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Research article

Weakly Gorenstein comodules over triangular matrix coalgebras

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Abstract: In this paper, we characterise weakly Gorenstein injective and weakly Gorenstein coflat comodules over triangular matrix coalgebras by introducing the class of weakly compatible bicomodules. In particular, Gorenstein injective and Gorenstein coflat comodules are investigated.

Keywords: triangular matrix coalgebra; weakly Gorenstein comodule; weakly compatible bicomodule
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1. Introduction

The homological theory of comodules over coalgebras and Hopf algebras was introduced by Doi [5]. Auslander and Bridger defined Gorenstein projective modules by G-dimensions for finitely generated modules in [2]. Enoch and Jenda [6] developed the relative homological algebra, especially the Gorenstein homological algebra. Since then, the Gorenstein homological algebra has been developed rapidly and has obtained fruitful results in this field [12, 18]. Asensio et al in [1] introduced Gorenstein injective comodules which is a generalization of injective comodules over any coalgebra. A coalgebra C is said to be right semiperfect [15] if the category $\mathcal{M}^C$ has enough projectives. Recently, Meng introduced weakly Gorenstein injective and weakly Gorenstein coflat comodules over any coalgebra in [16], which proved that, for a left semiperfect coalgebra, weakly Gorenstein injective comodules is equivalent with weakly Gorenstein coflat comodules.

Triangular matrix rings play a significant role in the study of classical ring theory and representation theory of algebras. Given two rings A, B, and A-B-bimodule M, one can form the upper triangular matrix ring $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. A number of researchers have investigated the triangular matrix rings (algebras). The readers can review [11, 20, 22–24] and references therein for more details. Zhang studied the structure of Gorenstein-projective modules over triangular matrix algebras in [24]. Under some mild conditions, Zhang and Xiong [22] described all the modules in ${}^{\perp}\Lambda$, and obtained criteria for the Gorensteinness of Λ. As applications, they determined all the Gorenstein-projective Λ-modules if Λ is Gorenstein. Dually, given coalgebras C and D, a C-D-bicomodule U, $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$ can be made into a coalgebra, which is called triangular matrix coalgebras. The reader may refer to [8,10,13,14] and references therein. The comodule representation category over the Morita-Takeuchi context coalgebra Γ was studied in [10]. Moreover, the authors explicitely determined all Gorenstein injective comodules over the Morita-Takeuchi context coalgebra Γ.

Motivated by the research mentioned above, we devote this paper to studying weakly Gorenstein injective and weakly Gorenstein coflat comodules over triangular matrix coalgebras by means of the relative homological theory in comodule categories.

The main theorems of this paper are the following:

Theorem 1.1 (Theorem 3.3). *Let* $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$ *be the triangular matrix coalgebra, and U be a weakly compatible C-D-bicomodule. The following conditions are equivalent:*

(1) (X, Y, φ) *is a weakly Gorenstein injective* Γ*-comodule.*
(2) *X is a weakly Gorenstein injective C-comodule,* $\ker\varphi$ *is a weakly Gorenstein injective D-comodule, and* $\varphi : Y \to X\Box_C U$ *is surjetive.*

Theorem 1.2 (Theorem 4.5). *Let* $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$ *be the triangular matrix coalgebra which is semiperfect, and U be a weakly compatible C-D-bicomodule. The following conditions are equivalent:*

(1) (X, Y, φ) *is a weakly Gorenstein coflat* Γ*-comodule.*
(2) *X is a weakly Gorenstein coflat C-comodule,* $\ker\varphi$ *is a weakly Gorenstein coflat D-comodule, and* $\varphi : Y \to X\Box_C U$ *is surjetive.*

2. Preliminaries

In this section, we include some details to establish notation and for sake of completeness.

Throughout this paper we fix an arbitrary field k. The reader is referred to [17] for the coalgebra and comodule terminology. Let C be a k-coalgebra with comultiplication Δ and counit ε. We recall that a let C-comodule is a k-vector space M together with a k-linear map $\rho_M : M \to C \otimes M$ such that $(\Delta \otimes id)\rho_M = (id \otimes \rho_M)\rho_M$ and $(\varepsilon \otimes id)\rho_M = id_M$. A k-linear map $f : M \to N$ between two left C-comodules M and N is a C-comodule homomorphism if $\rho_N f = (id \otimes f)\rho_M$. The k-vector space of all C-comodule homomorphisms from M to N is denoted by $\mathrm{Hom}^C(M, N)$. Similarly we can define a right C-comodule. Let $\mathcal{M}^C$ and ${}^C\mathcal{M}$ denotes the category of right and left C-comodules respectively. For any $M \in \mathcal{M}^C$ and $N \in {}^C\mathcal{M}$. Following [5,9], we recall that the cotensor product $M\Box_C N$ is the k-vector space

$$M\Box_C N = \mathrm{Ker}(\ M \otimes_k N \xrightarrow{\rho_M\otimes I_N - I_M\otimes\rho_N} M \otimes_k C \otimes_k N\ ),$$

where ρ_M and ρ_N are the structure maps of M and N, respectively.

