Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 15741–15758.
DOI: 10.3934/math.2022862
Received: 20 February 2022
Revised: 10 June 2022
Accepted: 16 June 2022
Published: 26 June 2022

Research article

New exploration of operators of fractional neutral integro-differential equations in Banach spaces through the application of the topological degree concept

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Abstract: In this paper, we analyze the behavior of the neutral integro-differential equations of fractional order including the Caputo-Hadamard fractional derivative. The results and solutions are obtained using the topological degree method. Furthermore, some specific numerical examples are given to ascertain the wide applicability and high efficiency of the suggested fixed point technique.

Keywords: fractional calculus; Caputo-Hadamard fractional derivative; integro-differential equation; neutral equation; topological degree method
Mathematics Subject Classification: 34A08, 37C25, 34K10, 34K37

1. Introduction

In last few decades, fractional calculus has become important tool to address real world problems in physics, chemistry, engineering etc. For reference, we reviewed the analytical and numerical results of fractional differential equation (FDE)-related studies in [2, 7, 15, 18, 19, 22, 25–28, 30, 32, 33, 38, 41, 47, 48]. The existence and uniqueness of the results of FDEs have been proved and solved using fixed point theory. Focusing o the difficulties of several limits in fixed point theory, Mawhin [34] established the topological degree theory (TDT) to solve the integral equations. Afterword, Isaia [23] described some integral equations using TDT. Some of the studies on TDT have been reported in [16, 44, 45, 51].

Ahmad and Sivasundaram [8], explained the nonlinear term of fractional neutral integro-differential equations(FIDEs) with boundary value problems. Zuo et al. [35] investigated the FIDEs with impulsive and antiperiodic boundary conditions. Agarwal et al. [3] described an initial value problem (IVP) based on neutral equations of the fractional type with bounded delay. Murugesu et al. and Santos et al. [29,42] determined the existence solutions for neutral equations and integro-differential equations with neutrality based on fractional derivatives. Ravichandran et al. [40] briefly discussed the fractional neutral systems with infinite delay. Yan and Zhang [49, 50] applied the neutral functions in fractional integro-differential equations with state-dependent delay.Zhou and Jiao [52–54] discussed the p-type fractional of neutral differential equations and fractional neutral evolution equations with infinite delay.

We have seen many works related to Riemann-Liouville (RL) or Caputo Fabrizio (CF) fractional derivatives in the analytical and numerical results for FDEs. Hadamard [21], introduced a new derivative containing a logarithmic kernel term, which is known as the Hadamard (HD) fractional derivative. Butzer et al. [12–14] used this HD fractional derivative to analyze the HD fractional integral by using Mellin transform method. Ardjouni [5] described the positive solution of FDEs with integral boundary conditions by using the HD derivative. FDEs and FIDEs with HD fractional derivatives were analyzed in [9, 10, 20, 31, 36, 37, 39, 43, 46]. After that, Jarad et al. [24] discussed the Caputo type HD derivative (CH-derivative) modified from the Caputo derivative. FDEs with boundary-value problems treated with the CH-derivative have been reported in [1, 4, 6, 11].

Here, we explore the existence results for neutral FIDEs with CH-derivatives by using the topological degree method

$$\begin{cases} {}^C_H D^{\rho}[\Xi(t) - g(t, \Xi(t), P\Xi(t))] = f(t, \Xi(t), S\,\Xi(t)), \quad t \in J := [0, L] \\ a\Xi(1) + b^C_H D^{\sigma}\Xi(1) = c_1 {}^H_1\mathcal{I}^{\rho_1}\Xi(\eta_1), \quad 1 < \eta_1 < L, \rho_1 > 0 \\ c\Xi(L) + d^C_H D^{\sigma}\Xi(L) = c_2 {}^H_2\mathcal{I}^{\rho_2}\Xi(\eta_2), \quad 1 < \eta_2 < L, \rho_2 > 0 \end{cases} \tag{1.1}$$

where ${}^C_H D^{\rho}$ and ${}^C_H D^{\sigma}$ are CH-derivatives of order ρ, σ respectively, with $1 < \rho \leq 2 \, and \, 0 < \sigma \leq 1$; ${}^H\mathcal{I}^{\rho_i}$ is the HD integral of fractional order $\rho_i, i = 1, 2$, $f \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is the continuous function and the linear operators P and Q are defined by

$$(P\Xi)(t) = \int_0^t k(t, s)\Xi(s)ds,$$

$$(S\,\Xi)(t) = \int_0^1 h(t, s)\Xi(s)ds,$$

where $t \in J, k \in C(D, \mathbb{R}), D = \{(t, s) \in J \times J : t \geq s\}$ and $h \in C(J \times J, \mathbb{R})$.

Let $a, b, c, d \in \mathbb{R}$ be constants such that

$$\Psi = \Big(a - \frac{c_1(\log \eta_1)^{\rho_1}}{\Gamma(\rho_1 + 1)}\Big)\Big(c \log L + \frac{d(\log L)^{1-\sigma}}{\Gamma(2 - \sigma)} - \frac{c_2(\log \eta_2)^{\rho_2+1}}{\Gamma(\rho_2 + 2)}\Big) + \frac{c_1(\log \eta_1)^{\rho_1+1}}{\Gamma(\rho_1 + 2)}\Big(c \log L - \frac{c_2(\log \eta_2)^{\rho_2}}{\Gamma(\rho_2 + 1)}\Big) \neq 0. \tag{1.2}$$

The structure of this article is as follows: We establish some basic results to prove the existence using TDT in Section 2. In Section 3, we discuss the existence results for FIDEs with boundary conditions by using the topological degree method. An illustration and the conclusions are respectively described in Sections 5 and 6.

2. Basic results

Let X be a Banach space and $\mathcal{B} \subset \mathcal{P}(X)$ be the family of all bounded subsets of X.

Definition 2.1. *[16] $\mu : \mathcal{B} \to \mathbb{R}_+$ is defined as*

$$\Xi(B) := \inf\{d > 0 : B \text{ admits a finite cover by sets of diameter} \leq d\},$$

where $B \in \mathcal{B}$ is called the Kuratowski measure of non compactness.

We shall briefly explain what this definition means. Let us start with a compact set B. A set B of a space X is compact if and only if it is complete (i.e., if every Cauchy sequence in B is convergent to a limit in B with respect to the norm $\|\cdot\|$) and totally bounded (for every $\epsilon > 0$ $\exists$ exists a finite cover for B each of which has a diameter that is less than or equal to ϵ). Thus if B is compact then it is totally bounded. Consequently, the value of $\mu(B)$ is 0. Thus, we can interpret the value μ to be a measure of the compactness of a set. If the value is zero, then the set is compact. The larger the value of μ, less it resembles a compact set.

With this definition in hand, we shall state the following results, the proofs of which can be found in [17].

