AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 15795–15811.
DOI: 10.3934/math.2022864
Received: 13 May 2022
Revised: 15 June 2022
Accepted: 18 June 2022
Published: 27 June 2022

Research article

Breather wave, resonant multi-soliton and M-breather wave solutions for a (3+1)-dimensional nonlinear evolution equation

Sixing Tao*

School of Mathematics and Statistics, Shangqiu Normal University, Shangqiu 476000, China

* **Correspondence:** Email:taosixing@163.com; Tel: +8613781680641.

Abstract: In this paper, a (3+1)-dimensional nonlinear evolution equation is considered. First, its bilinear formalism is derived by introducing dependent variable transformation. Then, its breather wave solutions are obtained by employing the extend homoclinic test method and related figures are presented to illustrate the dynamical features of these obtained solutions. Next, its resonant multi-soliton solutions are obtained by using the linear superposition principle. Meanwhile, 3D profiles and contour plots are presented to exhibit the process of wave motion. Finally, M-breather wave solutions such as one-breather, two-breather, three-breather and hybrid solutions between breathers and solitons are constructed by applying the complex conjugate method to multi-soliton solutions. Furthermore, their evolutions are shown graphically by choosing suitable parameters.

Keywords: breather wave solutions; rouge waves; resonant multi-soliton solutions; M-breather solutions; hybrid solutions
Mathematics Subject Classification: 35A25, 35G50, 35Q35, 37K10

1. Introduction

It is known that Hirota bilinear method [1] is a direct and effective method to solve a large number of nonlinear evolution equations, which can depict physical phenomenon in nonlinear science. Recently, Professor Ma considered the Hirota conditions of a (2+ 1)-dimensional combined equation and derived its the N-soliton solution in [2]. Furthermore, Ma presented N-soliton solutions of the combined pKP-BKP equation in [3] and soliton solutions to the B-type Kadomtsev-Petviashvili equation under general dispersion relations in [4] by using Hirota bilinear method. In recent literature, many types of exact solutions have been obtained by employing the Hirota bilinear method, such as soliton solutions [5–7], breather solutions [8, 9], lump wave solutions [10], interaction solutions [11, 12], rogue wave solutions [13, 14], resonant multi-soliton solutions [15–17], bifurction solitons [18, 19], bright and dark soliton solutions [20, 21] and so on.

In 2009, Dai proposed the homoclinic test approach and extended homoclinic test approach to search solitary-wave solution of high dimensional nonlinear equation in [22]. Then Xu proposed the homoclinic breather limit method for searching rogue wave solution to nonlinear evolution equation in [23]. In 2011, Ma constructed the resonant multi-soliton solutions of nonlinear equations by introducing introducing the linear superposition principle [24]. Afterwards the linear superposition principle was used to establish the resonant multiple wave solutions of a (3+1)-dimensional generalized Kadomtsev-Petviashvili equation by Lin in [25]. By comparing the linear superposition principle and the velocity resonant condition, Kuo [26, 27] concluded that the linear superposition principle is more effective, although they could result in the same results. In recent years, the complex conjugate method was applied to N-soliton solutions to construct M-breather solutions and corresponding hybrid solutions in [28–30]. As early as 1977, lump wave solutions were first proposed by Manakov et al. [31] to indicate that the type of wave does not decrease in the direction of (x, y)-plane. Later, Satsuma and Ablowitz [32] proposed the long wave limit method for seeking M-lump solutions based on the collision effects of lump and N-soliton. And it is proved to be the most effective method to construct M-lump solutions of nonlinear evolution equationsİn the past few years, a lot of literature on M-lump solutions and hybrid solutions between lump and solitons of nonlinear evolution equations has been appeared such as the (2+1)-dimensional combined pKP-BKP equation [33], the (3+1)-dimensional potential-YTSF equation [34], a (2+1)-dimensional generalized KDKK equation [35], the (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation [36], (2+1)-dimensional HSI equation [37], (3+1)-dimensional Kadomtsev-Petviashvili equation [38], (3+1)-dimensional gCH-KP equation [39], the (4+1)-dimensional Boiti-LeonManna-Pempinelli equation [40] and so on.

Recently, Zhang etc [41] considered a (3 +1)- dimensional nonlinear evolution equation which was written by

$$-4u_{xt} + u_{xxxz} + 3\alpha u_{yy} + 4u_x u_{xz} + 2u_{xx}u_z = 0. \tag{1.1}$$

Its M-lump and interactive solutions of Eq (1.1) were given in [41] by using long wave limit method. when $\alpha = 1$, Eq (1.1) can be reduced to the potential-YTSF equation [34], whose nontravelling wave solutions were discussed by Yan in [42]. Based on the above literature, we aim to consider the breather wave solutions and resonant multi-soliton solutions, M-breather solutions of Eq (1.1).

The rest of this paper is organized as follows. In Section 2, the bilinear formalism of the (3 + 1)-dimensional equation (1.1) is derived via introducing variable transformation; In Section 3, breather wave solutions are derived by using the extend homoclinic test approach. Meanwhile, a rouge wave solution is derived by using the homoclinic breather limit method ; In Section 4, resonant multi-soliton solutions are obtained by using linear superposition principle; In Section 5, multi-soliton solutions are derived by Hirota bilinear method, then its M-breather solutions and hybrid solutions are constructed based upon the obtained multi-soliton solutions. Finally, related remarks are given.

2. Bilinear formalism for the (3 + 1)-dimensional equation (1.1)

Through simple calculation, we can see that it is impossible to establish the Hirota bilinear formalism of the (3 + 1)-dimensional equation (1.1) directly. By introducing a dependent variable transformation $\xi = x + kz$ in (1.1), the (3 + 1)-dimensional equation (1.1) can be transformed into the following form

$$-4u_{\xi t} + ku_{\xi\xi\xi\xi} + 3\alpha u_{yy} + 6ku_\xi u_{\xi\xi} = 0, \tag{2.1}$$

where k is a real constant. In what follows, we introduce another potential transformation

$$u = cq_{\xi}. \tag{2.2}$$

By substituting (2.2) into (2.1), we can get

$$E(q) = -4cq_{\xi\xi t} + ckq_{\xi\xi\xi\xi\xi} + 3\alpha cq_{\xi yy} + 6kc^2 q_{\xi\xi}q_{\xi\xi\xi} = 0. \tag{2.3}$$

Integrating the Eq (2.3) with respect to ξ once, we have

$$-4cq_{\xi t} + ckq_{\xi\xi\xi\xi} + 3c\alpha q_{yy} + 3c^2 kq_{\xi\xi}^2 = 0. \tag{2.4}$$

According to the results between Bell polynomials and bilinear formalism in Ref. [43] and taking $c = 1$, the above expression (2.4) leads to the following bilinear form of (2.1)

$$\left[-4D_{\xi}D_t + kD_{\xi}^4 + 3\alpha D_y^2\right]\varphi \cdot \varphi = 0, \tag{2.5}$$

with the help of the following transformation relationship

$$q = 2\ln(\varphi) \Leftrightarrow u = q_{\xi} = 2\left(\ln\varphi\right)_{\xi}. \tag{2.6}$$

