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Research article

Inequalities of Simpson-Mercer-type including Atangana-Baleanu fractional operators and their applications

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Abstract: Fractional operators with integral inequalities have attracted the interest of several mathematicians. Fractional inequalities are best utilized in mathematical science with their features and wide range of applications in optimization, modeling, engineering and artificial intelligence. In this article, we consider new variants of Simpson-Mercer type inequalities involving the Atangana-Baleanu (A-B) fractional integral operator for s-convex functions. First, an integral identity, which acts as an auxiliary result for the main results is proved in the frame of fractional operator. Employing this new identity, some estimations of Simpson-Mercer type for s-convex functions in the second sense are discussed. In addition, we study various new applications on Modified Bessel functions, special means and q-digamma functions. These applications confirm the effectiveness and validity of the results and also bring a different dimension to the study.

Keywords: Jensen-Mercer inequality; Simpson's inequality; s-convex functions; Atangana-Baleanu fractional operator; Hölder's inequalities; q-digamma functions; Modified Bessel functions
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1. Introduction

Fractional analysis can be regarded as a generalization of classical analysis. Fractional analysis has been investigated and discussed by several mathematicians and they have explored the fractional derivatives and integrals in variant ways with numerous notations. Although the expressions of these generalized definitions can be transformed into each other, but have variant physical meanings. It is well known that the first fractional integral operator is the Riemann-Liouville fractional integral operator. Recently, fractional calculus has become one of the prestigious fields in research because of its inherent applications in different fields like numerical physical science [1], fluid mechanics [2], biological modeling [3, 4], and many more. Fractional calculus owes its starting point to whether or not the importance of the derivative of an integer order could be generalized to a fractional order which is not an integer. Analysis of fractional calculus is the generalization of classical analysis, it has been developed rapidly with the interesting convexity theory attributed to Jensen [5]. Its fertile applications in functional analysis, stochastic theory, finite element method, and optimization theory have made it a fascinating topic for research. Therefore, it acts as an incorporative subject among quantum theory, orthogonal polynomials, combinatorics, etc. These are the principal urges that led to the deep research and improvement of the theory of inequality in literature (see [6–8]).

Let $0 < \gamma_1 \leq \gamma_2 \leq ... \leq \gamma_{\mathcal{P}}$ and $\varphi = (\varphi_1, \varphi_2, ..., \varphi_{\mathcal{P}})$ be non-negative weights such that $\sum_{k=1}^{\mathcal{P}} \varphi_k = 1$. The Jensen's inequality (see [5]) in the literature, reads as:

If the convexity of Φ holds true on the interval $[\wp_1, \wp_2]$, then

$$\Phi\left(\sum_{k=1}^{\mathcal{P}} \varphi_k\, \gamma_k\right) \leq \sum_{k=1}^{\mathcal{P}} \varphi_k\, \Phi(\gamma_k), \tag{1.1}$$

for all $\gamma_k \in [\mathrm{d}, \mathrm{D}]$, $\varphi_k \in [0,1]$ and $(k = 1, 2, ..., \mathcal{P})$. This inequality helps to obtain new bounds for useful distances in information theory (see [9]).

Mercer [10] in the year 2003, investigated a generalized form of Jensen's inequality, which is famously known as Jensen-Mercer (J-M) inequality:

If Φ is a convex function on $[\mathrm{d}, \mathrm{D}]$, then

$$\Phi\left(\mathrm{d} + \mathrm{D} - \sum_{k=1}^{\mathcal{P}} \varphi_k\, \gamma_k\right) \leq \Phi(\mathrm{d}) + \Phi(\mathrm{D}) - \sum_{k=1}^{\mathcal{P}} \varphi_k\, \Phi(\gamma_k), \tag{1.2}$$

holds true for all $\gamma_k \in [\mathrm{d}, \mathrm{D}]$, $\varphi_k \in [0,1]$ and $(k = 1, 2, ..., \mathcal{P})$.

Some generalizations on J-M operator inequalities are proposed by Pečarić et al. in [11]. Later, Niezgoda in [12] proved several generalizations of Mercer's type inequalities to higher dimensions. Kian [13] contemplated the idea of Jensen inequality for superquadratic functions. Moreover, reverse Mercer's type operator inequalities and the aforesaid inequalities for super-quadratic functions have been proposed in [14, 15]. Also, the integral and integral mean version have been established in [16]. Cortez et al. (see Remark 3.4 [17]) presented the concept of s-convex variant of J-M inequality in a generalized form. For some recent generalizations on J-M type inequalities, we direct readers to go through [18–20] and the references cited therein.

Hudzik et al. [21], introduced the notion of s-convex functions in the first and second sense and also discussed some of their notable features.

Definition 1.1. *Let $s \in (0,1]$ and Φ be a real valued function on an interval $\mathcal{I} = [0,\infty)$. Then ϕ is said to be s-convex in the first sense if*

$$\phi(\varphi_1\gamma_1 + \varphi_2\gamma_2) \leq \varphi_1^s\phi(\gamma_1) + \varphi_2^s\phi(\gamma_2),$$

holds true for all $\gamma_1, \gamma_2 \in \mathcal{I}$, $\varphi_1, \varphi_2 \geq 0$ with $\varphi_1^s + \varphi_2^s = 1$.

Definition 1.2. *Let $s \in (0,1]$ and Φ be a real valued function on an interval $\mathcal{I} = [0,\infty)$. Then ϕ is said to be s-convex in the second sense if*

$$\phi(\varphi_1\gamma_1 + \varphi_2\gamma_2) \leq \varphi_1^s\phi(\gamma_1) + \varphi_2^s\phi(\gamma_2),$$

holds true for all $\gamma_1, \gamma_2 \in \mathcal{I}$, $\varphi_1, \varphi_2 \geq 0$ with $\varphi_1 + \varphi_2 = 1$.

The class of s-convex function in the first and second sense are denoted by ($\Phi \in K_s^1$) and ($\Phi \in K_s^2$) respectively. Moreover, they also studied that the class of s-convex function in the second sense is more stronger than the s-convexity in first sense for $s \in (0,1)$. Several features of s-convex function in both senses are discussed taking some examples into consideration. It is interesting to see that if $s \in (0,1)$ and $\Phi \in K_s^2$, then Φ is non negative.

For more detail on s-convex function (see [21–24]) and references cited therein. Chen [23] gave a nice relationship between convex functions and s-convex functions as:

Lemma 1.1. *(*see [23]*) Suppose $\Phi : [\mathrm{d},\mathrm{D}] \to \Re$ is a convex function. Then*
(i) If Φ is non-negative, then it is s-convex for $s \in (0,1]$.
(ii) If Φ is non-positive, then it is s-convex for $s \in [1,\infty)$.

Butt et al. [25] investigated the J-M inequality for s-convex functions in Breckner sense $\Big(s > 0\Big)$ as follows:

Theorem 1.1. *Let $\varphi_1, \varphi_2, ..., \varphi_{\mathcal{P}}$ be positive probability distributions and $\sum_{\jmath=1}^{\mathcal{P}}\varphi_\jmath^s \leq 1$. If $\Phi : [\mathrm{d},\mathrm{D}] \subset (0,\infty) \to \Re$ is s-Breckner sense convex function, then the following inequality is valid*

$$\Phi\left(\mathrm{d} + \mathrm{D} - \sum_{\jmath=1}^{\mathcal{P}}\varphi_\jmath\gamma_\jmath\right) \leq \Phi(\mathrm{d}) + \Phi(\mathrm{D}) - \sum_{\jmath=1}^{\mathcal{P}}\varphi_\jmath^s\Phi\left(\gamma_\jmath\right), \tag{1.3}$$

for any increasing sequence $\{\gamma_\jmath\}_{\jmath=1}^{\mathcal{P}} \in [\mathrm{d},\mathrm{D}]$.

The following conditions are necessary:

(i) By considering $s = 1$ in (1.3), we get the J-M inequality (1.1).

(ii) By considering $s = 1$ and $\varphi_1 = \mathrm{d}$ and $\varphi_2 = \mathrm{D}$, in (1.3) it gives the definition of the classical convex function.

The following inequality named as Simpson's inequality, is given in the literature as (see [26–28]).

$$\left|\frac{1}{3}\left\{\frac{\Phi(\mathrm{d})+\Phi(\mathrm{D})}{2}+2\Phi\left(\frac{\mathrm{d}+\mathrm{D}}{2}\right)-\frac{1}{\mathrm{D}-\mathrm{d}}\int_{\mathrm{d}}^{\mathrm{D}}\Phi(x)\,dx\right\}\right|\leq\frac{1}{2880}\left\|\Phi^4\right\|_\infty(\mathrm{D}-\mathrm{d})^4,$$

where $\Phi:[\mathrm{d},\mathrm{D}]\to\Re$ is a four times continuously differentiable mapping on (d,D) and $\left\|\Phi^4\right\|_\infty=\sup_{x\in(\mathrm{d},\mathrm{D})}\left|\Phi^4(x)\right|<\infty$.

Several authors have concentrated on Simpson-type inequalities for different classes of mappings. Precisely, a few mathematicians have dealt with Simpson-type inequalities for convex mappings, since convexity theory is an efficient and stable process for taking care of an extraordinary number of issues that emerge in various branches in pure and applied sciences. For some recent improvements on Simpson type inequality via different convexities and fractional operators see references [29–32] and cited therein.

Let $\Phi\in\mathcal{L}[\mathrm{d},\mathrm{D}]$. Then the left and right Riemann-Liouville (R-L) fractional integrals of order $\delta>0$ with $\mathrm{d}\geq 0$ are defined as follows:

$$J_{\mathrm{d}^+}^{\delta}\Phi(x)=\frac{1}{\Gamma(\delta)}\int_{\mathrm{d}}^{x}(x-\kappa)^{\delta-1}\ \Phi(\kappa)\ d\kappa,\quad x>\mathrm{d},$$

and

$$J_{\mathrm{D}^-}^{\delta}\Phi(x)=\frac{1}{\Gamma(\delta)}\int_{x}^{\mathrm{D}}(\kappa-x)^{\delta-1}\ \Phi(\kappa)\ d\kappa,\quad x<\mathrm{D}.$$

For further details one may (see [33, 34]).