Proposition 2.2. *[16] For bounded subsets $\mathcal{U}$, $\mathcal{U}_1$ and $\mathcal{U}_2$ on Banach space, we have that*

(1) $\mu(\mathcal{U}) = 0 \Leftrightarrow \overline{\mathcal{U}}$ *is compact.*

(2) $\mu(\lambda\mathcal{U}) = \|\lambda\|\mu(\mathcal{U}), \quad \lambda \in \mathbb{R}.$

(3) $\mu(\mathcal{U}_1 + \mathcal{U}_2) \leq \mu(\mathcal{U}_1) + \mu(\mathcal{U}_2), \quad \mathcal{U}_1, \mathcal{U}_2 \in \mathcal{B}.$

(4) $\mathcal{U}_1 \subset \mathcal{U}_2 \Rightarrow \mu(\mathcal{U}_1) \leq \mu(\mathcal{U}_2).$

(5) $\mu(\mathcal{U}_1 \cup \mathcal{U}_2) = \max\{\mu(\mathcal{U}_1), \mu(\mathcal{U}_2)\}.$

(6) $\mu(conv\, \mathcal{U}) = \mu(\mathcal{U}).$

(7) $\mu(\overline{\mathcal{U}}) = \mu(\mathcal{U}).$

We define $\Upsilon := \{\Xi : [0, T] \to \mathbb{R} : \Xi \in C(I')\}$. Also, $(\Upsilon, \|\cdot\|)$ is a Banach space under the super norm: $\|\Xi\| := \sup\{\|\Xi(t)\| : t \in [0, T]\}$.

Definition 2.3. *[16] Let $\mathcal{F} : \omega \to Y$ be a continuous bounded map and $\omega \subset Y$. If $\mathcal{F}$ is a $\Xi-$Lipschitz function $\exists$ $k \geq 0$ such that $\mu(\mathcal{F}(\mathcal{H})) \leq k\mu(\mathcal{H})$ for all $\mathcal{H} \subset \omega$. If $k < 1$ then $\mathcal{F}$ is a $\mu-$contraction.*

Proposition 2.4. *[16] If $T_1, T_2, T_3, T_4 : \omega \to Y$ are $\mu-$Lipschitz maps with the constants m_1, m_2, m_3, m_4 respectively, then $T_1, T_2, T_3, T_4 : \omega \to Y$ are $\mu-$Lipschitz maps with the constant $m_1 + m_2 + m_3 + m_4$.*

For $i \in \{1, 2, 3, 4\}$, we have,

$$\|\mathcal{F}_i(x) - \mathcal{F}_i(y)\| \leq m_i\|x - y\| \leq (m_1 + m_2 + m_3 + m_4)\|x - y\|.$$

Hence for each $i \in \{1, 2, 3, 4\}$, $\mathcal{F}_i$ represents a $\mu-$Lipschitz map with the constant $m_1 + m_2 + m_3 + m_4$.

Proposition 2.5. *[16] If $\mathcal{G} : \omega \to Y$ is a compact set, then $\mathcal{G}$ is $\mu-$Lipschitz with a constant value equal to zero.*

For any compact subset $\mathcal{H}$, from Proposition 2.2, it follows that $\mu(\mathcal{G}(\mathcal{H})) = 0$. Hence the result follows.

Proposition 2.6. *[16] If $\mathcal{G} : \omega \to Y$ is Lipschitz then $\mathcal{G}$ is $\mu-$Lipschitz with the same constant k.*

Definition 2.7. *[16] A map $\mathcal{F} : Y \to \mathcal{F}$ is said to be $\mu-$condensing if for all $A \in \mathcal{B}$, $\mu(\mathcal{F}(A)) \leq \mu(A)$. We denote $C_\mu(\Omega)$ to be the class of all $\mu-$condensing maps $\mathcal{F} : \Omega \to X$.*

Now we define D to be a degree function used in the TDT [17, 23].

Proposition 2.8. *[16] Let $\mathcal{T} := \{(I - \mathcal{F}, \Lambda, y) : \Lambda \subset X$ open and bounded, $\mathcal{F} \in C_\mu(\overline{\Lambda})$ and $y \in X \setminus (I - \mathcal{F})(\partial\Lambda)\}$. Then $\exists\, D : \mathcal{T} \to \mathbb{Z}$ which satisfies the following:*

(1) $D(I, \Lambda, y) = 1$, for every $y \in \Lambda$ (Normalization).

(2) $D(I-\mathcal{F}, \Lambda, y) = D(I-\mathcal{F}, \Lambda_1, y) + D(I-\mathcal{F}, \Lambda_2, y)$ where $\Lambda_1, \Lambda_2 \subset \Lambda$ and $y \notin (I-\mathcal{F})(\overline{\Lambda} \setminus (\Lambda_1 \cup \Lambda_2))$ (Additivity on domain).

(3) $\mu(\mathcal{H}([0, 1] \times B)) < \mu(B) \quad \forall B \subset \overline{\Lambda}$ and $\mu(B) > 0$ where $D(I - \mathcal{H}(t, \cdot), \Lambda$ and $y(t))$ are independent of $t \in [0, 1]$ for every continuous bounded map $\mathcal{H} : [0, 1] \times \overline{\Lambda} \to X$ and every continuous function $y : [0.1] \to X$ which satisfies $y(t) \neq x - \mathcal{H}(t, x) \quad \forall t \in [0, 1]$. $\forall x \in \partial\Lambda$ (Invariance under homotopy).

(4) $D(I - \mathcal{F}, \Lambda, y) \neq 0$ implies $y \in (I - \mathcal{F})(\Lambda)$ (Existence).

(5) $D(I - \mathcal{F}, \Lambda, y) = D(I - \mathcal{F}, \Lambda_1, y)$ for every open set $\Lambda_1 \subset \Lambda$ and every $y \notin (I - \mathcal{F})(\overline{\Lambda} \setminus \Lambda_1)$ (Excision).

Theorem 2.9. *[16] Let $\mathcal{F} : Y \to Y$ be a $\mu-$condensing map and*

$$\mathcal{H} = \{\Xi \in Y : \exists\, \lambda \in [1, L] \text{ s. t. } \Xi = \lambda T\Xi\}.$$

If $\mathcal{H}$ is a bounded set in Υ, $\exists\ r > 0$ such that $\mathcal{H} \subset B_r(0)$, then

$$D(I - \lambda T, B_r(0), 0) = 1, \quad \text{for all } \lambda \in [1, L].$$

Then T has at least one fixed point.

Definition 2.10. *[32] The HD integral of fractional order $\rho > 0$ for $\Xi \in L^1(J)$ is stated as*

$$^{H}\mathcal{I}^{\rho}\Xi(t) = \frac{1}{\Gamma(\rho)} \int_1^t \log(t/s)^{\rho-1} \Xi(s) \frac{ds}{s}$$

where

$$\Gamma(\rho) = \int_0^\infty e^{-t} t^{\rho-1} dt, \quad \rho > 0.$$

Let $\delta = t\dfrac{d}{dt}, \rho > 0$ and $n = [\rho] + 1$.

Definition 2.11. *[32] The HD derivative of $\rho > 0$ and $\Xi \in \Upsilon$ is stated as*

$$^{H}D^{\rho}\Xi(t) = \delta^{n}(^{H}\mathcal{I}^{n-\rho}\Xi(t)).$$

Definition 2.12. *[24, 32] The CH-derivative of $\rho > 0$ and $\Xi \in \Upsilon$ is stated as*

$$^{H}_{C}D^{\rho}\Xi(t) =^{H} \mathcal{I}^{n-\rho}\delta^{n}\Xi(t).$$

We refer to some standard lemmas that are used in the study of FDEs.

Lemma 2.13. *[24, 32] Let $\rho > 0, r > 0, n = [\rho] + 1$ and $a > 0$. Then we have,*

(1) $^{H}\mathcal{I}^{\rho}(\log \frac{t}{a})^{r-1} = \dfrac{\Gamma(r)}{\Gamma(\rho + r)}(\log \frac{t}{a})^{\rho+r+1}$.

