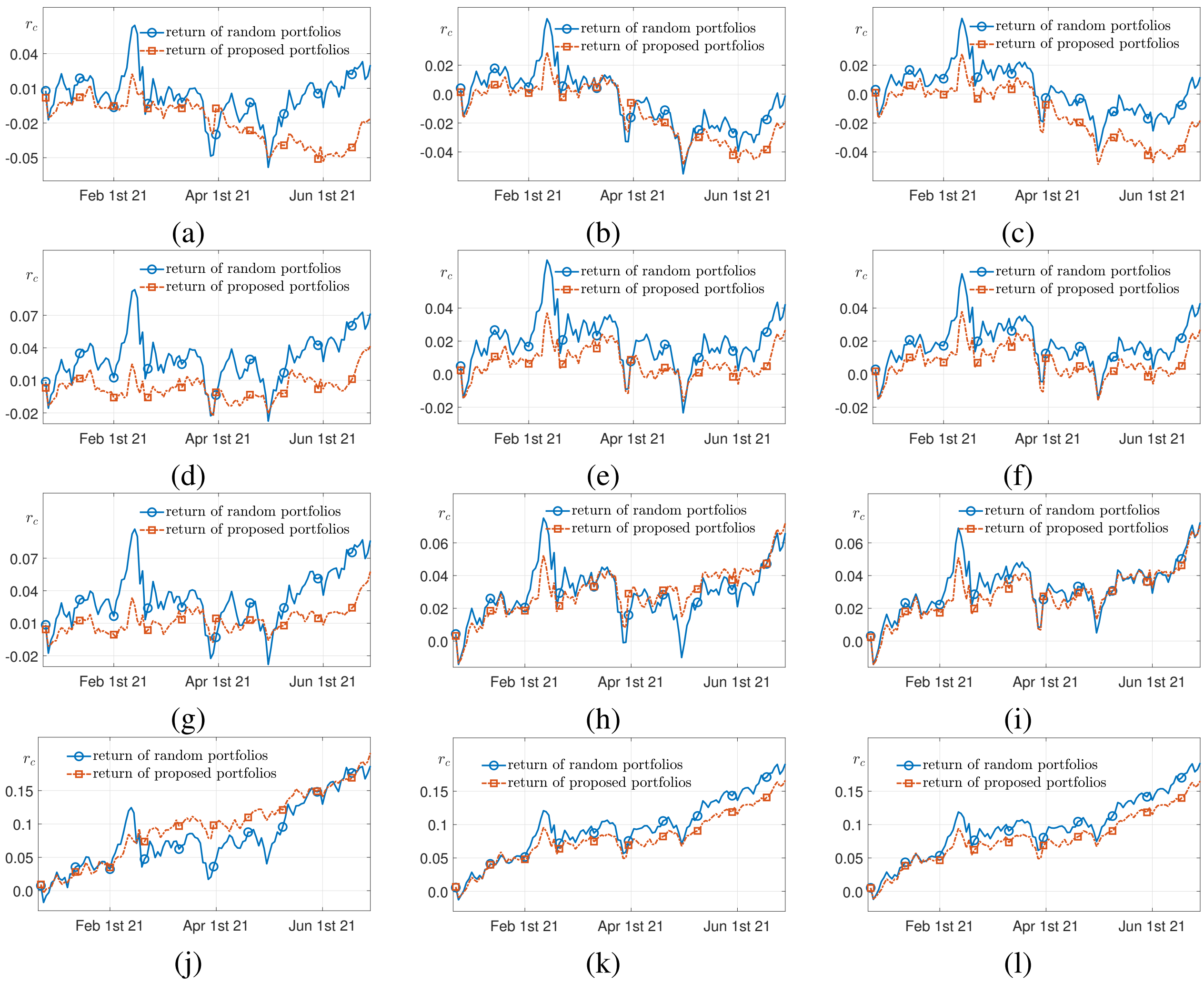


Figure 7. Cumulative return comparison simulation of daily risk by our daily-one asset replacement scheme and daily plot by randomly selecting assets scheme using 100 KOSPI stock assets in the bullish market 2021.

Although we confirmed that the cumulative return of proposed portfolio as seen in Figure 5 would be high, the performance with 2021 KOSPI data did not show us significant differences between two portfolio schemes. However, comparing with the average over 30% top-performing random portfolios in Figure 7, we see that the cumulative return of portfolio by our scheme maintains consistency in return. In other words, the portfolio by our scheme is applicable to make a profit more safely.