In [35], the idea of the J-M inequality has been used by Kian and Moslehian and the following Hermite-Hadamard-Mercer inequality was demonstrated:

$$\begin{aligned}&\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\leq\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\Phi(\mathrm{d}+\mathrm{D}-\kappa)\,d\kappa\\&\leq\frac{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)}{2}\leq\Phi(\mathrm{d})+\Phi(\mathrm{D})-\frac{\Phi(\omega_1)+\Phi(\omega_2)}{2}.\end{aligned}\tag{1.4}$$

Recently, the notions of fractional operators have drawn the attention of several researchers. Fractional calculus helps mathematicians solve many real-life problems in a convincing manner than classical calculus. Mainly, there are two types of nonlocal fractional derivatives, the classical Riemann-Liouville and Caputo derivatives with singular kernels and the others with non-singular kernels, which have been introduced recently, such as the Caputo-Fabrizio and Atangana-Baleanu derivatives. However, the fractional derivative operators with non-singular kernels are effective to solve the non-locality of real-world problems in an appropriate manner. The introduction of new notions of fractional operators is a journey to succeed momentum to the fractional calculus and to gain the most efficient operators to the discussion. Now, we recall the notion of the Caputo-Fabrizio fractional operator:

Definition 1.3. *Let $p\in[1,\infty)$ and $[\omega_1,\omega_2]$ be an open subset of $\Re$, the Sobolev space $\mathcal{H}^p(\omega_1,\omega_2)$ is defined by*

$$\mathcal{H}^p(\omega_1,\omega_2)=\left\{\Phi\in\mathcal{L}^2(\omega_1,\omega_2)\ :\ D^\mu\Phi\in\mathcal{L}^2(\omega_1,\omega_2),\quad for\ \ all\ |\mu|\leq p\right\}.$$

Definition 1.4. *[36] Let* $\Phi \in \mathcal{H}^1(0,\kappa)$*,* $\kappa > \omega$*,* $\delta \in [0,1]$ *then, the definition of the new Caputo fractional derivative is given as:*

$$^{CF}D^{\delta}\Phi(t) = \frac{\mathcal{M}(\delta)}{1-\delta}\int_{\omega}^{t}\Phi'(s)\exp\left[-\frac{\delta}{(1-\delta)}(\kappa-s)\right]ds, \tag{1.5}$$

where $\mathcal{M}(\delta)$ *is normalization function with* $\mathcal{M}(0) = \mathcal{M}(1) = 1$*.*

Moreover, the corresponding Caputo-Fabrizio fractional integral operator is given as:

Definition 1.5. *[37] Let* $\Phi \in \mathcal{H}^1(0,\kappa)$*,* $\kappa > \omega$*,* $\delta \in [0,1]$*. Then, the definition of Caputo-Fabrizio fractional integrals are given as:*

$$\left({}^{CF}_{\omega}\mathcal{I}^{\delta}\Phi\right)(t) = \frac{1-\delta}{\mathcal{M}(\delta)}\Phi(t) + \frac{\delta}{M(\delta)}\int_{\omega}^{t}\Phi(y)dy,$$

and

$$\left({}^{CF}\ \mathcal{I}^{\delta}_{\kappa}\Phi\right)(t) = \frac{1-\delta}{\mathcal{M}(\delta)}\Phi(t) + \frac{\delta}{\mathcal{M}(\delta)}\int_{t}^{\kappa}\Phi(y)dy,$$

where $\mathcal{M}(\delta)$ *is normalization function with* $\mathcal{M}(0) = \mathcal{M}(1) = 1$*.*

Recently, Atangana & Baleanu introduced a new fractional operator containing the Mittag-Leffler function in the kernel, that solves the problem of retrieving the original function (a clear advantage over the Caputo-Fabrizio operator). Mittag-Leffler's function is more suitable than a power law in modeling nature and physical phenomena. This attribute has made the A-B operator more effective and helpful. As a result, many researchers have shown keen interest in utilizing this operator. Atangana & Baleanu introduced the derivative operator both in Caputo and Riemann-Liouville sense:

Definition 1.6. *[38] Let* $\kappa > \omega$*,* $\delta \in [0,1]$ *and* $\Phi \in \mathcal{H}^1(\omega,\kappa)$*. The new fractional derivative is given as:*

$$^{ABC}_{\ \ \omega}D^{\delta}_{t}\left[\Phi(t)\right] = \frac{\mathcal{M}(\delta)}{1-\delta}\int_{\omega}^{t}\Phi'(x)E_{\delta}\left[-\delta\frac{(t-x)^{\delta}}{(1-\delta)}\right]dx, \tag{1.6}$$

where E_{δ} *is the Mittag-Leffler function and is defined as* $E_{\delta}(-t^{\delta}) = \sum_{k=0}^{\infty}\frac{(-t)^{\delta k}}{\Gamma(\delta k+1)}$*.*

Definition 1.7. *[38] Let* $\Phi \in \mathcal{H}^1(\omega,\kappa)$*,* $\omega > \kappa$*,* $\delta \in [0,1]$*. The new fractional derivative is given as:*

$$^{ABR}_{\ \ \omega}D^{\delta}_{t}\left[\Phi(t)\right] = \frac{\mathcal{M}(\delta)}{1-\delta}\frac{d}{dt}\int_{\omega}^{t}\Phi(x)E_{\delta}\left[-\delta\frac{(t-x)^{\delta}}{(1-\delta)}\right]dx, \tag{1.7}$$

where E_{δ} *is the Mittag-Leffler function.*

However, in the same paper the authors also presented corresponding A-B fractional integral operator as:

Definition 1.8. *[38] The fractional integral operator with non-local kernel of a function* $\Phi \in \mathcal{H}^1(\omega,\kappa)$ *is defined as:*

$$^{AB}_{\ \omega}\mathcal{I}_{t}^{\ \delta}\left\{\Phi(t)\right\} = \frac{1-\delta}{\mathcal{M}(\delta)}\Phi(t) + \frac{\delta}{\mathcal{M}(\delta)\Gamma(\delta)}\int_{\omega}^{t}\Phi(y)(t-y)^{\delta-1}dy,$$

where $\kappa > \omega, \delta \in [0,1]$*.*

In [39], the right hand side of A-B fractional integral operator is presented as follows:

$$ {}^{AB}_{\kappa}\mathcal{I}_t^{\delta}\{\Phi(t)\} = \frac{1-\delta}{\mathcal{M}(\delta)}\Phi(t) + \frac{\delta}{\mathcal{M}(\delta)\Gamma(\delta)}\int_t^{\kappa}\Phi(y)(y-t)^{\delta-1}dy. $$

Here, $\Gamma(\delta)$ is the Gamma function. The positivity of the normalization function $\mathcal{M}(\delta)$ implies that the fractional A-B integral of a positive function is positive. It is worth noticing that the case when the order $\delta \to 1$ yields the classical integral and the case when $\delta \to 0$ provides the initial function.

Sarikaya et al. [33] used Riemann-Liouville fractional integrals to generalise the Hermite-Hadamard inequality. Iscan [40] generalised the conclusions reached by Sarikaya et al. [33] to Hermite-Hadamard-Fejér type inequalities. Chen [41] employed the methods of Sarikaya et al. [33] and used the product of two convex functions to produce fractional Hermite-Hadamard type integral inequalities. Recently, Sun proved different variants of the Hermite-Hadamard inequality via generalized local fractional integral operators with Mittag-Leffler kernel for h-convex functions [42], s-preinvex functions [43] and generalized preinvex fuctions [44]. Fernandez and Mohammed [45] presented a nice relation between Riemann-Liouville fractional integrals and A-B fractional integral operators and also proved new type of Hermite-Hadamard inequalities for convex functions. Ogulmus et al. [46] established Hermite-Hadamard-Mercer type inequalities involving Riemann-Liouville fractional operator. Sahoo et al. [47] presented some mid-point versions of Hermite-Hadamard inequalities via Caputo-Fabrizio fractional operator. Latif et al. [48] worked on Hermite-Hadamard-Fejér type fractional inequalities relating to a convex harmonic function and a positive symmetric increasing function.

The main motivation of this study is to obtain an integral identity employing A-B fractional integral operator, which has a unique place among fractional integral operators, and to generate some new Simpson-Mercer's like inequalities for s-convex functions based on this identity. Considering many special cases of the main findings, the study also includes the applications of the results.

2. Main results

In this section, we study several Simpson-Mercer's like inequalities via the A-B fractional integral operator for differentiable functions on (d, D). For this, we establish a new A-B integral identity that will serve as an auxiliary result to produce subsequent results for improvements. Throughout the paper, let us assume that $\mathcal{P}_1 := \left(\frac{1+\kappa}{2}\right)^s + \left(\frac{1-\kappa}{2}\right)^s \leq 1$, for $\kappa \in [0, 1]$.

Lemma 2.1. *Suppose the mapping* $\Phi : \mathcal{I} = [\mathrm{d}, \mathrm{D}] \to \mathfrak{R}$ *is differentiable on* (d, D) *with* $\mathrm{D} > \mathrm{d}$. *If* $\Phi' \in \mathcal{L}[\mathrm{d}, \mathrm{D}]$, *then for all* $\omega_1, \omega_2 \in [\mathrm{d}, \mathrm{D}]$ *and* $\delta > 0$, *the following A-B fractional identity of Simpson-Mercer type holds true:*

$$ \begin{aligned} &\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1) + \Phi(\mathrm{d}+\mathrm{D}-\omega_2)\} + \frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \qquad (2.1) \\ &+ \frac{2^{\delta}(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \\ &- \frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\left\{ {}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \right. \\ &\left. + {}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right\} \end{aligned} $$

$$= \frac{\omega_2 - \omega_1}{2}\Big[\int_0^1 \left(\frac{\kappa^\delta}{2} - \frac{1}{3}\right)\Phi'\left(\mathrm{d} + \mathrm{D} - \left[\frac{1+\kappa}{2}\omega_1 + \frac{1-\kappa}{2}\omega_2\right]\right)d\kappa$$
$$+ \int_0^1 \left(\frac{1}{3} - \frac{\kappa^\delta}{2}\right)\Phi'\left(\mathrm{d} + \mathrm{D} - \left[\frac{1-\kappa}{2}\omega_1 + \frac{1+\kappa}{2}\omega_2\right]\right)d\kappa\Big],$$

for $\kappa \in [0, 1]$.

