AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(8): 15041–15063.
DOI: 10.3934/math.2022824
Received: 21 April 2022
Revised: 29 May 2022
Accepted: 09 June 2022
Published: 14 June 2022

Research article

Interval valued Hadamard-Fejér and Pachpatte Type inequalities pertaining to a new fractional integral operator with exponential kernel

Hari Mohan Srivastava[1,2,3,4], **Soubhagya Kumar Sahoo**[5], **Pshtiwan Othman Mohammed**[6], **Bibhakar Kodamasingh**[5], **Kamsing Nonlaopon**[7,*] **and Khadijah M. Abualnaja**[8]

[1] Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
[2] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[3] Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan
[4] Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy
[5] Department of Mathematics, Institute of Technical Education and Research, Siksha 'O' Anusandhan University, Bhubaneswar 751030, India
[6] Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Iraq
[7] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
[8] Department of Mathematics and Statistics, College of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia

* **Correspondence:** Email: nkamsi@kku.ac.th; Tel: +66866421582.

Abstract: The aim of this research is to combine the concept of inequalities with fractional integral operators, which are the focus of attention due to their properties and frequency of usage. By using a novel fractional integral operator that has an exponential function in its kernel, we establish a new Hermite-Hadamard type integral inequality for an LR-convex interval-valued function. We also prove new fractional-order variants of the Fejér type inequalities and the Pachpatte type inequalities in the setting of pseudo-order relations. By showing several numerical examples, we further validate the accuracy of the results that we have derived in this study. We believe that the results, presented in this article are novel and that they will be beneficial in encouraging future research in this field.

Keywords: LR-convex interval-valued function; fractional integral operator; Hermite–Hadamard type inequality; Pachpatte type inequality; Fejér type inequality; exponential kernel
Mathematics Subject Classification: 26A51, 26A33, 26D10

1. Introduction

Fractional calculus has become increasingly important in domains such as inequality theory, applied mathematics, sciences, and engineering in recent years. Fractional integrals are used to describe the attributes of many physical processes, such as those in physics and medicine [1], epidemiology [2], fluid mechanics [3], nanotechnology [4], economy [5], bioengineering [6], and so on.

It is worth emphasizing that the concept of fractional calculus was first put forward by Leibniz and L'Hôpital (1695). Nevertheless, additional contributions to the topic of fractional calculus and its numerous applications were made by such other mathematicians as Riemann, Liouville, Grunwald, Letnikov, Erdélyi, Kober, and others. Fractional calculus has drawn the attention of many physical and engineering professionals due to its behaviour and capacity to tackle a wide range of real-world problems (see, for details, [7]; see also the recent survey-cum-expository review articles [8, 9]).

For fractional integrals, various integral inequalities have been found. These integrals are useful for generalizing important and well-known integral inequalities. One kind of integral inequalities is the Hermite-Hadamard integral inequality. It is widely used in the literature, and it specifies both the necessary and sufficient conditions for a function to be convex. Sarikaya *et al.* [10] generalized the Hermite-Hadamard inequalities using Riemann-Liouville fractional integrals. The conclusions by Sarikaya *et al.* [10] were also extended to Hermite-Hadamard-Fejér type inequalities by Iscan [11]. Chen [12] used the methodology of Sarikaya *et al.* [10] and established fractional Hermite-Hadamard type integral inequalities by using the product of two convex functions. Guessab [13] proposed Jensen type inequalities on convex polytopes and investigated the error in convex function approximation. Guessab *et al.* [14] discovered a Korovkin-type theorem that shows that a series of operators does not have to have an identity limit. Guessab also worked on the concepts such as higher order convexity [15] and bivariate Hermite interpolant (see [16]). The presentation of several well-known integral inequalities by employing various novel concepts of fractional integral operators has grown increasingly popular among mathematicians in recent years. In this regard, one can consult the results described in [10, 17–22].

Interval analysis is subdivided into the set-valued analysis. There is no doubting the importance of interval analysis in both pure and applied research. One of the early uses of interval analysis was the error limits of numerical solutions of finite state machines. However, in recent times, interval analysis has been a crucial component of mathematical and computer models for addressing interval uncertainty.