6.2.5. NASDAQ and KOSPI

In the following simulation, we consider a 5-asset portfolio. A set of the total assets contains 100 KOSPI assets and 100 NASDAQ assets, i.e., we consider 200 assets in this simulation, and the each asset is denoted by $\{k_1, \ldots, k_{100}, n_1, \ldots, n_{100}\}$. The initial portfolio set consists of 5 assets: $\{p_1, p_2, p_3, p_4, p_5\}$, which are 5 KOSPI assets, and we replace one asset in portfolio with one of the reference assets daily. Everyday a portfolio is updated to a new portfolio that optimizes risk by replacing one asset. From March 20th 2020 to March 9th 2021, over the 230 days held in the market were calculated and evaluated in units of 115 days, forming one portfolio with the optimal risk every day for a total of 115 days.

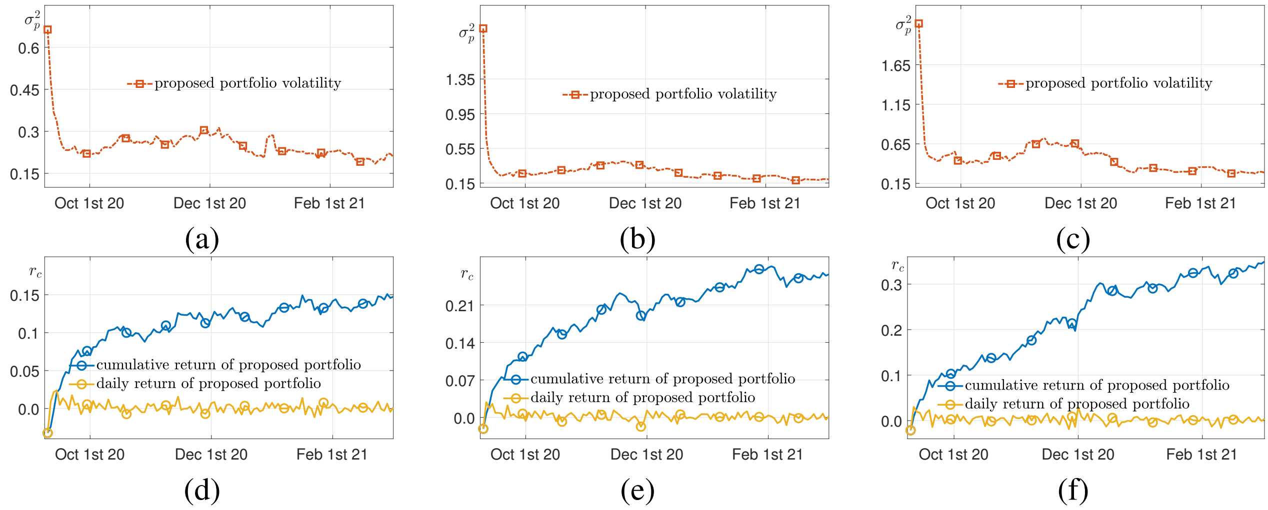


Figure 8. (a), (b), and (c) are daily risk by our daily-one-asset replacement scheme when the target return is $r_p = 0.05, 0.1, 0.15$. (d), (e), and (f) are daily cumulative return by our daily-one-asset replacement scheme when the target return is 0.05, 0.1 and 0.15.

This simulation, which includes the KOSPI market and the NASDAQ market, has daily returns from March 20th, 2020 to March 9th, 2021. In both the first and second rows, data for one year was calculated in units of 6 months, and represents the risk variations and cumulative return from September 14th and 2020 to March 20th and 2020, respectively. Figure 8 represents the risk and cumulative return measured by σ_p^2, r_c. The cumulative return r_c is daily calculated on each business day. In the above simulations, when the target return is 0.05, the assets and weights of portfolio on the last day measured by P_{115} and $\mathbf{x}_{115}$ are:

$$P_{115} = \{n_{69}, k_{99}, k_{56}, k_{30}, k_{100}\}, \quad \mathbf{x}_{115} = (0.5014, 0.1850, 0.1281, 0, 0.1855)^T.$$

When the target return is given by $r_p = 0.1$, P and $\mathbf{x}$ are as follows.

$$P_{115} = \{k_{99}, k_5, n_{69}, k_{54}, n_{38}\}, \quad \mathbf{x}_{115} = (0.1638, 0.0841, 0.5396, 0.0821, 0.1304)^T.$$