Let C, D and E be three coalgebras. If M is a left E-comodule with structure map $\rho_M^- : M \to E \otimes M$, and also a right C-comodule with structure map $\rho_M^+ : M \to M \otimes C$ such that $(I \otimes \rho_M^+)\rho_M^- = (\rho_M^- \otimes I)\rho_M^+$, we then say that M is an (E, C)-bicomodule. We let ${}^E\mathcal{M}^C$ denote the category of (E, C)-bicomodules. Let N be a (C, D)-bicomodule. Then $M\Box_C N$ acquires a structure of (E, D)-bicomodule with structure maps

$$\rho_M^- \Box_C I : M\Box_C N \to (E \otimes_k M)\Box_C N \cong E \otimes_k (M\Box_C N), \text{and}$$

$$I\Box_C \rho_N^+ : M\Box_C N \to M\Box_C(N \otimes_k D) \cong (M\Box_C N) \otimes_k D.$$

It is known that $M\Box_k N = M \otimes_k N,\ M\Box_C C \cong M, C\Box_C N \cong N, (M\Box_C N)\Box_D L \cong M\Box_C(N\Box_D L)$ for any $L \in {}^D\mathcal{M}$, i.e. the cotensor product is associative and the functors

$$M\Box_C - : {}^C\mathcal{M} \to \mathcal{M}_k \text{ and } -\Box_C N : \mathcal{M}^C \to \mathcal{M}_k$$

are left exact, commute with arbitrary direct sums.

For every right exact exact linear functor $F : \mathcal{M}^C \to \mathcal{M}^D$ preserving direct sums, there exists a (C, D)-bicomodule X, in fact $X = F(C)$, such that F is naturally isomorphic to the funtor $-\Box_C X$ (See [19, Proposition 2.1]). Since every comodule is the union of its finite-dimensional subcomodules, there is a functorial isomorphism

$$M\Box_C N \cong \mathrm{Hom}^C(N^*, M)$$

for any $M \in \mathcal{M}^C$ and finite-dimensional $N \in {}^C\mathcal{M}$, where $N^* = \mathrm{Hom}_k(N, k)$ is equipped with the k-dual right C-comodule structure map

$$N^* \to \mathrm{Hom}_K(N, C) \cong N^* \otimes C,\ \alpha \to (I \otimes \alpha)\rho_N,\ \alpha \in N^* \text{ (see [5, P.32]).}$$

This implies that the functor $M\Box_C -$ from ${}^C\mathcal{M}$ to $\mathcal{M}_k$ is exact if and only if so is the functor $\mathrm{Hom}^C(-, M)$ from $\mathcal{M}^C$ to $\mathcal{M}_k$, i.e., the functors $M\Box_C -$ (resp. $-\Box_C N$) is exact if and only if M (resp. N) is an injective right (resp. left) C-comodule.

Let U be a C-D-bicomodule, then we may consider the functor $-\Box_C U : \mathcal{M}^C \to \mathcal{M}^D$. Unfortunately, in general, $-\Box_C U$ does not have a left adjoint functor. However, Takeuchi proved the following results:

Theorem 2.1. *[19, Proposition 1.10] Let C and D be two coalgebras, and U be a C-D-bicomodule. Then the functor $-\Box_C U : \mathcal{M}^C \to \mathcal{M}^D$ has a left adjoint functor if and only if U is a quasi-finite right D-comodule, i.e.,* $\mathrm{Hom}^D(F, U)$ *is finite-dimensional for all finite-dimensional $F \in \mathcal{M}^D$.*

If U is a quasi-finite right D-comodule, we denote the left adjoint functor of $-\Box_C U$ by $\mathrm{h}_D(U, -)$. Then for any right C-comodule W and any D-comodule N, we have that

$$\mathrm{Hom}^D(N, W\Box_C U) \cong \mathrm{Hom}^C(\mathrm{h}_D(U, N), W),$$

that is, $(\mathrm{h}_D(U, -), -\Box_C U)$ is a adjoint pair with unit $\epsilon : \mathrm{Id}_{\mathcal{M}^D} \to \mathrm{h}_D(U, -)\Box_C U$ and counit $\delta : \mathrm{h}_D(U, -\Box_C U) \to \mathrm{Id}_{\mathcal{M}^C}$. The functor $\mathrm{h}_D(U, -)$ has a behavior similar to the usual Hom functor of algebras.