Proof. Let us assume that

$$\frac{\omega_2 - \omega_1}{2}\Big[\int_0^1 \left(\frac{\kappa^\delta}{2} - \frac{1}{3}\right)\Phi'\left(\mathrm{d} + \mathrm{D} - \left[\frac{1+\kappa}{2}\omega_1 + \frac{1-\kappa}{2}\omega_2\right]\right)d\kappa$$
$$+ \int_0^1 \left(\frac{1}{3} - \frac{\kappa^\delta}{2}\right)\Phi'\left(\mathrm{d} + \mathrm{D} - \left[\frac{1-\kappa}{2}\omega_1 + \frac{1+\kappa}{2}\omega_2\right]\right)d\kappa\Big] = \frac{\omega_2 - \omega_1}{2}\{\mathcal{J}_1 + \mathcal{J}_2\}. \tag{2.2}$$

Implies that

$$\mathcal{J}_1 = \int_0^1 \left(\frac{\kappa^\delta}{2} - \frac{1}{3}\right)\Phi'\left(\mathrm{d} + \mathrm{D} - \left[\frac{1+\kappa}{2}\omega_1 + \frac{1-\kappa}{2}\omega_2\right]\right)d\kappa.$$

Integrating by parts, we get

$$\mathcal{J}_1 = \left(\frac{\kappa^\delta}{2} - \frac{1}{3}\right)\frac{\Phi\left(\mathrm{d} + \mathrm{D} - [\frac{1+\kappa}{2}\omega_1 + \frac{1-\kappa}{2}\omega_2]\right)}{\frac{\omega_2-\omega_1}{2}}\Bigg|_0^1 - \int_0^1 \frac{\Phi\left(\mathrm{d} + \mathrm{D} - [\frac{1+\kappa}{2}\omega_1 + \frac{1-\kappa}{2}\omega_2]\right)}{\frac{\omega_2-\omega_1}{2}} \times \frac{\delta\kappa^{\delta-1}}{2}d\kappa.$$

By substituting the variables, we get

$$\mathcal{J}_1 = \frac{2}{\omega_2 - \omega_1}\left\{\frac{1}{6}\Phi\left(\mathrm{d} + \mathrm{D} - \omega_1\right) + \frac{1}{3}\Phi\left(\mathrm{d} + \mathrm{D} - \frac{\omega_1 + \omega_2}{2}\right)\right\} - \frac{2^\delta \mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2 - \omega_1)^{\delta+1}}$$
$$\times\left\{ {}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d} + \mathrm{D} - \frac{\omega_1 + \omega_2}{2}\right) - \frac{1-\delta}{\mathcal{M}(\delta)}\Phi\left(\mathrm{d} + \mathrm{D} - \frac{\omega_1 + \omega_2}{2}\right)\right\}.$$

Similarly,

$$\mathcal{J}_2 = \int_0^1 \left(\frac{1}{3} - \frac{\kappa^\delta}{2}\right)\Phi'\left(\mathrm{d} + \mathrm{D} - \left[\frac{1-\kappa}{2}\omega_1 + \frac{1+\kappa}{2}\omega_2\right]\right)d\kappa$$
$$= \frac{2}{\omega_2 - \omega_1}\left\{\frac{1}{6}\Phi\left(\mathrm{d} + \mathrm{D} - \omega_2\right) + \frac{1}{3}\Phi\left(\mathrm{d} + \mathrm{D} - \frac{\omega_1 + \omega_2}{2}\right)\right\}$$
$$- \frac{2^\delta \mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2 - \omega_1)^{\delta+1}}\left\{ {}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d} + \mathrm{D} - \frac{\omega_1 + \omega_2}{2}\right) - \frac{1-\delta}{\mathcal{M}(\delta)}\Phi\left(\mathrm{d} + \mathrm{D} - \frac{\omega_1 + \omega_2}{2}\right)\right\}.$$

By using the values of $\mathcal{J}_1$ and $\mathcal{J}_2$ in (2.2), we get (2.1). □

Corollary 2.1. *If we consider* $\delta = 1$ *in* (2.1)*, then we have the following Simpson-Mercer type equality*

$$\frac{1}{6}\left\{\Phi\left(\mathrm{d} + \mathrm{D} - \omega_1\right) + 4\Phi\left(\mathrm{d} + \mathrm{D} - \frac{\omega_1 + \omega_2}{2}\right) + \Phi\left(\mathrm{d} + \mathrm{D} - \omega_2\right)\right\} - \frac{1}{(\omega_2 - \omega_1)}\int_{\mathrm{d}+\mathrm{D}-\omega_2}^{\mathrm{d}+\mathrm{D}-\omega_1}\Phi(x)\,dx$$
$$= \frac{\omega_2 - \omega_1}{2}\Big[\int_0^1 \left(\frac{\kappa}{2} - \frac{1}{3}\right)\Phi'\left(\mathrm{d} + \mathrm{D} - \left[\frac{1+\kappa}{2}\omega_1 + \frac{1-\kappa}{2}\omega_2\right]\right)d\kappa$$
$$+ \int_0^1 \left(\frac{1}{3} - \frac{\kappa}{2}\right)\Phi'\left(\mathrm{d} + \mathrm{D} - \left[\frac{1-\kappa}{2}\omega_1 + \frac{1+\kappa}{2}\omega_2\right]\right)d\kappa\Big].$$

Remark 2.1. *If we set* $\omega_1 = \mathrm{d}$ *and* $\omega_2 = \mathrm{D}$ *in Corollary 2.1, we obtain Lemma 1 in [27].*

Theorem 2.1. *Taking all the conditions as defined in Lemma 2.1 and assumption* $\mathcal{P}_1$ *into consideration and let* $|\Phi'|$ *be* s*-convex function on* $[\mathrm{d}, \mathrm{D}]$, *for* $s > 0$*. Then, for all* $\delta > 0$, *the following Simpson-Mercer type inequality for A-B fractional integral holds true for* $\kappa \in [0, 1]$*:*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\\
&+\frac{2^{\delta}(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\\
&-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\Big\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\\
&+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\Big\}\Bigg|\\
&\leq\frac{\omega_2-\omega_1}{2}\Bigg[2\left\{\left(\frac{2}{3}\right)^{1+\frac{1}{\delta}}\left(\frac{\delta}{\delta+1}\right)+\frac{1-2\delta}{6(\delta+1)}\right\}\Big[|\Phi'(\mathrm{d})|+|\Phi'(\mathrm{D})|\Big]\\
&-\frac{1}{2^{s}}\Big[|\Phi'(\omega_1)|+|\Phi'(\omega_2)|\Big]\Big\{\mathcal{J}_3(\delta)+\mathcal{J}_4(\delta)\Big\}\Bigg],
\end{aligned}\tag{2.3}$$

where

$$\mathcal{J}_3(\delta)=\int_0^{\left(\frac{2}{3}\right)^{\frac{1}{\delta}}}\left(\frac{1}{3}-\frac{\kappa^{\delta}}{2}\right)[(1+\kappa)^{s}+(1-\kappa)^{s}]d\kappa,$$

and

$$\mathcal{J}_4(\delta)=\int_{\left(\frac{2}{3}\right)^{\frac{1}{\delta}}}^{1}\left(\frac{\kappa^{\delta}}{2}-\frac{1}{3}\right)[(1+\kappa)^{s}+(1-\kappa)^{s}]d\kappa.$$

Proof. Employing Lemma 2.1 with the J-M inequality and the s-convexity of $|\Phi'|$, we have

$$\begin{aligned}
&\Bigg|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\\
&+\frac{2^{\delta}(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\\
&-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\Big\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\\
&+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\Big\}\Bigg|\\
&\leq\frac{\omega_2-\omega_1}{2}\Bigg[\int_0^1\left|\frac{\kappa^{\delta}}{2}-\frac{1}{3}\right|\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1+\kappa}{2}\omega_1+\frac{1-\kappa}{2}\omega_2\right]\right)\right|d\kappa\\
&+\int_0^1\left|\frac{1}{3}-\frac{\kappa^{\delta}}{2}\right|\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1-\kappa}{2}\omega_1+\frac{1+\kappa}{2}\omega_2\right]\right)\right|d\kappa\Bigg]\\
&\leq\frac{\omega_2-\omega_1}{2}\Bigg[\int_0^1\left|\frac{\kappa^{\delta}}{2}-\frac{1}{3}\right|\left\{|\Phi'(\mathrm{d})|+|\Phi'(\mathrm{D})|-\left[\left(\frac{1+\kappa}{2}\right)^{s}|\Phi'(\omega_1)|+\left(\frac{1-\kappa}{2}\right)^{s}|\Phi'(\omega_2)|\right\}d\kappa\\
&+\int_0^1\left|\frac{1}{3}-\frac{\kappa^{\delta}}{2}\right|\left\{|\Phi'(\mathrm{d})|+|\Phi'(\mathrm{D})|-\left[\left(\frac{1-\kappa}{2}\right)^{s}|\Phi'(\omega_1)|+\left(\frac{1+\kappa}{2}\right)^{s}|\Phi'(\omega_2)|\right\}d\kappa\Bigg]
\end{aligned}\tag{2.4}$$

$$\leq \frac{\omega_2-\omega_1}{2}\Big[2\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|\Big\{|\Phi'(\mathrm{d})|+|\Phi'(\mathrm{D})|\Big\}d\kappa$$

$$-\frac{1}{2^s}\Big\{|\Phi'(\omega_1)|+|\Phi'(\omega_2)|\Big\}\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|\Big[(1+\kappa)^s+(1-\kappa)^s\Big]d\kappa$$

$$=\frac{\omega_2-\omega_1}{2}\Big[2\Big\{\int_0^{\left(\frac{2}{3}\right)^{\frac{1}{\delta}}}\left(\frac{1}{3}-\frac{\kappa^\delta}{2}\right)d\kappa+\int_{\left(\frac{2}{3}\right)^{\frac{1}{\delta}}}^1\left(\frac{\kappa^\delta}{2}-\frac{1}{3}\right)d\kappa\Big\}\Big\{|\Phi'(\mathrm{d})|+|\Phi'(\mathrm{D})|\Big\}$$