3. Breather wave and rouge wave solutions

In the beginning, we aim to investigate the breather wave solutions of Eq (1.1) by using the extend homoclinic test method [22]. We choose the test function as the following form

$$\varphi = \exp(-p_1(\xi + m_1 y + c_1 t)) + \delta_1\cos(p_2(\xi + m_2 y + c_2 t)) + \delta_2\exp(p_1(\xi + m_1 y + c_1 t)), \tag{3.1}$$

where $p_1, p_2, m_1, m_2, c_1, c_2, \delta_1, \delta_2$ are undetermined real constants. By substituting Eq (3.1) into Eq (2.5), an algebraic equation can be obtained. Setting all coefficients for the powers of $\exp(\pm p_1(\xi + m_1 y + c_1 t)), \sin(p_2(\xi + m_2 y + c_2 t)), \cos(p_2(\xi + m_2 y + c_2 t))$ to be zero, some algebraic equations can be obtained. Taking $p_1 = p_2 = p$, we can get

$$\begin{cases} 2p^2(-6\alpha\delta_1 m_1 m_2 + 4c_1\delta_1 + 4c_2\delta_1) = 0, \\ 2p^2(3\alpha\delta_1 m_1^2 - 3\alpha\delta_1 m_2^2 - 4p^2k\delta_1 - 4c_1\delta_1 + 4c_2\delta_1) = 0, \\ 2p^2(3\alpha\delta_1\delta_2 m_1^2 - 3\alpha\delta_1\delta_2 m_2^2 - 4p^2k\delta_1\delta_2 - 4c_1\delta_1\delta_2 + 4c_2\delta_1\delta_2) = 0, \\ 2p^2(6\alpha\delta_1\delta_2 m_1 m_2 - 4c_1\delta_1\delta_2 - 4c_2\delta_1\delta_2) = 0, \\ 2p^2(-3\alpha\delta_1^2 m_2^2 + 4p^2k\delta_1^2 + 12\alpha\delta_2 m_1^2 + 16p^2k\delta_2 + 4c_2\delta_1^2 - 16c_1\delta_2) = 0. \end{cases} \tag{3.2}$$

Solving the obtained equations (3.2) by means of symbolic computation, two cases are obtained:

Case 1.

$$\begin{aligned} &\delta_1^2 = \frac{4((m_1 - m_2)^2\alpha + 4p^2k)\delta_2}{(m_1 - m_2)^2\alpha - 4p^2k}, c_1 = \frac{(3m_1^2 + 6m_1 m_2 - 3m_2^2)\alpha}{8} - \frac{p^2k}{2}, \\ &c_2 = \frac{(-3m_1^2 + 6m_1 m_2 + 3m_2^2)\alpha}{8} + \frac{p^2k}{2}. \end{aligned} \tag{3.3}$$

in which p, δ_2, m_1, m_2 are arbitrary real numbers with $\frac{4((m_1-m_2)^2\alpha+4p^2k)\delta_2}{(m_1-m_2)^2\alpha-4p^2k} > 0$.

Case 2.

$$\delta_1^2 = -4\delta_2, c_1 = \frac{3m_2^2\alpha}{4} - \frac{p^2k}{2}, c_2 = \frac{3m_2^2\alpha}{4} + \frac{p^2k}{2}, \tag{3.4}$$

in which δ_2, p, m_1, m_2 are arbitrary real numbers with $\delta_2 < 0$.

For Case 1, substituting (3.3) along with $\xi = x + kz$ into (3.1), we obtain the following results

$$\varphi = \exp(-p(x+m_1y+kz+\Omega_1t)) \pm \sqrt{\Omega_2}\cos(p(x+m_2y+kz+\Omega_3t)) + \delta_2\exp(p(x+m_1y+kz+\Omega_1t)), \tag{3.5}$$

where $\Omega_1 = \frac{(3m_1^2+6m_1m_2-3m_2^2)\alpha}{8} - \frac{p^2k}{2}, \Omega_2 = \frac{4((m_1-m_2)^2\alpha+4p^2k)\delta_2}{(m_1-m_2)^2\alpha-4p^2k}, \Omega_3 = \frac{(-3m_1^2+6m_1m_2+3m_2^2)\alpha}{8} + \frac{p^2k}{2}$.

When $\delta_2 > 0$, (3.5) can be reduced to the following equation

$$\varphi = 2\sqrt{\delta_2}\cosh(p(x + m_1y + kz + \Omega_1t) + \frac{1}{2}\ln(\delta_2)) \pm \sqrt{\Omega_2}\cos(p_2(x + m_2y + kz + \Omega_3t)). \tag{3.6}$$

According to the variable transformation $u = 2\ln(\varphi)_\xi$, we obtain the exact wave solutions for (1.1) as

$$u_1 = \frac{4p\sqrt{\delta_2}\sinh(p(x + m_1y + kz + \Omega_1t) + \frac{1}{2}\ln(\delta_2)) - 2p\sqrt{\Omega_2}\sin(p(x + m_2y + kz + \Omega_3t))}{2\sqrt{\delta_2}\cosh(p(x + m_1y + kz + \Omega_1t) + \frac{1}{2}\ln(\delta_2)) + \sqrt{\Omega_2}\cos(p(x + m_2y + kz + \Omega_3t))}, \tag{3.7}$$

$$u_2 = \frac{4p\sqrt{\delta_2}\sinh(p(x + m_1y + kz + \Omega_1t) + \frac{1}{2}\ln(\delta_2)) + 2p\sqrt{\Omega_2}\sin(p(x + m_2y + kz + \Omega_3t))}{2\sqrt{\delta_2}\cosh(p(x + m_1y + kz + \Omega_1t) + \frac{1}{2}\ln(\delta_2)) - \sqrt{\Omega_2}\cos(p(x + m_2y + kz + \Omega_3t))}, \tag{3.8}$$

where u_1 and u_2 are two homoclinic breather wave solutions of (1.1). Their dynamical properties are much similar, so we only discuss u_2 for an example. By selecting appropriate parameters, the profile of u_2 is shown in Figure 1, which is of great value in understanding the dynamical behaviors of the breathers (3.7) and (3.8).

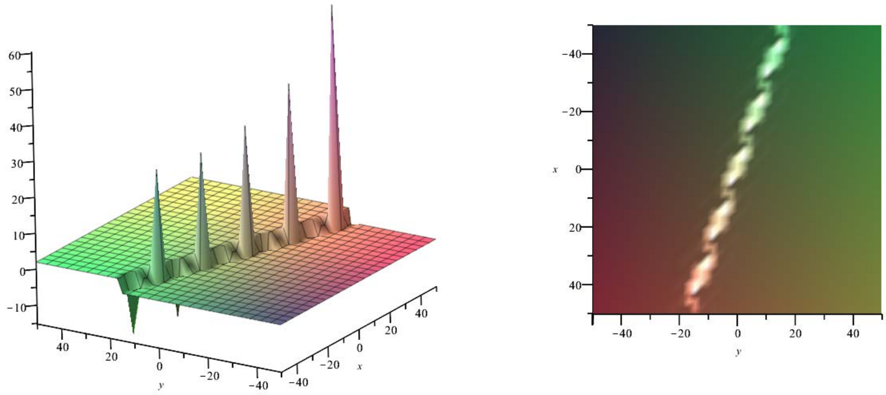
For u_2 in Eq (3.8) at $p = 0$, taking $\delta_2 = 1$, we can obtain the following Taylor expansions

$$\begin{aligned}
&\sinh(p(x + m_1y + kz + \Omega_1t)) = p(x + m_1y + kz + \Omega_1t) + O(p^2),\\
&\sin(p(x + m_2y + kz + \Omega_3t)) = p(x + m_2y + kz + \Omega_3t) + O(p^2),\\
&\cosh(p(x + m_1y + kz + \Omega_1t)) = 1 + \frac{1}{2}(p(x + m_1y + kz + \Omega_1t))^2 + O(p^3),\\
&\cos(p(x + m_1y + kz + \Omega_1t)) = 1 - \frac{1}{2}(p(x + m_2y + kz + \Omega_3t))^2 + O(p^3).
\end{aligned} \tag{3.9}$$

Supposing an arbitrary constant u_0 is the equilibrium solution of Eq (1.1), substituting above Taylor expansions into Eq (3.8), then the following rouge wave solution for Eq (1.1) can be obtained

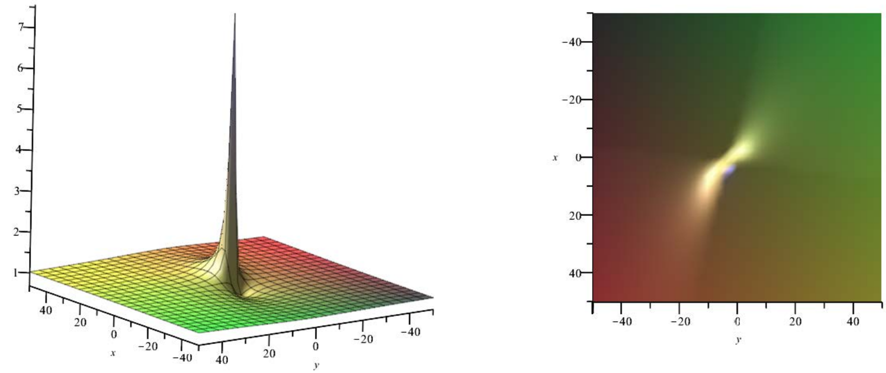
$$u_3 = u_0 + \frac{8x + 4(m_1 + m_2)y + 8kz + 6m_1m_2t}{\left(x + m_1y + kz + \frac{(3m_1^2+6m_1m_2-3m_2^2)\alpha t}{8}\right)^2 + \left(x + m_2y + kz + \frac{(-3m_1^2+6m_1m_2+3m_2^2)\alpha t}{8}\right)^2}. \tag{3.10}$$

By choosing the same parameters as in Figure 1, we exhibit the profile of rouge wave solution (3.10) in Figure 2.



