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Research article

Soliton solutions for some nonlinear models in mathematical physics via conservation laws

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Abstract: In this paper, we derive the soliton solutions from conserved quantities for the Benjamin-Bona-Mahoney equation with dual-power law nonlinearity (BBM), modified regularized long wave (MRLW) equation, modified nonlinearly dispersive KdV equations 2K(2,2,1) and 3K(3,2,2) equation, which are constructed by the multiplier approach (variational derivative method). Finally, we give the numerical simulations to illustrate this method.

Keywords: multiplier; Benjamin-Bona-Mahoney equation with dual-power law nonlinearity; modified regularized long wave equation; nonlinearly dispersive KdV equations 2K(2,2,1) and 3K(3,2,2) equation; soliton solutions
Mathematics Subject Classification: 70S10, 35Q53, 76M60

1. Introduction

In all areas of physics, conservation laws [1, 2] are essential and important since they allow us to draw conclusions of a physical system under study in an efficient way. Knowledge of conservation laws are important for numerical integration of partial differential equations (PDEs). The existence of a large number of conservation laws for a nonlinear partial differential equation (system) is a strong indication of its integrability. Many powerful methods have been developed for the construction of conservation laws, such as the Noethers theorem [3] for variational problems, Laplaces Direct method [4], characteristic form introduced by Stuedel [5], multiplier approach [6, 7], Kara and Mahomed [8] symmetry condition, partial Noether approach [9–11] and Cheviakov [12] developed powerful software packages to compute conservation laws for partial differential equations.

The multiplier approach (also known as the variational derivative method) was successfully applied to the construction of conservation laws for nonlinear partial differential equations [13–16].

Soliton solutions for nonlinear partial differential equations (PDEs) are play a significant role in soliton theory [17]

In [18] the authors obtained the novel exact solutions in form of dark, bright, combined dark-bright, combined singular and other soliton solutions solitons for the meta materials model having third and fourth order dispersions, the authors in [19] obtained investigation for the optical solitons solution for the nonlinear Schrödinger equation in magneto-optic waveguides with anti-cubic nonlinearity and in [20] the author obtained the new exact soliton solutions for the two deformed nonlinear Schrödinger (NLS) type equations.

Based on He's variational approach [21, 22], which depends on the Lagrange function and the finding of conservation laws of system of differential equations (DEs), which is often the first step towards finding the solution the variational formula from the conserved quantities for Benjamin-Bona-Mahoney equation with dual-power law nonlinearity (BBM), modified regularized long wave (MRLW) equation and modified nonlinearly dispersive KdV equations 2K(2,2,1) and 3K(3,2,2) equation and new kinds of solitary wave solutions for these models and numerical simulations were obtained in Sections 3–6. in Section 5. This method helps to explore the new soliton solutions for new phenomena of nonlinear evolution equations. So we proposed a new method to derive the functional from conserved quantities since it depends on the Lagrange function and its derivatives [23] and then find the soliton solutions by Ritz method [24].

The Benjamin-Bona-Mahoney equation with dual-power law nonlinearity (BBM) equation describes the dynamics of shallow water waves in the presence of advection, the modified regularized long wave (MRLW) equation was first introduced by Peregrine [25] to describe the development of an undular bore, the modified nonlinearly dispersive KdV equations 2K(2,2,1) equation and the modified nonlinearly dispersive KdV equations 3K(3,2,2) equation are derived from the well-known KdV equation which describes the mathematical modeling of traveling wave solutions, known as solitary water waves (also called solitons) in a shallow water domain.

The proposed method derived the soliton solutions from the conserved quantities for the models while the existing method derived the soliton solutions from the Lagrangian for the models. In [26], He's variational iteration method was applied to obtain the numerical solution for various kinds of Newell-Whitehead-Segel nonlinear diffusion equation. This paper is organized as follows: In Section 2, we give brief definitions related to the multiplier approach and main steps for deriving the soliton solutions. In Sections 3–6, we apply this method to Benjamin-Bona-Mahoney equation with dual-power law nonlinearity (BBM), modified regularized long wave (MRLW) equation and modified nonlinearly dispersive KdV equations 2K(2,2,1) and 3K(3,2,2) equation and give the soliton solutions by using a proposed method and the figures for each models to illustrate the properties of these solitons solutions. Finally, results are summarized in Section 7.

2. The proposed method

For giving an independent variable t, x and dependent variable u

(1) The total derivative operator D with respect to t, x are

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{txx} \frac{\partial}{\partial u_{xx}} + ...$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + u_{xxt}\frac{\partial}{\partial u_{xt}} + ... \tag{2.1}$$

(2) The Euler operator is defined

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_xD_t\frac{\partial}{\partial u_{xt}} - D_t^3\frac{\partial}{\partial u_{ttt}} - ..... \tag{2.2}$$

Consider a mth-order partial differential equation of t, x

$$E(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, ...) = 0. \tag{2.3}$$

(3) A vector $T = (T^1, T^2)$ such that

$$D_tT^1 + D_xT^2 = 0, \tag{2.4}$$

holds for all solutions of (2.3) is known as the conserved vector of (2.3).
(4) The multiplier Λ for (2.3) has the property

$$D_tT^1 + D_xT^2 = \Lambda E, \tag{2.5}$$

for arbitrary function $u(x, t)$ [5, 6]
(5) The determining equations for multiplier are obtained by taking the variational derivative of (2.5) [6]

$$\frac{\delta}{\delta u}(\Lambda E) = 0. \tag{2.6}$$

Equation (2.6) holds for arbitrary function $u(x, t)$ not only for solutions of (2.3). We can derive the conserved vectors using (2.5) after computing the multiplier from (2.6)
The main steps of the proposed method are given as follows:

For giving a nonlinear equation
Step 1. By using the wave transformation $u(x, t) = u(\zeta)$, $\zeta = kx - \omega t + \zeta_0$, we can convert Eq (2.3) into an ordinary differential equation (ODE)

$$H(u, -\omega u_\zeta, ku_\zeta, \omega^2u_{\zeta\zeta}, -k\omega u_{\zeta\zeta}, k^2u_{\zeta\zeta}, ...) = 0. \tag{2.7}$$

Step 2. Equation (2.4) can be converted into

$$D_\zeta T^\zeta = 0, \tag{2.8}$$

Step 3. The multiplier Λ for (2.7) has the property

$$D_\zeta T^\zeta = \Lambda(\zeta, u(\zeta))H, \tag{2.9}$$

for arbitrary function $U(\zeta)$.
Step 4. The determining equations for multiplier are obtained by taking the variational derivative of (2.9)

$$\frac{\delta}{\delta u}(\Lambda(\zeta, u(\zeta))H) = 0. \tag{2.10}$$

Equation (2.10) holds for arbitrary function $u(\zeta)$ not only for solutions of (2.7). We can derive the conserved vectors using (2.9) after computing the multiplier from (2.10)
Step 5. We construct the following functional

$$J = \int_0^\infty T^\zeta d\zeta, \qquad or \qquad J = \int_{-\infty}^\infty T^\zeta d\zeta, \tag{2.11}$$

where T^ζ is the conserved quantity for the Eq (2.7).
Step 6. By a Ritz method [24], we can obtain solitary wave solutions, such as $u(\zeta) = Asech(\zeta), u(\zeta) = Asech(\zeta)^2$, where A is a constant to be determined.
Substituting $u(\zeta) = Asech(\zeta), u(\zeta) = Asech(\zeta)^2$ into Eq (2.11) and making J stationary with respect to A we obtain

$$\frac{\partial J}{\partial A} = 0. \tag{2.12}$$

Solving Eq (2.12) we obtain the constant A . Hence the solitary wave solution is well determined.