$$-\frac{1}{2^s}\Big\{|\Phi'(\omega_1)|+|\Phi'(\omega_2)|\Big\}\Big\{\int_0^{\left(\frac{2}{3}\right)^{\frac{1}{\delta}}}\left(\frac{1}{3}-\frac{\kappa^\delta}{2}\right)\Big[(1+\kappa)^s+(1-\kappa)^s\Big]d\kappa$$

$$+\int_{\left(\frac{2}{3}\right)^{\frac{1}{\delta}}}^1\left(\frac{\kappa^\delta}{2}-\frac{1}{3}\right)\Big[(1+\kappa)^s+(1-\kappa)^s\Big]d\kappa\Big\}\Big]$$

$$=\frac{\omega_2-\omega_1}{2}\Big[2\Big\{\left(\frac{2}{3}\right)^{1+\frac{1}{\delta}}\left(\frac{\delta}{\delta+1}\right)+\frac{1-2\delta}{6(\delta+1)}\Big\}\Big[|\Phi'(\mathrm{d})|+|\Phi'(\mathrm{D})|\Big]$$

$$-\frac{1}{2^s}\Big[|\Phi'(\omega_1)|+|\Phi'(\omega_2)|\Big]\Big\{\mathcal{J}_3(\delta)+\mathcal{J}_4(\delta)\Big\}\Big],$$

which completes the proof. □

Remark 2.2. *If we consider $\omega_1=\mathrm{d}$ and $\omega_2=\mathrm{D}$ and $\delta=1$ in Theorem 2.1, we obtain Theorem 7 in [27].*

Corollary 2.2. *Taking the same assumptions as defined in Theorem 2.1 with $s=1$ into consideration, we get the following Simpson-Mercer type inequality*

$$\Big|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \tag{2.5}$$

$$+\frac{2^\delta(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)$$

$$-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Big\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)$$

$$+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\Big\}\Big|$$

$$\leq\frac{\omega_2-\omega_1}{2}\Big[2\Big\{\left(\frac{2}{3}\right)^{1+\frac{1}{\delta}}\left(\frac{\delta}{\delta+1}\right)+\frac{1-2\delta}{6(\delta+1)}\Big\}\Big[|\Phi'(\mathrm{d})|+|\Phi'(\mathrm{D})|\Big]$$

$$-\frac{1}{2}\Big[|\Phi'(\omega_1)|+|\Phi'(\omega_2)|\Big]\Big\{\mathcal{J}_3(\delta)+\mathcal{J}_4(\delta)\Big\}\Big].$$

Corollary 2.3. *If we consider $\delta=1$ in Theorem 2.1, then we have the following Simpson-Mercer type inequality*

$$\left|\frac{1}{6}\Big\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+4\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\Big\}-\frac{1}{\omega_2-\omega_1}\int_{\mathrm{d}+\mathrm{D}-\omega_2}^{\mathrm{d}+\mathrm{D}-\omega_1}\Phi(x)\,dx\right|$$

$$\leq\frac{\omega_2-\omega_1}{2}\Big[\frac{5}{18}\left(|\Phi'(\mathrm{d})|+|\Phi'(\mathrm{D})|\right)$$

$$
- \frac{1}{2^s}\left(|\Phi'(\omega_1)| + |\Phi'(\omega_2)|\right)\left\{\frac{1}{6(s+1)}\left(\frac{5^{s+2}-1}{3^{s+1}}-6\right) - \frac{1}{2(s+2)}\left(\frac{5^{s+2}+1}{3^{s+2}}-2\right) + \frac{5^{s+2}+1}{2(s+1)(s+2)3^{s+2}} - \frac{2^s(2s+7)}{3(s+1)(s+2)}\right\}\Bigg].
$$

In the results to follow hereon, we will give the new upper bounds for the right hand side of the Simpson-Mercer type inequality via s-convexity involving A-B fractional integrals operator.

Theorem 2.2. *Taking all the conditions as defined in Lemma 2.1 and assumption $\mathcal{P}_1$ into consideration, if $|\Phi'|^q$ is s-convex function on $[\mathrm{d},\mathrm{D}]$, for $s>0$, then for all $\delta>0$, the following inequality for A-B fractional integral holds true for $\kappa\in[0,1]$:*

$$
\begin{aligned}
&\Bigg|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \\
&+\frac{2^\delta(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \\
&-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\left\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right. \\
&\left.+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right\}\Bigg| \\
&\le \frac{\omega_2-\omega_1}{2}\left(\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\left\{\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{|\Phi'(\omega_1)|^q+\left|\Phi'\left(\frac{\omega_1+\omega_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right. \\
&\left.+\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{|\Phi'(\omega_2)|^q+\left|\Phi'\left(\frac{\omega_1+\omega_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right\},
\end{aligned}
\tag{2.6}
$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $q>1$.

Proof. Employing Lemma 2.1 and the Hölder's inequality, we have

$$
\begin{aligned}
&\Bigg|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \\
&+\frac{2^\delta(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \\
&-\frac{2^{\delta-1}M(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\left\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right. \\
&\left.+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right\}\Bigg| \\
&\le \frac{\omega_2-\omega_1}{2}\Bigg[\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1+\kappa}{2}\omega_1+\frac{1-\kappa}{2}\omega_2\right]\right)\right|d\kappa \\
&+\int_0^1\left|\frac{1}{3}-\frac{\kappa^\delta}{2}\right|\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1-\kappa}{2}\omega_1+\frac{1+\kappa}{2}\omega_2\right]\right)\right|d\kappa\Bigg]
\end{aligned}
$$

$$\leq \frac{\omega_2-\omega_1}{2}\Bigg[\left(\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1+\kappa}{2}\omega_1+\frac{1-\kappa}{2}\omega_2\right]\right)\right|^q d\kappa\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1-\kappa}{2}\omega_1+\frac{1+\kappa}{2}\omega_2\right]\right)\right|^q d\kappa\right)^{\frac{1}{q}}\Bigg]. \tag{2.7}$$

Since $|\Phi'|^q$ is s-convex on $[\mathrm{d},\mathrm{D}]$, with the J-M inequality, we obtain

$$\int_0^1\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1+\kappa}{2}\omega_1+\frac{1-\kappa}{2}\omega_2\right]\right)\right|^q d\kappa \tag{2.8}$$

$$\leq |\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{|\Phi'(\omega_1)|^q+\left|\Phi'\left(\frac{\omega_1+\omega_2}{2}\right)\right|^q}{s+1}\right],$$

and

$$\int_0^1\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1-\kappa}{2}\omega_1+\frac{1+\kappa}{2}\omega_2\right]\right)\right|^q d\kappa \tag{2.9}$$

$$\leq |\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{|\Phi'(\omega_2)|^q+\left|\Phi'\left(\frac{\omega_1+\omega_2}{2}\right)\right|^q}{s+1}\right].$$

By using the Eqs (2.8) and (2.9) with (2.7), we get the (2.6). This concludes the proof. □

Remark 2.3. *If we set* $\omega_1=\mathrm{d}$, $\omega_2=\mathrm{D}$ *and* $\delta=1$ *in Theorem 2.2, we obtain Theorem 8 in [27].*

Corollary 2.4. *Under the same assumptions as defined in Theorem 2.2 with* $s=1$, *we get the following A-B fractional inequality of Simpson-Mercer type:*

$$\Bigg|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \tag{2.10}$$

$$+\frac{2^\delta(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)$$

$$-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Bigg\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)$$

$$+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\Bigg\}\Bigg|$$

$$\leq \frac{\omega_2-\omega_1}{2}\left(\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\Bigg\{\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{|\Phi'(\omega_1)|^q+\left|\Phi'\left(\frac{\omega_1+\omega_2}{2}\right)\right|^q}{2}\right]\right)^{\frac{1}{q}}$$

$$+\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{|\Phi'(\omega_2)|^q+\left|\Phi'\left(\frac{\omega_1+\omega_2}{2}\right)\right|^q}{2}\right]\right)^{\frac{1}{q}}\Bigg\}.$$

Corollary 2.5. *Under the same assumptions as defined in Theorem 2.2 with* $\delta=1$, *we get the following Simpson-Mercer type inequality*

$$\left|\frac{1}{6}\left\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+4\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\right\}-\frac{1}{\omega_2-\omega_1}\int_{\mathrm{d}+\mathrm{D}-\omega_2}^{\mathrm{d}+\mathrm{D}-\omega_1}\Phi(x)\,dx\right|$$
$$\leq \frac{\omega_2-\omega_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{|\Phi'(\omega_1)|^q+\left|\Phi'\left(\frac{\omega_1+\omega_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{|\Phi'(\omega_2)|^q+\left|\Phi'\left(\frac{\omega_1+\omega_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right\}.$$

Theorem 2.3. *Taking all the conditions as defined in Lemma 2.1 and assumption $\mathcal{P}_1$ into consideration, if $|\Phi'|^q$ is s-convex function on $[\mathrm{d},\mathrm{D}]$, for $s>0$ and $q>1$, then for all $\delta>0$, the following inequality for A-B fractional integral holds true for $\kappa\in[0,1]$:*

$$\left|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right.$$
$$+\frac{2^\delta(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)$$
$$-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\left\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right\}\right|$$
$$\leq \frac{\omega_2-\omega_1}{2}\left(\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}$$
$$\times\left\{\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{2^{s+1}-1}{(s+1)2^s}|\Phi'(\omega_1)|^q+\frac{1}{(s+1)2^s}|\Phi'(\omega_2)|^q\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{1}{(s+1)2^s}|\Phi'(\omega_1)|^q+\frac{2^{s+1}-1}{(s+1)2^s}|\Phi'(\omega_2)|^q\right]\right)^{\frac{1}{q}}\right\}, \tag{2.11}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $q>1$.