(a) (b)

Figure 1. (Color online) Plots of breather solution (3.8) with $\alpha = 2, p = 1, \delta_2 = 1, m_1 = 3, m_2 = 1, k = -1, z = 1$ at time $t = 0$. (a) 3D plot; (b) Overhead view of the wave.



(a) (b)

Figure 2. (Color online) Rouge wave for Eq (1.1) described by (3.10) with $u_0 = 1, \alpha = 2, k = 3, m_1 = 2, m_2 = -1, z = 1$ at $t = 2$. (a) 3D profile; (b) Vertical view of the wave.

We can see that there is one peak and one trough in Figure 2(a). For any given time t, when $x^2 + y^2 + z^2 \to +\infty, u_3$ tends to the equilibrium solution u_0. If taking $u_0 = 0$, when p tends to 0, u_3 is exactly the limit of u_2 . From the mathematical point of view, the rogue wave solution (3.10) is a limit behavior of breather wave solution (3.8). From the physical point of view, we may think perhaps the energy collection and superposition of breather waves in many periods generate a rogue wave.

For Case 2, when $\delta_2 < 0$, (3.5) can be reduced to the following equation

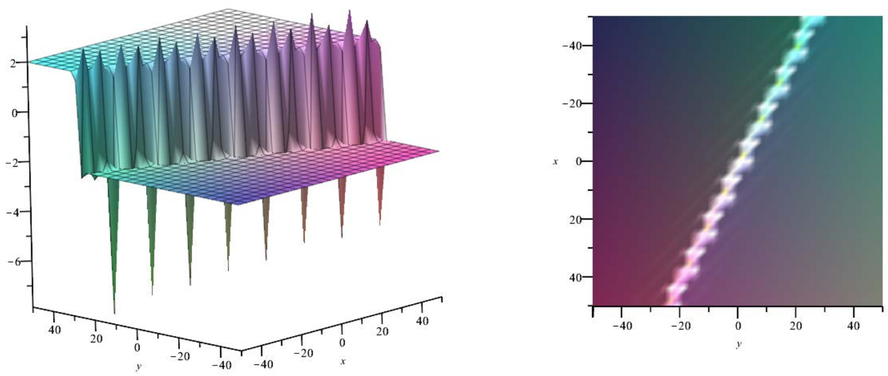
$$\varphi = -2\sqrt{-\delta_2}\sinh(p(x + m_1 y + kz + \Omega_1 t) + \frac{1}{2}\ln(-\delta_2)) \pm \sqrt{\Omega_2}\cos(p_2(x + m_2 y + kz + \Omega_3 t)). \quad (3.11)$$

According to the variable transformation $u = 2\ln(\varphi)_\xi$, we obtain the exact wave solutions for (1.1) as

$$u_4 = \frac{4p\sqrt{-\delta_2}\cosh(p(x + m_1 y + kz + \Omega_1 t) + \frac{1}{2}\ln(-\delta_2)) + 2p\sqrt{\Omega_2}\sin(p(x + m_2 y + kz + \Omega_3 t))}{2\sqrt{-\delta_2}\sinh(p(x + m_1 y + kz + \Omega_1 t) + \frac{1}{2}\ln(-\delta_2)) - \sqrt{\Omega_2}\cos(p(x + m_2 y + kz + \Omega_3 t))}, \quad (3.12)$$

$$u_5 = \frac{4p\sqrt{-\delta_2}\cosh(p(x + m_1 y + kz + \Omega_1 t) + \frac{1}{2}\ln(-\delta_2)) - 2p\sqrt{\Omega_2}\sin(p(x + m_2 y + kz + \Omega_3 t))}{2\sqrt{-\delta_2}\sinh(p(x + m_1 y + kz + \Omega_1 t) + \frac{1}{2}\ln(-\delta_2)) + \sqrt{\Omega_2}\cos(p(x + m_2 y + kz + \Omega_3 t))}, \quad (3.13)$$

where u_4 and u_5 are also two homoclinic breather wave solutions of (1.1). They also have similar dynamical properties , here we only choose u_4 for an example. By selecting proper parameters, we present the profile of u_5 in Figure 3, which will help us better understand the dynamical behaviors of the breather wave solutions (3.12) and (3.13).



(a) (b)

Figure 3. (Color online) Breather wave for Eq.(1.1) described by (3.13) with $\alpha = -2, p = -1, \delta_2 = -1, m_1 = 2, m_2 = 1, k = -1, z = 2$ at time $t = 1$. (a) Front view of the wave. (b) Overhead view of the wave.

Substituting (3.4) along with $\xi = x + kz$ into (3.1), we have the following expression

$$\begin{aligned}\varphi =& \exp(-p(x + m_1 y + kz + \Phi_1 t)) \pm 2\sqrt{-\delta_2}\cos(p(x + m_2 y + kz + \Phi_2 t) \\ &+ \delta_2\exp(p_1(\xi + m_1 y + kz + \Phi_1 t)),\end{aligned} \quad (3.14)$$

where $\Phi_1 = \frac{3m_2^2\alpha}{4} - \frac{p^2 k}{2}, \Phi_2 = \frac{3m_2^2\alpha}{4} + \frac{p^2 k}{2}$. When $\delta_2 < 0$, (3.14) can be reduced to the following equation

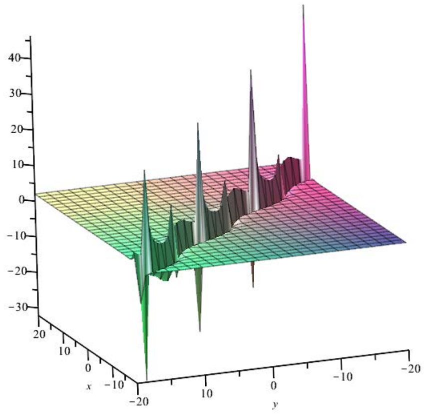
$$\varphi = -2\sqrt{-\delta_2}\sinh(p(x + m_1 y + kz + \Phi_1 t)) \pm 2\sqrt{-\delta_2}\cos(p(x + m_2 y + kz + \Phi_2 t)). \quad (3.15)$$

According to the variable transformation $u = 2\ln(\varphi)_\xi$, we obtain the breather wave solution for (1.1) as

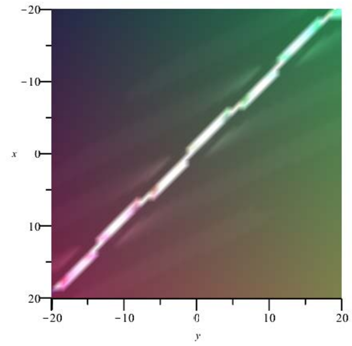
$$u_6 = \frac{2p\cosh(p(x + m_1y + kz + \Phi_1t)) + 2p\sin(p(x + m_2y + kz + \Phi_2t)}{\sinh(p(x + m_1y + kz + \Phi_1t)) - \cos(p(x + m_2y + kz + \Phi_2t)}, \tag{3.16}$$

$$u_7 = \frac{2p\sinh(p(x + m_1y + kz + \Phi_1t)) - 2p\sin(p(x + m_2y + kz + \Phi_2t)}{\sinh(p(x + m_1y + kz + \Phi_1t)) + \cos(p(x + m_2y + kz + \Phi_2t)}, \tag{3.17}$$

where u_6 and u_7 are another two homoclinic breather wave solutions of (1.1). They also have similar dynamical properties , here we only choose u_7 for an example. By selecting appropriate parameters, the profile of u_7 is presented in Figure 4. By comparing the other two pairs of breather wave solutions, we can better comprehending the dynamical behaviors of the breather wave solutions (3.16) and (3.17).