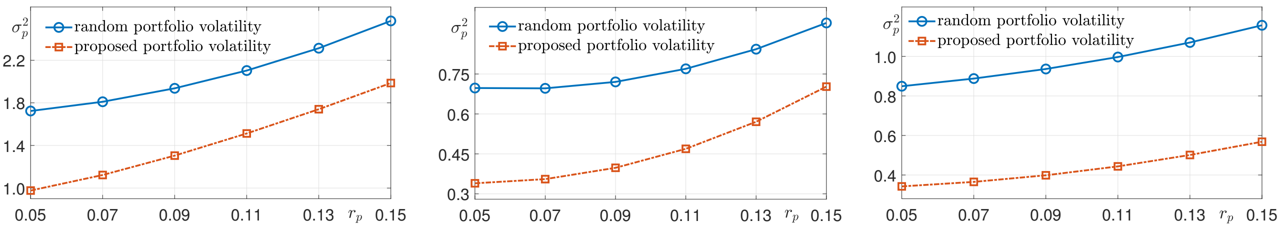
Also, P and $\mathbf{x}$ are composed as follows for a given target return $r_p = 0.15$.

$$P_{115} = \{n_{69}, k_{67}, k_{100}, k_{99}, n_{142}\}, \quad \mathbf{x}_{115} = (0.5513, 0.0935, 0.1704, 0.1847, 0)^T.$$

Thus, a risk averse investor with a target return of 0.05, an investor with a target return of 0.1, and a risk tolerance with a target return of 0.15 all started with portfolio of $\{k_1, k_2, k_3, k_4, k_5\}$. However, As an asset in portfolio changes daily, investors have different portfolios depending on their investment propensity.

6.2.6. The efficient frontier

A risk averse investor chooses to preserve capital over the potential for a higher than average return. Modern portfolio theory assumes that investors are risk averse, meaning that they prefer the minimum volatility portfolio for a given target return. In this simulation, the minimum risks for each target return given in the KOSPI market, NASDAQ market, and the combined market are shown. The graph as seen in Figure 9 is known as the efficient frontier. In the efficient frontier, portfolio corresponding to each point in the curve is called the minimum variation portfolio. It is possible to create different portfolio in the area above the efficient frontier. However, the risk of the other portfolios is greater than the risk of the minimum variance portfolio for the same target return, so no one invests in portfolio, which belongs to the area above the efficient frontier.



(a) (b) (c)

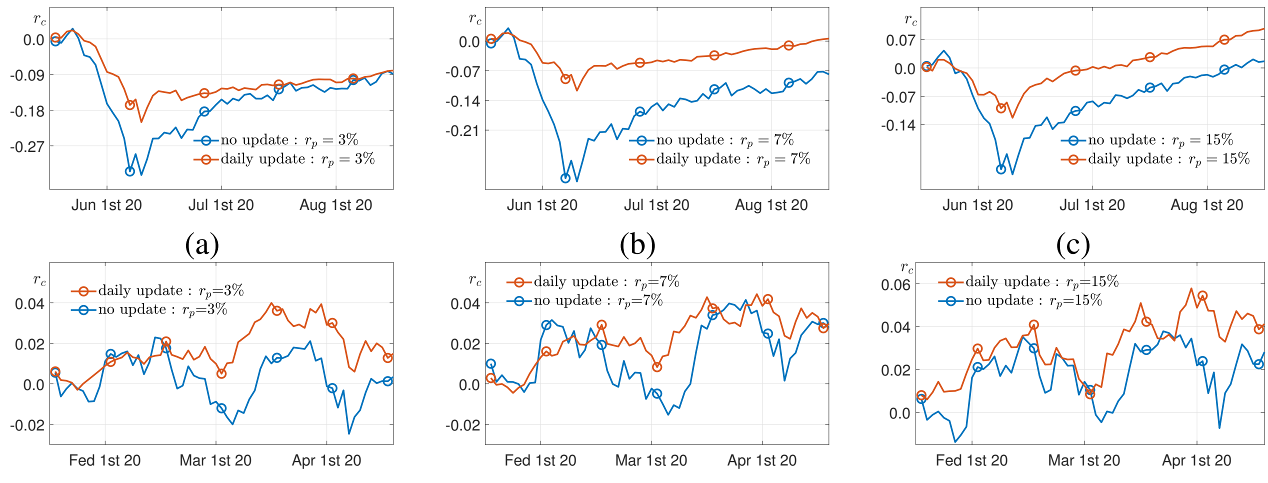
Figure 9. The efficient frontier.

Figure 9 represents the efficient frontiers of portfolios, which consist of randomly chosen assets and proposed portfolios by using our method. The graph shows the risk of efficient portfolio ${\sigma_p}^2$ for target return r_p, we can confirm that the risk of portfolio made through our method is always lower than the risk of portfolio made from top 30% random portfolio. Figure 9(a) shows the effective frontier using the KOSPI market data of 2020, also Figure 9(b) implies the effective frontier using the KOSPI market data of 2021, and last Figure 9(c) shows the effective frontier analyzed by the KOSPI market and the NASDAQ market data from 2020 to 2021.