Proof. Employing Lemma 2.1 and the Hölder inequality, we have

$$\left|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right.$$
$$+\frac{2^\delta(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)$$
$$-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\left\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right\}\right|$$
$$\leq \frac{\omega_2-\omega_1}{2}\left[\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1+\kappa}{2}\omega_1+\frac{1-\kappa}{2}\omega_2\right]\right)\right|d\kappa\right.$$

$$+\int_0^1\left|\frac{1}{3}-\frac{\kappa^\delta}{2}\right|\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1-\kappa}{2}\omega_1+\frac{1+\kappa}{2}\omega_2\right]\right)\right|d\kappa\Big]$$

$$\leq \frac{\omega_2-\omega_1}{2}\Big[\left(\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1+\kappa}{2}\omega_1+\frac{1-\kappa}{2}\omega_2\right]\right)\right|^q d\kappa\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1-\kappa}{2}\omega_1+\frac{1+\kappa}{2}\omega_2\right]\right)\right|^q d\kappa\right)^{\frac{1}{q}}\Big]. \tag{2.12}$$

Since $|\Phi'|^q$ is s-convex on $[\mathrm{d},\mathrm{D}]$, with the J-M inequality, we obtain

$$\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1+\kappa}{2}\omega_1+\frac{1-\kappa}{2}\omega_2\right]\right)\right|^q \tag{2.13}$$

$$\leq |\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\left(\frac{1+\kappa}{2}\right)^s|\Phi'(\omega_1)|^q+\left(\frac{1-\kappa}{2}\right)^s|\Phi'(\omega_2)|^q\right],$$

and

$$\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1-\kappa}{2}\omega_1+\frac{1+\kappa}{2}\omega_2\right]\right)\right|^q \tag{2.14}$$

$$\leq |\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\left(\frac{1-\kappa}{2}\right)^s|\Phi'(\omega_1)|^q+\left(\frac{1+\kappa}{2}\right)^s|\Phi'(\omega_2)|^q\right].$$

Upon combining the Eqs (2.13) and (2.14) in (2.12) and using the tool of calculus, we get (2.11). This completes the proof. □

Remark 2.4. *If we set* $\omega_1=\mathrm{d}$, $\omega_2=\mathrm{D}$ *and* $\delta=1$ *in Theorem 2.3, we obtain Theorem 9 in [27].*

Corollary 2.6. *Under the same assumptions as defined in Theorem 2.3 with* $s=1$, *we get the following A-B fractional inequality of Simpson-Mercer type:*

$$\Big|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \tag{2.15}$$

$$+\frac{2^\delta(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)$$

$$-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Big\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)$$

$$+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\Big\}\Big|$$

$$\leq \frac{\omega_2-\omega_1}{2}\left(\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|^p d\kappa\right)^{\frac{1}{p}}\Big\{\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{3|\Phi'(\omega_1)|^q+|\Phi'(\omega_2)|^q}{4}\right]\right)^{\frac{1}{q}}$$

$$+\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{|\Phi'(\omega_1)|^q+3|\Phi'(\omega_2)|^q}{4}\right]\right)^{\frac{1}{q}}\Big\}.$$

Corollary 2.7. *Under the same assumptions as defined in Theorem 2.3 with $\delta = 1$, we get the following Simpson-Mercer type inequality:*

$$\left|\frac{1}{6}\left\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+4\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\right\}-\frac{1}{\omega_2-\omega_1}\int_{\mathrm{d}+\mathrm{D}-\omega_2}^{\mathrm{d}+\mathrm{D}-\omega_1}\Phi(x)\,dx\right|$$
$$\leq \frac{\omega_2-\omega_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{2^{s+1}-1}{(s+1)2^s}|\Phi'(\omega_1)|^q+\frac{1}{(s+1)2^s}|\Phi'(\omega_2)|^q\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\frac{1}{(s+1)2^s}|\Phi'(\omega_1)|^q+\frac{2^{s+1}-1}{(s+1)2^s}|\Phi'(\omega_2)|^q\right]\right)^{\frac{1}{q}}\right\}.$$

Theorem 2.4. *Taking all the conditions as defined in Lemma 2.1 and assumption $\mathcal{P}_1$ into consideration, if $|\Phi'|^q$ is s-convex function on $[\mathrm{d},\mathrm{D}]$, for $s>0$, $q\geq 1$, then for all $\delta>0$, the following Simpson-Mercer type inequality for A-B fractional integral holds true for $\kappa\in[0,1]$:*

$$\left|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right.$$
$$+\frac{2^\delta(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \tag{2.16}$$
$$-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\left\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right.$$
$$\left.\left.+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^\delta_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right\}\right|$$
$$\leq\frac{\omega_2-\omega_1}{2}\mathcal{J}_5(\delta)\times\left\{\mathcal{J}_6(\delta)^{\frac{1}{q}}+\mathcal{J}_7(\delta)^{\frac{1}{q}}\right\}, \tag{2.17}$$

where

$$\mathcal{J}_5(\delta)=\left(\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|d\kappa\right)^{1-\frac{1}{q}}=\left(\int_0^1\left|\frac{1}{3}-\frac{\kappa^\delta}{2}\right|d\kappa\right)^{1-\frac{1}{q}},$$
$$\mathcal{J}_6(\delta)=\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|\left\{|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\left(\frac{1+\kappa}{2}\right)^s|\Phi'(\omega_1)|^q+\left(\frac{1-\kappa}{2}\right)^s|\Phi'(\omega_2)|^q\right]\right\}d\kappa,$$

and

$$\mathcal{J}_7(\delta)=\int_0^1\left|\frac{\kappa^\delta}{2}-\frac{1}{3}\right|\left\{|\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\left(\frac{1-\kappa}{2}\right)^s|\Phi'(\omega_1)|^q+\left(\frac{1+\kappa}{2}\right)^s|\Phi'(\omega_2)|^q\right]\right\}d\kappa.$$

Proof. Employing Lemma 2.1 and well-known Power mean inequality, we have

$$\left|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right.$$
$$+\frac{2^\delta(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^\delta}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) \tag{2.18}$$

$$
\begin{aligned}
&- \frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\left\{ {}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right.\\
&\left.\left. + {}^{AB}_{(\mathrm{D}+\mathrm{D}-\omega_2)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right\}\right|\\
&\le \frac{\omega_2-\omega_1}{2}\Bigg[\int_0^1\left|\frac{\kappa^{\delta}}{2}-\frac{1}{3}\right|\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1+\kappa}{2}\omega_1+\frac{1-\kappa}{2}\omega_2\right]\right)\right|d\kappa\\
&+\int_0^1\left|\frac{1}{3}-\frac{\kappa^{\delta}}{2}\right|\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1-\kappa}{2}\omega_1+\frac{1+\kappa}{2}\omega_2\right]\right)\right|d\kappa\Bigg]\\
&\le \frac{\omega_2-\omega_1}{2}\Bigg[\left(\int_0^1\left|\frac{\kappa^{\delta}}{2}-\frac{1}{3}\right|d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{\kappa^{\delta}}{2}-\frac{1}{3}\right|\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1+\kappa}{2}\omega_1+\frac{1-\kappa}{2}\omega_2\right]\right)\right|^{q}d\kappa\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\frac{\kappa^{\delta}}{2}-\frac{1}{3}\right|d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{\kappa^{\delta}}{2}-\frac{1}{3}\right|\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1-\kappa}{2}\omega_1+\frac{1+\kappa}{2}\omega_2\right]\right)\right|^{q}d\kappa\right)^{\frac{1}{q}}\Bigg].
\end{aligned} \tag{2.19}
$$

Since $|\Phi'|^q$ is s-convex on $[\mathrm{d},\mathrm{D}]$, with the J-M inequality, we obtain

$$
\begin{aligned}
&\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1+\kappa}{2}\omega_1+\frac{1-\kappa}{2}\omega_2\right]\right)\right|^{q}\\
&\le |\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\left(\frac{1+\kappa}{2}\right)^{s}|\Phi'(\omega_1)|^q+\left(\frac{1-\kappa}{2}\right)^{s}|\Phi'(\omega_2)|^q\right],
\end{aligned} \tag{2.20}
$$

and

$$
\begin{aligned}
&\left|\Phi'\left(\mathrm{d}+\mathrm{D}-\left[\frac{1-\kappa}{2}\omega_1+\frac{1+\kappa}{2}\omega_2\right]\right)\right|^{q}\\
&\le |\Phi'(\mathrm{d})|^q+|\Phi'(\mathrm{D})|^q-\left[\left(\frac{1-\kappa}{2}\right)^{s}|\Phi'(\omega_1)|^q+\left(\frac{1+\kappa}{2}\right)^{s}|\Phi'(\omega_2)|^q\right].
\end{aligned} \tag{2.21}
$$

By using the Eqs (2.20) and (2.21) in (2.19), we get

$$
\begin{aligned}
&\left|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}\right.\\
&+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)+\frac{2^{\delta}(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\\
&-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\left\{ {}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right.\\
&\left.\left.+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right\}\right|\\
&\le\frac{\omega_2-\omega_1}{2}\mathcal{J}_5(\delta)\times\left\{\mathcal{J}_6(\delta)^{\frac{1}{q}}+\mathcal{J}_7(\delta)^{\frac{1}{q}}\right\},
\end{aligned}
$$

which completes the proof. □

Remark 2.5. *If we consider $\omega_1=\mathrm{d}$, $\omega_2=\mathrm{D}$ and $\delta=1$ in Theorem 2.4, we obtain Theorem 10 in [27].*

Corollary 2.8. *Under the same assumptions as defined in Theorem 2.4 with $s = 1$, we get the following A-B fractional inequality of Simpson-Mercer type:*

$$\begin{aligned}&\Bigg|\frac{1}{6}\{\Phi(\mathrm{d}+\mathrm{D}-\omega_1)+\Phi(\mathrm{d}+\mathrm{D}-\omega_2)\}\\&+\frac{2}{3}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)+\frac{2^{\delta}(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\\&-\frac{2^{\delta-1}\mathcal{M}(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\left\{{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_1)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right.\\&\left.+{}^{AB}_{(\mathrm{d}+\mathrm{D}-\omega_2)}\mathcal{I}^{\delta}_{\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right)\right\}\Bigg|\\&\le\frac{\omega_2-\omega_1}{2}\left(\mathcal{J}_5(\delta)\times\left\{\mathcal{J}_6(\delta)^{\frac{1}{q}}+\mathcal{J}_7(\delta)^{\frac{1}{q}}\right\}\right).\end{aligned}$$