Proposition 2.2. *Let C, D and E be three coalgebras, M and N be a (D, C)-bicomodule and an (E, C)-bicomodule, respectively, such that M is quasi-finite as right C-comodule. Then the following assertions hold:*

(a) *We have* $\mathrm{h}_C(M,N) = \varinjlim \mathrm{Hom}^C(N_\lambda, M)^* = \varinjlim (M\square_C N_\lambda^*)^*$, *where* $\{N_\lambda\}$ *is the family of finite dimensional subcomodules of* C*-comodule* N *(See the proof of [19, Proposition 1.3, P.633] and [5, P.32]).*

(b) *The vector space* $\mathrm{h}_C(M,N)$ *is an* (E,D)*-bicomodule (See [19, 1.7–1.9, P.634]).*

(c) *The functor* $\mathrm{h}_C(M,-)$ *is right exact and preserves all direct limits and direct sums (See [19, 1.6, P.634]).*

(d) *The functor* $\mathrm{h}_C(M,-)$ *is exact if and only if* M *is injective as right* C*-comodule (See [19, 1.12, P.635]).*

Remark 2.3. The set $\mathrm{Coend}_C(M) = \mathrm{h}_C(M,M)$ has an structure of k-coalgebra and then M becomes a $(\mathrm{Coend}_C(M), C)$-bicomodule, see [19, Proposition 2.1] for details. Symmetrically, $M \in {}^D\mathcal{M}^C$ is quasi-finite as left D-comodule if and only if the functor $M\square_C- : {}^C\mathcal{M} \to {}^D\mathcal{M}$ has a left adjoint functor. In this case we denote by $h_D(-,M)$ that functor.

For two k-coalgebras C and D, let U be a C-D-bicomodule with the left C-coaction on U, $u \mapsto u_{[-1]} \otimes u_{[0]}$, and the right D-coaction on U, $u \mapsto u_{[0]} \otimes u_{[1]}$ (using Sweedler's convention with the summations symbol omitted). We recall from [4, 13, 14, 21] that $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$ can be made into a coalgebra by defining the comutiplication $\Delta : \Gamma \to \Gamma \otimes \Gamma$ and the counit $\varepsilon : \Gamma \to K$ as follows

$$\begin{aligned} \Delta(\begin{pmatrix} c & u \\ 0 & d \end{pmatrix}) &= \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} u_{[-1]} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & u_{[0]} \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & u_{[0]} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & u_{[1]} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d_1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix} \\ \varepsilon(\begin{pmatrix} c & u \\ 0 & d \end{pmatrix}) &= \varepsilon_C(c) + \varepsilon_D(d). \end{aligned}$$

The coalgebra Γ is called a triangular matrix coalgebra.

We know from [14] that the right comodule category $\mathcal{M}^\Gamma$ and the comodule representation category $\mathcal{R}(\Gamma)$ are equivalent. The objects of $\mathcal{R}(\Gamma)$ are the triples (X, Y, φ), where X is a right C-comodule, Y is a right D-comodule,and $\varphi \in \mathrm{Hom}^D(Y, X\square_C U)$ is the right D-comodule morphism. For any two objects (X, Y, φ) and (X', Y', φ') in $\mathcal{R}(\Gamma)$, the morphism from (X, Y, φ) to (X', Y', φ') in $\mathcal{R}(\Gamma)$ is a pair of homomorphisms

$$\alpha{=}(\alpha_1, \alpha_2) : (X, Y, \varphi) \to (X', Y', \varphi'),$$

where $\alpha_1 \in \mathrm{Hom}^C(X, X')$ and $\alpha_2 \in \mathrm{Hom}^D(Y, Y')$ such that the following diagram is commutative

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "\alpha_2"'] & X\square_C U \arrow[d, "\alpha_1\square 1_U"] \\ Y' \arrow[r, "\varphi'"] & X'\square_C U \end{tikzcd}$$

Next we define some exact functors between the right comodule category $\mathcal{M}^\Gamma$ and the comodule representation category $\mathcal{R}(\Gamma)$.

(1) For any objects X and X' in $\mathcal{M}^C$, and any right C-comodule morphism $\alpha : X \to X'$, the functor $T_C : \mathcal{M}^C \to \mathcal{R}(\Gamma)$ is given by $T_C(X) = (X, X\square_C U, \mathrm{Id}_{X\square_C U})$ and $T_C(\alpha) = (\alpha, \alpha\square\mathrm{Id}_U)$.
(2) The functor $U_C : \mathcal{R}(\Gamma) \to \mathcal{M}^C$ is defined by $U_C(X, Y, \varphi) = X$, $U_C(\alpha, \beta) = \alpha$ for any objects (X, Y, φ) and (X', Y', φ') in $\mathcal{R}(\Gamma)$ and any right Γ-comodule morphism $(\alpha, \beta) : (X, Y, \varphi) \to (X', Y', \varphi')$.
(3) The functor $U_D : \mathcal{R}(\Gamma) \to \mathcal{M}^D$ is defined by $U_D(X, Y, \varphi) = Y$, $U_D(\alpha, \beta) = \beta$ for any objects (X, Y, φ) and (X', Y', φ') in $\mathcal{R}(\Gamma)$ and any right Γ-comodule morphism $(\alpha, \beta) : (X, Y, \varphi) \to (X', Y', \varphi')$.
(4) The functor $H_D : \mathcal{M}^D \to \mathcal{R}(\Gamma)$ is given by $H_D(Y) = (\mathrm{Hom}^D(U, Y), Y, \epsilon_Y)$, $H_D(\beta) = (\mathrm{Hom}^D(U, \beta), \beta)$ for any right D-comodule morphism $\beta : Y \to Y'$.