(a)



(b)

Figure 4. (Color online) Breather depicted by(3.17) with $\alpha = 2, p = 1, \delta_2 = -1, m_1 = 1, m_2 = \frac{1}{2}, k = 1, z = 1$ at $t = 1$. (a) Perspective view of the wave. (b) Overhead view of the wave.

4. Resonant multi-soliton solutions of Eq (1.1)

In what follows, we would like to construct the resonant multi-soliton solution of Eq (1.1). For the bilinear formula Eq (2.5), we choose N-exponential wave function as the following form

$$\varphi = \sum_{j=1}^{N} \mu_j\varphi_j = \sum_{j=1}^{N} \mu_j\exp\zeta_j = \mu_1\exp\zeta_1 + \mu_2\exp\zeta_2 + \cdots + \mu_N\exp\zeta_N. \tag{4.1}$$

in which the resonant multi-soliton variables $\zeta_j = h_j\xi + l_jy + r_jt$ and μ_j are nonzero ($1 \leq j \leq N$). On the basis of the properties of differential operator to exponential functions, substituting the expression (4.1) into the bilinear equation (2.5), we have

$$\left[-4D_\xi D_t + kD_\xi^4 + 3\alpha D_y^2\right]\varphi \cdot \varphi = \sum_{j,i=1}^{N} \mu_j\mu_i P(h_j - h_i, l_j - l_i, r_j - r_i)\exp(\zeta_j + \zeta_i) = 0. \tag{4.2}$$

therefore the Eq (4.2) holds if and only if $P(h_j - h_i, l_j - l_i, r_j - r_i) = 0$. Hence the resonant multi-soliton condition can be written as

$$kh_i^4 - 4kh_i^3h_j + 6kh_i^2h_j^2 - 4kh_ih_j^3 + kh_j^4 + 3\alpha l_i^2 - 6\alpha l_il_j + 3\alpha l_j^2 - 4h_ir_i + 4r_jh_i + 4h_jr_i - 4r_jh_j = 0. \quad (4.3)$$

By balancing the power of h_i, l_j, r_j, we conjecture the following wave related numbers

$$h_j = h_j, l_j = c_1h_j^2, r_j = c_2h_j^3. \quad (4.4)$$

Substituting (4.4) into (4.2), we can obtain the following solution for bilinear equation (2.5)

$$\varphi = \mu_1\exp(h_1\xi + c_1h_1^2y + c_2h_1^3t) + \mu_2\exp(h_2\xi + c_1h_2^2y + c_2h_2^3t) + \cdots + \mu_N\exp(h_N\xi + c_1h_N^2y + c_2h_N^3t), \quad (4.5)$$

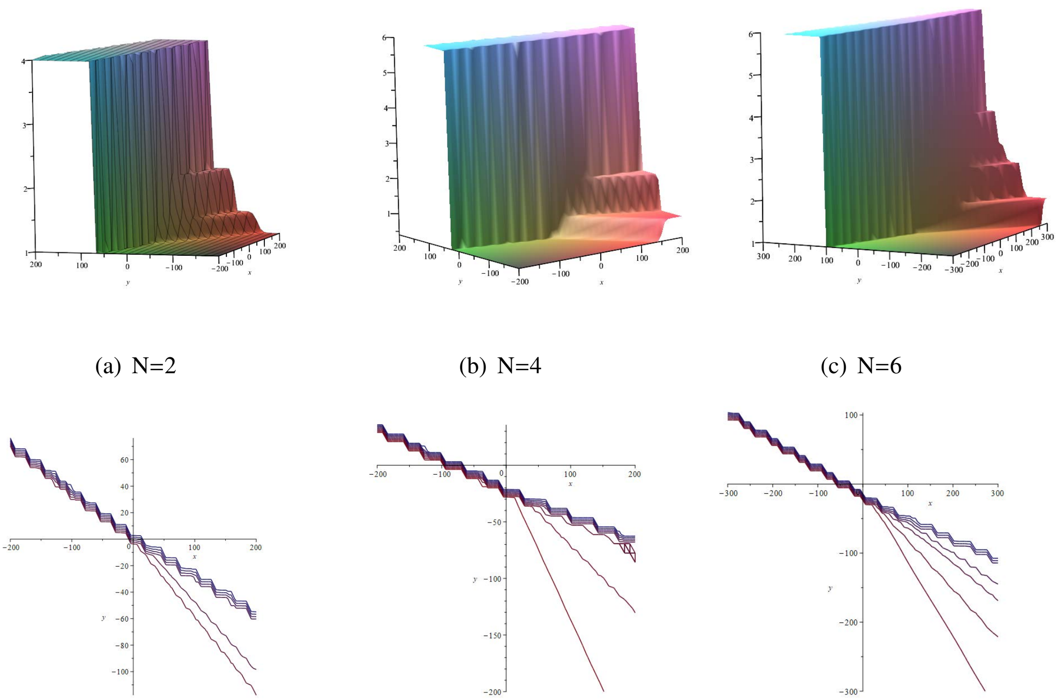
with the condition $c_1 = \sqrt{\frac{k}{\alpha}}, c_2 = k$. Through the transformation $u = 2\ln(\varphi)_\xi$ and $\xi = x + kz$, we can derive the resonant multi-soliton solution of the equation (1.1), which can be written as

$$u = 2\frac{\partial}{\partial x}\ln\left(\sum_{j=1}^{N}\mu_j\exp(h_jx + \sqrt{\frac{k}{\alpha}}h_j^2y - kh_jz + kh_j^3t)\right). \quad (4.6)$$

Especially, the 2-kink solution is written as

$$u_{kink} = \frac{2h_1\mu_1\exp(h_1x + \sqrt{\frac{k}{\alpha}}h_1^2y + kh_1z + kh_1^3t) + 2h_2\mu_2\exp(h_2x + \sqrt{\frac{k}{\alpha}}h_2^2y + kh_2z + kh_2^3t)}{\mu_1\exp(h_1x + \sqrt{\frac{k}{\alpha}}h_1^2y + kh_1z + kh_1^3t) + \mu_2\exp(h_2x + \sqrt{\frac{k}{\alpha}}h_2^2y + kh_2z + kh_2^3t)}. \quad (4.7)$$

In Figure 5, the physical behaviors at $t = 2$ are vividly presented by selecting proper parameters. For (a) and (d) with the following parameters: $N = 2, \mu_1 = 1, \mu_2 = 3, h_1 = 0.5, h_2 = 1, k = 4, \alpha = 3, z = 1$; For (b) and (e) with the following parameters $N = 4, \mu_1 = 1, \mu_2 = 3, \mu_3 = 0.5, \mu_4 = 2, h_1 = 0.5, h_2 = 1, h_3 = 0.2, h_4 = 3, k = 4, \alpha = 3, z = 1$; For(c) and (f) with the following parameters: $N = 6, \mu_1 = 0.5, \mu_2 = 1.5, \mu_3 = 2, \mu_4 = 0.6, \mu_5 = 0.7, \mu_6 = 1.4, h_1 = 0.8, h_2 = 3, h_3 = 1.2, h_4 = 0.5, h_5 = 1.4, h_6 = 1.8, k = 2, \alpha = 4, z = 0$. As a matter of fact, the waves shown in Figure 5 are inelastic. It is clear that two convex waves appeared in (a) and (d), three convex waves appeared in (b) and (e), while four convex waves appeared in (c) and (f). Their fashions and motion trend in (x, y)-plane are vividly exhibited separately. Consequently, the resonant solutions expressed by (4.6) emerge a novel type of interactions.



(a) N=2 (b) N=4 (c) N=6

(d) N=2 (e) N=4 (f) N=6

Figure 5. (Color online) 3D profiles and contour plots for solutions (4.6) at $t = 2$ with parameters mentioned above.