3. Conservation laws and soliton solutions for the Benjamin-Bona-Mahoney equation with dual-power law nonlinearity (BBM) equation

We consider the Benjamin-Bona-Mahoney equation with dual-power law nonlinearity (BBM) equation [27]

$$u_t + au_x + (bu^n + cu^{2n})u_x + lu_{xxx} = 0. \tag{3.1}$$

The first term represents the evolution term while b and c represent the coefficients of dual-power law nonlinearity. Then a and l are the coefficients of dispersion terms. The parameter n is the power law parameter while u is the wave profile. The independent variables x and t represent spatial and temporal variables respectively.
The two conserved vectors [28]

$$\begin{aligned} T_1^t &= \frac{u^2}{2}, \\ T_1^x &= \frac{a}{2}u^2 + \frac{b}{n+2}u^{n+2} + \frac{c}{2(n+1)}u^{2(n+1)} + luu_{xx} - \frac{l}{2}u_x^2, \\ T_2^t &= u, \\ T_2^x &= au + \frac{b}{n+1}u^{n+1} + \frac{c}{2n+1}u^{2n+1} + lu_{xx}. \end{aligned} \tag{3.2}$$

Using the wave variable transformation $u(x,t) = u(\zeta)$, $\zeta = kx - \omega t + \zeta_0$ to convert Eq (3.1) into ODE

$$(ka - \omega)u' + k(bu^n + cu^{2n})u' + k^3 lu''' = 0. \tag{3.3}$$

The determining equation for multiplier $\Lambda(\zeta, u(\zeta))$, from (2.10), is

$$\Lambda_u((ka - \omega)u' + k(bu^n + cu^{2n})u' + k^3 lu''') + k(nbu^{n-1} + 2ncu^{2n-1})u'\Lambda$$

$$-D_\zeta((ka-\omega)+k(bu^n+cu^{2n}))\Lambda - k^3lD_\zeta^3\Lambda = 0. \tag{3.4}$$

Equation (3.4) is separated according to different combinations of derivatives of u and and an overdetermined system of equations for multiplier Λ is obtained which gives

$$\Lambda(\zeta,u) = c_1u + c_2. \tag{3.5}$$

From (2.9) and (3.5), we obtain the conserved quantities

$$T^\zeta = (ak-\omega)\left(\frac{1}{2}c_1u(\zeta)^2 + c_2u(\zeta)\right) + \frac{bc_2ku(\zeta)^{n+1}}{n+1} + \frac{bc_1ku(\zeta)^{n+2}}{n+2} + \frac{c_2du(\zeta)^{2n+1}}{2n+1}$$
$$+\frac{c_1du(\zeta)^{2n+2}}{2n+2} + c_2k^3lu''(\zeta) + c_1k^3l\left(u(\zeta)u''(\zeta) - \frac{1}{2}u'(\zeta)^2\right). \tag{3.6}$$

We can construct the following functional

$$J = \int_0^\infty ((ak-\omega)\left(\frac{1}{2}c_1u(\zeta)^2 + c_2u(\zeta)\right) + \frac{bc_2ku(\zeta)^{n+1}}{n+1} + \frac{bc_1ku(\zeta)^{n+2}}{n+2} + \frac{c_2du(\zeta)^{2n+1}}{2n+1}$$
$$+\frac{c_1du(\zeta)^{2n+2}}{2n+2} + c_2k^3lu''(\zeta) + c_1k^3l\left(u(\zeta)u''(\zeta) - \frac{1}{2}u'(\zeta)^2\right))d\zeta. \tag{3.7}$$

According to the Ritz-like method , we look for a solitary wave solution in the form

$$u(\zeta) = Asech(\zeta). \tag{3.8}$$

Substituting Eq (3.8) into Eq (3.7) , we have for $n = 1$

$$J = \frac{1}{12}A\left(Ac_1\left(6ak + 2A^2d + \pi Abk - 6\left(k^3l+\omega\right)\right) + c_2\left(6\pi ak + \pi A^2d + 6Abk - 6\pi\omega\right)\right). \tag{3.9}$$

To find the constant A , we solve the following equation

$$\frac{\partial J}{\partial A} = \frac{1}{4}c_2\left(2\pi ak + \pi A^2d + 4Abk - 2\pi\omega\right) + c_1\left(A\left(ak + k^3(-l) - \omega\right) + \frac{2A^3d}{3} + \frac{1}{4}\pi A^2bk\right) = 0. \tag{3.10}$$

From (3.10) , we get

$$A = \begin{cases} \frac{-3\pi bk \mp \sqrt{3(3\pi^2b^2k^2 - 128adk + 128dk^3l + 128d\omega)}}{16d}, & c_1 = 1, \quad c_2 = 0, \\ \frac{-\sqrt{2(2b^2k^2 - \pi^2adk + \pi^2d\omega)} - 2bk}{\pi d}, & c_1 = 0, \quad c_2 = 1. \end{cases} \tag{3.11}$$

Therefore, the solitary wave solutions for the BBM equation with dual-power law nonlinearity are

$$u(x,t) = \begin{cases} \frac{-3\pi bk \mp \sqrt{3(3\pi^2b^2k^2 - 128adk + 128dk^3l + 128d\omega)}}{16d} sech(kx - \omega t + \zeta_0), & c_1 = 1, c_2 = 0, \\ \frac{-\sqrt{2(2b^2k^2 - \pi^2adk + \pi^2d\omega)} - 2bk}{\pi d} sech(kx - \omega t + \zeta_0), & c_1 = 0, c_2 = 1. \end{cases} \tag{3.12}$$

The 3D graphs in Figures 1 and 2 illustrate the trough propagating to the right, while 2D graphs show the movement of the wave along x-direction as time increases for Eq (3.12), a contour plot could also be useful for describing the movement of the solitons.

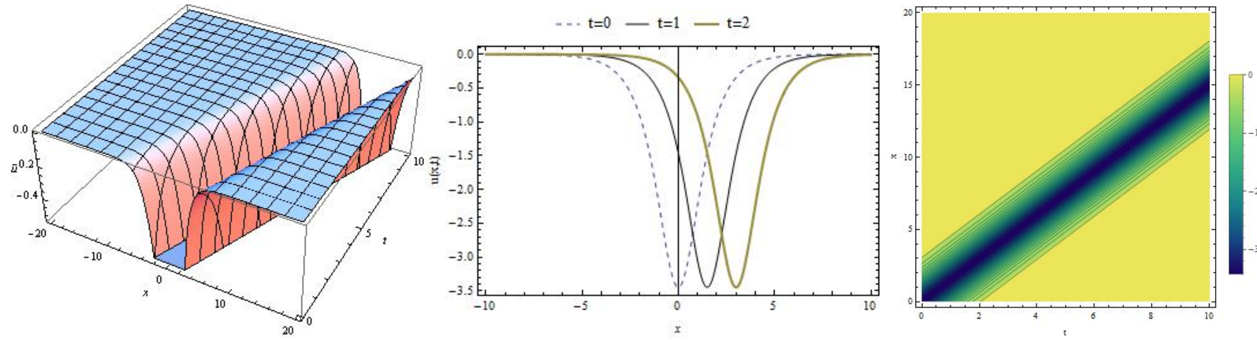

Figure 1. Shows the soliton solution of Eq (3.12) for $c_1 = 1, c_2 = 0$.