To generate more generalised results, we used a new fractional integral operator in this study. This is due to the fact that this fractional operator has an exponential kernel. The aforementioned fractional analogues of functional inequalities do not follow from our conclusions, which is a distinction between our results and existing generalisations. Many experts have presented extensions of the Hermite-Hadamard inequality with various fractional integral operators, but there is no exponential characteristic in their results. This research sparked an interest in creating more generalised fractional inequalities with an exponential function as the kernel. Moreover, the application of interval valued analysis to the main findings brings a new direction to the field of inequalities. We have incorporated the concepts of a Pseudo-order relation with Interval valued analysis to present new inequalities. Although we can find many studies on the growth of fractional order integral inequalities involving convex functions, there are still numerous gaps to be filled for fractional integral inequalities involving various types of convex functions. As a result, the primary goal of this study is to establish novel Hermite-Hadamard, Pachppate and Fejér type inequalities for LR-convex interval-valued function utilizing fractional integral operators.

2. Preliminaries

We denote by $\mathcal{M}_C$ the collection of all closed and bounded intervals of $\mathbb{R}$, given by

$$\mathcal{M}_C = \{[\mathit{\Lambda}_*, \mathit{\Lambda}^*] : \mathit{\Lambda}_*, \mathit{\Lambda}^* \in \mathbb{R} \text{ and } \mathit{\Lambda}_* \leqq \mathit{\Lambda}^*\}.$$

If $\mathit{\Lambda}_* \geqq 0$, then we say the inerval $[\mathit{\Lambda}_*, \mathit{\Lambda}^*]$ is positive interval. The set of all positive intervals is denoted by $\mathcal{M}_C^+$ and is given as

$$\mathcal{M}_C^+ = \{[\mathit{\Lambda}_*, \mathit{\Lambda}^*] : \mathit{\Lambda}_*, \mathit{\Lambda}^* \in \mathcal{M}_C \text{ and } \mathit{\Lambda}_* \geqq 0\}.$$

Let us now study how the intervals behave under some arithmetic operations such as addition, multiplication and scalar multiplication. If $[\Lambda_*, \Lambda^*], [\mathit{\Lambda}_*, \mathit{\Lambda}^*] \in \mathcal{M}_C$ and $\kappa \in \mathbb{R}$, then

$$[\Lambda_*, \Lambda^*] + [\mathit{\Lambda}_*, \mathit{\Lambda}^*] = [\Lambda_* + \mathit{\Lambda}_*, \Lambda^* + \mathit{\Lambda}^*]$$

$$\begin{aligned}&[\Lambda_*, \Lambda^*] \times [\mathit{\Lambda}_*, \mathit{\Lambda}^*]\\ &= [\min\{\Lambda_*\mathit{\Lambda}_*, \Lambda^*\mathit{\Lambda}_*, \Lambda_*\mathit{\Lambda}^*, \Lambda^*\mathit{\Lambda}^*\}, \max\{\Lambda_*\mathit{\Lambda}_*, \Lambda^*\mathit{\Lambda}_*, \Lambda_*\mathit{\Lambda}^*, \Lambda^*\mathit{\Lambda}^*\}]\end{aligned}$$

$$\kappa.[\Lambda_*, \Lambda^*] = \begin{cases} [\kappa\Lambda_*, \kappa\Lambda^*] & (\kappa > 0) \\ \{0\} & (\kappa = 0) \\ [\kappa\Lambda^*, \kappa\Lambda_*] & (\kappa < 0), \end{cases}$$

respectively.

For $[\Lambda_*, \ \Lambda^*], \ [\mathit{\Lambda}_*, \ \mathit{\Lambda}^*] \in \mathcal{M}_C$, the inclusion " $\subseteq$ " is given as follows

$[\Lambda_*, \ \Lambda^*] \subseteq [\mathit{\Lambda}_*, \ \mathit{\Lambda}^*]$, if and only if, $\mathit{\Lambda}_* \leqq \Lambda_*, \Lambda^* \leqq \mathit{\Lambda}^*$.

Khan *et al.* [23] presented the following properties of the newly introduced idea (LR-conex interval valued functions).