In particular, for $a = 1$ and $r = 1$, the above result reduces to

$$^{H}\mathcal{I}^{\rho}(1)(\rho) = \frac{1}{\Gamma(\rho + 1)}(\log(t))^{\rho+1}.$$

(2) $^{H}_{C}D^{\rho}(\log \frac{t}{a})^{r-1} = \begin{cases} \dfrac{\Gamma(r)}{\Gamma(r-\rho)}(\log \frac{t}{a})^{r-\rho-1}, & r > n \\ 0, \quad r \in \{0, 1, \ldots, n-1\}. \end{cases}$

Lemma 2.14. *[24, 32] Let $nu_1, \rho_2 > 0$ and $\Xi \in \Upsilon$. Then we have,*

(1) $^{H}\mathcal{I}^{\rho_1}(^{H}\mathcal{I}^{\rho_2}\Xi(t)) = (^{H}\mathcal{I}^{\rho_1+\rho_2}\Xi(t))$.

(2) $^{H}_{C}D^{\rho_1}(^{H}\mathcal{I}^{\rho_2}\Xi(t)) = (^{H}\mathcal{I}^{\rho_2-\rho_1}\Xi(t))$.

(3) $^{H}_{C}D^{\rho_1}(^{H}\mathcal{I}^{\rho_1}\Xi(t)) = \Xi(t)$.

Lemma 2.15. *[24, 32] Let $\rho > 0$. Define $n = [\rho] + 1$. For any $\Xi \in \Upsilon$ the CH-derivative of the FDE*

$$^{H}_{C}D^{\rho}\Xi(t) = 0$$

has the solution,

$$\Xi(t) = \sum_{i=0}^{n} c_i(\log t)^i$$

and $c_i, i = 0, 1, \ldots, n-1$ are real constants Also, the following result follows immediately:

$$^{H}I^{\rho}(^{H}_{C}D^{\rho}\Xi(t)) = \Xi(t) + \sum_{i=0}^{n-1} c_i(\log t)^i.$$

3. Main results

We shall define some hypotheses in which we make the following assumptions:

(A1) There exist constant values of $\mathcal{E}, \tilde{\mathcal{E}} > 0$ and $p \in [0, 1)$ such that

$$\|g(t, \Xi(t), P\Xi(t))\| \leq (\mathcal{E} + \tilde{\mathcal{E}}\|\Xi\|^{p})$$

for all $\Xi \in \Upsilon$.

(A2) There exist the constants $\mathcal{E}_1$ *and* $\mathcal{E}_2$ such that

$$\|g(t,\Xi_1(t),P\Xi_1(t)) - g(t,\Xi_2(t),P\Xi_2(t))\| \le \mathcal{E}_1\|\Xi_1 - \Xi_2\| + \mathcal{E}_2\|P\Xi_1 - P\Xi_2\|.$$

(A3) There exist constant values of $\mathcal{F}, \tilde{\mathcal{F}} > 0$ and $p \in [0,1)$ such that

$$\|f(t,\Xi(t),S\Xi(t))\| \le (\mathcal{F} + \tilde{\mathcal{F}}\|\Xi\|^p)$$

for all $\Xi \in \Upsilon$.

(A4) There exist constants $\mathcal{F}_1$ *and* $\mathcal{F}_2$ such that

$$\|f(t,\Xi_1(t),S\Xi_1(t)) - f(t,\Xi_2(t),S\Xi_2(t))\| \le \mathcal{F}_1\|\Xi_1 - \Xi_2\| + \mathcal{F}_2\|S\Xi_1 - S\Xi_2\|.$$

(A5) Let $k_{\max} = \sup\limits_{t\in J}\int_0^t \|k(t,s)\|ds$ and $h_{\max} = \sup\limits_{t\in J}\int_0^1 \|h(t,s)\|ds$.

Lemma 3.1. *[16] For a continuous function V on J, the first order differential equation (FODE) is*

$$\begin{cases} {}_H^C D^{\rho}[\Xi(t) - W(t)] = V(t), \quad t \in J := [0,L]. \\ a\Xi(1) + b\,{}_H^C D^{\sigma}\Xi(1) = c_1^H \mathcal{I}^{\rho_1}\Xi(\eta_1), \quad 1 < \eta_1 < L, \rho_1 > 0. \\ c\Xi(L) + d\,{}_H^C D^{\sigma}\Xi(L) = c_2^H \mathcal{I}^{\rho_2}\Xi(\eta_2), \quad 1 < \eta_2 < L, \rho_2 > 0. \end{cases} \tag{3.1}$$

Then, there is a unique solution known as

$$\begin{aligned} \Xi(t) = {}^H\mathcal{I}^{\rho}V(t) + W(t) + K_1(t)({}^H\mathcal{I}^{\rho_1+\rho}V(\eta_1) + W(\eta_1)) + K_2(t)(c_2({}^H\mathcal{I}^{\rho_2+\rho}V(\eta_2) + W(\eta_2)) \\ - c({}^H\mathcal{I}^{\rho}V(L) + W(L)) + d({}^H\mathcal{I}^{\rho-\sigma}V(L) + W(L)) \end{aligned} \tag{3.2}$$

where,

- $K_1(t) = c_1(\Psi_1 - \Psi_2 t), \quad K_2(t) = c_1\Psi_3 + \Psi_4 t,$
- $\Psi_1 = \dfrac{1}{\Psi}\Big(c\log L + \dfrac{d(\log L)^{1-\sigma}}{\Gamma(2-\sigma)} - \dfrac{c_2(\log\eta_2)^{\rho_2+1}}{\Gamma(\rho_2+2)}\Big),$
- $\Psi_2 = \dfrac{1}{\Psi}\Big(c\log L - \dfrac{c_2(\log\eta_2)^{\rho_2}}{\Gamma(\rho_2+1)}\Big),$
- $\Psi_3 = \dfrac{1}{\Psi}\Big(\dfrac{c_1(\log\eta_1)^{\rho_1+1}}{\Gamma(\rho_1+2)}\Big),$
- $\Psi_4 = \dfrac{1}{\Psi}\Big(a - \dfrac{c_1(\log\eta_1)^{\rho_1}}{\Gamma(\rho_1+1)}\Big),$

and Ψ *is given by Eq* (1.2).