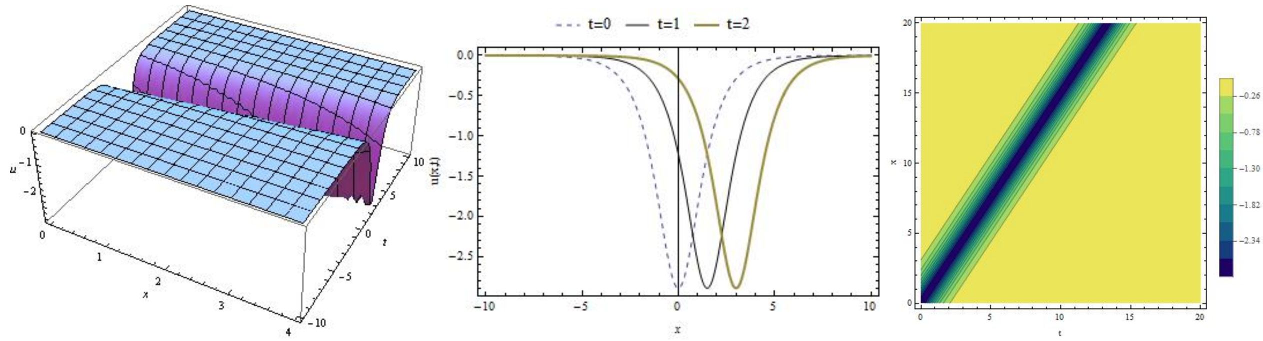

Figure 2. Shows the soliton solution of Eq (3.12) for $c_1 = 0, c_2 = 1$.

For another soliton solution in the form [29]

$$u(\zeta) = F sech^2(\zeta). \tag{3.13}$$

where F is a constant to be determined. Substituting Eq (3.13) into Eq (3.9), we obtain

$$\begin{aligned} J = &\frac{F}{315}(Fc_1\left(21\left(5ak - 12k^3 l - 5\omega\right) + 36F^2 d + 56Fbk\right)) \\ &+\frac{F}{315}(7c_2\left(45ak + 8F^2 d + 15Fbk - 45\omega\right)). \end{aligned} \tag{3.14}$$

To find the constant F, we solve the following equation

$$\begin{aligned} \frac{\partial J}{\partial F} = &\frac{1}{105}(2Fc_1\left(35ak + 24F^2 d + 28Fbk - 84k^3 l - 35\omega\right)) \\ &+\frac{1}{105}(7c_2\left(15ak + 8F^2 d + 10Fbk - 15\omega\right)) = 0. \end{aligned} \tag{3.15}$$

From Eq (3.15), we have

$$F = \begin{cases} \frac{-7bk\mp\sqrt{7(72dk^3l-30adk+7b^2k^2+30d\omega)}}{12d}, & c_1 = 1, \quad c_2 = 0, \\ \frac{-5bk\mp\sqrt{5(5b^2k^2-24adk+24d\omega)}}{8d}, & c_1 = 0, \quad c_2 = 1. \end{cases} \tag{3.16}$$

Therefore, the solitary wave solutions for the BBM equation with dual-power law nonlinearity are as follows:

$$u(x,t)=\begin{cases}\frac{-7bk\mp\sqrt{7(72dk^3l-30adk+7b^2k^2+30d\omega)}}{12d}sech^2(kx-\omega t+\zeta_0), & c_1=1, c_2=0,\\ \frac{-5bk\mp\sqrt{5(5b^2k^2-24adk+24d\omega)}}{8d}sech^2(kx-\omega t+\zeta_0), & c_1=0, c_2=1.\end{cases} \tag{3.17}$$

The 3D graphs in Figures 3 and 4 describe nonlinear wave propagation of soliton solutions, while 2D graphs depict the movement of the solitons obtained from Eq (3.17) for different times and a contour plot describes the movement of the solitons.

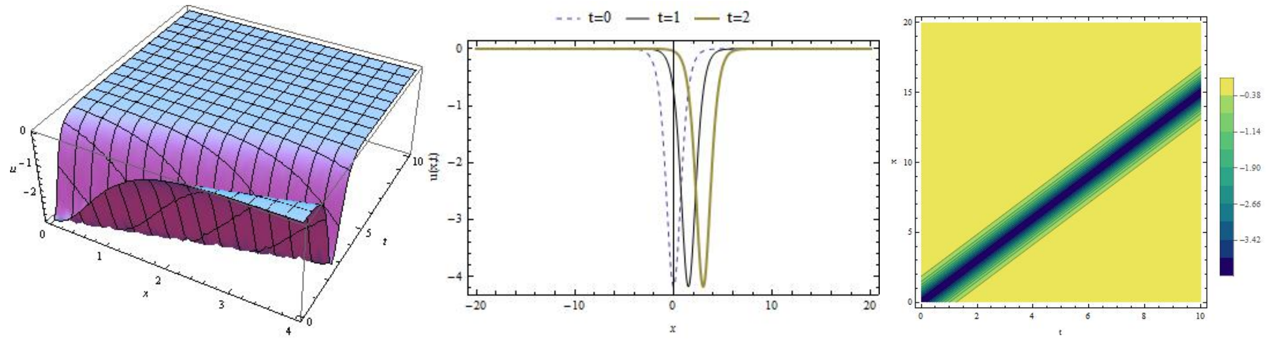

Figure 3. Shows the soliton solution of Eq (3.17) for $c_1 = 1, c_2 = 0$.

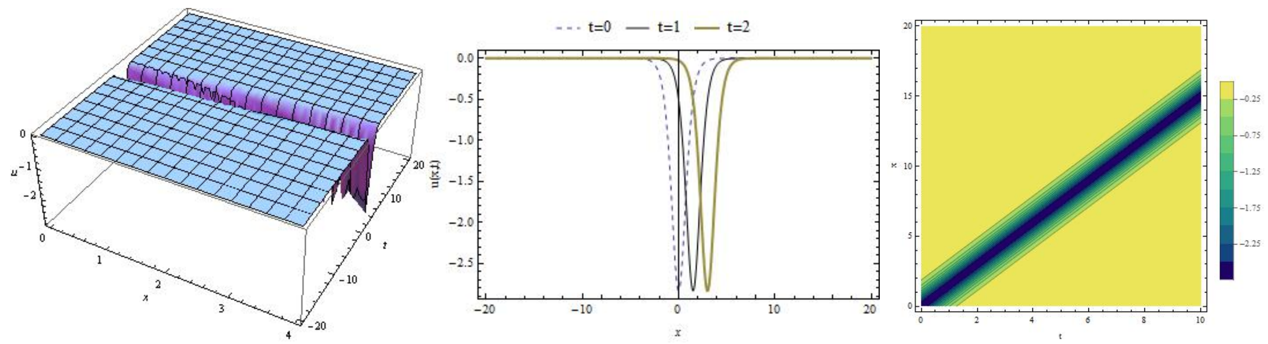

Figure 4. Shows the soliton solution of Eq (3.17) for $c_1 = 0, c_2 = 1$.