Remark 2.4. **(i)** If I is an indecomposable injective right C-comodule, then $T_C(I)$ is an indecomposable injective right Γ-comodule.

(ii) If P is an indecomposable projective right D-comodule, then $H_D(P)$ is an indecomposable projective right Γ-comodule.

(iii) (T_C, U_C) and (U_D, H_D) are adjoint pairs.

Lemma 2.5. *Let* $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$ *be a triangular matrix coalgebra, which is semiperfect. Then*

(1) $U_C L_n H_D(-) \cong \mathrm{Ext}^n_D(U, -)$, $U_D L_n H_D(-) = 0$, *for the left derived functor* $L_n H_D$ *(*$\forall n \geq 1$*).*

(2) *If* $\mathrm{Ext}^i_D(U, Y) = 0$ *(*$1 \leq i \leq n$*), then there exists an isomorphism*

$$\mathrm{Tor}^i_D(Y, Y') \cong \mathrm{Tor}^i_\Gamma(H_D(Y), (X', Y', \varphi'))$$

for any right D-comodule Y and left D-comodule Y′ and any $1 \leq i \leq n$.

Proof. (1) By [10, Theorem 2], if the triangular matrix coalgebra $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$ is semiperfect, then coalgebras C and D are semiperfect. So, for any $Y \in \mathcal{M}^D$, there exists an exact sequence in $\mathcal{M}^D$

$$0 \longrightarrow \ker\alpha \xrightarrow{\pi} P \xrightarrow{\alpha} Y \longrightarrow 0$$

with P a projective right D-comodule. Then $H_D(P)$ is a right Γ-projective comodule. Applying the left derived functor to the above exact sequence, we have $L_1 H_D(P) = 0$. Thus we get the following exact sequence

$$0 \longrightarrow (L_1 H_D)Y \longrightarrow H_D(\ker\alpha) \xrightarrow{H_D(\pi)} H_D(P) \xrightarrow{H_D(\alpha)} H_D(Y) \longrightarrow 0\ .$$

So

$$(L_1 H_D)Y \cong \ker(H_D(\pi)) = (\ker(\mathrm{Hom}^D(U, \pi)), 0, 0) = (\mathrm{Ext}^1_D(U, Y), 0, 0),$$

and $(L_n H_D)Y = (\mathrm{Ext}^n_D(U, Y), 0, 0)$, for any positive integer n. Hence,

$$U_C(L_n H_D)Y = \mathrm{Ext}^n_D(U, Y), U_D(L_n H_D)Y = 0$$

for any positive integer n.

(2) Assume that

$$\cdots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow Y \longrightarrow 0$$

is a projective resolution of Y. Since $\mathrm{Ext}^i_D(U, Y) = 0 (1 \leq i \leq n)$, it follows from (1) that $U_C L_n H_D(Y) = 0$, $U_D L_n H_D(Y) = 0$, and $L_n H_D(Y) = 0$. This implies that the following sequence

$$\cdots \longrightarrow H_D(P_2) \longrightarrow H_D(P_1) \longrightarrow H_D(P_0) \longrightarrow H_D(Y) \longrightarrow 0$$

is a projective resolution of $H_D(Y)$. For any left Γ-comodule (X', Y', φ'), its dual space is $(X', Y', \varphi')^* \in \mathcal{M}^\Gamma$. Then we get $H_D(P^i)\Box_\Gamma(X', Y', \varphi') \cong \mathrm{Hom}^\Gamma((X', Y', \varphi')^*, H_D(P^i))(1 \leq i \leq n)$ and $P^i\Box_D Y' \cong \mathrm{Hom}^D((Y')^*, P^i)(1 \leq i \leq n)$.

For brevity, we denote $\mathrm{Hom}^\Gamma((X', Y', \varphi')^*, H_D(P^i))$ by $((X', Y', \varphi')^*, H_D(P^i))$. Since (U_D, H_D) is an adjoint pair, we have the commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & ((X', Y', \varphi')^*, H_D(P^1)) \arrow[r] \arrow[d, "\cong"] & ((X', Y', \varphi')^*, H_D(P^0)) \arrow[r] \arrow[d, "\cong"] & ((X', Y', \varphi')^*, H_D(Y)) \arrow[r] \arrow[d, "\cong"] & 0 \\
\cdots \arrow[r] & \mathrm{Hom}^D((Y')^*, P^1) \arrow[r] & \mathrm{Hom}^D((Y')^*, P^0) \arrow[r] & \mathrm{Hom}^D((Y')^*, Y) \arrow[r] & 0
\end{tikzcd}$$