Proof. From Lemma 2.15, Eq (3.1) becomes,

$$\Xi(t) = {}^H\mathcal{I}^{\rho}V(t) + W(t) + k_0 + k_1\log(t), \quad k_0, k_1 \in \mathbb{R}.$$

Applying the boundary conditions, we have

$${}^H\mathcal{I}^{\rho_i}\Xi(\eta_i) = {}^H\mathcal{I}^{\rho_1+\rho}V(\eta_i) + W(\eta_i) + k_0\frac{(\log\eta_i)^{\rho_i}}{\Gamma(\rho_i+1)} + k_1\frac{(\log\eta_i)^{\rho_i+1}}{\Gamma(\rho_i+2)}, \quad i = 1,2\ldots$$

$$ {}^C_H D^{\sigma}\Xi(L) = {}^H \mathcal{I}^{\rho-\sigma}V(L) + W(L) + k_1\frac{\Gamma(2)(\log L)^{1-\sigma}}{\Gamma(2-\sigma)}. $$

Solving for k_0 *and* k_1 we get the following solutions:

$$ k_0 = c_1\Psi_1^H\mathcal{I}^{\rho+\rho_1}\Xi(\eta_1) + \Psi_3(c_2^H\mathcal{I}^{\rho+\rho_2}\Xi(\eta_2) - (c({}^H\mathcal{I}^{\rho}V(L) + W(L)) + d({}^H\mathcal{I}^{\rho-\sigma}V(L) + W(L)))) $$

and

$$ \begin{aligned} k_1 = {} & c_1\Psi_4(({}^H\mathcal{I}^{\rho+\rho_2}V(\eta_2) + W(\eta_2)) - (c({}^H\mathcal{I}^{\rho}V(L) + W(L)) + d({}^H\mathcal{I}^{\rho-\sigma}V(L) + W(L)))) \\ & - c_1\Psi_2({}^H\mathcal{I}^{\rho}V(\eta_1) + W(\eta_1)). \end{aligned} $$

Substituting for k_0 and k_1 we get Eq (3.2). □

In view of the problem defined by Eq (1.1), using the above Lemma 3.1, the solution is

$$ \begin{aligned} \Xi(t) = {}^H \mathcal{I}^{\rho}f_{\Xi}(t) + g_{\Xi}(t) + K_1(t)({}^H\mathcal{I}^{\rho_1+\rho}f_{\Xi}(\eta_1) + g_{\Xi}(\eta_1)) + K_2(t)(c_2({}^H\mathcal{I}^{\rho_2+\rho}f_{\Xi}(\eta_2) + g_{\Xi}(\eta_2)) \\ - c({}^H\mathcal{I}^{\rho}f_{\Xi}(L) + g_{\Xi}(L)) + d({}^H\mathcal{I}^{\rho-\sigma}f_{\Xi}(L) + g_{\Xi}(L))), \end{aligned} \tag{3.3} $$

where for brevity, we denote $g(t, \Xi(t), P\Xi(t))$ by g_{Ξ} and $f(t, \Xi(t), S\,\Xi(t))$ by f_{Ξ} and we have

$$ \begin{aligned} & K_1(t) = c_1(\Psi_1 - \Psi_2 t), \quad K_2(t) = c_1\Psi_3 + \Psi_4 t, \\ & \Psi_1 = \frac{1}{\Psi}\Big(c\log L + \frac{d(\log L)^{1-\sigma}}{\Gamma(2-\sigma)} - \frac{c_2(\log\eta_2)^{\rho_2+1}}{\Gamma(\rho_2+2)}\Big), \;\; \Psi_2 = \frac{1}{\Psi}\Big(c\log L - \frac{c_2(\log\eta_2)^{\rho_2}}{\Gamma(\rho_2+1)}\Big), \\ & \Psi_3 = \frac{1}{\Psi}\Big(\frac{c_1(\log\eta_1)^{\rho_1+1}}{\Gamma(\rho_1+2)}\Big), \quad \Psi_4 = \frac{1}{\Psi}\Big(a - \frac{c_1(\log\eta_1)^{\rho_1}}{\Gamma(\rho_1+1)}\Big), \end{aligned} $$

where Ψ is given by Eq (1.2).

We shall define the following three operators:

(1) Define $T_1 : \Upsilon \to \Upsilon$ as $T_1\Xi(t) = {}^H \mathcal{I}^{\rho}f_{\Xi}(t) + g_{\Xi}(t)$.

(2) Define $T_2 : \Upsilon \to \Upsilon$ as $T_2\Xi(t) = K_1(t)({}^H\mathcal{I}^{\rho_1+\rho}f_{\Xi}(\eta_1) + g_{\Xi}(\eta_1)) + K_2(t)(c_2({}^H\mathcal{I}^{\rho_2+\rho}f_{\Xi}(\eta_2) + g_{\Xi}(\eta_2)))$.

(3) Define $T_3 : \Upsilon \to \Upsilon$ as $T_3\Xi(t) = K_2(t)(c({}^H\mathcal{I}^{\rho}f_{\Xi}(L) + g_{\Xi}(L)) + d({}^H\mathcal{I}^{\rho-\sigma}f_{\Xi}(L) + g_{\Xi}(L)))$.

Let $T : \Upsilon \to \Upsilon$ given $T = T_1 + T_2 + T_3$.

Theorem 3.2. *If T_1 is Lipschitz with the Lipschitz constant $\frac{(\log L)^{\rho}}{\Gamma(\rho+1)}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + \mathcal{E}_1 + \mathcal{E}_2 k_{\max}$, it also satisfies the following growth relation:*

$$ \|T_1\Xi(t)\| \le \frac{(\log L)^{\rho}}{\Gamma(\rho+1)}(\mathcal{F} + \tilde{\mathcal{F}}\|\Xi\|^p) + \mathcal{E} + \tilde{\mathcal{E}}\|\Xi\|^p. $$

Proof. Let $\Xi_1, \Xi_2 \in \Upsilon$. We have,

$$ \begin{aligned} \|T_1\Xi_1(t) - T_1\Xi_2(t)\| & \le \|{}^H\mathcal{I}^{\rho}f_{\Xi_1}(t) - {}^H \mathcal{I}^{\rho}f_{\Xi_2}(t)\| + \|g_{\Xi_1}(t) - g_{\Xi_2}(t)\| \\ & \le {}^H \mathcal{I}^{\rho}\|f_{\Xi_1} - f_{\Xi_2}\|(t) + (\mathcal{E}_1 + \mathcal{E}_2 k_{\max})\|\Xi_1 - \Xi_2\| \end{aligned} $$

$$\leq [{}^{H}\mathcal{I}^{\rho}(1)(L)(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + \mathcal{E}_1 + \mathcal{E}_2 k_{\max}]||\Xi_1 - \Xi_2||$$
$$= \Big[\frac{(\log L)^{\rho}}{\Gamma(\rho+1)}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + \mathcal{E}_1 + \mathcal{E}_2 k_{\max}\Big]||\Xi_1 - \Xi_2||.$$

Hence,

$$||T_1\Xi_1(t) - T_1\Xi_2(t)|| \leq \Big[\frac{(\log L)^{\rho}}{\Gamma(\rho+1)}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + \mathcal{E}_1 + \mathcal{E}_2 k_{\max}\Big]||\Xi_1 - \Xi_2||.$$

Hence T_1 is Lipschitz with a constant given by

$$\Big[\frac{(\log L)^{\rho}}{\Gamma(\rho+1)}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + \mathcal{E}_1 + \mathcal{E}_2 k_{\max}\Big].$$

For the growth relation of $t \in J$, we have

$$\begin{aligned} ||T_1\Xi(t)|| &\leq ||{}^{H}\mathcal{I}^{\rho} f_{\Xi}(t) + g_{\Xi}(t)|| \\ &\leq^{H} \mathcal{I}^{\rho}||f_{\Xi}(t)|| + ||g_{\Xi}(t)|| \\ &\leq^{H} \mathcal{I}^{\rho}(\mathcal{F} + \tilde{\mathcal{F}}||\Xi||^{p}) + \mathcal{E} + \tilde{\mathcal{E}}||\Xi||^{p} \\ &= \frac{(\log L)^{\rho}}{\Gamma(\rho+1)}(\mathcal{F} + \tilde{\mathcal{F}}||\Xi||^{p}) + \mathcal{E} + \tilde{\mathcal{E}}||\Xi||^{p}. \end{aligned}$$