3. Applications

3.1. Modified Bessel functions

Example 3.1. *Let the function $\mathcal{J}_{\varrho}:\mathfrak{R}\to[1,\infty)$ be defined [54] as*

$$\mathcal{J}_{\varrho}(u)=2^{\varrho}\Gamma(\varrho+1)u^{-\varrho}\mathrm{I}_{\varrho}(u),\quad u\in\mathfrak{R}.$$

Here, we consider the modified Bessel function of first kind given in

$$\mathcal{J}_{\varrho}(u)=\sum_{n=0}^{\infty}\frac{\left(\frac{u}{2}\right)^{\varrho+2n}}{n!\Gamma(\varrho+n+1)}.$$

The first and second order derivative are given as

$$\mathcal{J}'_{\varrho}(u)=\frac{u}{2(\varrho+1)}\mathcal{J}_{\varrho+1}(u).$$

$$\mathcal{J}''_{\varrho}(u)=\frac{1}{4(\varrho+1)}\left[\frac{u^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(u)+2\mathcal{J}_{\varrho+1}(u)\right].$$

If we use, $\Phi(u)=\mathcal{J}'_{\varrho}(u)$ and the above functions in Corollary 2.3, we have

$$\begin{aligned}&\left|\frac{1}{6}\left\{\frac{(d+D-\omega_1)}{2(\varrho+1)}\mathcal{J}_{\varrho+1}(d+D-\omega_1)+4\left(\frac{d+D-\frac{\omega_1+\omega_2}{2}}{2(\varrho+1)}\mathcal{J}_{\varrho+1}\left(d+D-\frac{\omega_1+\omega_2}{2}\right)\right)\right.\right.\\&\left.\left.+\frac{d+D-\omega_2}{2(\varrho+1)}\mathcal{J}_{\varrho+1}(d+D-\omega_2)\right\}-\frac{1}{\omega_2-\omega_1}\left[\mathcal{J}_{\varrho}(d+D-\omega_1)-\mathcal{J}_{\varrho}(d+D-\omega_2)\right]\right|\\&\le\frac{\omega_2-\omega_1}{2}\left[\frac{5}{18}\left\{\left|\frac{1}{4(\varrho+1)}\left[\frac{d^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(d)+2\mathcal{J}_{\varrho+1}(d)\right]\right|+\left|\frac{1}{4(\varrho+1)}\left[\frac{D^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(D)+2\mathcal{J}_{\varrho+1}(D)\right]\right|\right\}\right.\\&-\frac{1}{2^s}\left[\left|\frac{1}{4(\varrho+1)}\left[\frac{\omega_1^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(\omega_1)+2\mathcal{J}_{\varrho+1}(\omega_1)\right]\right|+\left|\frac{1}{4(\varrho+1)}\left[\frac{\omega_2^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(\omega_2)+2\mathcal{J}_{\varrho+1}(\omega_2)\right]\right|\right]\\&\left.\times\left(\frac{1}{6(s+1)}\left(\frac{5^{s+2}-1}{3^{s+1}}-6\right)-\frac{1}{2(s+2)}\left(\frac{5^{s+2}+1}{3^{s+2}}-2\right)+\frac{5^{s+2}+1}{2(s+1)(s+2)3^{s+2}}-\frac{2^s(2s+7)}{3(s+1)(s+2)}\right)\right].\end{aligned}$$

Example 3.2. *If we use, $\Phi(u) = \mathcal{J}'_{\varrho}(u)$ and the above functions in Corollary 2.5, we have*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Bigg\{\frac{(d+D-\omega_1)}{2(\varrho+1)}\mathcal{J}_{\varrho+1}(d+D-\omega_1)+4\left(\frac{d+D-\frac{\omega_1+\omega_2}{2}}{2(\varrho+1)}\mathcal{J}_{\varrho+1}\left(d+D-\frac{\omega_1+\omega_2}{2}\right)\right)\\
&+\frac{d+D-\omega_2}{2(\varrho+1)}\mathcal{J}_{\varrho+1}(d+D-\omega_2)\Bigg\}-\frac{1}{\omega_2-\omega_1}\left[\mathcal{J}_{\varrho}(d+D-\omega_1)-\mathcal{J}_{\varrho}(d+D-\omega_2)\right]\Bigg|\\
&\leq\frac{\omega_2-\omega_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\Bigg\{\left|\frac{1}{4(\varrho+1)}\left[\frac{d^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(d)+2\mathcal{J}_{\varrho+1}(d)\right]\right|^q\\
&+\left|\frac{1}{4(\varrho+1)}\left[\frac{D^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(D)+2\mathcal{J}_{\varrho+1}(D)\right]\right|^q\\
&-\left[\frac{\left|\frac{1}{4(\varrho+1)}\left[\frac{\omega_1^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(\omega_1)+2\mathcal{J}_{\varrho+1}(\omega_1)\right]\right|^q+\left|\frac{1}{4(\varrho+1)}\left[\frac{(\frac{\omega_1+\omega_2}{2})^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}\left(\frac{\omega_1+\omega_2}{2}\right)+2\mathcal{J}_{\varrho+1}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|^q}{s+1}\right]\Bigg\}^{\frac{1}{q}}\\
&+\Bigg\{\left|\frac{1}{4(\varrho+1)}\left[\frac{d^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(d)+2\mathcal{J}_{\varrho+1}(d)\right]\right|^q+\left|\frac{1}{4(\varrho+1)}\left[\frac{D^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(D)+2\mathcal{J}_{\varrho+1}(D)\right]\right|^q\\
&-\left[\frac{\left|\frac{1}{4(\varrho+1)}\left[\frac{\omega_2^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}(\omega_2)+2\mathcal{J}_{\varrho+1}(\omega_2)\right]\right|^q+\left|\frac{1}{4(\varrho+1)}\left[\frac{(\frac{\omega_1+\omega_2}{2})^2}{(\varrho+1)}\mathcal{J}_{\varrho+2}\left(\frac{\omega_1+\omega_2}{2}\right)+2\mathcal{J}_{\varrho+1}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|^q}{s+1}\right]\Bigg\}^{\frac{1}{q}}.
\end{aligned}$$

3.2. *Special means*

Consider the following two special means for $0 < \mathrm{d} < \mathrm{D}$.
The arithmetic mean:

$$A(\mathrm{d},\mathrm{D}) = \frac{\mathrm{d}+\mathrm{D}}{2}.$$

The generalized logarithmic-mean:

$$L_k(\mathrm{d},\mathrm{D}) = \left[\frac{\mathrm{D}^{k+1}-\mathrm{d}^{k+1}}{(k+1)(\mathrm{D}-\mathrm{d})}\right]^{\frac{1}{k}};\quad k\in\mathfrak{R}\setminus\{-1,0\}.$$

Proposition 3.1. *Suppose that* $\mathrm{d},\mathrm{D}\in\mathfrak{R}$, $s\in(0,1]$, $0<\mathrm{d}<\mathrm{D}$, $0\notin[\mathrm{d},\mathrm{D}]$. *Then,*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Big\{2A\left((\mathrm{d}+\mathrm{D}-\omega_1)^s,(\mathrm{d}+\mathrm{D}-\omega_2)^s\right)\Big\}\\
&+4\left(2A(\mathrm{d},\mathrm{D})-A(\omega_1,\omega_2)\right)^s-L_s^s(\mathrm{d}+\mathrm{D}-\omega_1,\mathrm{d}+\mathrm{D}-\omega_2)\Bigg|\\
&\leq\frac{s(\omega_2-\omega_1)}{2}\Bigg[\left(\frac{5}{9}A\left(\mathrm{d}^{s-1},\mathrm{D}^{s-1}\right)-\frac{1}{2^{s-1}}A\left(\omega_1^{s-1},\omega_2^{s-1}\right)\right)\Big(\mathcal{J}_3(1)+\mathcal{J}_4(1)\Big)\Bigg].
\end{aligned}$$

Proof. The proof follows an immediate consequences from Corollary 2.3 by considering $\Phi(u) = u^s$. □

Proposition 3.2. *Suppose that* $\mathrm{d}, \mathrm{D} \in \Re$, $s \in (0,1]$, $0 < \mathrm{d} < \mathrm{D}$, $0 \notin [\mathrm{d}, \mathrm{D}]$. *Then for all* $q > 1$, *the following inequality holds true:*

$$\Bigg|\frac{1}{6}\Big\{2A\left((\mathrm{d}+\mathrm{D}-\omega_1)^{-s}, (\mathrm{d}+\mathrm{D}-\omega_2)^{-s}\right)\Big\}$$
$$+ 4\left(2A(\mathrm{d},\mathrm{D}) - A(\omega_1,\omega_2)\right)^{-s} - L_{-s}^{-s}(\mathrm{d}+\mathrm{D}-\omega_1, \mathrm{d}+\mathrm{D}-\omega_2)\Bigg|$$
$$\leq \frac{s(\omega_2-\omega_1)}{12}\left(\frac{1+2^{p+1}}{(3p+3)}\right)^{\frac{1}{p}}\left\{\left(2A\left(\mathrm{d}^{-(s+1)q}, \mathrm{D}^{-(s+1)q}\right) - \left[\frac{\omega_1^{-(s+1)q} + A(\omega_1,\omega_2)^{-(s+1)q}}{s+1}\right]\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(2A\left(\mathrm{d}^{-(s+1)q}, \mathrm{D}^{-(s+1)q}\right) - \left[\frac{\omega_2^{-(s+1)q} + A(\omega_1,\omega_2)^{-(s+1)q}}{s+1}\right]\right)^{\frac{1}{q}}\right\}.$$

Proof. The proof follows an immediate consequence from Corollary 2.5 by considering $\Phi(u) = u^{-s}$. □