6.2.7. Black swan event

In the previous simulations, it was confirmed that the risk of daily update portfolio was lower than that of the randomly constructed portfolio. In this simulation, we observe how to affect the returns when unexpected, "Black Swan", event occurs. We compare the daily updated portfolio and portfolio that has not been updated. Noting not updated portfolio implies that all asset weights are fixed and portfolio maintenance is not available in an unexpected occasion.

There is no doubt that COVID-19 was a "Black Swan" in financial markets. One of the first countries to be affected by COVID-19 was Republic of Korea, where the first confirmed case was reported on January 20th, 2020, detected at Incheon International Airport. After fears of the COVID-19 caused the KOSPI to fall below more than 1000 points for the first time in ten years. As shown in



(d) (e) (f)

Figure 10. First day, Jan. 20th, 2020, COVID-19 one confirmed case up to three months afterward, proposed portfolio's returns and no maintenance portfolio's returns are compared.

Figure 10, unexpected event lowers the most of portfolio's returns. Even though COVID-19 confirmed cases increased as return by proposed scheme fell, our returns overrun the graph of no update scheme's return.

7. Conclusions

Portfolio optimization is a cornerstone of financial theory and practice. However, it is still not easy to allocate assets in a portfolio. Furthermore, we do not allow any negative weight value on the allocations. i.e., the optimized portfolio weights are to be found without short positions. The optimized weights should be re-balanced due to the stock fluctuations. In this paper, we developed the efficiently updating scheme for the optimized portfolio. Given the difficulties of re-calculating a large-scale covariance matrix, our proposed scheme simply performed rank-2 modification of the matrix.

Based on Markowiz's portfolio theory, we assume that investors are risk averse. In other words, investors choose the portfolio with the lowest risk when portfolios with the same expected return are provided. Also, there is a trade-off between expected return and risk, and investors take different risks according to their investment propensity. Therefore, in this paper, we have dealt with various the target returns, and all simulation results have shown that the risks of our proposed portfolio are always low.

The year 2020 indeed, has turned out to be quite impressive for most stock markets. All major economic indices have declined, and reached their lowest levels in recent periods. The lowest indicators were caused by economic effects of the COVID19 pandemic crisis, however, most simulation results by our proposed scheme outperformed the average of the top 30% of tests on random portfolios, in both KOSPI and NASDAQ markets. On the other hand, the year 2021 has had the upward tendency, due to the Bank of Korea's quantitative easing, and economic revival by COVID-19 vaccination, and so on. These economic boosts put the KOSPI and NASDAQ indices in bullish market.

The main contribution of this work is to build and update (or maintain) the optimal portfolio weights daily subject to the nonnegative constraint, $\mathbf{x} \geq \mathbf{0}$. We assume that portfolio consists of fixed assets, i.e. merely asset replacement occurs to find the best portfolio under the given low-risk and high-return conditions. Our dynamic portfolio optimization allows for minutely, hourly, daily, weekly or monthly maintenance, but we only consider daily stock closing prices in light of limits on available market data. Furthermore, we obtain a profitable portfolio by adjusting one of assets and renewing the portfolio as time goes by. Therefore, we plan on efficiently solving the dynamic portfolio selection problem.

Numerical tests have shown to have high validity for the portfolio optimization using Monte-Carlo simulation and empirical application on the stock prices of KOSPI and NASDAQ. We also demonstrated the robustness of our proposed scheme with a large number of assets. We applied our proposed scheme to KOSPI and NASDAQ markets, and considered the returns and risks in bearish and bullish periods. These finding illustrated the validity of our proposed portfolio scheme.

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Conflict of interest

The authors declare no conflict of interest.