4. Conservation laws and soliton solutions for the modified regularized long wave (MRLW) equation

We consider the modified regularized long wave (MRLW) equation [30]

$$u_t + u_x + 6u^2u_x - \mu u_{xxt} = 0, \tag{4.1}$$

where μ is a positive constants, was originally introduced to describe the behavior of the undular bore by Peregrine [31], and later by Benjamin et al. [32]. This equation is very important in physics media since it describes a phenomenon with weak nonlinearity and dispersion waves, including nonlinear transverse waves in shallow water, ion-acoustic and magneto hydrodynamic waves in plasma and

phonon packets in nonlinear crystals.
The two conserved vectors [28]

$$\begin{aligned} T_1^t &= \frac{u^2}{2} - \mu(uu_{xx} - \frac{1}{2}u_x^2), \\ T_1^x &= \frac{1}{2}u^2 + \mu u_x u_t + \frac{3}{2}u^4, \\ T_2^t &= u, \qquad\qquad T_2^x = u + 2u^3. \end{aligned} \tag{4.2}$$

To find the soliton solutions, we convert Eq (4.1) into an ODE

$$(k-\omega)u' + 6ku^2u' + \mu k^2\omega u''' = 0. \tag{4.3}$$

The multiplier $\Lambda(\zeta, u(\zeta))$, can be obtained as follows

$$\Lambda_u((k-\omega)u' + 6ku^2u' + \mu k^2\omega u''') + 12kuu'\Lambda - D_\zeta((k-\omega) + 6ku^2)\Lambda - \mu\omega k^2 D_\zeta^3\Lambda = 0. \tag{4.4}$$

Solving Eq (4.4) for multiplier Λ, we obtain

$$\Lambda(\zeta, u) = c_1 u + c_2. \tag{4.5}$$

From (2.9) and (4.5), we obtain the conserved quantities

$$\begin{aligned} T^\zeta = &-k^2\mu\omega\left(c_2u''(\zeta) + c_1\left(u(\zeta)u''(\zeta) - \frac{1}{2}u'(\zeta)^2\right)\right) + \frac{1}{2}(k-\omega)u(\zeta)\left(c_1u(\zeta) + 2c_2\right) \\ &+\frac{3}{2}ku(\zeta)^3\left(c_1u(\zeta) + \frac{4c_2}{3}\right). \end{aligned} \tag{4.6}$$

From (4.6), we can obtain the functional

$$\begin{aligned} J = &\int_0^\infty (-k^2\mu\omega\left(c_2u''(\zeta) + c_1\left(u(\zeta)u''(\zeta) - \frac{1}{2}u'(\zeta)^2\right)\right) + \frac{1}{2}(k-\omega)u(\zeta)\left(c_1u(\zeta) + 2c_2\right) \\ &+\frac{3}{2}ku(\zeta)^3\left(c_1u(\zeta) + \frac{4c_2}{3}\right))d\zeta. \end{aligned} \tag{4.7}$$

According to the Ritz-like method , we search a solitary wave solution in the form

$$u(\zeta) = Asech(\zeta). \tag{4.8}$$

Substituting Eq (4.8) into Eq (4.7) , we have

$$J = \frac{1}{2}A\left(Ac_1\left(2A^2k + k^2\mu\omega + k - \omega\right) + \pi c_2\left(A^2k + k - \omega\right)\right). \tag{4.9}$$

To find the constant A , we solve the following equation

$$\frac{\partial J}{\partial A} = \frac{1}{2}\left(2Ac_1\left(4A^2k + k^2\mu\omega + k - \omega\right) + \pi c_2\left(3A^2k + k - \omega\right)\right) = 0. \tag{4.10}$$

From (4.10) , we obtain

$$A = \begin{cases} \mp\sqrt{\frac{\omega-k-\mu\omega k^2}{4k}}, & c_1 = 1, \quad c_2 = 0, \\ \mp\sqrt{\frac{\omega-k}{3k}}, & c_1 = 0, \quad c_2 = 1. \end{cases} \tag{4.11}$$

Therefore, the solitary wave solutions for the MRLW equation are

$$u(x,t) = \begin{cases} \mp\sqrt{\frac{\omega-k-\mu\omega k^2}{4k}} sech(kx-\omega t+\zeta_0), & c_1 = 1, c_2 = 0, \\ \mp\sqrt{\frac{\omega-k}{3k}} sech(kx-\omega t+\zeta_0), & c_1 = 0, c_2 = 1. \end{cases} \tag{4.12}$$

The 3D graphs in Figures 5 and 6 describe the shape of solitons for waves $u(x,t)$, while 2D graphs depict the movement of the soliton waves along x-direction as time increases for Eq (4.12), a contour plot can also be useful for deseribing the movement of the solitons for (4.12).

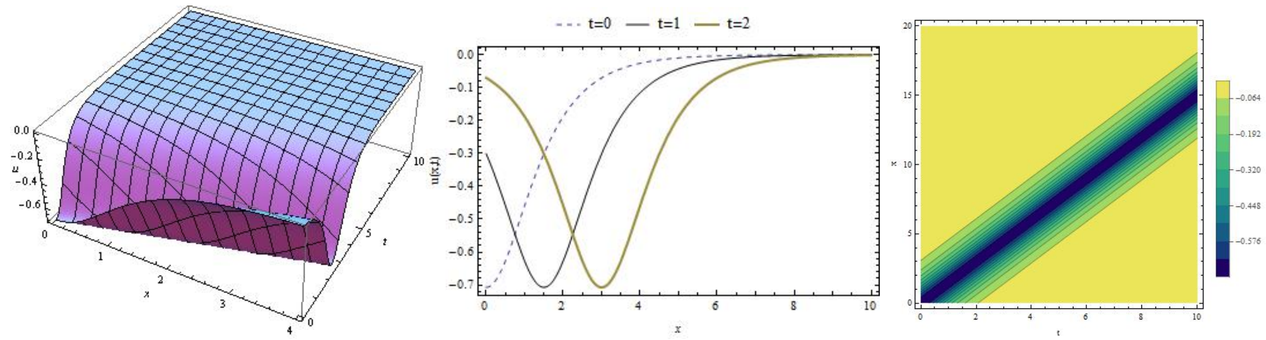

Figure 5. Shows the soliton solution of Eq (4.12) for $c_1 = 1, c_2 = 0$.

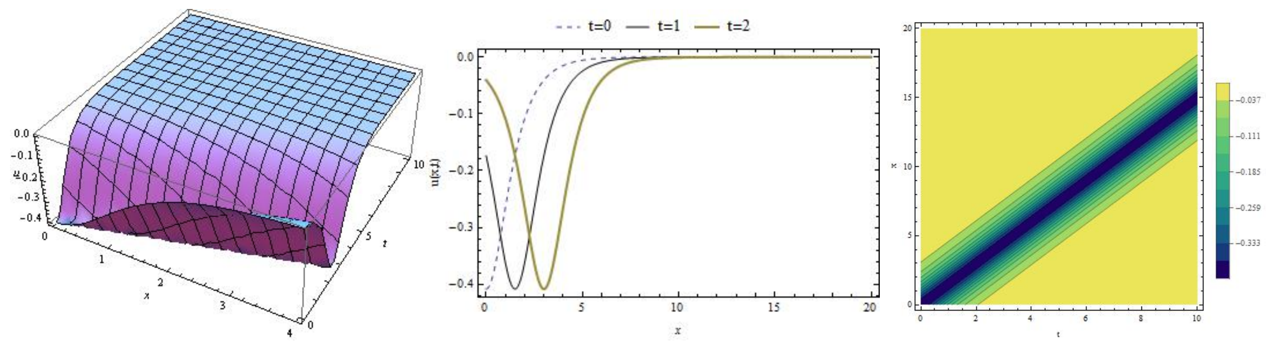

Figure 6. Shows the soliton solution of Eq (4.12) for $c_1 = 0, c_2 = 1$.