Hence, we have

$$||T_1\Xi(t)|| \leq \Big[\frac{(\log L)^{\rho}}{\Gamma(\rho+1)}(\mathcal{F} + \tilde{\mathcal{F}}||\Xi||^{p})\Big] + \mathcal{E} + \tilde{\mathcal{E}}||\Xi||^{p}.$$

□

Theorem 3.3. *T_2 is continuous and satisfies the following growth relation:*

$$||T_2\Xi(t)|| \leq A_{T_2}(\mathcal{F} + \tilde{\mathcal{F}}||\Xi||^{p}) + B_{T_2}(\mathcal{E} + \tilde{\mathcal{E}}||\Xi||^{p})$$

where,

$$\begin{aligned} A_{T_2} &= ||c_1||||\Psi_1|| + ||c_2||||\Psi_2||L\frac{(\log \eta_1)^{\rho+\rho_1}}{\Gamma(\rho_1+\rho+1)} + (||c_1||c_2||||\Psi_3|| + ||\Psi_4||L)\frac{(\log \eta_2)^{\rho+\rho_2}}{\Gamma(\rho_2+\rho+1)}, \\ B_{T_2} &= (||c_1||||\Psi_1|| + ||c_2||||\Psi_2||L) + (||c_1||c_2||||\Psi_3|| + ||\Psi_4||L). \end{aligned}$$

Proof. Let Ξ_n be a sequence in Υ that converges to $\Xi \in \Upsilon$. Since f_{Ξ} *and* g_{Ξ} are continuous, it follows that $f_{\Xi_n} \to f_{\Xi}$ and $g_{\Xi_n} \to g_{\Xi}$. Thus by Lebesgue's dominated convergence theorem, it follows that T_2 is continuous.

We have,

$$\begin{aligned} ||T_2\Xi(t)|| &= ||K_1(t)[{}^{H}\mathcal{I}^{\rho_1+\rho} f_{\Xi}(\eta_1) + g_{\Xi}(\eta_1)] + K_2(t)(c_2[{}^{H}\mathcal{I}^{\rho_2+\rho} g_{\Xi}(\eta_2) + g_{\Xi}(\eta_2)])|| \\ &\leq ||K_1(t)||[{}^{H}\mathcal{I}^{\rho_1+\rho}||f_{\Xi}(\eta_1)|| + ||g_{\Xi}(\eta_1)||] + ||K_2(t)c_2||[{}^{H}\mathcal{I}^{\rho_2+\rho}||f_{\Xi}(\eta_2)|| + ||g_{\Xi}(\eta_2)||] \\ &\leq A_{T_2}(\mathcal{F} + \tilde{\mathcal{F}}||\Xi||^{p}) + B_{T_2}(\mathcal{E} + \tilde{\mathcal{E}}||\Xi||^{p}). \end{aligned}$$

Hence, we have,

$$||T_2\Xi(t)|| \leq A_{T_2}(\mathcal{F} + \tilde{\mathcal{F}}||\Xi||^{p}) + B_{T_2}(\mathcal{E} + \tilde{\mathcal{E}}||\Xi||^{p}).$$

□

Theorem 3.4. *T_3 is continuous and satisfies the following growth relation:*

$$\|T_3\Xi(t)\| \leq A_{T_3}(\mathcal{F}+\tilde{\mathcal{F}}\|\Xi\|^p)+B_{T_3}(\mathcal{E}+\tilde{\mathcal{E}}\|\Xi\|^p),$$

where

$$\begin{aligned} A_{T_3} &= (\|c_1\|\|\Psi_3\|+\|\Psi_4\|)\Big[\frac{\|c\|(\log L)^{\rho}}{\Gamma(\rho+1)}+\frac{\|d\|(\log L)^{\rho-\sigma}}{\Gamma(\rho-\sigma+1)}\Big], \\ B_{T_3} &= (\|c_1\|\|\Psi_3\|+\|\Psi_4\|L)[\|c\|+\|d\|]. \end{aligned}$$

Proof. Let Ξ_n be a sequence in Υ that converges to $\Xi \in \Upsilon$. Since f_Ξ *and* g_Ξ are continuous, it follows that $f_{\Xi_n} \to f_\Xi$ and $g_{\Xi_n} \to g_\Xi$. Thus by Lebesgue's dominated convergence theorem, it follows that T_3 is continuous.

We have,

$$\begin{aligned} \|T_3\Xi(t)\| &= \|K_2(t)(c[{}^{H}\mathcal{I}^{\rho}f_\Xi(L)+g_\Xi(L)]+d[{}^{H}\mathcal{I}^{\rho-\sigma}f_\Xi(L)+g_\Xi(L))]\| \\ &\leq \|K_1(t)\|\Big(\|c\|{}^{H}\mathcal{I}^{\rho}\|f_\Xi(L)\|+\|c\|\|g_\Xi(L)\|+\|d\|{}^{H}\mathcal{I}^{\rho-\sigma}\|f_\Xi(L)\|+\|d\|\|g_\Xi(L)\|\Big) \\ &\leq A_{T_3}(\mathcal{F}+\tilde{\mathcal{F}}\|\Xi\|^p)+B_{T_3}(\mathcal{E}+\tilde{\mathcal{E}}\|\Xi\|^p). \end{aligned}$$

Hence, we have,

$$\|T_3\Xi(t)\| \leq A_{T_3}(\mathcal{F}+\tilde{\mathcal{F}}\|\Xi\|^p)+B_{T_3}(\mathcal{E}+\tilde{\mathcal{E}}\|\Xi\|^p).$$

□

Theorem 3.5. *If T_2 is compact then T_2 is Lipschitz with a constant value equal to zero.*

Proof. Let $\Lambda \subset B(r)$ be a bounded set. To prove T_2 is a compact map, we have to prove that $T_2(\Lambda)$ is relatively compact, i.e., $\overline{T_2(\Lambda)}$ is compact in Υ. From Theorem 3.3, we have, for $\Xi \in \Lambda$,

$$\|T_2\Xi(t)\| \leq A_{T_2}(\mathcal{F}+\tilde{\mathcal{F}}r^p)+B_{T_2}(\mathcal{E}+\tilde{\mathcal{E}}r^p).$$

Hence $T_2(\Lambda)$ is uniformly bounded.