By the above isomorphism, we furthermore get the following commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H_D(P^1)\Box_\Gamma(X', Y', \varphi') \arrow[r] \arrow[d, "\cong"] & H_D(P^0)\Box_\Gamma(X', Y', \varphi') \arrow[r] \arrow[d, "\cong"] & H_D(Y)\Box_\Gamma(X', Y', \varphi') \arrow[r] \arrow[d, "\cong"] & 0 \\
\cdots \arrow[r] & P^1\Box_D Y' \arrow[r] & P^0\Box_D Y' \arrow[r] & Y\Box_D Y' \arrow[r] & 0.
\end{tikzcd}$$

Consequently,

$$\mathrm{Tor}^i_D(Y, Y') \cong \mathrm{Tor}^i_\Gamma(H_D(Y), (X', Y', \varphi'))$$

for any $1 \leq i \leq n$. $\square$

3. Weakly Gorenstein injective comodules over triangular matrix coalgebras

Recall from [1, 10, 16] that for an exact sequence of injective right C-comodules

$$\mathcal{E}_C = \ \cdots \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots,$$

if $\mathrm{Hom}^C(I, \mathcal{E}_C)$ is also exact for any injective right C-comodule I, then $\mathcal{E}_C$ is said to be complete. For a right C-comodule M, if $M \cong \ker(E^0 \to E^1)$, then M is called Gorenstein injective. If there exists an exact sequence of right C-comodules

$$\mathcal{M}_C = \ 0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots$$

with $E^i (i \geq 0)$ injectives and which remains exact whenever $\mathrm{Hom}^C(E, -)$ is applied for any injective right C-comodule E, then we call M is weakly Gorenstein injective. From now on, we denote by $\mathcal{GI}(\Gamma)$ and $\mathcal{WGI}(\Gamma)$ the category of Gorenstein injective comodules and weakly Gorenstein injective comodules over Γ, respectively.

As a generalization of compatible bicomodules, we now show the "weak analogue" of compatible bicomodules as follows.

A C-D-bicomodule U is weakly compatible if the following two conditions hold:

(1) If $\mathcal{M}_C$ is an exact sequence of injective right C-comodules, then $\mathcal{M}_C\Box U$ is exact.

(2) If $\mathcal{M}_D$ is a complete exact sequence of injective right D-comodules, then $\mathrm{Hom}^D(U, \mathcal{M}_D)$ is exact.

Lemma 3.1. *[7] For the triangular matrix coalgebra* $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$*, if* $(X, Y, \varphi) \in \mathcal{GI}(\Gamma)$*, then*

$$\mathrm{Ext}^i_C(I, (X, Y, \varphi)) = 0$$

for any injective right Γ*-comodule I and any* $i \geq 1$.

Lemma 3.2. *[16] For the right C-comodule M, the followings are equivalent:*

(1) M is weakly Gorenstein injective;

(2) $\mathrm{Ext}^i_C(E, M) = 0$ *for any injective comodule E and any* $i \geq 1$*;*

(3) $\mathrm{Ext}^i_C(L, M) = 0$ *for any finite-dimensional right C-comodule L and any* $i \geq 1$.

Theorem 3.3. *Let* $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$ *be the triangular matrix coalgebra, and U be a weakly compatible C-D-bicomodule. The following conditions are equivalent:*

(1) (X, Y, φ) *is a weakly Gorenstein injective* Γ*-comodule.*
(2) X is a weakly Gorenstein injective C-comodule, $\ker\varphi$ *is a weakly Gorenstein injective D-comodule, and* $\varphi : Y \to X\square_C U$ *is surjetive.*

Proof. (2) $\Rightarrow$ (1) If $X \in \mathcal{WGI}(C)$, then there exists the following exact sequence

$$\mathcal{X}_C = \ 0 \longrightarrow X \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots$$

with each $E^i (i \geq 0)$ injective. By the assumption that U is weakly compatible, it follows that the sequence $\mathcal{X}_C \square_C U$ is exact and $E^i \square_C U (i \geq 0)$ are injective. Here $\ker\varphi \in \mathcal{WGI}(D)$. Thus there exists an exact sequence as follows

$$\mathcal{K}_D = \ 0 \longrightarrow \ker\varphi \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

with each $I^i (i \geq 0)$ injective. By using "the Generalized Horseshoe Lemma", we get the following exact sequence

$$0 \longrightarrow Y \longrightarrow I^0 \oplus (E^0 \square_C U) \longrightarrow I^1 \oplus (E^1 \square_C U) \longrightarrow \cdots .$$

So we have the following commutative diagrams with exact rows

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, "\varphi"] & I^0 \oplus (E^0 \square_C U) \arrow[r] \arrow[d] & I^1 \oplus (E^1 \square_C U) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & X \square_C U \arrow[r] & (E^0 \square_C U) \arrow[r] & (E^1 \square_C U) \arrow[r] & \cdots \end{tikzcd}$$

Hence there exists an exact sequence

$$\bar{\mathcal{L}}_\Gamma = \ 0 \longrightarrow (X, Y, \varphi) \longrightarrow (E^0, I^0 \oplus (E^0 \square_C U), (0, id)) \longrightarrow (E^1, I^1 \oplus (E^1 \square_C U), (0, id)) \longrightarrow \cdots .$$

Next we only need to prove that $\mathrm{Hom}^\Gamma(E, \bar{\mathcal{L}}_\Gamma)$ is exact for any injective right Γ-comodule E.