5. M-breather and hybrid solutions

In the end, we aim to derive M-breather and hybrid solutions of Eq (1.1) by using complex conjugate method. To achieve this goal, we consider its multi-soliton solutions. Based upon the bilinear formalism (2.5), we expand function $\varphi(\xi, y, t)$ into a series of infinitesimal ε,

$$\varphi = 1 + \varphi^{(1)}\varepsilon + \varphi^{(2)}\varepsilon^2 + \varphi^{(3)}\varepsilon^3 + \cdots . \tag{5.1}$$

Substituting Eq (5.1) into Eq (2.5), we obtain the following bilinear equations by comparing the coefficients of the same power with respect to ε

$$-4\varphi^{(1)}_{\xi t} + k\varphi^{(1)}_{\xi\xi\xi\xi} + 3\alpha\varphi^{(1)}_{yy} = 0. \tag{5.2a}$$

$$\left[-4D_\xi D_t + kD_\xi^4 + 3\alpha D_y^2\right]\varphi^{(1)} \cdot \varphi^{(1)} = 2\left(4\varphi^{(2)}_{\xi t} - k\varphi^{(2)}_{\xi\xi\xi\xi} - 3\alpha\varphi^{(2)}_{yy}\right). \tag{5.2b}$$

$$-4\varphi^{(3)}_{\xi t} + k\varphi^{(3)}_{\xi\xi\xi\xi} + 3\alpha\varphi^{(3)}_{yy} = \left[4D_\xi D_t - kD_\xi^4 - 3\alpha D_y^2\right]\varphi^{(1)} \cdot \varphi^{(2)}. \tag{5.2c}$$

From the (5.2a), we can get a solution of $\varphi^{(1)}$ in this form $\varphi^{(1)} = \exp(\eta_1), \eta_1 = l_1(\xi + m_1 y + n_1 t) + \eta_{01}, n_1 = \frac{kl_1^2 + 3\alpha m_1^2}{4}$. Substituting $\varphi^{(1)}$ into (5.2b), we have $4\varphi^{(2)}_{\xi t} - k\varphi^{(2)}_{\xi\xi\xi\xi} - 3\alpha\varphi^{(2)}_{yy} = 0$. If we take $\varphi^{(2)} = 0$, we can derive $-4\varphi^{(3)}_{\xi t} + k\varphi^{(3)}_{\xi\xi\xi\xi} + 3\alpha\varphi^{(3)}_{yy} = 0$ from (5.2c) such that $\varphi^{(3)} = 0$. Continuing to study like this, we can further derive $\varphi^{(4)} = \varphi^{(5)} = \cdots = 0$. So a finite form of series (5.1) can be derived by truncation. If taking $\varepsilon = 1$, we have $\varphi = 1 + \exp(\eta_1)$. By means of transformation $u = 2\ln(\varphi)_\xi$, we derive the one-soliton solution $u = 2\frac{\partial}{\partial x}\ln(1 + \exp(\eta_1))$ of equation (1.1), and $\eta_1 = l_1(x + m_1 y + kz + n_1 t) + \eta_{01}$. On account of the linear property for Eq (5.2a), $\varphi^{(1)} = \exp(\eta_1)+\exp(\eta_2)$ is also its solution. By substituting $\varphi^{(1)} = \exp(\eta_1)+\exp(\eta_2)$ into (5.2b), according to the identity of Hirota bilinear operators, it is clear that $\varphi^{(2)} = \exp(\eta_1 + \eta_2 + A_{12})$ is a solution of (5.2b), where $\mathrm{A}_{12} = -\frac{4(l_1-l_2)(l_1n_1-l_2n_2)-k(l_1-l_2)^4-3\alpha(l_1m_1-l_2m_2)^2}{(4(l_1+l_2)(l_1n_1+l_2n_2)-k(l_1+l_2)^4-3\alpha(l_1m_1+l_2m_2)^2}, \eta_i = l_i(\xi+m_i y+n_i t)+\eta_{0i}$ and n_i is defined as before. From (5.2b), we can further obtain $\varphi^{(4)} = \varphi^{(5)} = \cdots = 0$. Thus, the truncation solution at $\varepsilon = 1$ with $\varphi = 1+\exp(\eta_1)+\exp(\eta_2)+\exp(\eta_1+\eta_2+A_{12})$ can be derived. In addition the two-soliton solution $u = 2\frac{\partial}{\partial x}(\ln(1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12}))$ can also be obtained. According to the existence theorem of N-soliton solution, with the help of the mathematical induction, N-soliton solution for the (3+1)-dimensional equation (1.1) can be derived. Therefore, we have the following conclusions

$$\varphi_N = \sum_{\lambda=0,1} \exp\left(\sum_{i=1}^{N} \lambda_i\eta_i + \sum_{i<j}^{N} \lambda_i\lambda_j A_{ij}\right). \tag{5.3}$$

through transformation $u = 2\ln(\varphi)_\xi$ and $\xi = x + kz$, N-soliton solutions of equation (1.1) can be written as

$$u = 2\frac{\partial}{\partial x}\ln\left(1 + \sum_{\lambda_i=0,1} \exp\left(\sum_{i=1}^{N} \lambda_i\eta_i + \sum_{i<j}^{N} \lambda_i\lambda_j A_{ij}\right)\right). \tag{5.4}$$

in which $\exp(A_{ij}) = -\frac{4(l_i-l_j)(l_in_i-l_jn_j)-k(l_i-l_j)^4-3\alpha(l_im_i-l_jm_j)^2}{4(l_i+l_j)(l_in_i+l_jn_j)-k(l_i+l_j)^4-3\alpha(l_im_i+l_jm_j)^2}, \eta_i = l_i(\xi + m_i y + n_i t) + \eta_{0i}, n_i = \frac{kl_i^2+3\alpha m_i^2}{4}$, $\sum_{\lambda=0,1}$ is all possible sums with regard to $\lambda_i = 0$ or 1 under the condition $1 \le i < j \le N$. Furthermore, l_i, m_i are non-zero constants.

In what follows, we would like to construct M-breather based on multi-soliton solutions. By taking $N = 2$ in (5.4), we can obtain the following result

$$u = 2\frac{\partial}{\partial x}(\ln(1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12})). \tag{5.5}$$

By selecting $l_1 = l_2^* = a_1 + a_2 i, m_1 = m_2$ in solution (5.5), we can derive the one-breather solution. Hence the function φ can be expressed as

$$\varphi = 1 + 2\cosh(\theta)\cos(\rho) + 2\sinh(\theta)\cos(\rho) + \exp(A_{12})(\cosh(2\theta) + \cosh(2\theta)). \tag{5.6}$$

where

$$\begin{aligned} \theta &= a_1\left(x + m_1 y + kz + \frac{kl_1^2 + 3\alpha m_1^2}{4}t\right), \\ \rho &= b_1\left(x + m_1 y + kz + \frac{kl_1^2 + 3\alpha m_1^2}{4}t\right). \end{aligned} \tag{5.7}$$

Therefore, we obtain the 1-breather solution of Eq (1.1) as the following form

$$u = \frac{2(\cosh(\theta) + \sinh(\theta))(a_1\cos(\rho) - b_1\sin(\rho)) + 2a_1\exp(A_{12})(\cosh(2\theta) + \cosh(2\theta))}{1 + 2\cosh(\theta)\cos(\rho) + 2\sinh(\theta)\cos(\rho) + \exp(A_{12})(\cosh(2\theta) + \cosh(2\theta))}. \tag{5.8}$$