Remark 2.1 (see [23])**.** *1. The pseudo order relation " $\leqq_p$ " defined on $\mathcal{M}_C$ by $[\Lambda_*, \Lambda^*] \leqq_p [\mathit{\Lambda}_*, \mathit{\Lambda}^*]$ holds true if and only if $\Lambda_* \leqq \mathit{\Lambda}_*, \Lambda^* \leqq \mathit{\Lambda}^*$, for all $[\Lambda_*, \Lambda^*], [\mathit{\Lambda}_*, \mathit{\Lambda}^*] \in \mathcal{M}_C$,. The relation $[\Lambda_*, \Lambda^*] \leqq_p [\mathit{\Lambda}_*, \mathit{\Lambda}^*]$ similar to $[\Lambda_*, \Lambda^*] \leqq [\mathit{\Lambda}_*, \mathit{\Lambda}^*]$ on $\mathcal{M}_C$.*

*2. It is easily seen that " $\leqq_p$ " looks same as that of "left and right" on the real line $\mathbb{R}$, and hence can also be called as "*LR*" order.*

Moore [24] is credited for introducing the concept of the Riemann integral for interval valued functions, which is defined as follows:

Theorem 2.1. *Let* $\mathtt{S} : [\mathfrak{g}, \mathfrak{p}] \subset \mathbb{R} \to \mathcal{M}_C$ *be an I-V-F on such that*

$$\mathtt{S}(\upsilon) = [\mathtt{S}_*(\upsilon), \mathtt{S}^*(\upsilon)].$$

Then $\mathtt{S}$ *is Riemann integrable over* $[\mathfrak{g}, \mathfrak{p}]$ *iff both* $\mathtt{S}_*$ *and* $\mathtt{S}^*$ *are Riemann integrable over* $[\mathfrak{g}, \mathfrak{p}]$.

$$(IR)\int_{\mathfrak{g}}^{\mathfrak{p}} \mathtt{S}(\upsilon)\, d\upsilon = [(R)\int_{\mathfrak{g}}^{\mathfrak{p}} \mathtt{S}_*(\upsilon)\, d\upsilon, (R)\int_{\mathfrak{g}}^{\mathfrak{p}} \mathtt{S}^*(\upsilon)\, d\upsilon].$$

Let us denote the Riemann integrable functions and Riemann integrable interval valued functions by $R_{[\mathfrak{g},\mathfrak{p}]}$ and $IR_{[\mathfrak{g},\mathfrak{p}]}$, respectively.

Definition 2.1 (see [10]; see also [7, 25])**.** *Let* $\mathtt{S} \in \mathcal{L}\left([\mathfrak{g}, \mathfrak{p}], \mathcal{M}_C^+\right)$. *Then the left- and right- interval Riemann-Liouville fractional integral of order* $\alpha > 0$ *is defined by*

$$\mathcal{I}_{\mathfrak{g}^+}^{\alpha} \mathtt{S}(\upsilon) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{g}}^{\upsilon} (\upsilon - \mathtt{r})^{\alpha-1} \mathtt{S}(\mathtt{r}) d\mathtt{r}, \qquad (\upsilon > \mathfrak{g})$$

and

$$\mathcal{I}_{\mathfrak{p}^-}^{\alpha} \mathtt{S}(\upsilon) = \frac{1}{\Gamma(\alpha)} \int_{\upsilon}^{\mathfrak{p}} (\mathtt{r} - \upsilon)^{\alpha-1} \mathtt{S}(\mathtt{r}) d\mathtt{r}, \qquad (\upsilon < \mathfrak{p}),$$

respectively, where

$$\Gamma(\upsilon) = \int_0^{\infty} \mathtt{r}^{\upsilon-1} e^{-\mathtt{r}}\, d\mathtt{r},$$

is the Euler gamma function.

We now recall a potentially useful fractional integral operator, introduced by Ahmad *et al.* [26], that has an exponential function in its kernel.