Now, for any $\Xi \in \Upsilon$, we have,

$$\begin{aligned} \|T_2\Xi'(t)\| &= \|K_1'(t)({}^{H}\mathcal{I}^{\rho_1+\rho}f_\Xi(\eta_1)+g_\Xi(\eta_1))+K_2'(t)(c_2({}^{H}\mathcal{I}^{\rho_2+\rho}f_\Xi(\eta_2)+g_\Xi(\eta_2))\| \\ &\leq \|K_1'(t)\|[{}^{H}\mathcal{I}^{\rho_1+\rho}\|f_\Xi(\eta_1)\|+\|g_\Xi(\eta_1)\|]+\|K_2'(t)c_2\|[{}^{H}\mathcal{I}^{\rho_2+\rho}\|f_\Xi(\eta_2)\|+\|g_\Xi(\eta_2)\|] \\ &\leq \Big(\|c_1\|\Psi_2\|\frac{(\log\eta_1)^{\rho+\rho_1}}{\Gamma(\rho_1+\rho+1)}+\|\Psi_4\|\|c_2\|\frac{(\log\eta_2)^{\rho+\rho_2}}{\Gamma(\rho_2+\rho+1)}\Big)(\mathcal{F}+\tilde{\mathcal{F}}\|\Xi\|^p) \\ &\quad +(\|c_1\|\Psi_2\|+\|\Psi_4\|\|c_2\|)(\mathcal{E}+\tilde{\mathcal{E}}\|\Xi\|^p)) \\ &\leq \overline{\omega_1}(\mathcal{F}+\tilde{\mathcal{F}}\|\Xi\|^p)+\overline{\omega_2}(\mathcal{E}+\tilde{\mathcal{E}}\|\Xi\|^p). \end{aligned}$$

Now, for $t_1, t_2 \in J$, we have,

$$\|T_2\Xi(t_2)-T_2\Xi(t_1)\| \leq \int_{t_1}^{t_2}\|T_2\Xi'(t)\|dt \leq [\overline{\omega_1}(\mathcal{F}+\tilde{\mathcal{F}}\|\Xi\|^p)+\overline{\omega_2}(\mathcal{E}+\tilde{\mathcal{E}}\|\Xi\|^p)](t_2-t_1).$$

Thus, as $t_2 \to t_1$, $\|T_2\Xi(t_2)-T_2\Xi(t_1)\| \to 0 \Rightarrow T_2$ is equicontinuous. By the Arzel$\grave{a}$-Ascoli theorem, T_2 is compact.

$\therefore$ T_2 is Lipschitz with a constant value of zero according to Proposition 2.5. □

Theorem 3.6. *If T_3 is compact then T_3 is Lipschitz with a constant value of zero.*

Proof. Let $\Lambda \subset B(r)$ be a bounded set. To prove that T_3 is a compact map, we have to prove that $T_3(\Lambda)$ is relatively compact, i.e., $\overline{T_3(\Lambda)}$ is compact in Υ. From Theorem 3.4, we have, for $\Xi \in \Lambda$,

$$||T_3\Xi(t)|| \leq A_{T_3}(\mathcal{F} + \tilde{\mathcal{F}} r^p) + B_{T_3}(\mathcal{E} + \tilde{\mathcal{E}} r^p).$$

Hence $T_3(\Lambda)$ is uniformly bounded.

Now, for any $\Xi \in \Upsilon$, we have,

$$\begin{aligned} ||T_3\Xi'(t)|| &\leq ||K_2'(t)||\Big(||c||[^H\mathcal{I}^\rho||f_\Xi(L)|| + ||g_\Xi(L)||] + ||d||[^H\mathcal{I}^\rho||f_\Xi(L)|| + ||g_\Xi(L)||]\Big) \\ &\leq ||\Psi_4||\Big(||c||\frac{(\log L)^\rho}{\Gamma(\rho+1)} + ||d||\frac{(\log L)^{\rho-\sigma}}{\Gamma(\rho-\sigma+1)}\Big)(\mathcal{F} + \tilde{\mathcal{F}}||\Xi||^p) + ||\Psi_4||(||c|| + ||d||)(\mathcal{E} + \tilde{\mathcal{E}}||\Xi||^p) \\ &\leq \overline{\omega_3}(\mathcal{F} + \tilde{\mathcal{F}}||\Xi||^p) + \overline{\omega_4}(\mathcal{E} + \tilde{\mathcal{E}}||\Xi||^p). \end{aligned}$$

Now, for $t_1, t_2 \in J$, we have,

$$||T_3\Xi(t_2) - T_3\Xi(t_1)|| \leq \int_{t_1}^{t_2} ||T_2\Xi'(t)||dt \leq [\overline{\omega_3}(\mathcal{F} + \tilde{\mathcal{F}}||\Xi||^p) + \overline{\omega_4}(\mathcal{E} + \tilde{\mathcal{E}}||\Xi||^p)](t_2 - t_1).$$

Thus, as $t_2 \to t_1$, $||T_3\Xi(t_2) - T_3\Xi(t_1)|| \to 0 \Rightarrow T_3$ is equicontinuous. By Arzel$\grave{a}$-Ascoli theorem, T_3 is compact.

$\therefore$ T_3 is Lipschitz with a constant value of zero according to Proposition 2.5. □

Since $T = T_1+T_2+T_3$, where T_1 is Lipschitz with the Lipschitz constant $\frac{(\log L)^\rho}{\Gamma(\rho+1)}(\mathcal{F}_1+\mathcal{F}_2 h_{\max})+\mathcal{E}_1+\mathcal{E}_2 k_{\max}$ and T_2 *and* T_3 are Lispchitz with a Lipschitz constant equal to 0, it follows that T is Lipschitz with the Lipschitz constant $\frac{(\log L)^\rho}{\Gamma(\rho+1)}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + \mathcal{E}_1 + \mathcal{E}_2 k_{\max}$.

If we assume that

$$\frac{(\log L)^\rho}{\Gamma(\rho+1)}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + \mathcal{E}_1 + \mathcal{E}_2 k_{\max} < 1,$$

then by Definition 2.7, it follows that T is μ-condensing.

We shall now prove the existence theorem using the TDT.

Theorem 3.7 (Existence). *The FODE Eq* (1.1) *has at least one solution if*

$$\Big(\frac{(\log L)^\rho}{\Gamma(\rho+1)} + A_{T_2} + A_{T_3}\Big) + (1 + B_{T_2} + B_{T_3}) < 1.$$

Proof. Consider the set

$$\mathcal{H} = \{\Xi \in \Upsilon \,:\, \exists\, \lambda \in [0,1] \text{ s. t. } \lambda T\Xi = \Xi\}.$$

We claim that $\mathcal{H}$ is bounded.

Let $\Xi \in \mathcal{H}$. Then $\exists\, \lambda \in [0,1]$ such that $\lambda T\Xi = \Xi$, i.e.,

$$\Xi = \lambda(T_1(\Xi) + T_2(\Xi) + T_3(\Xi)).$$

Taking $\|\cdot\|$ on both sides, we have,

$$\begin{aligned}\|\Xi\| &= \lambda\|T_1(\Xi)\| + \|T_2(\Xi)\| + \|T_3(\Xi)\| \\ &\le \Big(\frac{(\log L)^{\rho}}{\Gamma(\rho+1)} + A_{T_2} + A_{T_3}\Big)(\mathcal{F} + \tilde{\mathcal{F}}\|\Xi\|^{p}) \\ &\quad + (1 + B_{T_2} + B_{T_3})(\mathcal{E} + \tilde{\mathcal{E}}\|\Xi\|^{p}).\end{aligned}$$

If $\mathcal{H}$ is not bounded, dividing by $\|\Xi\|$ on both sides, we have

$$1 \le \Big(\frac{(\log L)^{\rho}}{\Gamma(\rho+1)} + A_{T_2} + A_{T_3}\Big)\frac{(\mathcal{F} + \tilde{\mathcal{F}}\|\Xi\|^{p})}{\|\Xi\|} + (1 + B_{T_2} + B_{T_3})\frac{(\mathcal{E} + \tilde{\mathcal{E}}\|\Xi\|^{p})}{\|\Xi\|}.$$

Letting $\|\Xi\| \to \infty$, and using the fact that $p \in [0,1)$, the right hand side of the above inequality goes to 0. Thus we get $1 \le 0$ which is a contradiction. This proves that H is bounded. By Theorem 2.9, T has at least one fixed point. Thus, the problem given by Eq (1.1) has at least one solution. □

Now we shall prove the uniqueness theorem for the considered problem.