Proposition 3.3. *Suppose that* $\mathrm{d}, \mathrm{D} \in \Re$, $s \in (0,1]$, $0 < \mathrm{d} < \mathrm{D}$, $0 \notin [\mathrm{d}, \mathrm{D}]$. *Then,*

$$\Bigg|\frac{1}{6}\Big\{2A\left((\mathrm{d}+\mathrm{D}-\omega_1)^{s}, (\mathrm{d}+\mathrm{D}-\omega_2)^{s}\right)\Big\} + 4\left(2A(\mathrm{d},\mathrm{D}) - A(\omega_1,\omega_2)\right)^{s} - L_{s}^{s}(\mathrm{d}+\mathrm{D}-\omega_1, \mathrm{d}+\mathrm{D}-\omega_2)\Bigg|$$
$$\leq \frac{s(\omega_2-\omega_1)}{12}\left(\frac{1+2^{p+1}}{(3p+3)}\right)^{\frac{1}{p}}\left\{\left(2A\left(\mathrm{d}^{-(s+1)q}, \mathrm{D}^{-(s+1)q}\right) - \left[\frac{(2^{s+1}-1)\omega_1^{(s-1)q} + \omega_2^{(s-1)q}}{2^s(s+1)}\right]\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(2A\left(\mathrm{d}^{-(s+1)q}, \mathrm{D}^{-(s+1)q}\right) - \left[\frac{(2^{s+1}-1)\omega_2^{(s-1)q} + \omega_1^{(s-1)q}}{(s+1)2^s}\right]\right)^{\frac{1}{q}}\right\}.$$

Proof. The proof follows an immediate consequences from Corollary 2.7 by considering $\Phi(u) = u^{s}$. □

3.3. *Q-digamma function*

The q-digamma(psi) function ϱ_ρ, is the ρ-analogue of the digamma function ϱ (see [54, 55]) given as:

$$\varrho_\rho = -\ln(1-\rho) + \ln\rho \sum_{k=0}^{\infty} \frac{\rho^{k+\gamma}}{1-\rho^{k+\gamma}}$$
$$= -\ln(1-\rho) + \ln\rho \sum_{k=0}^{\infty} \frac{\rho^{k\gamma}}{1-\rho^{k\gamma}}.$$

For $\rho > 1$ and $\gamma > 0$, ρ-digamma function ϱ_ρ can be given as:

$$\varrho_\rho = -\ln(\rho-1) + \ln\rho\left[\gamma - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{\rho^{-(k+\gamma)}}{1-\rho^{-(k+\gamma)}}\right]$$
$$= -\ln(\rho-1) + \ln\rho\left[\gamma - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{\rho^{-k\gamma}}{1-\rho^{-k\gamma}}\right].$$

Proposition 3.4. *Suppose* d, D, ρ *be real numbers such that* $0 < \mathrm{d} < \mathrm{D}$, $0 < \rho < 1$, $s \in (0,1]$ *and* $\frac{1}{p} + \frac{1}{q} = 1, q > 1$. *Then the following inequality holds true:*

$$\left|\frac{1}{6}\left\{\varrho_\rho(\mathrm{d}+\mathrm{D}-\omega_1) + \varrho_\rho(\mathrm{d}+\mathrm{D}-\omega_2)\right\} + \frac{2}{3}\varrho_\rho\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) - \frac{1}{\omega_2-\omega_1}\int_{\mathrm{d}+\mathrm{D}-\omega_2}^{\mathrm{d}+\mathrm{D}-\omega_1}\varrho_\rho(\gamma)\,d\gamma\right|$$
$$\leq \frac{\omega_2-\omega_1}{12}\left(\frac{1+2^{p+1}}{(3p+3)}\right)^{\frac{1}{p}}\left\{\left(\left|\varrho'_\rho(\mathrm{d})\right|^q + \left|\varrho'_\rho(\mathrm{D})\right|^q - \left[\frac{2^{s+1}-1}{(s+1)2^s}\left|\varrho'_\rho(\omega_1)\right|^q + \frac{1}{(s+1)2^s}\left|\varrho'_\rho(\omega_2)\right|^q\right]\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\left|\varrho'_\rho(\mathrm{d})\right|^q + \left|\varrho'_\rho(\mathrm{D})\right|^q - \left[\frac{1}{(s+1)2^s}\left|\varrho'_\rho(\omega_1)\right|^q + \frac{2^{s+1}-1}{(s+1)2^s}\left|\varrho'_\rho(\omega_2)\right|^q\right]\right)^{\frac{1}{q}}\right\}.$$

Proof. By employing the definition of ρ-digamma function $\varrho_\rho(\gamma)$, it is easy to notice that ρ -digamma function $\Phi(\gamma) = \varrho_\rho(\gamma)$ is completely monotonic on $(0,\infty)$. This ensures that the function $\Phi'(\gamma) = \varrho'_\rho(\gamma)$ is s-convex on $(0,\infty)$ for each $\rho \in (0,1)$. Now by applying Corollary 2.7, we get the desired result. □

Proposition 3.5. *Suppose* d, D, q, ρ *be real numbers such that* $0 < \mathrm{d} < \mathrm{D}$, $\frac{1}{p} + \frac{1}{q} = 1, q > 1$ *and* $0 < \rho < 1$, $s \in (0,1]$. *Then the following inequality holds true:*

$$\left|\frac{1}{6}\left\{\varrho_\rho(\mathrm{d}+\mathrm{D}-\omega_1) + \varrho_\rho(\mathrm{d}+\mathrm{D}-\omega_2)\right\} + \frac{2}{3}\varrho_\rho\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) - \frac{1}{\omega_2-\omega_1}\int_{\mathrm{d}+\mathrm{D}-\omega_2}^{\mathrm{d}+\mathrm{D}-\omega_1}\varrho_\rho(\gamma)\,d\gamma\right|$$
$$\leq \frac{\omega_2-\omega_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\left|\varrho'_\rho(\mathrm{d})\right|^q + \left|\varrho'_\rho(\mathrm{D})\right|^q - \left[\frac{\left|\varrho'_\rho(\omega_1)\right|^q + \left|\varrho'_\rho\left(\frac{\omega_1+\omega_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\left|\varrho'_\rho(\mathrm{d})\right|^q + \left|\varrho'_\rho(\mathrm{D})\right|^q - \left[\frac{\left|\varrho'_\rho(\omega_2)\right|^q + \left|\varrho'_\rho\left(\frac{\omega_1+\omega_2}{2}\right)\right|^q}{s+1}\right]\right)^{\frac{1}{q}}\right\}.$$

Proof. By employing the definition of ρ-digamma function $\varrho_\rho(\gamma)$, it is easy to notice that ρ -digamma function $\Phi(\gamma) = \varrho_\rho(\gamma)$ is completely monotonic on $(0,\infty)$. This ensures that the function $\Phi'(\gamma) = \varrho'_\rho(\gamma)$ is s-convex on $(0,\infty)$ for each $\rho \in (0,1)$. Now by applying Corollary 2.5, we get the desired result. □

Proposition 3.6. *Suppose* d, D, ρ *be real numbers such that* $0 < \mathrm{d} < \mathrm{D}$, *and* $0 < \rho < 1$, $s \in (0,1]$. *Then the following inequality holds true:*

$$\left|\frac{1}{6}\left\{\varrho_\rho(\mathrm{d}+\mathrm{D}-\omega_1) + \varrho_\rho(\mathrm{d}+\mathrm{D}-\omega_2)\right\} + \frac{2}{3}\varrho_\rho\left(\mathrm{d}+\mathrm{D}-\frac{\omega_1+\omega_2}{2}\right) - \frac{1}{\omega_2-\omega_1}\int_{\mathrm{d}+\mathrm{D}-\omega_2}^{\mathrm{d}+\mathrm{D}-\omega_1}\varrho_\rho(\gamma)\,d\gamma\right|$$
$$\leq \frac{\omega_2-\omega_1}{2}\left[\frac{5}{18}\left(\left|\varrho'_\rho(\mathrm{d})\right| + \left|\varrho'_\rho(\mathrm{D})\right|\right)\right.$$
$$- \frac{1}{2^s}\left(\left|\varrho'_\rho(\omega_1)\right| + \left|\varrho'_\rho(\omega_2)\right|\right)\left\{\frac{1}{6(s+1)}\left(\frac{5^{s+2}-1}{3^{s+1}} - 6\right) - \frac{1}{2(s+2)}\left(\frac{5^{s+2}+1}{3^{s+2}} - 2\right)\right.$$
$$\left.\left. + \frac{5^{s+2}+1}{2(s+1)(s+2)3^{s+2}} - \frac{2^s(2s+7)}{3(s+1)(s+2)}\right\}\right].$$

Proof. By employing the definition of ρ-digamma function $\varrho_\rho(\gamma)$, it is easy to notice that ρ -digamma function $\Phi(\gamma) = \varrho_\rho(\gamma)$ is completely monotonic on $(0,\infty)$. This ensures that the function $\Phi'(\gamma) = \varrho'_\rho(\gamma)$ is s-convex on $(0,\infty)$ for each $\rho \in (0,1)$. Now by applying Corollary 2.3, we get the desired result. □

4. Conclusions

One can see that the recent developments in the field of inequalities are related to finding generalized versions and new bounds of various well-known inequalities employing different fractional integral operators. Researchers use new methodologies, new applications, and new operators to add originality to this subject. In this article, the incorporation of the Simpson-Mercer inequalities and the Atangana-Baleanu (A-B) fractional integral operator is studied for differentiable mappings, whose derivatives in absolute value at certain powers are s-convex in the second sense. Some novel cases of the established results are discussed as well. Moreover, the study also have applications to special means, modified Bessel functions, and q-digamma functions. An interesting scope is to check whether the methodology of this paper can be utilized to refine the Simpson-Mercer inequality via concepts such as interval analysis, quantum calculus, and coordinates. In future readers can work on modified A-B fractional operators as given in manucripts (see [49–51]) and modified Caputo-Fabrizio fractional operators (see [52, 53]).

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Conflict of interest

The authors declare no conflicts of interest.

References

1. H. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Chen, A new collection of real world applications of fractional calculus in science and engineering, *Commun. Nonlinear Sci. Numer. Simul.*, **64** (2018), 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019
2. V. V. Kulish, J. L. Lage, Application of fractional calculus to fluid mechanics, *J. Fluids Eng.*, **124** (2002), 803–806. https://doi.org/10.1115/1.1478062
3. M. A. El-Shaed, Fractional calculus model of the semilunar heart valve vibrations, *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, 2003, 711–714. https://doi.org/10.1115/DETC2003/VIB-48384
4. A. Atangana, Application of fractional calculus to epidemiology, In: *Fractional dynamics*, De Gruyter Open Poland, 2015, 174–190. https://doi.org/10.1515/9783110472097-011
5. D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Classical and new inequalities in analysis*, Springer Dordrecht, 1993. https://doi.org/10.1007/978-94-017-1043-5
6. S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, *Science direct working paper*, 2003. Available from: `https://ssrn.com/abstract=3158351`.
7. M. K. Bakula, J. Pečarić, J. Perić, Extensions of the Hermite-Hadamard inequality with applications, *Math. Inequal. Appl.*, **12** (2012), 899–921.