Definition 2.2 (see, for details, [26])**.** *Let* $\mathtt{S} \in \mathcal{L}[\mathfrak{g}, \mathfrak{p}]$. *Then the fractional integrals* $\mathcal{I}_{\mathfrak{g}^+}^{\alpha}$ *and* $\mathcal{I}_{\mathfrak{p}^-}^{\alpha}$ *of order* $\alpha > 0$ *are defined as*

$$\mathcal{I}_{\mathfrak{g}^+}^{\alpha} \mathtt{S}(\upsilon) := \frac{1}{\alpha} \int_{\mathfrak{g}}^{\upsilon} e^{-\frac{1-\alpha}{\alpha}(\upsilon-\mathtt{r})} \mathtt{S}(\mathtt{r})\, d\mathtt{r} \qquad (0 \leqq \mathfrak{g} < \upsilon < \mathfrak{p}),$$

and

$$\mathcal{I}_{\mathfrak{p}^-}^{\alpha} \mathtt{S}(\upsilon) := \frac{1}{\alpha} \int_{\upsilon}^{\mathfrak{p}} e^{-\frac{1-\alpha}{\alpha}(\mathtt{r}-\upsilon)} \mathtt{S}(\mathtt{r})\, d\mathtt{r} \qquad (0 \leqq \mathfrak{g} < \upsilon < \mathfrak{p}),$$

respectively.

Definition 2.3 (see [27])**.** *The interval valued function* $\mathtt{S} : \mathrm{J} \to \mathcal{M}_C^+$ *is said to be Left-Right-convex interval valued function on convex set* J *iff*

$$\mathtt{S}(\mathtt{r}\mathfrak{g} + (1-\mathtt{r})\,\mathfrak{p}) \leqq_p \mathtt{r}\mathtt{S}(\mathfrak{g}) + (1-\mathtt{r})\mathtt{S}(\mathfrak{p}), \tag{2.1}$$

holds true for all $\mathfrak{g},\ \mathfrak{p} \in \mathrm{J}$ and $\mathtt{r} \in [0,\ 1]$.

$\mathtt{S}$ is said to be LR-concave on J, if inequality (2.1) is reversed, and if $\mathtt{S}$ is both LR-convex and LR-concave function, then it is said to be affine.

Theorem 2.2 (see [27]). *Let* J *be a convex set and* $\mathtt{S} : \mathrm{J} \to \mathcal{M}_C^+$ *be an interval valued function such that*

$$\mathtt{S}(\upsilon) = [\mathtt{S}_*(\upsilon), \ \mathtt{S}^*(\upsilon)], \ \forall \ \upsilon \in \mathrm{J},$$

for all $\upsilon \in \mathrm{J}$. *Then* S *is said to be left-right-convex interval valued function on* J, *if and only if both* $\mathtt{S}_*(\upsilon)$ *and* $\mathtt{S}^*(\upsilon)$ *are convex functions.*

Budak *et al.* [25] examined the Hermite-Hadamard and Pachpatte type inequalities for interval-valued convex functions via fractional integrals. The following are some connected outcomes:

Theorem 2.3. *Let* $\mathtt{S} : [\mathfrak{g}, \mathfrak{p}] \longrightarrow \mathbb{R}_I^+$ *be an interval-valued convex function with*

$$\mathtt{S}(\upsilon) = [\mathtt{S}_*(\upsilon), \mathtt{S}^*(\upsilon)].$$

Then, the fractional-order H-H *integral inequality of order* $\alpha > 0$ *for interval-valued functions is given by*

$$\mathtt{S}\left(\frac{\mathfrak{g}+\mathfrak{g}}{2}\right) \supseteq \frac{\Gamma(\alpha+1)}{2(\mathfrak{p}-\mathfrak{g})^\alpha}\left[\mathcal{I}^\alpha_{(\mathfrak{g})^+}\mathtt{S}(\mathfrak{p}) + \mathcal{I}^\alpha_{(\mathfrak{p})^-}\mathtt{S}(\mathfrak{g})\right] \supseteq \frac{\mathtt{S}(\mathfrak{g})+\mathtt{S}(\mathfrak{p})}{2}.$$

Theorem 2.4. *If* $\mathtt{S}, \Upsilon : [\mathfrak{g}, \mathfrak{p}] \longrightarrow \mathbb{R}_I^+$ *are two interval-valued convex functions with*

$$\mathtt{S}(\upsilon) = [\mathtt{S}_*(\upsilon), \mathtt{S}^*(\upsilon)],$$

and

$$\Upsilon(\upsilon) = [\Upsilon_*(\upsilon), \Upsilon^*(\upsilon)],$$

then the fractional-order H-H *type inequality for* $\alpha > 0$ *holds true as follows*:

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(\mathfrak{p}-\mathfrak{g})^\alpha}\left[\mathcal{I}^\alpha_{(\mathfrak{g})^+}\mathtt{S}(\mathfrak{p})\Upsilon(\mathfrak{p}) + \mathcal{I}^\alpha_{(\mathfrak{p})^-}\mathtt{S}(\mathfrak{g})\Upsilon(\mathfrak{g})\right]\\ &\supseteq \left[\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\Psi(\mathfrak{g},\mathfrak{p}) + \left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\Omega(\mathfrak{g},\mathfrak{p}),\end{aligned}$$

where

$$\Psi(\mathfrak{g},\mathfrak{p}) = [\mathtt{S}(\mathfrak{g})\Upsilon(\mathfrak{g}) + \mathtt{S}(\mathfrak{p})\Upsilon(\mathfrak{p})],$$

and

$$\Omega(\mathfrak{g},\mathfrak{p}) = [\mathtt{S}(\mathfrak{g})\Upsilon(\mathfrak{p}) + \mathtt{S}(\mathfrak{p})\Upsilon(\mathfrak{g})].$$

Theorem 2.5. *Let* $\mathtt{S}, \Upsilon : [\mathfrak{g}, \mathfrak{p}] \longrightarrow \mathbb{R}_I^+$ *be two interval-valued convex functions with*

$$\mathtt{S}(\upsilon) = [\mathtt{S}_*(\upsilon), \mathtt{S}^*(\upsilon)],$$

and

$$\Upsilon(\upsilon) = [\Upsilon_*(\upsilon), \Upsilon^*(\upsilon)].$$

Then the fractional-order H-H *type inequality for* $\alpha > 0$ *is given as*

$$\begin{aligned}2\mathtt{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\Upsilon\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) &\supseteq \frac{\Gamma(\alpha+1)}{2(\mathfrak{p}-\mathfrak{g})^\alpha}\left[\mathcal{I}^\alpha_{(\mathfrak{g})^+}\mathtt{S}(\mathfrak{p})\Upsilon(\mathfrak{p}) + \mathcal{I}^\alpha_{(\mathfrak{p})^-}\mathtt{S}(\mathfrak{g})\Upsilon(\mathfrak{g})\right]\\ &+\left[\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\Omega(\mathfrak{g},\mathfrak{p}) + \left[\frac{\alpha}{(\alpha+1)(\alpha+2)}\right]\Psi(\mathfrak{g},\mathfrak{p}).\end{aligned}$$

Many scientists have linked integral inequalities with interval-valued functions (IVFs) in recent decades, giving numerous important findings. Costa [28] postulated Opial-type inequalities as a basis for IVF inequalities. The generalized Hukuhara derivative was used by Chalco-Cano [29] to look into Ostrowski type inequalities for interval-valued functions. In [30], Roman-Flores deduced the Minkowski type inequalities and Beckenbach's type inequalities for interval-valued functions. Zhao *et al.* [31] recently improved the concept by introducing $\mathrm{H}-\mathrm{H}$ type inequalities for interval-valued coordinated functions. The idea of interval-valued analysis was applied to reinforce this inequality via some innovative concepts such as coordinated convexity [33, 34], and preinvex functions [35–37]. Recently, Lai *et al.* [38] extended the concept of interval valued preinvex functions to interval valued coordinated preinvex function. Kalsoom *et al.* [39] proved some Hermite-Hadamard-Fejér type fractional inequalities for h-convex and Harmonically h-convex interval valued functions. Shi *et al.* [40] investigated some related inequalities employing coordinated log-h-convex interval valued functions.

Khan and his colleagues recently extended this (apparently new) concept to include LR-convex interval valued functions and fuzzy convex interval-valued functions, both of which take a pseudo-order relation into account. To illustrate inequalities of the Hermite-Hadamard, Hermite-Hadamard-Fejér, and Pachpatte types, his team utilised LR-h-convex interval-valued functions (see [41]), LR-χ-preinvex functions (see [42]), and LR-(h_1, h_2)-convex interval-valued functions (see [43]). For various recent achievements related to the notion of fuzzy interval-valued analysis of some well-known integral inequalities, we refer to Khan *et al.* [44–46].