Theorem 3.8 (**Uniqueness**). *The FODE given by Eq* (1.1) *has a unique solution if*

$$\Big(\frac{(\log L)^{\rho}}{\Gamma(\rho+1)} + A_{T_2} + A_{T_3}\Big)(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + (1 + B_{T_2} + B_{T_3})((\mathcal{E}_1 + \mathcal{E}_2 h_{\max})) < 1.$$

Proof. Let $\Xi_1, \Xi_2 \in \Upsilon$ be arbitrary. For an arbitrary $t \in J$,

$$\begin{aligned}\|T\Xi_1(t) - T\Xi_2(t)\| &= \|(T_1\Xi_1(t) - T_1\Xi_2(t)) + (T_2\Xi_1(t) - T_2\Xi_2(t)) + (T_3\Xi_1(t) - T_3\Xi_2(t))\| \\ &\le \|(T_1\Xi_1 - T_1\Xi_2)\| + \|(T_2\Xi_1 - T_2\Xi_2)\| + \|(T_3\Xi_1 - T_3\Xi_2)\|.\end{aligned}$$

Taking the supremum over all $t \in J$ gives

$$\|T\Xi_1 - T\Xi_2\| \le \|(T_1\Xi_1 - T_1\Xi_2)\| + \|(T_2\Xi_1 - T_2\Xi_2)\| + \|(T_3\Xi_1 - T_3\Xi_2)\|.$$

From Theorem 3.1, we have

$$\|T_1\Xi_1(t) - T_1\Xi_2(t)\| \le \Big[\frac{(\log L)^{\rho}}{\Gamma(\rho+1)}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + \mathcal{E}_1 + \mathcal{E}_2 k_{\max}\Big]\|\Xi_1 - \Xi_2\|.$$

We get the following from the definition of T_2:

$$\begin{aligned}\|T_2\Xi_1(t) - T_2\Xi_2(t)\| &\le \|K_1(t)\|[{}^{H}\mathcal{I}^{\rho_1+\rho}\|f_{\Xi_1}(\eta_1) - f_{\Xi_2}(\eta_1)\| + \|g_{\Xi_1}(\eta_1) - g_{\Xi_2}(\eta_1)\|] \\ &\quad + \|K_2(t)\|c_2\|[{}^{H}\mathcal{I}^{\rho_2+\rho}\|f_{\Xi_1}(\eta_1) - f_{\Xi_2}(\eta_1)\| + \|g_{\Xi_1}(\eta_2) - g_{\Xi_2}(\eta_2)\|] \\ &\le \Big(\|K_1(t)\|^{H}\mathcal{I}^{\rho_1+\rho}(1)(\rho_1+\rho) + \|K_2(t)\|c_2\|^{H}\mathcal{I}^{\rho_2+\rho}(1)(\rho_2+\rho)\Big) \\ &\quad \times (\mathcal{F}_1 + \mathcal{F}_2 h_{\max})\|\Xi_1 - \Xi_2\| \\ &\qquad (\|K_1(t)\| + \|K_2(t)\|c_2\|)(\mathcal{E}_1 + \mathcal{E}_2 h_{\max})\|\Xi_1 - \Xi_2\| \\ &= [A_{T_2}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + B_{T_2}(\mathcal{E}_1 + \mathcal{E}_2 h_{\max})]\|\Xi_1 - \Xi_2\|.\end{aligned}$$

Thus,

$$\|T_2\Xi_1 - T_2\Xi_2\| \leq [A_{T_2}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + B_{T_2}(\mathcal{E}_1 + \mathcal{E}_2 h_{\max})]\|\Xi_1 - \Xi_2\|.$$

We get the following from the definition of T_3:

$$\begin{aligned}
\|T_3\Xi_1(t) - T_3\Xi_2(t)\| &\leq \|K_2(t)\|\Big(\|c\|[{}^H\mathcal{I}^{\rho}\|f_{\Xi_1}(L) - f_{\Xi_2}(L)\| + \|g_{\Xi_1}(L) - g_{\Xi_2}(L)\|] \\
&\quad + \|d\|[{}^H\mathcal{I}^{\rho-\sigma}\|f_{\Xi_1}(L) - f_{\Xi_2}(L)\| + \|g_{\Xi_1}(L) - g_{\Xi_2}(L)\|]\Big) \\
&\leq \|K_2(t)\|\Big(\|c\|^H\mathcal{I}^{\rho}(1)(\rho) + \|d\|c_2\|^H\mathcal{I}^{\rho-\sigma}(1)(\rho-\sigma)\Big) \\
&\quad \times (\mathcal{F}_1 + \mathcal{F}_2 h_{\max})\|\Xi_1 - \Xi_2\| + \|K_2(t)(\|c\| + \|d\|)(\mathcal{E}_1 + \mathcal{E}_2 h_{\max})\|\Xi_1 - \Xi_2\| \\
&= [A_{T_3}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + B_{T_3}(\mathcal{E}_1 + \mathcal{E}_2 h_{\max})]\|\Xi_1 - \Xi_2\|.
\end{aligned}$$

Thus,

$$\|T_3\Xi_1 - T_3\Xi_2\| \leq [A_{T_3}(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + B_{T_3}(\mathcal{E}_1 + \mathcal{E}_2 h_{\max})]\|\Xi_1 - \Xi_2\|.$$

Hence, we have,

$$\begin{aligned}
\|T\Xi_1 - T\Xi_2\| &\leq |(T_1\Xi_1 - T_1\Xi_2)\| + |(T_2\Xi_1 - T_2\Xi_2)\| + |(T_3\Xi_1 - T_3\Xi_2)\| \\
&\leq [\Big(\frac{(\log L)^{\rho}}{\Gamma(\rho+1)} + A_{T_2} + A_{T_3}\Big)(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + (1 + B_{T_2} + B_{T_3})((\mathcal{E}_1 + \mathcal{E}_2 h_{\max}))]\Xi_1 - \Xi_2\|.
\end{aligned}$$

Since

$$\Big(\frac{(\log L)^{\rho}}{\Gamma(\rho+1)} + A_{T_2} + A_{T_3}\Big)(\mathcal{F}_1 + \mathcal{F}_2 h_{\max}) + (1 + B_{T_2} + B_{T_3})(\mathcal{E}_1 + \mathcal{E}_2 h_{\max}) < 1,$$

by the Banach contraction theorem, T has a unique fixed point. Hence the FODE given by Eq (1.1) has a unique solution. □