Because $E \cong \oplus_{i\in\Lambda} T_C(E^i)$ with finite-dimensional indecomposable injective right C-comodule $E^i (i \in \Lambda)$, where Λ is a finite index set. Then

$$\begin{aligned}\mathrm{Hom}^{\Gamma}(E, \bar{\mathcal{L}}_{\Gamma}) &= \mathrm{Hom}^{\Gamma}(\oplus_{i\in\Lambda} T_C(E^i), \bar{\mathcal{L}}_{\Gamma}) \\ &\cong \oplus_{i\in\Lambda} \mathrm{Hom}^{\Gamma}(T_C(E^i), \bar{\mathcal{L}}_{\Gamma}) \\ &\cong \oplus_{i\in\Lambda} \mathrm{Hom}^{C}(E^i, U_C(\bar{\mathcal{L}}_{\Gamma})).\end{aligned}$$

So $\mathrm{Hom}^{\Gamma}(E, \bar{\mathcal{L}}_{\Gamma})$ is exact, that is, $(X, Y, \varphi) \in \mathcal{WGI}(\Gamma)$.

(1) $\Rightarrow$ (2) If $(X, Y, \varphi) \in \mathcal{WGI}(\Gamma)$, then there exists an exact sequence of right Γ-comodule

$$\bar{\mathcal{L}}_{\Gamma} = \ 0 \longrightarrow (X, Y, \varphi) \longrightarrow (E^0, I^0 \oplus (E^0 \square_C U), (0, id)) \longrightarrow (E^1, I^1 \oplus (E^1 \square_C U), (0, id)) \longrightarrow \cdots .$$

Applying U_C to $\bar{\mathcal{L}}_{\Gamma}$, we get an exact sequence

$$\mathcal{X}_C = \ 0 \longrightarrow X \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots .$$

Hence $\mathrm{Hom}^{C}(E, \mathcal{X}_C) \cong \mathrm{Hom}^{\Gamma}(T_C(E), \bar{\mathcal{L}}_{\Gamma})$ for any injective right C-comodule E. Therefore, $X \in \mathcal{WGI}(C)$.

By applying the exact functor U_D to $\bar{\mathcal{L}}_{\Gamma}$ again, we also get an exact sequence

$$0 \longrightarrow Y \longrightarrow I^0 \oplus (E^0 \square_C U) \longrightarrow I^1 \oplus (E^1 \square_C U) \longrightarrow \cdots$$

Hence, we get the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker\varphi \arrow[d, "\pi"] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & Y \arrow[r] \arrow[d, "\varphi"] & I^0 \oplus (E^0 \square_C U) \arrow[r] \arrow[d] & I^1 \oplus (E^1 \square_C U) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & X \square_C U \arrow[r] \arrow[d] & E^0 \square_C U \arrow[r] \arrow[d] & E^1 \square_C U \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Thus we conclude that φ is surjective, and $\ker\varphi \in \mathcal{WGI}(D)$. □

The following result can be viewed as an application of the above theorem on Gorenstein injective comodules.

Corollary 3.4. *Let* $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$ *be the triangular matrix coalgebra with U a weakly compatible C-D-bicomodule, we have the following equivalence:*

$$(X, Y, \varphi) \in \mathcal{GI}(\Gamma) \Leftrightarrow X \in \mathcal{GI}(C),\ \ker\varphi \in \mathcal{GI}(D),\ \textit{and}\ \varphi : Y \to X \square_C U\ \textit{is surjective.}$$

4. Weakly Gorenstein coflat comodules over triangular matrix coalgebras

In this section, we first have the following key observation, which is very important for the proof of our main result. The reader may refer to [16] for more details.

Definition 4.1. A right C-comodule M is called Gorenstein coflat if there is an exact sequence of injective right C-comodules

$$\mathbb{E}_C = \cdots \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

such that $M = \ker(E^0 \to E^1)$, and $\mathbb{E}_C \square_C Q$ is exact for any projective left C-comodule Q.

Definition 4.2. A right C-comodule M is called weakly Gorenstein coflat if there is an exact sequence of right C-comodules

$$\mathcal{M}_C = 0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots$$

with each $E^i (i \geq 0)$ injective, and $\mathcal{M}_C \square_C Q$ is exact for any projective left C-comodule Q.

We write $\mathcal{WGC}(\Gamma)$ and $\mathcal{GC}(\Gamma)$ for the category of weakly Gorenstein coflat and Gorenstein coflat comodules over Γ, respectively. Under the assumption of right semiperfect, the following result establishes the relation between weakly Gorenstein injective right C-comodules and weakly Gorenstein coflat right C-comodules.