For another soliton solution (3.13), we obtain

$$J = \frac{1}{105} F \left(F c_1 \left(\left(72F^2 + 35 \right) k + 84k^2\mu\omega - 35\omega \right) + 7c_2 \left(\left(16F^2 + 15 \right) k - 15\omega \right) \right). \tag{4.13}$$

Solving the following equation to find F

$$\frac{\partial J}{\partial F} = \frac{2}{105} F c_1 \left(\left(144F^2 + 35 \right) k + 84k^2\mu\omega - 35\omega \right) + \frac{1}{5} c_2 \left(\left(16F^2 + 5 \right) k - 5\omega \right) = 0. \tag{4.14}$$

From Eq (4.14), we have

$$F = \begin{cases} \mp\frac{1}{12}\sqrt{\frac{7(5\omega-12k^2\mu\omega-5k)}{k}}, & c_1 = 1, \quad c_2 = 0, \\ \mp\frac{1}{4}\sqrt{\frac{5(\omega-k)}{k}}, & c_1 = 0, \quad c_2 = 1. \end{cases} \tag{4.15}$$

Therefore, the solitary wave solutions for the MRLW equation are as follows

$$u(x,t) = \begin{cases} \mp\frac{1}{12}\sqrt{\frac{7(5\omega-12k^2\mu\omega-5k)}{k}}sech^2(kx-\omega t+\zeta_0), & c_1 = 1, c_2 = 0, \\ \mp\frac{1}{4}\sqrt{\frac{5(\omega-k)}{k}}sech^2(kx-\omega t+\zeta_0), & c_1 = 0, c_2 = 1. \end{cases} \tag{4.16}$$

The 3D graphs in Figures 7 and 8 describe nonlinear wave propagation of soliton solutions to the right, while 2D graphs depict the movement of the solitons obtained from Eq (4.16) for different times and a contour plot describes the movement of the solitons.

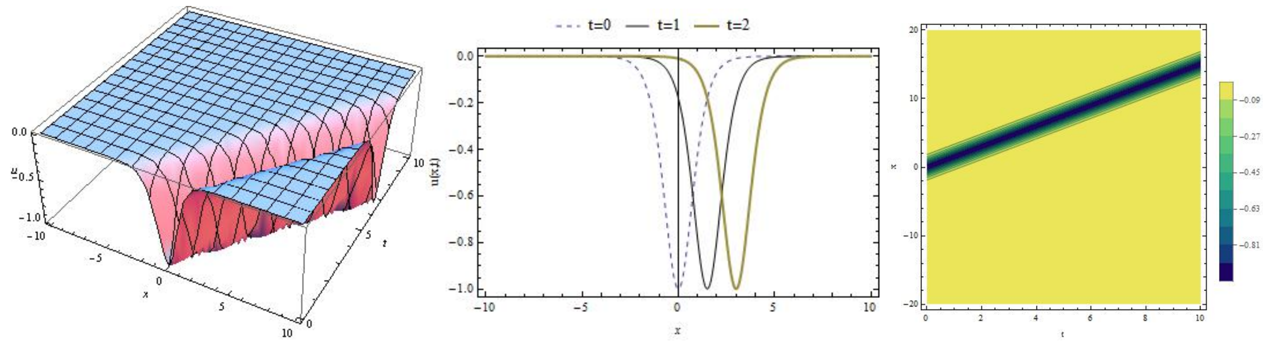

Figure 7. Shows the soliton solution of Eq (4.16) for $c_1 = 1, c_2 = 0$.

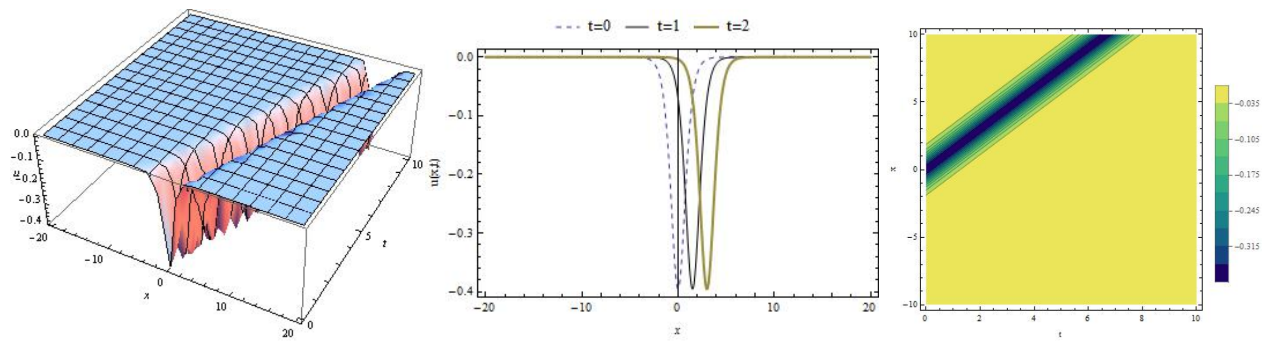

Figure 8. Shows the soliton solution of Eq (4.16) for $c_1 = 0, c_2 = 1$.

5. Conservation laws and soliton solutions for the modified nonlinearly dispersive KdV equations 2K(2,2,1) equation

The modified nonlinearly dispersive KdV equations 2K(2,2,1) equation [33] takes the form

$$uu_t + a(u^2)_x + bu_{xxx} = 0, \tag{5.1}$$

where a and b are constants.
The three conserved vectors [28]

$$
\begin{aligned}
T_1^t &= \frac{u^2}{2}(t - \frac{x}{2a}),\\
T_1^x &= u^2(at - \frac{x}{2}) + u_{xx}(bt - \frac{x}{2}) + \frac{1}{2a}u_x,\\
T_2^t &= \frac{u^3}{3}, \qquad T_2^x = \frac{2}{3}au^3 - \frac{b}{2}(u_x)^2 + buu_{xx},\\
T_3^t &= \frac{u^2}{2}, \qquad T_3^x = au^2 + bu_{xx}.
\end{aligned} \tag{5.2}
$$

By converting Eq (5.1) into an ODE

$$(2ak - \omega)uu' + bk^3u''' = 0. \tag{5.3}$$

The multiplier $\Lambda(\zeta, u(\zeta))$, can be obtained as follows

$$\Lambda_u((2ak - \omega)uu' + bk^3u''') + (2ak - \omega)(u'\Lambda - D_\zeta(u\Lambda)) - bk^3D_\zeta^3\Lambda = 0. \tag{5.4}$$

Solving Eq (5.4) for multiplier Λ, we obtain

$$\Lambda(\zeta, u) = c_1u + c_2. \tag{5.5}$$

From (2.9) and (5.5), we obtain the conserved quantities

$$T^\zeta = \frac{1}{6}u(\zeta)^2(2ak - \omega)\left(2c_1u(\zeta) + 3c_2\right) + bk^3\left(c_2u''(\zeta) + c_1\left(u(\zeta)u''(\zeta) - \frac{1}{2}u'(\zeta)^2\right)\right). \tag{5.6}$$

From (5.6), we can obtain the functional

$$J = \int_0^\infty (\frac{1}{6}u(\zeta)^2(2ak - \omega)\left(2c_1u(\zeta) + 3c_2\right) + bk^3\left(c_2u''(\zeta) + c_1\left(u(\zeta)u''(\zeta) - \frac{1}{2}u'(\zeta)^2\right)\right))d\zeta. \tag{5.7}$$

According to the Ritz-like method, we search a solitary wave solution in the form

$$u(\zeta) = Asech(\zeta). \tag{5.8}$$

Substituting Eq (5.8) into Eq (5.7) , we have

$$J = \frac{1}{12}A^2\left(c_1\left(-\pi A(\omega - 2ak) - 6bk^3\right) + 6c_2(2ak - \omega)\right). \tag{5.9}$$

To find the constant A , we solve the following equation

$$\frac{\partial J}{\partial A} = \frac{1}{4}A\left(c_1\left(-\pi A(\omega - 2ak) - 4bk^3\right) + 4c_2(2ak - \omega)\right) = 0. \tag{5.10}$$

From (5.10) , we obtain

$$A = \begin{cases} \frac{4bk^3}{\pi(2ak-\omega)}, & c_1 = 1, \quad c_2 = 0,\\ \frac{4\left(-2ak+bk^3+\omega\right)}{\pi(2ak-\omega)}, & c_1 = 1, \quad c_2 = 1. \end{cases} \tag{5.11}$$

Therefore, the solitary wave solutions for the 2K(2,2,1) equation are

$$u(x,t)=\begin{cases}\frac{4bk^3}{\pi(2ak-\omega)}sech(kx-\omega t+\zeta_0), & c_1=1, c_2=0,\\ \frac{4\left(-2ak+bk^3+\omega\right)}{\pi(2ak-\omega)}sech(kx-\omega t+\zeta_0), & c_1=1, c_2=1.\end{cases} \tag{5.12}$$

The 3D graphs in Figures 9 and 10 describe the shape of solitons for waves $u(x,t)$, while 2D graphs depict the movement of the soliton waves along x-direction as time increases for Eq (5.12), a contour plot can also be useful for describing the movement of the solitons for (5.12).