4. Example

Let the neutral type FIDEs have the following boundary value conditions

$$\begin{cases}
{}^C_H D^{\frac{3}{2}}[\Xi(t) - \frac{1}{8}\int_0^1 e^{2s-2}\Xi(s)ds] = \frac{1}{12\pi}\sin(2\pi\Xi) + \frac{1}{30}\Xi + t^2 + 2 + \frac{1}{16}\int_0^1 e^{2s-2}\Xi(s)ds, \quad t \in [0,1] \\
\Xi(0) + 3({}^C_H D^{\frac{1}{3}}\Xi(0)) = \int_0^{\frac{1}{4}} \frac{(\frac{1}{4}-s)^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})}\Xi(s)ds, \\
\Xi(1) + 2({}^C_H D^{\frac{1}{3}}\Xi(1)) = 3\int_0^{\frac{3}{4}} \frac{(\frac{3}{4}-s)^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})}\Xi(s)ds,
\end{cases} \tag{4.1}$$

where $g(t,\Xi(t),P\Xi(t)) = \frac{1}{8}\int_0^1 e^{2s-2}\Xi(s)ds$ and

$$f(t,\Xi(t),S\,\Xi(t)) = \frac{1}{12\pi}\sin(2\pi\Xi) + \frac{1}{30}\Xi + t^2 + 2 + \frac{1}{16}\int_0^1 e^{2s-2}\Xi(s)ds.$$

Here $\rho = \frac{3}{2}, \sigma = \frac{1}{3}, a = 1, b = 3, c = 1, d = 2, c_1 = 1, c_1 = 3, \rho_1 = \frac{1}{4}, \rho_2 = \frac{3}{4}, \eta_1 = \frac{1}{4}, \eta_2 = \frac{3}{4}$ *and* $L = e$; additionally,

$$
\begin{aligned}
A_{T_2} &= ||c_1||||\Psi_1|| + ||c_2||||\Psi_2||L\frac{(\log \eta_1)^{\rho+\rho_1}}{\Gamma(\rho_1 + \rho + 1)} + (||c_1||c_2||||\Psi_3|| + ||\Psi_4||L)\frac{(\log \eta_2)^{\rho+\rho_2}}{\Gamma(\rho_2 + \rho + 1)},\\
B_{T_2} &= (||c_1||||\Psi_1|| + ||c_2||||\Psi_2||L) + (||c_1||c_2||||\Psi_3|| + ||\Psi_4||L),\\
A_{T_3} &= (||c_1||||\Psi_3|| + ||\Psi_4||)\Big[\frac{||c||(\log L)^{\rho}}{\Gamma(\rho + 1)} + \frac{||d|(\log L)^{\rho-\sigma}}{\Gamma(\rho - \sigma + 1)}\Big],\\
B_{T_3} &= (||c_1||||\Psi_3|| + ||\Psi_4||L)[||c|| + ||d||].
\end{aligned}
$$

Applying the above parameters in $A_{T_2}, B_{T_2}, A_{T_3}$ and B_{T_3}, we get

$$
\Big(\frac{(\log L)^{\rho}}{\Gamma(\rho + 1)} + A_{T_2} + A_{T_3}\Big) + (1 + B_{T_2} + B_{T_3}) = 8.0427.
$$

To prove Theorem 3.6, we apply

$$
\begin{aligned}
f(t, \Xi, S(\Xi)) &= \frac{1}{12\pi}\sin(2\pi\Xi) + \frac{1}{30}\Xi + t^2 + 2 + \frac{1}{16}\int_0^1 e^{2s-2}\Xi(s)ds,\\
g(t, \Xi, P(\Xi)) &= \frac{1}{8}\int_0^1 e^{2s-2}\Xi(s)ds
\end{aligned}
$$

in Eq (1.1); then,

$$
\begin{aligned}
|f(t, \Xi_1, S(\Xi_1)) - f(t, \Xi_2, S(\Xi_2))| &= \frac{1}{12\pi}|\sin(2\pi\Xi_1) - \sin(2\pi\Xi_2)| + \frac{1}{30}|\Xi_1 - \Xi_2| + 0.054|\Xi_1 - \Xi_2|\\
&= 0.087|\Xi_1 - \Xi_2|
\end{aligned}
$$

and

$$
|g(t, \Xi_1, P(\Xi_1)) - g(t, \Xi_2, P(\Xi_2))| = 0.1081|\Xi_1 - \Xi_2|.
$$

Hence, Conditions (A2) and (A4) hold with $U_1 = 0.087$ *and* $U_2 = 0.1081$, where $U_1 = \mathcal{F}_1 + \mathcal{F}_2 h_{\max}$ *and* $U_2 = \mathcal{E}_1 + \mathcal{E}_2 k_{\max}$.

Calculating t_f from the given data yields

$$
t_f = \frac{U_1(\log L)^{\rho}}{\Gamma(\rho + 1)} = 0.0655.
$$

Otherwise, for any $t \in J$ *and* $\Xi \in \mathbb{R}$, we have

$$
\begin{aligned}
|f(t, \Xi, S(\Xi))| &= |\frac{1}{12\pi}\sin(2\pi\Xi) + \frac{1}{30}\Xi + t^2 + 2 + \frac{1}{16}\int_0^1 e^{2s-2}\Xi(s)ds|\\
&= 3 + 0.0873|\Xi|
\end{aligned}
$$

and

$$
|g(t, \Xi, P(\Xi))| = 0.1081|\Xi|.
$$

Hence, Conditions (A1) and (A3) hold with $\mathcal{F} = 3, \tilde{\mathcal{F}} = 0.0873 \, and \, \mathcal{E} = 0, \tilde{\mathcal{E}} = 0.1081$. By Theorem 3.6,

$$\mathcal{H} = \{\Xi \in \Upsilon : \exists\, \lambda \in [0,1] \text{ s. t. } \lambda T \Xi = \Xi\}$$

is the solution set; then

$$\begin{aligned}\|\Xi\| &= \|\lambda(T_1(\Xi) + T_2(\Xi) + T_3(\Xi))\| \\ &\le \Big(\frac{(\log L)^{\rho}}{\Gamma(\rho+1)} + A_{T_2} + A_{T_3}\Big)(\mathcal{F} + \tilde{\mathcal{F}}\|\Xi\|^{p}) \\ &\quad + (1 + B_{T_2} + B_{T_3})(\mathcal{E} + \tilde{\mathcal{E}}\|\Xi\|^{p}).\end{aligned}$$

Thus,

$$\|\Xi\| \le \frac{\Big(\frac{(\log L)^{\rho}}{\Gamma(\rho+1)} + A_{T_2} + A_{T_3}\Big)\mathcal{F} + (1 + B_{T_2} + B_{T_3})\mathcal{E}}{1 - \Big(\frac{(\log L)^{\rho}}{\Gamma(\rho+1)} + A_{T_2} + A_{T_3}\Big)\tilde{\mathcal{F}} - (1 + B_{T_2} + B_{T_3})\tilde{\mathcal{E}}} = 49.1897.$$

By Theorem 3.6, applying System (1.1) with the values obtained using Eq (4.1) results in at least one solution Ξ in $C(J, \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Moreover,

$$\Big(\frac{(\log L)^{\rho}}{\Gamma(\rho+1)} + A_{T_2} + A_{T_3}\Big)U_1 + (1 + B_{T_2} + B_{T_3})U_2 = 0.8036 < 1.$$

Hence by Theorem 3.7, applying System (1.1) with the values obtained using Eq (4.1) results in a unique solution.

5. Conclusions

We investigated the existence and uniqueness results of a neutral type of FIDEs with CH-derivatives by using integral boundary conditions. The analytical results have been described by using the topological degree method. A suitable example has been illustrated for the obtained theory.

Acknowledgments

The authors extend their appreciation to the Deputyship for Research & Innovation 'within the' Ministry of Education in Saudi Arabia for funding this research work through the project number IF-PSAU-2021/01/18769.

Conflict of interest

This work does not have any conflicts of interest.

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