References

1. R. O. Michaud, The Markowitz Optimization Enigma: Is 'Optimized' Optimal? *Financ. Anal. J.,* **45** (1989), 31–42. https://doi.org/10.2469/faj.v45.n1.31
2. L. Jiang, S. Wang, Robust multi-period and multi-objective portfolio selection, *J. Ind. Manag. Optim.,* **17** (2021), 695–709. https://doi.org/10.3934/jimo.2019130
3. L. Lao, Portfolio selection based on uncertain fractional differential equation, *AIMS Math.,* **7** (2022), 4304–4314. https://doi.org/10.3934/math.2022238
4. H. Huang, Z. Zhang, An intrinsic robust rank-one-approximation approach for currency portfolio optimization, *AIMS QFE.,* **2** (2018), 160–189. https:// 10.3934/QFE.2018.1.160
5. L. Wu, Y. Yang, H. Liu, Non-negative-lasso and application in index tracking, *Comput. Stat. Data Anal.,* **70** (2014), 116–126. https://doi.org/10.1016/j.csda.2013.08.012
6. Y. M. Yen, T. J. Yen, Solving norm constrained portfolio optimization via coordinate-wise descent algorithms, *Comput. Stat. Data Anal.,* **76** (2014), 737–759. https://doi.org/10.1016/j.csda.2013.07.010
7. Z. Dai, J. Kang, F. Wen, Predicting stock returns: a risk measurement perspective, *Int. Rev. Financ. Anal.,* **74** (2021), 101676. https://doi.org/10.1016/j.irfa.2021.101676
8. Z. Dai, J. Kang, Some new efficient mean variance portfolio selection models, *International J. Financ. Econ.,* **74** (2021), 1–13. https://doi.org/10.1002/ijfe.2400
9. R. C. Green, B. Hollifield, When will mean-variance efficient portfolios be well diversified?, *J. Finance,* **47** (1992), 1785–1809. https://doi.org/10.2307/2328996

10. R. Mansini, W. Ogryczak, M. G. Speranza, Twenty years of linear programming based portfolio optimization, *Eur. J. Oper. Res.,* **234** (2014), 518–535. https://doi.org/10.1016/j.ejor.2013.08.035

11. C. Aranha, H. Iba, The Memetic Tree-based genetic algorithm and its application to Portfolio Optimization, *Menet. Comput.,* **1** (2009), 139–151. https://doi.org/10.1007/s12293-009-0010-2

12. X. Zeng, A stochastic volatility model and optimal portfolio selection, *Quant. Financ.,* **13** (2013), 1547–1558. https://doi.org/10.1080/14697688.2012.740568

13. A. Buraschi, P. Porchia, F. Trojani, Correlation risk and optimal portfolio choice, *J. Financ.,* **65** (2010), 393–420. http://dx.doi.org/10.2139/ssrn.908664

14. W. Sun, Y. X. Yuan, *Optimization Theory and Methods: Nonlinear Programming*, Springer, 2006.

15. X. Lin, M. Wang, C. H. Lai, A modification term for Black-Scholes model based on discrepancy calibrated with real market data, *AIMS DSFE.,* **1** (2021), 313–326. https://10.3934/DSFE.2021017

16. S. Boyd, *Convex optimization*, Cambridge University Press, 2004.

17. R. H. Kwon, *Introduction to Linear Optimization and Extensions with MATLAB*, CRC Press, 2014.

18. D. J. Higham, *An Introduction to Financial Option Valuation*, Cambridge University Press, 2004.

Appendix: Mathematical definition and more detailed explanation

We assume that n financial assets are available. For the asset index $1 \le i \le n$ and the number of total unit times l, e.g. n assets and l days, let us first define the following $l \times 1$ vectors $R_i = (r_{1,i}, \dots, r_{k,i}, \dots, r_{l,i})^T$ containing unit time's the asset returns, where $r_{k,i}$ represents the return of i-th asset on k-th day. The i-th asset's mean return in the portfolio is also defined as $\mu_{R_i} := \mathbb{E}(R_i)$. Together with with $R_1,\ R_2, \dots, R_n$, we build the variancecovariance (or covariance) matrix Σ:

$$\Sigma = \begin{pmatrix} var(R_1) & cov(R_1,R_2) & \cdots & cov(R_1,R_n) \\ cov(R_2,R_1) & var(R_2) & \cdots & cov(R_2,R_n) \\ \vdots & \vdots & \ddots & \vdots \\ cov(R_n,R_1) & cov(R_n,R_2) & \cdots & var(R_n) \end{pmatrix}.$$

The two vectors $u,\ v$ in section 3 are orthogonal and it is readily confirmed as follows:

$$\begin{aligned} u^T v &= \frac{1}{\beta_1\beta_2}(m_{1,i}^2 + \cdots + m_{i-1,i}^2 + \sigma_1\sigma_2 + m_{i+1,i}^2 \cdots + m_{n,i}^2) \\ &= \frac{1}{\beta_1\beta_2}\left[m_{1,i}^2 + \cdots + m_{i-1,i}^2 + \frac{1}{4}\left(m_{i,i}^2 - 4\sum_{j=1}^{n} m_{j,i}^2 + 3m_{i,i}^2\right) + m_{i+1,i}^2 \cdots + m_{n,i}^2\right] = 0. \end{aligned}$$