Remark 4.3. (1) The class of weakly Gorenstein injective right C-comodules is closed under extensions, cokernels of monomorphisms and direct summands. If C is a right semiperfect coalgebra, then the class of weakly Gorenstein injective right C-comodules is closed under direct products.

(2) The class of weakly Gorenstein coflat right C-comodules is closed under extensions, cokernels of monomorphisms, direct sums, direct summands and direct limit.

(3) Let C be a right semiperfect coalgebra and M a right C-comodule, then M is weakly Gorenstein coflat if and only if M is weakly Gorenstein injective.

Lemma 4.4. *[16] For a right C-comodule M, the following statements are equivalent:*

(1) M is Gorenstein coflat;

(2) There is an exact sequence of injective right C-comodules

$$\mathbb{E}_C = \cdots \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

such that $M = \ker(E^0 \to E^1)$, and $\mathbb{E}_C \square_C Q$ is also exact for any finite-dimensional projective left C-comodule Q.

Lemma 4.5. *Let C be a semiperfect coalgebra, then the following statements are equivalent for any right C-comodule M:*

(1) *M is weakly Gorenstein coflat;*

(2) $\mathrm{Tor}^i_C(M, P) = 0$ *for any projective left C-comodule P and any $i \geq 1$;*

(3) $\mathrm{Tor}^i_C(M, Q) = 0$ *for any finite-dimensional projective left C-comodule P and any* $i \geq 1$.

Proof. It is obvious for (1) $\Leftrightarrow$ (2) by the definition. We only need to show (3) $\Rightarrow$ (1). For any projective left C-comodule Q, then from the proof of [3, Corollary 2.4.22, P.100] we know that $Q \cong \oplus_{\lambda\in\Lambda} Q_\lambda$, where each Q_λ is a finite-dimensional projective. Choose an exact sequence

$$\mathcal{M}_C = \ 0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots$$

with each E^i injective. Since

$$\begin{aligned}\mathcal{M}_C \Box_C Q &\cong \mathcal{M}_C \Box_C (\oplus_{\lambda\in\Lambda} Q_\lambda)\\ &\cong \oplus_{\lambda\in\Lambda} (\mathcal{M}_C \Box_C Q_\lambda)\end{aligned}$$

and $\mathrm{Tor}^i_C(M, Q_\lambda) = 0$ for all $\lambda \in \Lambda$ and $i \geq 1$, it follows that $\mathcal{M}_C \Box_C Q$ is exact. That is, M is weakly Gorenstein coflat.

(1) $\Rightarrow$ (3) is evident. $\square$

Theorem 4.6. *Let* $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$ *be the triangular matrix coalgebra which is semiperfect, and U be a weakly compatible C-D-bicomodule. The following conditions are equivalent:*

(1) (X, Y, φ) *is a weakly Gorenstein coflat* Γ*-comodule.*
(2) X is a weakly Gorenstein coflat C-comodule, $\ker\varphi$ *is a weakly Gorenstein coflat D-comodule, and* $\varphi : Y \to X \Box_C U$ *is surjetive.*

Proof. (2) $\Rightarrow$ (1) Let $X \in \mathcal{WGC}(C)$. Since U is weakly compatible, it follows that there exists an exact sequence of right C-comodules with each I^i injective

$$\mathcal{X}_C = \ 0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

and $\mathcal{X}_C \Box_C U$ is exact. Suppose that $\ker\varphi \in \mathcal{WGC}(D)$, there exists the following exact sequence

$$\mathcal{K}_D = \ 0 \longrightarrow \ker\varphi \longrightarrow K^0 \longrightarrow K^1 \longrightarrow K^2 \longrightarrow \cdots .$$

The Generalized Horseshoe Lemma yields the following exact sequence

$$\mathcal{Y}_D = \ 0 \longrightarrow K^0 \oplus (I^0 \Box_C U) \longrightarrow K^1 \oplus (I^1 \Box_C U) \longrightarrow K^2 \oplus (I^2 \Box_C U) \longrightarrow \cdots .$$

This gives rise to the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, "\varphi"] & K^0 \oplus (I^0 \Box_C U) \arrow[r] \arrow[d, "(0,id)"] & K^1 \oplus (I^1 \Box_C U) \arrow[r] \arrow[d, "(0,id)"] & K^2 \oplus (I^2 \Box_C U) \arrow[r] \arrow[d, "(0,id)"] & \cdots . \\ 0 \arrow[r] & X \Box_C U \arrow[r] & I^0 \Box_C U \arrow[r] & I^1 \Box_C U \arrow[r] & I^2 \Box_C U \arrow[r] & \cdots \end{tikzcd}$$

Thus we obtain the following exact sequence of right Γ-comodules

$$\mathcal{L}_\Gamma = \ 0 \longrightarrow (X, Y, \varphi) \longrightarrow (I^0, K^0 \oplus (I^0 \Box_C U), (0, id)) \longrightarrow (I^1, K^1 \oplus (I^1 \Box_C U), (0, id)) \longrightarrow \cdots .$$