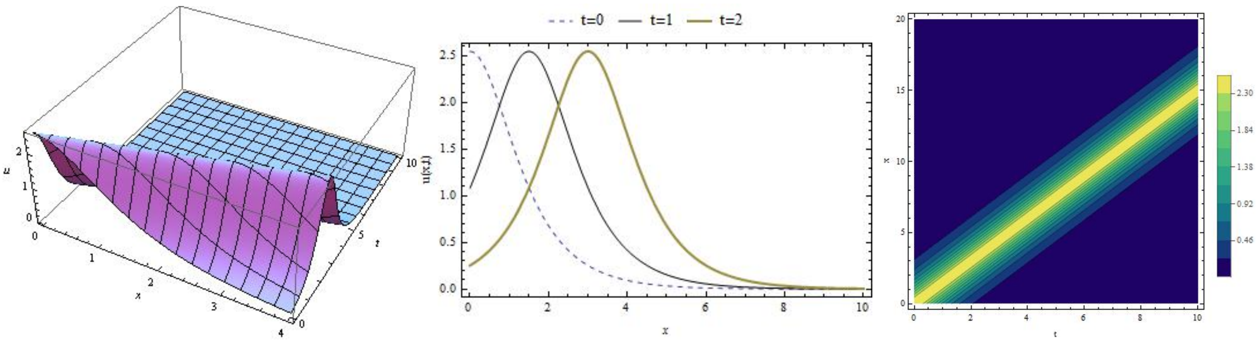

Figure 9. Shows the soliton solution of Eq (5.12) for $c_1=1, c_2=0$.

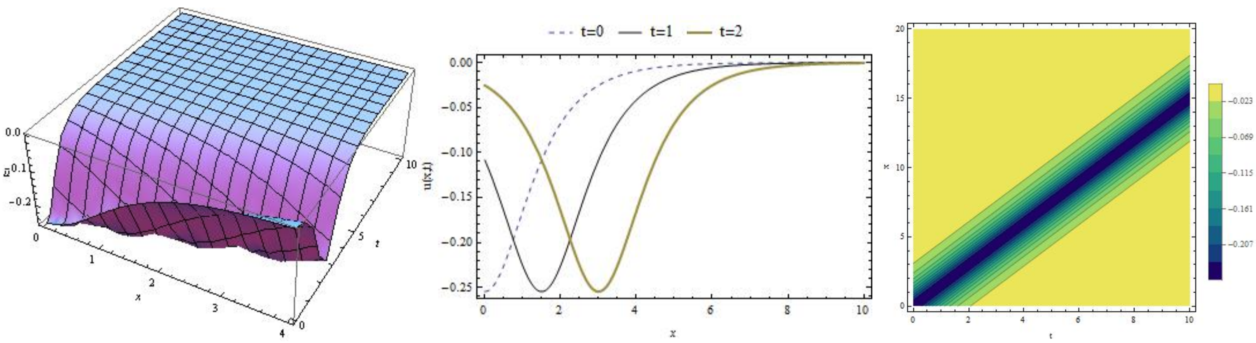

Figure 10. Shows the soliton solution of Eq (5.12) for $c_1=1, c_2=1$.

For another soliton solution (3.13), we obtain

$$J=\frac{1}{45}F^2\left(4c_1\left(4aFk-2F\omega-9bk^3\right)+15c_2(2ak-\omega)\right). \tag{5.13}$$

Solving the following equation to find F

$$\frac{\partial J}{\partial F}=\frac{2}{15}F\left(4c_1\left(2aFk-F\omega-3bk^3\right)+5c_2(2ak-\omega)\right)=0. \tag{5.14}$$

From Eq (5.14), we have

$$F=\begin{cases}\frac{3bk^3}{2ak-\omega}, & c_1=1, & c_2=0,\\ \frac{-10ak+12bk^3+5\omega}{4(2ak-\omega)}, & c_1=1, & c_2=1.\end{cases} \tag{5.15}$$

Therefore, the solitary wave solutions for the 2K(2,2,1) equation are as follows:

$$u(x,t) = \begin{cases} \frac{3bk^3}{2ak-\omega} sech^2(kx - \omega t + \zeta_0), & c_1 = 1, c_2 = 0, \\ \frac{-10ak+12bk^3+5\omega}{4(2ak-\omega)} sech^2(kx - \omega t + \zeta_0), & c_1 = 1, c_2 = 1. \end{cases} \tag{5.16}$$

The 3D graphs in Figures 11 and 12 describe nonlinear wave propagation of soliton solutions to the right, while 2D graphs depict the movement of the solitons obtained from Eq (5.16) for different times and a contour plot describes the movement of the solitons.

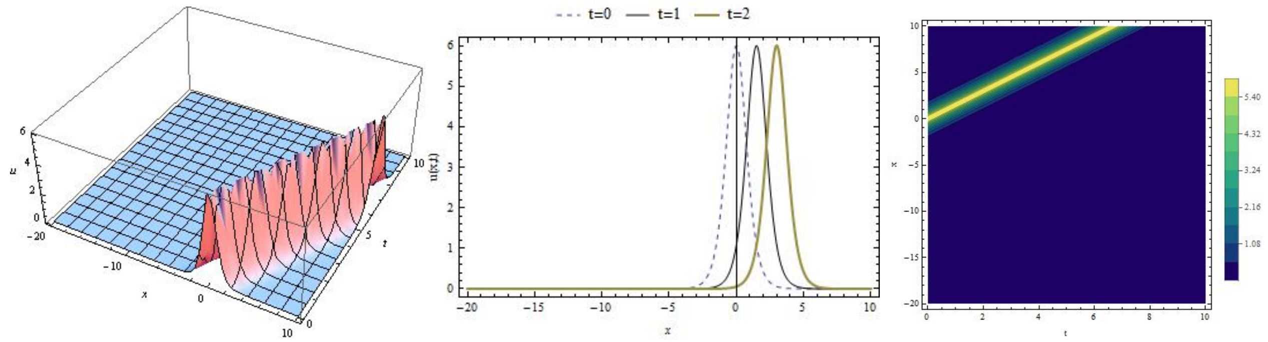

Figure 11. Shows the soliton solution of Eq (5.16) for $c_1 = 1, c_2 = 0$.