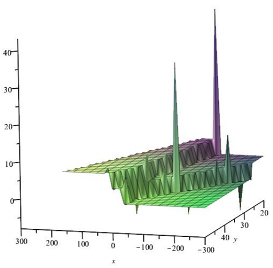
If taking $N = 4$ in (5.4), the function φ can be written as the following form

$$\begin{aligned}\varphi = {} & 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + \exp(\eta_4) + \exp(\eta_1 + \eta_2 + A_{12}) + \exp(\eta_2 + \eta_3 + A_{23}) \\ & + \exp(\eta_3 + \eta_4 + A_{34}) + \exp(\eta_1 + \eta_3 + A_{13}) + \exp(\eta_1 + \eta_4 + A_{14}) + \exp(\eta_2 + \eta_4 + A_{24}) \\ & + \exp(\eta_1 + \eta_2 + \eta_3 + A_{12} + A_{13} + A_{23}) + \exp(\eta_2 + \eta_3 + \eta_4 + A_{23} + A_{24} + A_{34}) \\ & + \exp(\eta_1 + \eta_2 + \eta_4 + A_{12} + A_{14} + A_{24}) + \exp(\eta_1 + \eta_3 + \eta_4 + A_{13} + A_{14} + A_{34}) \\ & + \exp(\eta_1 + \eta_2 + \eta_3 + \eta_4 + A_{12} + A_{13} + A_{14} + A_{23} + A_{24} + A_{34}).\end{aligned} \tag{5.9}$$

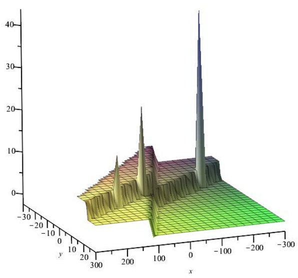
which is exactly the four-soliton solution for bilinear equation (2.5). According to the complex conjugate method, taking special selections $l_1 = l_3^* = a_1+b_1i, l_2 = l_4^* = a_2+b_2i, m_1 = m_3^* = c_1+d_1\mathrm{i}, m_2 = m_4^* = c_2 + d_2\mathrm{i}$ in (5.4), then the two-breather solution can be derived. By taking $N = 6$ and selecting $l_1 = l_4^* = a_1+b_1\mathrm{i}, l_2 = l_5^* = a_2+b_2\mathrm{i}, l_3 = l_6^* = a_3+b_3\mathrm{i}, m_1 = m_4^* = c_1+d_1\mathrm{i}, m_2 = m_5^* = c_2+d_2\mathrm{i}, m_3 = m_6^* = c_3 + d_3\mathrm{i}$ in (5.4), hence the three-breather solution will be obtained. As the expressions of two-breather solution and three-breather solution are too complex, so we omit their expressions here.

In order to describe their evolution process, appropriate parameters for two-breather are selected with $N = 4, k = 1, \alpha = 2, a_1 = 1, b_1 = 2, a_2 = 1, b_2 = 3, c_1 = 2, d_1 = 3, c_2 = 2, d_2 = 5$ at $t = -5, 0, 5$. The motion pattern of the wave is shown in Figure 6. By the similar way, we select suitable parameters for three-breather solution with $N = 6, k = 2, \alpha = 3, a_1 = 0, a_2 = 0, a_3 = 0, b_1 = 4, b_2 = 3, b_3 = 2, c_1 = 1, c_2 = 2, c_3 = 2, d_1 = 2, d_2 = 3, d_3 = 1$ at $t = -30, 0, 30$. Its fluctuation form is shown in Figure 7.

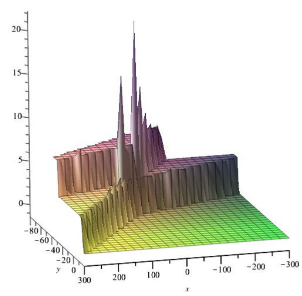
By analyzing these pictures, several column waves emerge for two-breather while many column waves emerge for three-breather. In Figure 6, when $t = -5$, it appears three peaks, two of them are taller than the third one. Then their heights are different from each other at $t = 0$. At $t = 5$, it appears four or more peaks. As can be seen from Figure 7, when $t = -30$, it appears many peaks and troughs with different heights. Then their numbers decrease at $t = 0$. At $t = 30$, it appears many peaks and troughs, but their height differences are not obvious. They maintain a froward posture for both the two-breather and three-breather waves.



(a) t=-5

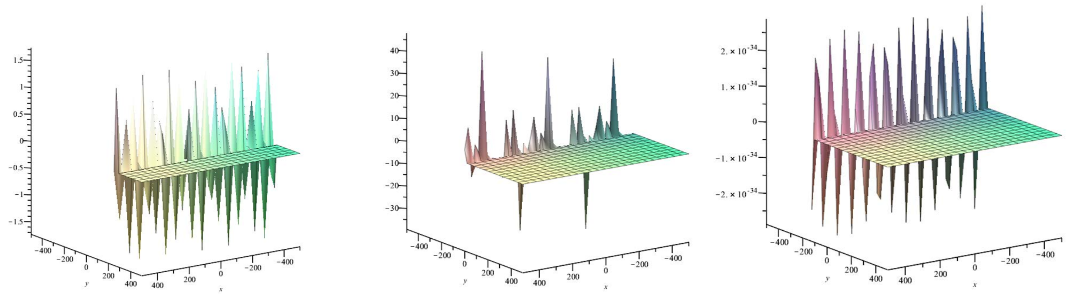


(b) t=0



(c) t=5

Figure 6. (Color online)Profiles of two-breather solution at different times with parameters described above:(a) At $t = -5$ with $z = 1$; (b) At $t = 0$ with $z = 1$; (c) At $t = 5$ with $z = 1$.



(a) t=-30 (b) t=0 (c) t=30

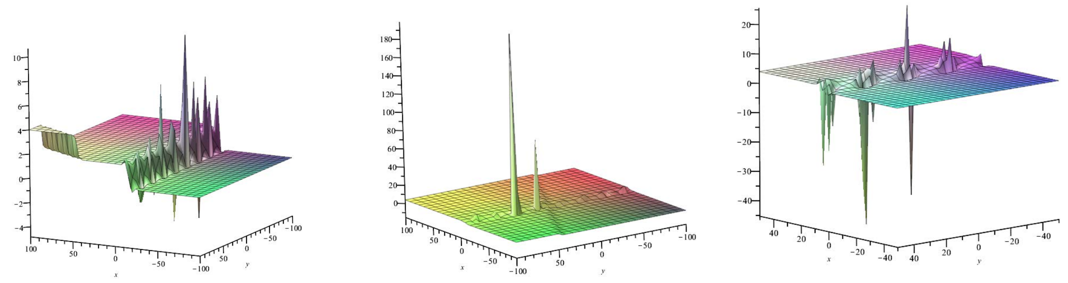
Figure 7. (Color online)Profiles of three-breather solution at different times:(a) At $t = -30$ with $z = 4$; (b) At $t = 0$ with $z = 4$; (c) At $t = 30$ with $z = 4$.

In the next, we would like to construct hybrid solutions based on multi-soliton solutions. Taking $N = 3$ in (5.5), the function φ can be written as the following form

$$\begin{aligned}\varphi = {} & 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + \exp(\eta_1 + \eta_2 + A_{12}) + exp(\eta_1 + \eta_3 + A_{13}) \\ & + \exp(\eta_2 + \eta_3 + A_{23}) + \exp(\eta_1 + \eta_2 + \eta_3 + A_{12} + A_{13} + A_{23}),\end{aligned} \tag{5.10}$$

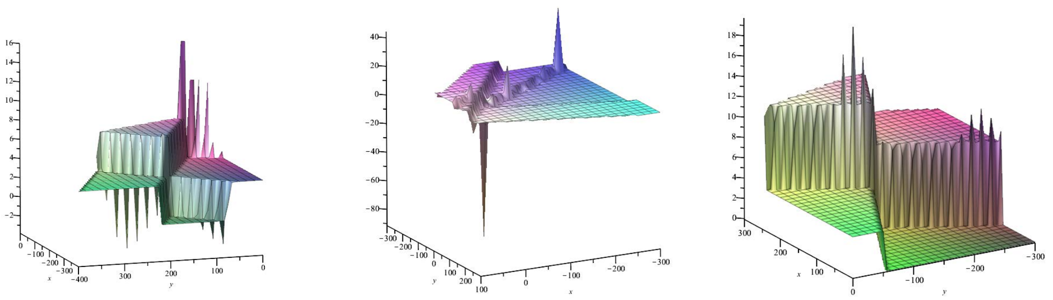
which is exactly the three-soliton solution for bilinear equation (2.5). By selecting $l_1 = l_3^* = a_1 + b_1 i, m_1 = m_3^* = c_1 + d_1 i$ in (5.10), we can derive the hybrid solution between one-soliton and one-breather. Similarly, by selecting $N = 5$ and $k_1 = k_3^* = a_1 + b_1 i, k_2 = k_4^* = a_2 + b_2 i, l_1 = l_3^* = c_1 + d_1 i, l_2 = l_4^* = c_2 + d_2 i$ in (5.4), the hybrid solution between one-soliton and two-breather can also be obtained. As their expressions are very long, we omit them here in order to save space.