Motivated by the above articles, we generalize the Hermite-Hadamard inequality for LR-convex interval valued functions and product of two LR-convex interval valued functions. Moreover, we also prove Hermite-Hadamard-Fejér type inequalities.We believe that our findings will motivate more researchers to work in this subject, particularly in the area of interval valued concepts.

The following is a breakdown of our current investigation. After reviewing the pre-requisite and relevant facts regarding the related inequalities and the interval-valued analysis in Section 2, we derive some new versions of the interval-valued H-H type inequalities and Pachpatte type inequalities in Section 3. Inequalities of the Hermite-Hadamard-Fejér type in the frame of LR interval-valued functions are also established in Section 4. Some examples are also reviewed to determine whether the established consequences are useful. In Section 5, a brief conclusion is offered, as well as potential areas for future research that are related to the findings presented in this paper.

3. Left and Right interval valued fractional integral inequalities

This section is devoted towards the main results of our manuscript, where different new versions of the Hermite-Hadamard type inequalities incorporated with LR-convex interval valued functions are discussed.

For brevity we will denote $\mu = \frac{1-\alpha}{\alpha}(\mathfrak{p} - \mathfrak{g})$.

Theorem 3.1. *Let* $\mathsf{S} : [\mathfrak{g}, \mathfrak{p}] \to \mathcal{M}_C^+$ *be an* LR*-convex I-V-F on* $[\mathfrak{g}, \mathfrak{p}]$*, which is given by*

$$\mathsf{S}(\upsilon) = [\mathsf{S}_*(\upsilon),\ \mathsf{S}^*(\upsilon)],$$

for all $\upsilon \in [\mathfrak{g}, \mathfrak{p}]$. *If* $\mathsf{S} \in \mathcal{L}\left([\mathfrak{g}, \mathfrak{p}], \mathcal{M}_C^+\right)$, *then*

$$\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) \leqq_p \frac{1-\alpha}{2(1-e^{-\mu})}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}(\mathfrak{g})\right] \leqq_p \frac{\mathsf{S}(\mathfrak{g})+\mathsf{S}(\mathfrak{p})}{2}.$$

Proof. Since S is an LR-convex I-V-F, we have

$$2\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) \leqq_p \mathsf{S}\left(\mathtt{r}\mathfrak{g} + (1-\mathtt{r})\,\mathfrak{p}\right) + \mathsf{S}\left((1-\mathtt{r})\,\mathfrak{g} + \mathtt{r}\mathfrak{p}\right).$$

Therefore, we have

$$2\mathsf{S}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) \leqq \mathsf{S}_*\left(\mathtt{r}\mathfrak{g} + (1-\mathtt{r})\,\mathfrak{p}\right) + \mathsf{S}_*\left((1-\mathtt{r})\,\mathfrak{g} + \mathtt{r}\mathfrak{p}\right), \tag{3.1}$$

and

$$2\mathsf{S}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) \leqq \mathsf{S}^*\left(\mathtt{r}\mathfrak{g} + (1-\mathtt{r})\,\mathfrak{p}\right) + \mathsf{S}^*\left((1-\mathtt{r})\,\mathfrak{g} + \mathtt{r}\mathfrak{p}\right). \tag{3.2}$$

Multiplying both sides of the equations (3.1) and (3.2) by $e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}$ and then integrating with respect to $\mathtt{r}$ over $[0, 1]$, we obtain

$$\begin{aligned}
&2\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathsf{S}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)d\mathtt{r} \\
&\qquad \leqq \int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathsf{S}_*\left(\mathtt{r}\mathfrak{g} + (1-\mathtt{r})\,\mathfrak{p}\right)d\mathtt{r} + \int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathsf{S}_*\left((1-\mathtt{r})\,\mathfrak{g} + \mathtt{r}\mathfrak{p}\right)d\mathtt{r},
\end{aligned}$$

and

$$\begin{aligned}
&2\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathsf{S}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)d\mathtt{r} \\
&\qquad \leqq \int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathsf{S}^*\left(\mathtt{r}\mathfrak{g} + (1-\mathtt{r})\,\mathfrak{p}\right)d\mathtt{r} + \int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathsf{S}^*\left((1-\mathtt{r})\,\mathfrak{g} + \mathtt{r}\mathfrak{p}\right)d\mathtt{r},
\end{aligned}$$

respectively.