Let Q be any finite-dimensional projective left Γ-comodule. Then $Q \cong \oplus_{i\in\Lambda} Q_i$, Q_i is indecomposable and projective. Here Q_i^* is indecomposable and injective. Thus there is an indecomposable and injective right C-comodule $E_i (i \in \Lambda)$ such that $T_C(E_i) \cong Q_i^* (i \in \Lambda)$. Thus

$$\begin{aligned}
\mathcal{L}_\Gamma \square_C Q &\cong \mathcal{L}_\Gamma \square_C (\oplus_{i\in\Lambda} Q_i)\\
&\cong \oplus_{i\in\Lambda} (\mathcal{L}_\Gamma \square_C Q_i)\\
&\cong \oplus_{i\in\Lambda} \operatorname{Hom}^\Gamma(Q_i^*, \mathcal{L}_\Gamma)\\
&\cong \oplus_{i\in\Lambda} \operatorname{Hom}^\Gamma(T_C(E_i), \mathcal{L}_\Gamma)\\
&\cong \oplus_{i\in\Lambda} \operatorname{Hom}^C(E_i, U_C(\mathcal{L}_\Gamma))\\
&\cong \oplus_{i\in\Lambda} \operatorname{Hom}^C(E_i, \mathcal{X}_C).
\end{aligned}$$

Therefore, $\mathcal{L}_\Gamma \square_C Q$ is exact since $\oplus_{i\in\Lambda}\operatorname{Hom}^C(E_i, \mathcal{X}_C)$ is exact. That is, $(X, Y, \varphi) \in \mathcal{WGC}(\Gamma)$.

(1) $\Rightarrow$ (2) If $(X, Y, \varphi) \in \mathcal{WGC}(\Gamma)$, then there is the following exact sequence

$$\mathcal{L}_\Gamma = \ 0 \longrightarrow (X, Y, \varphi) \longrightarrow (I^0, K^0 \oplus (I^0 \square_C U), (0, id)) \longrightarrow (I^1, K^1 \oplus (I^1 \square_C U), (0, id)) \longrightarrow \cdots$$

with $(I^i, K^i \oplus (I^i \square_C U), (0, id))$ injectives. By applying the functor U_C to $\mathcal{L}_\Gamma$, we get the exact sequence

$$\mathcal{X}_C = \ 0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots .$$

For any finitely dimensional projective left C-comodule P, P^* is injective right C-comodule. Thus $T_C(P^*)$ is injective. This yields that

$$\begin{aligned}
\mathcal{X}_C \square_C P &\cong \operatorname{Hom}^C(P^*, \mathcal{X}_C)\\
&= \operatorname{Hom}^C(P^*, U_C(\mathcal{L}_\Gamma))\\
&\cong \operatorname{Hom}^\Gamma(T_C(P^*), \mathcal{L}_\Gamma).
\end{aligned}$$

Thus $\operatorname{Hom}^\Gamma(T_C(P^*), \mathcal{L}_\Gamma)$ is exact since $(X, Y, \varphi) \in \mathcal{WGC}(\Gamma)$. So $\mathcal{X}_C \square_C P$ is also exact. That is, $X \in \mathcal{WGC}(C)$.

Similarly, applying U_D to $\mathcal{L}_\Gamma$, we get an exact sequence as follows

$$0 \longrightarrow Y \longrightarrow K^0 \oplus (I^0 \square_C U) \longrightarrow K^1 \oplus (I^1 \square_C U) \longrightarrow K^2 \oplus (I^2 \square_C U) \longrightarrow \cdots .$$

This yields the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker\varphi \arrow[d, "\pi"] & K^0 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & Y \arrow[r] \arrow[d, "\varphi"] & K^0 \oplus (I^0 \square_C U) \arrow[r] \arrow[d] & K^1 \oplus (I^1 \square_C U) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & X \square_C U \arrow[r] \arrow[d] & I^0 \square_C U \arrow[r] \arrow[d] & I^1 \square_C U \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Therefore, φ is surjective, and $\ker\varphi \in \mathcal{WGC}(D)$. $\square$

It is clearly that Gorenstein coflat comodules is weakly Gorenstein coflat comodules. Thus the above result holds for Gorenstein coflat comodules.

Corollary 4.7. *Let* $\Gamma = \begin{pmatrix} C & U \\ 0 & D \end{pmatrix}$ *be the triangular matrix coalgebra which is semiperfect, and U be a weakly compatible C-D-bicomodule. Then*

$$(X, Y, \varphi) \in \mathcal{GC}(\Gamma) \Leftrightarrow X \in \mathcal{GC}(C),\ \mathrm{ker}\varphi \in \mathcal{GC}(D),\ \textit{and}\ \varphi : Y \to X\square_C U\ \textit{is surjective.}$$

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Conflict of interest

The authors declare no conflicts of interest.

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