In order to depict the evolution progress of hybrid solutions between breather and soliton, proper parameters are selected with $N = 3, k = 2, \alpha = 4, a_1 = \frac{1}{2}, b_1 = \frac{1}{3}, c_1 = \frac{1}{4}, d_1 = \frac{1}{3}$ at $t = -20, 0, 20$ to exhibit its wave motions. Meanwhile, proper parameters are selected with $N = 5, k = 1, \alpha = 3, a_1 = \frac{1}{10}, a_2 = \frac{1}{20}, b_1 = \frac{1}{2}, b_2 = \frac{1}{4}, c_1 = \frac{3}{5}, c_2 = \frac{3}{10}, d_1 = \frac{4}{5}, d_2 = \frac{9}{10}, k_5 = \frac{3}{2}, l_5 = 3$ at $t = -5, 0, 10$ to depict its fluctuations. As is shown in Figure 8, it appear many breathers at $t = -20$. When colliding occurs at t=0 between the breather and soliton, we can see that the breather changes a lot at $t = 20$. It keeps moving forward over time. It can be seen from Figure 9, two parts constitute the hybrid wave solution. It is clear that one part is soliton wave and the other part is two parallel breathers. Both the two parallel breathers maintain their original propagation path. They gradually separate from each other after colliding with solitons.



(a) t=-20 (b) t=0 (c) t=20

Figure 8. (Color online)Evolution progress of hybrid solution between one-soliton and one-breather:(a) Profile of hybrid solution at $t = -20$ with $z = 3$; (b) Profile of hybrid solution at $t = 0$ with $z = 3$; (c) Profile of hybrid solution at $t = 20$ with $z = 3$.



(a) t=-5 (b) t= 0 (c) t= 10

Figure 9. (Color online)Evolution progress of hybrid solution between one-soliton and two-breather:(a) Profile of hybrid solution at $t = -5$ with $z = 2$; (b) Profile of hybrid solution at $t = 0$ with $z = 2$; (c) Profile of hybrid solution at $t = 10$ with $z = 2$.

6. Conclusions

In order to establish the bilinear formalism of a (3+1)-dimensional equation, a dependent variable transformation was introduced. This bilinear expression plays a key role in this paper. Then we derived three pairs of breather solutions by employing the homoclinic test method. Among these breather solutions, the rouge wave solution can only be derived from a from u_2 . Next we obtained resonant multi-soliton solutions by employing the linear superposition principle. Obviously, the resonant multi-soliton solutions which we obtained don't depend on dispersion relation. By choosing $N = 2, 4, 6$, the physical shape of the waves were shown graphically. In what follows, M-breather solutions including one-breather, two-breather and three-breather were obtained by applying the complex conjugate method to the multi-soliton solutions, and their evolution progress were exhibited by choosing appropriate parameters. Compared their dynamical behaviors, we can see that breather solutions which were derived by using the homoclinic test method are different from the ones which were derived by using the complex conjugate method. Finally, the hybrid solutions between breathers and solitons were constructed and their dynamic properties were exhibited in the form of plotting.Hence, these solutions would amend the existing literature on the exact solutions of nonlinear evolution equation. Furthermore, the method in this paper can be more effectively used in other nonlinear evolution equations.

Acknowledgments

This Project was supported by the Natural Science Foundation of Henan Province (No.162300410075), the Science and Technology Key Research Foundation of the Education Department of Henan Province (No.14A110010).

Conflict of interest

The authors declare that they have no conflict of interest concerning the publication of this manuscript.

References

1. R. Hirota, J. Satsuma, Soliton solutions of a coupled Korteweg-de Vries equation, *Phys. Lett. A,* **85** (1981), 407–408. https://doi.org/10.1016/0375-9601(81)90423-0
2. W. X. Ma, N-soliton solution and the Hirota condition of a (2+ 1)-dimensional combined equation, *Math. Comput. Simulat.,* **190** (2021), 270–279. https://doi.org/10.1016/j.matcom.2021.05.020
3. W. X. Ma, N-soliton solution of a combined pKP-BKP equation, *J. Geom. Phys.,* **165** (2021), 104191. https://doi.org/10.1016/j.geomphys.2021.104191
4. W. X. Ma, X. L. Yong, X. Lü, Soliton solutions to the B-type Kadomtsev-Petviashvili equation under general dispersion relations, *Wave Motion,* **103** (2021), 102719. https://doi.org/10.1016/j.wavemoti.2021.102719
5. J. X. Luo, Exact analytical solution of a novel modified nonlinear Schrödinger equation: Solitary quantum waves on a lattice, *Stud. Appl. Math.*, **146** (2021), 550–562. https://doi.org/10.1111/sapm.12355
6. B. Q. Li, A. M. Wazwaz, Y. L. Ma, Two new types of nonlocal Boussinesq equations in water waves: Bright and dark soliton solutions, *Chin. J. Phys.,* **7**7 (2022), 1782–1788. https://doi.org/10.1016/j.cjph.2021.11.008
7. A. A. Hamed, S. Shamseldeen, M. S. Abdel Latif , H. M. Nour, Analytical soliton solutions and modulation instability for a generalized (3 + 1)-dimensional coupled variable-coefficient nonlinear Schroödinger equations in nonlinear optics, *Mod. Phys. Lett. B,* **35** (2021), 2050407. https://doi.org/10.1142/S0217984920504072

8. B. F. Feng, L. M. Ling, D. A. Takahashi, Multi-breather and high-order rogue waves for the nonlinear Schrödinger equation on the elliptic function background, *Stud. Appl. Math.,* **1**44 (2020), 46–101. https://doi.org/10.1111/sapm.12287

9. B. Q. Li, Loop-like kink breather and its transition phenomena for the Vakhnenko equation arising from high-frequency wave propagation in electromagnetic physics, *Appl. Math. Lett.,* **112** (2021), 106822. https://doi.org/10.1016/j.aml.2020.106822

10. J. Y. Yang, W. X. Ma, C. M. Khalique, Determining lump solutions for a combined soliton equation in (2+ 1)-dimensions, *Eur. Phys. J. Plus,* **135** (2020), 1–13. https://doi.org/10.1140/epjp/s13360-020-00463-z

11. S. F. Tian, D. Guo, X. B. Wang, T. T. Zhang, Traveling wave, lump wave, rogue wave, multi-kink solitary wave and interaction solutions in a (3+ 1)-dimensional Kadomtsev-Petviashvili equation with Bäcklund transformation, *J. Appl. Anal. Comput.,* **11** (2021), 45–58. https://doi.org/10.11948/20190086

12. Y. L. Ma, A. M. Wazwaz, B. Q. Li, New extended Kadomtsev-Petviashvili equation: multiple soliton solutions, breather, lump and interaction solutions, *Nonlinear Dyn. ,* **104** (2021), 1581–1594. https://doi.org/10.1007/s11071-021-06357-8

13. Y. Y. Feng, S. D. Bilige, Multiple rogue wave solutions of (2+ 1)-dimensional YTSF equation via Hirota bilinear method, *Wave. Random Complex,* **31** (2021), 1–17. https://doi.org/10.1080/17455030.2021.1900625

14. A. A. Hamed, A. H. Abdel Kader, M. S. Abdel Latif, Solitons, rogue waves and breather solutions for the (2+ 1)-dimensional nonlinear Schrödinger equation with variable coefficients, *Optik,* **216** (2020), 164768. https://doi.org/10.1016/j.ijleo.2020.164768