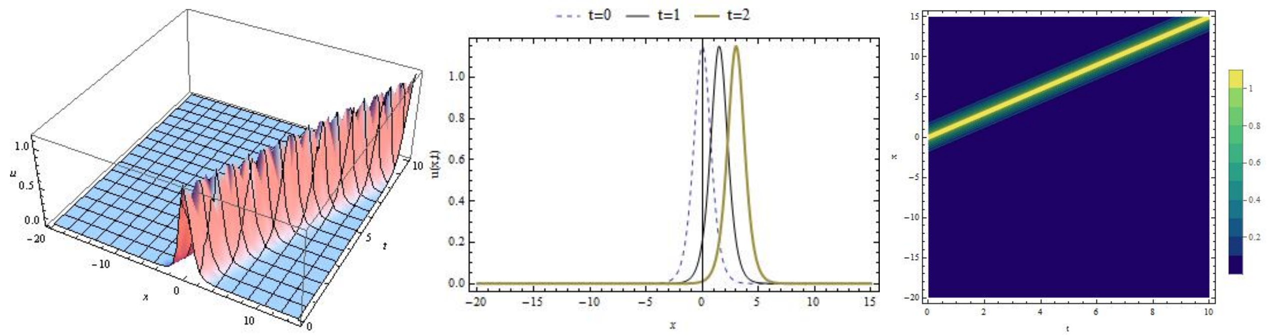

Figure 12. Shows the soliton solution of Eq (5.16) for $c_1 = 1, c_2 = 1$.

6. Conservation laws and soliton solutions for the modified nonlinearly dispersive KdV equations 3K(3,2,2) equation

The modified nonlinearly dispersive KdV equations 3K(3,2,2) equation [33] take the form

$$u^2u_t + a(u^2)_x + b(u^2)_{xxx} = 0, \tag{6.1}$$

where a and b are constants.
The four conserved vectors [28]

$$T_1^t = \frac{u^5}{5},$$
$$T_1^x = \frac{a}{2}u^4 + bu^2(u^2)_{xx} - 2bu^2u_x^2,$$

$$T_2^t = au^2, \qquad T_2^x = au^2 + 2(uu_{xx} + u_x^2),$$
$$T_3^t = \frac{1}{3}u^3 \sin(\sqrt{\frac{a}{b}}x), \qquad T_3^x = b(2(uu_{xx} + u_x^2)\sin(\sqrt{\frac{a}{b}}x) - 2\sqrt{\frac{a}{b}}uu_x\cos(\sqrt{\frac{a}{b}}x)),$$
$$T_4^t = \frac{1}{3}u^3 \cos(\sqrt{\frac{a}{b}}x), \qquad T_4^x = b(2(uu_{xx} + u_x^2)\cos(\sqrt{\frac{a}{b}}x) + 2\sqrt{\frac{a}{b}}uu_x\sin(\sqrt{\frac{a}{b}}x)). \quad (6.2)$$

By converting Eq (6.1) into an ODE

$$(2ak - \omega u)uu' + 2k^3(uu''' + 3u'u'') = 0. \quad (6.3)$$

The multiplier $\Lambda(\zeta, u(\zeta))$, can be obtained as follows

$$\Lambda_u((2ak - \omega u)uu' + 2k^3(uu''' + 3u'u'')) + ((2ak - 2\omega u)u' + 2k^3u''')\Lambda$$
$$-D_\zeta((2ak - \omega u)u + 6k^3u'')\Lambda) - 2k^3D_\zeta^3(u\Lambda) = 0. \quad (6.4)$$

Solving Eq (6.4) for multiplier Λ, we obtain

$$\Lambda(\zeta, u) = c_1 + c_2u^2. \quad (6.5)$$

From (2.9) and (6.5), we obtain the conserved quantities

$$\begin{aligned} T^\zeta = {} & \frac{1}{2}aku(\zeta)^2\left(c_2u(\zeta)^2 + 2c_1\right) \\ & +2k^3\left(c_2u(\zeta)^3u''(\zeta) + \frac{3}{2}c_1u'(\zeta)^2 + c_1\left(u(\zeta)u''(\zeta) - \frac{1}{2}u'(\zeta)^2\right)\right) \\ & -\frac{1}{15}\omega u(\zeta)^3\left(3c_2u(\zeta)^2 + 5c_1\right). \end{aligned} \quad (6.6)$$

From (6.6), we can get the functional

$$\begin{aligned} J = {} & \int_0^\infty (\frac{1}{2}aku(\zeta)^2\left(c_2u(\zeta)^2 + 2c_1\right) \\ & +2k^3\left(c_2u(\zeta)^3u''(\zeta) + \frac{3}{2}c_1u'(\zeta)^2 + c_1\left(u(\zeta)u''(\zeta) - \frac{1}{2}u'(\zeta)^2\right)\right) \\ & -\frac{1}{15}\omega u(\zeta)^3\left(3c_2u(\zeta)^2 + 5c_1\right))d\zeta. \end{aligned} \quad (6.7)$$

According to the Ritz-like method , we search a solitary wave solution in the form

$$u(\zeta) = Asech(\zeta). \quad (6.8)$$

Substituting Eq (6.8) into Eq (6.7) , we have

$$J = c_1\left(aA^2k - \frac{1}{12}\pi A^3\omega\right) - \frac{1}{240}A^4c_2\left(-80ak + 9\pi A\omega + 192k^3\right). \quad (6.9)$$

To find the constant A , we solve the following equation

$$\frac{\partial J}{\partial A} = -\frac{1}{240}A^3c_2\left(-320ak + 45\pi A\omega + 768k^3\right) - \frac{1}{4}Ac_1(\pi A\omega - 8ak) = 0. \quad (6.10)$$

From (6.10) , we obtain

$$A = \begin{cases} \frac{8ak}{\pi\omega}, & c_1 = 1, \quad c_2 = 0, \\ \frac{64\left(5ak-12k^3\right)}{45\pi\omega}, & c_1 = 0, \quad c_2 = 1. \end{cases} \tag{6.11}$$

Therefore, the solitary wave solutions for the 3K(3,2,2) equation are

$$u(x,t) = \begin{cases} \frac{8ak}{\pi\omega} sech(kx - \omega t + \zeta_0), & c_1 = 1, c_2 = 0, \\ \frac{64\left(5ak-12k^3\right)}{45\pi\omega} sech(kx - \omega t + \zeta_0), & c_1 = 0, c_2 = 1. \end{cases} \tag{6.12}$$

The 3D graphs in Figures 13 and 14 describe the shape of solitons for waves $u(x,t)$, while 2D graphs depict the movement of the soliton waves along x-direction as time increases for Eq (6.12), a contour plot can also be useful for describing the properties of the solitary waves for (6.12).

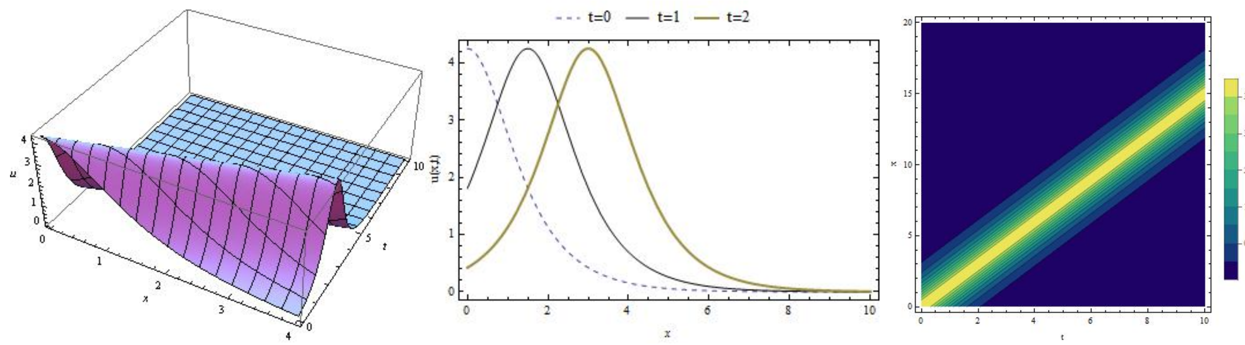

Figure 13. Shows the soliton solution of Eq (6.12) for $c_1 = 1, c_2 = 0$.