Now, if we let $\tau = \mathtt{r}\mathfrak{g} + (1-\mathtt{r})\,\mathfrak{p}$ and $\upsilon = (1-\mathtt{r})\,\mathfrak{g} + \mathtt{r}\mathfrak{p}$, then we have

$$\begin{aligned}
\frac{1-e^{-\mu}}{\mu}\mathsf{S}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) &\leqq \frac{1}{2\,(\mathfrak{p}-\mathfrak{g})}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\tau)}\mathsf{S}_*\left(\tau\right)d\tau + \frac{1}{2\,(\mathfrak{p}-\mathfrak{g})}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})}\mathsf{S}_*(\upsilon)d\upsilon \\
&= \frac{\alpha}{2(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}_*(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}_*(\mathfrak{g})\right],
\end{aligned}$$

and

$$\begin{aligned}
\frac{1-e^{-\mu}}{\mu}\mathsf{S}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) &\leqq \frac{1}{2\,(\mathfrak{p}-\mathfrak{g})}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\tau)}\mathsf{S}^*\left(\tau\right)d\tau + \frac{1}{2\,(\mathfrak{p}-\mathfrak{g})}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})}\mathsf{S}^*(\upsilon)d\upsilon \\
&= \frac{\alpha}{2(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}^*(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}^*(\mathfrak{g})\right].
\end{aligned}$$

That is

$$\frac{1-e^{-\mu}}{\mu}\left[\mathsf{S}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right),\mathsf{S}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\right]\leqq_p \frac{\alpha}{2(\mathfrak{p}-\mathfrak{g})}\left[\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\mathsf{S}_*(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\mathsf{S}_*(\mathfrak{g})\right],\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\mathsf{S}^*(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathsf{S}^*(\mathfrak{g})\right]\right],$$

which readily yields

$$\frac{1-e^{-\mu}}{\mu}\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\ \leqq_p \frac{\alpha}{2(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathsf{S}(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathsf{S}(\mathfrak{g})\right]. \tag{3.3}$$

Similarly, we have

$$\frac{\alpha}{2(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathsf{S}(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathsf{S}(\mathfrak{g})\right]\leqq_p \frac{\mathsf{S}(\mathfrak{g})+\mathsf{S}(\mathfrak{p})}{2}. \tag{3.4}$$

From the Eqs (3.3) and (3.4), we have

$$\frac{1-e^{-\mu}}{\mu}\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\ \leqq_p \frac{\alpha}{2(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathsf{S}(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathsf{S}(\mathfrak{g})\right]\leqq_p \frac{\mathsf{S}(\mathfrak{g})+\mathsf{S}(\mathfrak{p})}{2}.$$

This concludes the proof of Theorem 3.1. □

Corollary 3.1. *Let* $\mathsf{S}:[\mathfrak{g},\mathfrak{p}]\to\mathcal{M}_C^+$ *be an* LR*-concave I-V-F on* $[\mathfrak{g},\mathfrak{p}]$, *which is given by*

$$\mathsf{S}(\upsilon)=[\mathsf{S}_*(\upsilon),\ \mathsf{S}^*(\upsilon)],$$

for all $\upsilon\in[\mathfrak{g},\mathfrak{p}]$. *If* $\mathsf{S}\in\mathcal{L}\left([\mathfrak{g},\mathfrak{p}],\mathcal{M}_C^+\right)$, *then*

$$\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\geqq_p \frac{1-\alpha}{2(1-e^{-\mu})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathsf{S}(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathsf{S}(\mathfrak{g})\right]\geqq_p \frac{\mathsf{S}(\mathfrak{g})+\mathsf{S}(\mathfrak{p})}{2}.$$

Remark 3.1. *If we choose* $\alpha\to 1$, *then Theorem 3.1 reduces to the following inequality for* LR*-convex interval valued function given in [47]:*

$$\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\leqq_p \frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}}\mathsf{S}(\upsilon)\,d\upsilon\leqq_p \frac{\mathsf{S}(\mathfrak{g})+\ \mathsf{S}(\mathfrak{p})}{2}.$$