15. W. H. Liu, Y. F. Zhang, Resonant multiple wave solutions, complexiton solutions and rogue waves of a generalized (3+ 1)-dimensional nonlinear wave in liquid with gas bubbles, *Wave. Random Complex,* **30** (2020), 470–480. https://doi.org/10.1080/17455030.2018.1528026

16. H. M. Ahmed, W. B. Rabie, M. A. Ragusa, Optical solitons and other solutions to Kaup-Newell equation with Jacobi elliptic function expansion method, *Anal. Math. Phys.,* **11** (2021), 1–16. https://doi.org/10.1007/s13324-020-00464-2

17. B. Q. Li, Y. L. Ma, Solitons resonant behavior for a waveguide directional coupler system in optical fibers, *Opt. Quant. Electron.,* **50** (2018), 1–14. https://doi.org/10.1007/s11082-018-1536-7

18. Y. L. Ma, B. Q. Li, Bifurcation solitons and breathers for the nonlocal Boussinesq equations, *Appl. Math. Lett.,* **124** (2022), 107677. https://doi.org/10.1016/j.aml.2021.107677

19. Y. L. Ma, A. M. Wazwaz, B. Q. Li, Novel bifurcation solitons for an extended Kadomtsev-Petviashvili equation in fluids, *Appl. Math. Lett.,* **413** (2021), 127585. https://doi.org/10.1016/j.physleta.2021.127585

20. A. H. Abdel Kader, M. S. Abdel Latif, D. Baleanu, $W-$ Shaped, bright, and dark soliton solutions for a generalized Quasi-1D Bose-Einstein condensate system with local M-derivative, *Braz. J. Phys.,* **52** (2022), 1–8. https://doi.org/10.1007/s13538-021-01015-1

21. H. Elzehri, A. H. Abdel Kader, M. S. Abdel Latif, Kink wave, dark and bright soliton solutions for complex Ginzburg-Landau equation using Lie symmetry method, *Optik,* **241** (2022), 167048. https://doi.org/10.1016/j.ijleo.2021.167048

22. Z. D. Dai, J. Liu, D. L. Li, Applications of HTA and EHTA to YTSF equation, *Appl. Math. Comput.,* **207** (2009), 360–364. https://doi.org/10.1016/j.amc.2008.10.042

23. Z. H. Xu, H. L. Chen, Z. D. Dai, Rogue wave for the (2+ 1)-dimensional Kadomtsev-Petviashvili equation, *Appl. Math. Lett.,* **37** (2014), 34–38. https://doi.org/10.1016/j.aml.2014.05.005

24. W. X. Ma, E. G. Fan, Linear superposition principle applying to Hirota bilinear equations, *Comput. Math. Appl.,* **61** (2011), 950–959. https://doi.org/10.1016/j.camwa.2010.12.043

25. F. H. Lin, S. T. Chen, Q. X. Qu, J. P. Wang, X. W. Zhou, X. Lü, Resonant multiple wave solutions to a new (3+ 1)-dimensional generalized Kadomtsev-Petviashvili equation: Linear superposition principle, *Appl. Math. Lett.,* **78** (2018), 112–117. https://doi.org/10.1016/j.aml.2017.10.013

26. C. K. Kuo, W. X. Ma, A study on resonant multi-soliton solutions to the (2+1)-dimensional Hirota-Satsuma-Ito equations via the linear superposition principle, *Nonlinear Anal.,* **190** (2020), 111592. https://doi.org/10.1016/j.na.2019.111592

27. C. K. Kuo, Novel resonant multi-soliton solutions and inelastic interactions to the (3+1)- and (4+1)- dimensional Boiti-Leon-Manna-Pempinelli equations via the simplified linear superposition principle, *Eur. Phys. J. Plus,* **136** (2021), 1–11. https://doi.org/10.1140/epjp/s13360-020-01062-8

28. B. Q. Li, Y. L. Ma, Interaction dynamics of hybrid solitons and breathers for extended generalization of Vakhnenko equation, *Nonlinear Dyn.,* **102** (2020), 1787–1799. https://doi.org/10.1007/s11071-020-06024-4

29. B. Q. Li, Y. L. Ma, N-order rogue waves and their novel colliding dynamics for a transient stimulated Raman scattering system arising from nonlinear optics, *Nonlinear Dyn.,* **101** (2021), 2449–2461. https://doi.org/10.1007/s11071-020-05906-x

30. B. Q. Li, Y. L. Ma, Extended generalized Darboux transformation to hybrid rogue wave and breather solutions for a nonlinear Schrödinger equation, *Appl. Math. Comput.,* **386** (2020), 125469. https://doi.org/10.1016/j.amc.2020.125469

31. S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its, V. B. Matveev, Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction, *Phys. Lett. A,* **62** (1977), 205–206. https://doi.org/10.1016/0375-9601(77)90875-1

32. J. Satsuma, M. J. Ablowitz, Two-dimensional lumps in nonlinear dispersive systems, *J. Math. Phys.,* **20** (1979), 1496–1503. https://doi.org/10.1063/1.524208

33. Y. Y. Feng, S. D. Bilige, Resonant multi-soliton, M-breather, M-lump and hybrid solutions of a combined pKP-BKP equation, *J. Geom. Phys.,* **169** (2021), 104322. https://doi.org/10.1016/j.geomphys.2021.104322

34. H. D. Guo, T. C. Xia, B. B. Hu, Dynamics of abundant solutions to the (3+ 1)-dimensional generalized Yu-Toda-Sasa-Fukuyama equation, *Appl. Math. Lett.,* **105** (2020), 106301. https://doi.org/10.1016/j.aml.2020.106301

35. X. J. Zhou, O. A. Ilhan, J. Manafian, G. Singh, N. S. Tuguz, N-lump and interaction solutions of localized waves to the (2+ 1)-dimensional generalized KDKK equation, *J. Geom. Phys.,* **168** (2021), 104312. https://doi.org/10.1016/j.geomphys.2021.104312

36. J. Manafian, O. A. Ilhan, L. Avazpour, A. Alizadeh, N-lump and interaction solutions of localized waves to the (2+ 1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation arise from a model for an incompressible fluid, *Math. Method. Appl. Sci.,* **43** (2020), 9904–9927. https://doi.org/10.1002/mma.6665

37. Z. L. Zhao, L. C. He, M-lump and hybrid solutions of a generalized (2+ 1)-dimensional Hirota-Satsuma-Ito equation, *Appl. Math. Lett.,* **111** (2021), 106612. https://doi.org/10.1016/j.aml.2020.106612

38. Y. L. Ma, A. M. Wazwaz, B. Q. Li, A new (3+ 1)-dimensional Kadomtsev-Petviashvili equation and its integrability, multiple-solitons, breathers and lump waves, *Math. Comput. Simul.,* **187** (2021), 125469. https://https://doi.org/10.1016/j.matcom.2021.03.012

39. Y. Y. Feng, X. M. Wang, S. D. Bilige, Evolutionary behavior and novel collision of various wave solutions to (3 + 1)-dimensional generalized Camassa-Holm Kadomtsev-Petviashvili equation, *Nonlinear Dyn.,* **104** (2021), 4265–4275. https://doi.org/10.1007/s11071-021-06463-7

40. P. F. Han, T. Bao, Interaction of multiple superposition solutions for the (4+ 1)-dimensional Boiti-LeonManna-Pempinelli equation, *Nonlinear Dyn.,* **105** (2021), 717–734. https://doi.org/10.1007/s11071-021-06603-z

41. Y. Zhang, Y. P. Liu, X. Y. Tang, M-lump and interactive solutions to a (3+1)- dimensional nonlinear system, *Nonlinear Dyn.,* **93** (2018), 2533–2541. https://doi.org/10.1007/s11071-018-4340-9

42. Z. Y. Yan, New families of nontravelling wave solutions to a new (3+ 1)-dimensional potential-YTSF equation, *Phys. Lett. A*, **318** (2003), 78–83. https://doi.org/10.1016/j.physleta.2003.08.073

43. E. G. Fan, K. W. Chow, Darboux covariant lax pairs and infinite conservation laws of the (2+ 1)-dimensional breaking soliton equation, *J. Math. Phys.,* **52** (2011), 023504. https://doi.org/10.1063/1.3545804

AIMS Press