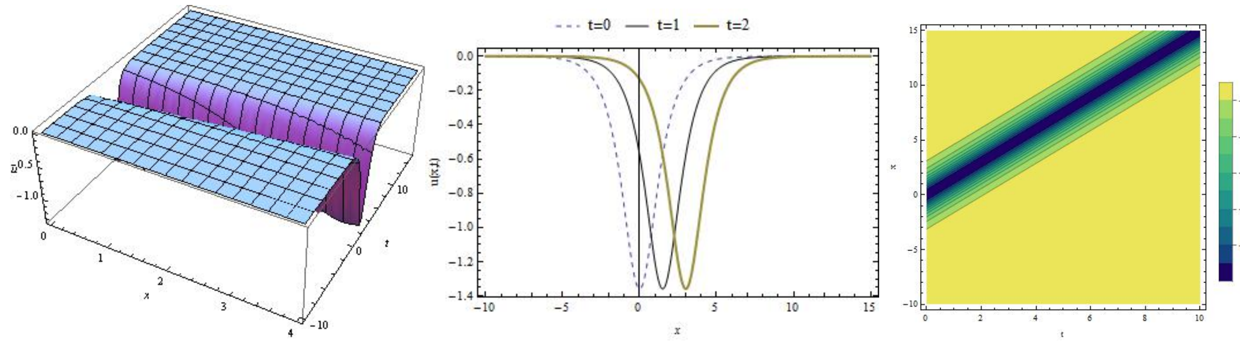

Figure 14. Shows the soliton solution of Eq (6.12) for $c_1 = 0, c_2 = 1$.

For another soliton solution (3.13), we obtain

$$J = \frac{2F^2\left(35c_1(15ak - 4F\omega) - 4F^2c_2\left(-45ak + 16F\omega + 240k^3\right)\right)}{1575}. \tag{6.13}$$

Solving the following equation to find F

$$\frac{\partial J}{\partial F} = \frac{4}{315}A\left(21c_1(5ak - 2A\omega) - 8A^2c_2\left(-9ak + 4A\omega + 48k^3\right)\right) = 0. \tag{6.14}$$

From Eq (6.14), we have

$$F = \begin{cases} \frac{5ak}{2\omega}, & c_1 = 1, \quad c_2 = 0, \\ \frac{3\left(3ak-16k^3\right)}{4\omega}, & c_1 = 0, \quad c_2 = 1. \end{cases} \tag{6.15}$$

Therefore, the solitary wave solutions for the 3K(3,2,2) equation are as follows:

$$u(x,t) = \begin{cases} \frac{5ak}{2\omega} sech^2(kx - \omega t + \zeta_0), & c_1 = 1, c_2 = 0, \\ \frac{3(3ak-16k^3)}{4\omega} sech^2(kx - \omega t + \zeta_0), & c_1 = 0, c_2 = 1. \end{cases} \quad (6.16)$$

The 3D graphs in Figures 15 and 16 describe nonlinear wave propagation of soliton solutions, while 2D graphs depict the movement of the solitons obtained from Eq (6.16) for different times and a contour plot describes the movement of the solitons.

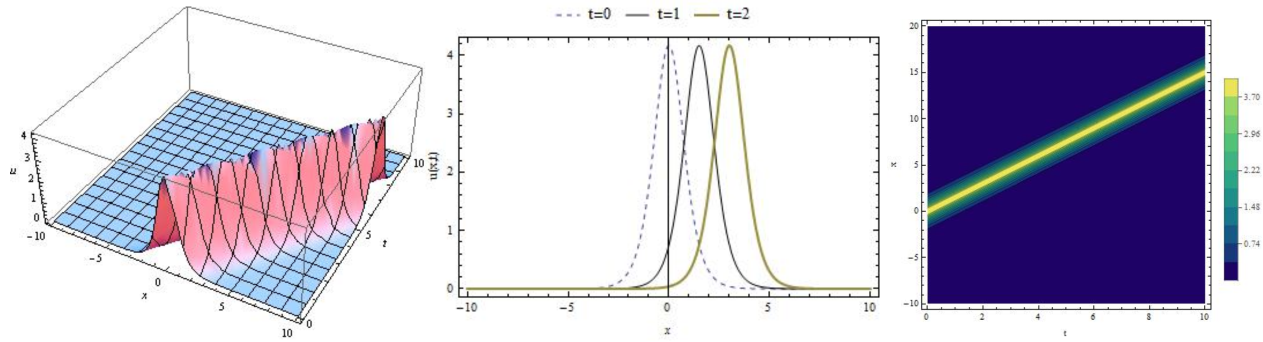

Figure 15. Shows the soliton solution of Eq (6.16) for $c_1 = 1, c_2 = 0$.

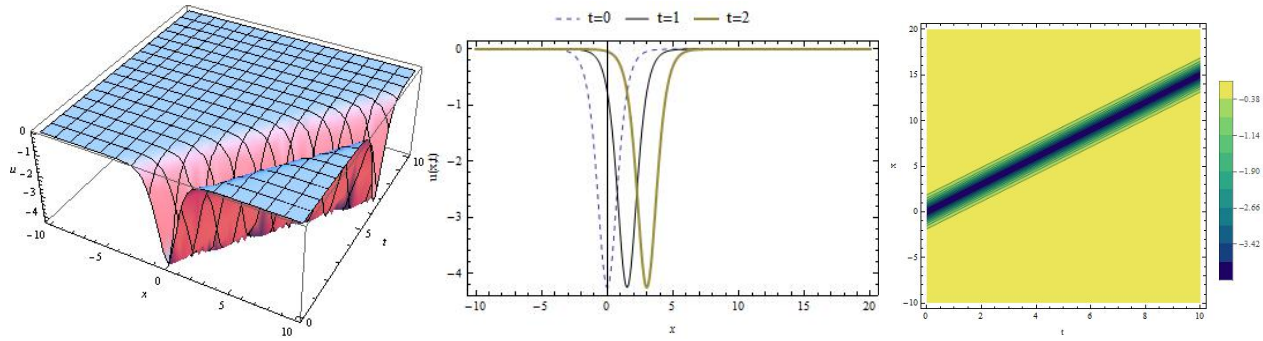

Figure 16. Shows the soliton solution of Eq (6.16) for $c_1 = 0, c_2 = 1$.

7. Conclusions

In this paper we derive the soliton solutions directly from the conserved quantities for Benjamin-Bona-Mahoney equation with dual-power law nonlinearity (BBM), modified regularized long wave (MRLW) equation and modified nonlinearly dispersive KdV equations 2K(2,2,1) and 3K(3,2,2) equation, which are constructed by multiplier approach (variational derivative method). The multiplier approach on Benjamin-Bona-Mahoney equation with dual-power law nonlinearity (BBM) and modified regularized long wave (MRLW) equation gave two multipliers of $\Lambda(x,t,u)$, and thus two conserved vectors obtained for each case. For the modified nonlinearly dispersive KdV equations 2K(2,2,1) equation, the multiplier approach yielded three multipliers and thus three conserved vectors were obtained. For the modified nonlinearly dispersive KdV equations 3K(3,2,2) equation, the multiplier approach yielded four multipliers and thus four conserved vectors were obtained. So, we can say that the proposed method is effective and an alternative mathematical tool for generating soliton solutions. The main results is derived the functional (2.11) using the conserved quantities and then the soliton solutions for the models. This is another method for deriving the variational principles for the nonlinear evolution equations and we can easily extend this work to all soliton equations in future works. Finally, we give the numerical simulations for these solutions.

Conflict of interests

The author declares no conflicts of interest in this paper.

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