8. K. Mehrez, P. Agarwal, New Hermite-Hadamard type integral inequalities for convex functions and their applications, *J. Comput. Appl. Math.*, **350** (2019), 274–285. https://doi.org/10.1016/j.cam.2018.10.022
9. S. Khan, M. A. Khan, S. I. Butt, Y. M. Chu, A new bound for the Jensen gap pertaining twice differentiable functions with applications, *Adv. Differ. Equ.*, **2020** (2020), 333. https://doi.org/10.1186/s13662-020-02794-8
10. A. M. Mercer, A variant of Jensens inequality, *J. Ineqal. Pure Appl. Math.*, 2003.
11. A. Matkovic, J. Pečarić, I. Perić, A variant of Jensens inequality of Mercers type for operators with applications, *Linear Algebra Appl.*, **418** (2016), 551–564. https://doi.org/10.1016/j.laa.2006.02.030
12. M. Niezgoda, A generalization of Mercer's result on convex functions, *Nonlinear Anal.*, **71** (2009).
13. M. Kian, Operator Jensen inequality for superquadratic functions, *Linear Algebra Appl.*, **456** (2014), 82–87. https://doi.org/10.1016/j.laa.2012.12.011
14. E. Anjidani, M. R. Changalvaiy, Reverse Jensen-Mercer type operator inequalities, *Electron. J. Linear Algebra*, **31** (2016), 87–99. https://doi.org/10.13001/1081-3810.3058
15. E. Anjidani, Jensen-Mercer operator inequalities involving superquadratic functions, *Mediterr. J. Math.*, **15** (2018), 31. https://doi.org/10.1007/s00009-017-1058-8
16. M. Ali, A. R. Khan, Generalized integral Mercer's inequality and integral means, *J. Inequal. Spec. Funct.*, **10** (2019), 60–76.
17. M. J. V. Cortez, J. E. H. Hernandez, Una Variante de la desigualdad de Jensen-Mercer para funciones h-convexas y funciones de operadores h-convexas, *MATUA*, **4** (2017), 62–76.
18. Q. Kang, S. I. Butt, W. Nazeer, M. Nadeem, J. Nasir, H. Yang, New variant of Hermite-Jensen-Mercer inequalities via Riemann-Liouville fractional integral operators, *J. Math.*, **2020** (2020), 4303727. https://doi.org/10.1155/2020/4303727
19. M. Vivas-Cortez, M. S. Saleem, S. Sajid, M. S. Zahoor, A. Kashuri, Hermite-Jensen-Mercer-type inequalities via Caputo-Fabrizio fractional integral for h-convex function, *Fractal Fract.*, **5** (2021), 269. https://doi.org/10.3390/fractalfract5040269
20. M. Vivas-Cortez, M. U. Awan, M. Z. Javed, A. Kashuri, M. A. Noor, K. I. Noor, Some new generalized κ-fractional Hermite-Hadamard-Mercer type integral inequalities and their applications, *AIMS Math.*, **7** (2022), 3203–3220. https://doi.org/10.3934/math.2022177
21. H. Hudzik, L. Maligranda, Some remarks on s-convex functions, *Aequationes Math.*, **48** (1994), 100–111. https://doi.org/10.1007/BF01837981
22. S. S. Dragomir, S. Fitzpatrick, The Jensen inequality for s-Breckner convex functions in linear spaces, *Demonstr. Math.*, **33** (2000), 43–50 https://doi.org/10.1515/dema-2000-0106
23. X. Chen, New convex functions in linear spaces and Jensen's discrete inequality, *J. Inequal. Appl.*, **2013** (2013), 472. https://doi.org/10.1186/1029-242X-2013-472
24. P. Burai, A. Házy, T. Juhász, On approximately Breckner s-convex functions, *Control Cybern.*, **40** (2011), 91–99.
25. S. I. Butt, J. Nasir, S. Qaisar, K. M. Abualnaja, k-fractional variants of Hermite-Mercer-type inequalities via convexity with applications, *J. Funct. Spaces*, **2021** (2021), 5566360. https://doi.org/10.1155/2021/5566360

26. M. Alomari, M. Darus, S. S. Dragomir, New inequalities of Simpson's type for s-convex functions with applications, *Research report collection*, 2009.
27. M. Z. Sarikaya, E. Set, M. E. Özdemir, On new inequalities of simpson's type for s-convex functions, *Comput. Math. Appl.*, **60** (2010), 2191–2199. https://doi.org/10.1016/j.camwa.2010.07.033
28. J. Chen, X. Huang, Some new inequalities of Simpson's type for s-convex functions via fractional integrals, *Filomat*, **31** (2017), 4989–4997. https://doi.org/10.2298/FIL1715989C
29. F. Hezenci, H. Budak, H. Kara, New version of fractional Simpson type inequalities for twice differentiable functions, *Adv. Differ. Equ.*, **2021** (2021), 460. https://doi.org/10.1186/s13662-021-03615-2
30. İ. İşcan, Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions, *J. Math.*, **2014** (2014), 346305. https://doi.org/10.1155/2014/346305
31. İ. İşcan, S. Turhan, S. Maden, Hermite-Hadamard and Simpson-like type inequalities for differentiable p-quasi-convex functions, *Filomat*, **31** (2017), 5945–5953. https://doi.org/10.2298/FIL1719945I
32. M. Iqbal, S. Qaisar, S. Hussain, On Simpson's type inequalities utilizing fractional integrals, *J. Comput. Anal. Appl.*, **23** (2017), 1137–1145.
33. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, **57** (2013), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048
34. I. Podlubni, *Fractional differential equations*, San Diego: Academic press, 1999.
35. M. Kian, M. S. Moslehian, Refinements of the operator Jensen-Mercer inequality, *Electron. J. Linear Algebra*, **26** (2013), 742–753.
36. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85. http://dx.doi.org/10.12785/pfda/010201
37. T. Abdeljawad, D. Baleanu, On fractional derivatives with exponential kernel and their discrete versions, *Rep. Math. Phys.*, **80** (2017), 11–27. https://doi.org/10.1016/S0034-4877(17)30059-9
38. A. Atangana, D. Baleanu, New fractional derivatices with non-local and non-singular kernel: Theory and application to heat transfer model, *Thermal Sci.*, **20** (2016), 763–769. https://doi.org/10.2298/TSCI160111018A
39. T. Abdeljawad, D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, *J. Nonlinear Sci. Appl.*, **10** (2017), 1098–1107. http://dx.doi.org/10.22436/jnsa.010.03.20
40. İ. İşcan, Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals, *Studia Univ. Babeş-Bolyai Sect. A Math.*, **60** (2015), 355–366.
41. F. Chen, A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals, *Ital. J. Pure Appl. Math.*, **33** (2014), 299–306.
42. W. B. Sun, Some new inequalities for generalized h-convex functions involving local fractional integral operators with Mittag-Leffler kernel, *Math. Meth. Appl. Sci.*, **44** (2021), 4985–4998. https://doi.org/10.1002/mma.7081

43. W. B. Sun, Hermite-Hadamard type local fractional integral inequalities for generalized s-preinvex functions and their generalization, *Fractals*, **29** (2021), 2150098. https://doi.org/10.1142/S0218348X21500985
44. W. B. Sun, Hermite-Hadamard type local fractional integral inequalities with Mittag-Leffler kernel for generalized preinvex functions, *Fractals*, **29** (2021), 2150253. https://doi.org/10.1142/S0218348X21502534
45. A. Fernandez, P. O. Mohammed, Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels, *Math. Meth. Appl. Sci.*, **44** (2021), 8414–8431. https://doi.org/10.1002/mma.6188
46. H. Öğülmüş, M. Z. Sarikaya, Hermite-Hadamard-Mercer type inequalities for fractional integrals, *Filomat*, **35** (2021), 2425–2436. https://doi.org/10.2298/FIL2107425O
47. S. K. Sahoo, P. O. Mohammed, B. Kodamasingh, M. Tariq, Y. S. Hamed, New fractional integral inequalities for convex functions pertaining to Caputo-Fabrizio operator, *Fractal Fract.*, **6** (2022), 171. https://doi.org/10.3390/fractalfract6030171
48. M. A. Latif, H. Kalsoom, Z. A. Khan, Hermite-Hadamard-Fejér type fractional inequalities relating to a convex harmonic function and a positive symmetric increasing function, *AIMS Math.*, **7** (2022), 4176–4198. https://doi.org/10.3934/math.2022232
49. M. Al-Refai, D. Baleanu, On an extension of the operator with Mittag-Leffler kernel, *Fractals*, 2022. https://doi.org/10.1142/S0218348X22401296
50. K. Velusamy, M. M. Arjunan, D. Baleanu, Non-instantaneous impulsive fractional-order delay differential systems with Mittag-Leffler kernel, *AIMS Math.*, **7** (2022), 9353–9372. https://doi.org/10.3934/math.2022519
51. B. Ahmed, Application of Shehu transform to Atangana-Baleanu derivatives, *J. Math. Computer Sci.*, **20** (2019), 101–107. https://doi.org/10.22436/jmcs.020.02.03
52. M. Caputo, M. Fabrizio. On the singular kernels for fractional derivatives. Some applications to partial differential equations, *Progr. Fract. Differ. Appl.*, **7** (2021), 79–82. http://dx.doi.org/10.18576/pfda/0070201
53. J. Losada, J. J. Nieto, Fractional integral associated to fractional derivatives with nonsingular kernels, *Progr. Fract. Differ. Appl.*, **7** (2021), 137–143. http://dx.doi.org/10.18576/pfda/070301
54. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1995.
55. S. Jain, K. Mehrez, D. Baleanu, P. Agarwal, Certain Hermite-Hadamard inequalities for logarithmically convex functions with applications, *Mathematics*, **7** (2009), 163. https://doi.org/10.3390/math7020163

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