If we choose $\mathsf{S}_*(\upsilon)=\mathsf{S}^*(\upsilon)$ in Theorem 3.1, we recapture the the following fractional integral Hermite-Hadamard type inequality, presented by Ahmad *et al.* (see [26]).

$$\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\ \leqq \frac{1-\alpha}{2(1-e^{-\mu})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathsf{S}(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathsf{S}(\mathfrak{g})\right]\leqq \frac{\mathsf{S}(\mathfrak{g})+\mathsf{S}(\mathfrak{p})}{2}.$$

If we choose $\alpha\to 1$ and $\mathsf{S}_*(\upsilon)=\mathsf{S}^*(\upsilon)$ in Theorem 3.1. Then the classical Hermite-Hadamard type inequality is recovered and is given as follows:

$$\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\leqq \frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}}\mathsf{S}(\upsilon)\,d\upsilon\leqq \frac{\mathsf{S}(\mathfrak{g})+\ \mathsf{S}(\mathfrak{p})}{2}.$$

Example 3.1. *If we take* $\alpha = \frac{1}{2}, \upsilon \in [0, 2]$ *and the following interval valued function* $\mathsf{S}(\upsilon) = [1,\ 2]\left(\upsilon^2\right)$. *Since, the left and right end points* $\mathsf{S}_*(\upsilon) = \upsilon^2$, $\mathsf{S}^*(\upsilon) = 2\upsilon^2$ *are* LR *convex interval valued functions, then* $\mathsf{S}(\upsilon)$ *is* LR-*convex I-V-F. Thus, we obtain*

$$\mathsf{S}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) = 1,$$

$$\mathsf{S}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) = 2,$$

$$\frac{\mathsf{S}_*(\mathfrak{g}) + \mathsf{S}_*(\mathfrak{p})}{2} = 2,$$

and

$$\frac{\mathsf{S}^*(\mathfrak{g}) + \mathsf{S}^*(\mathfrak{p})}{2} = 4.$$

We also note that

$$\frac{1-\alpha}{2(1-e^{-\mu})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathsf{S}_*(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathsf{S}_*(\mathfrak{g})\right] \approx 1.373,$$

and

$$\frac{1-\alpha}{2(1-e^{-\mu})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathsf{S}^*(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathsf{S}^*(\mathfrak{g})\right] \approx 2.746.$$

Therefore, we have

$$[1, 2] \leqq_p [1.373, 2.746] \leqq_p [2, 4].$$

This evidently verifies Theorem 3.1.

Pachpatte type fractional inequalities in interval valued settings

In the next two theorems, we establish Hermite-Hadamard related inequalities employing product of two LR-convex interval valued functions.

Theorem 3.2. *Let* $\mathsf{S}, \Upsilon : [\mathfrak{g}, \mathfrak{p}] \to \mathcal{M}^+_C$ *be two* LR-*convex I-V-Fs on* $[\mathfrak{g}, \mathfrak{p}]$, *such that*

$$\mathsf{S}(\upsilon) = [\mathsf{S}_*(\upsilon),\ \mathsf{S}^*(\upsilon)] \text{ and } \Upsilon(\upsilon) = [\Upsilon_*(\upsilon),\ \Upsilon^*(\upsilon)],$$

for all $\upsilon \in [\mathfrak{g}, \mathfrak{p}]$. *If* $\mathsf{S}.\Upsilon \in \mathcal{L}\left([\mathfrak{g}, \mathfrak{p}], \mathcal{M}^+_C\right)$, *then*

$$\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathsf{S}(\mathfrak{p})\Upsilon(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\mathsf{S}(\mathfrak{g})\Upsilon(\mathfrak{g})\right]$$
$$\leqq_p \frac{\mu^2 - 2\mu + 4 - (\mu^2 + 2\mu + 4)e^{-\mu}}{\mu^3}\aleph(\mathfrak{g}, \mathfrak{p}) + \frac{2\mu - 4 + (2\mu + 4)e^{-\mu}}{\mu^3}\nabla(\mathfrak{g}, \mathfrak{p}),$$

where

$$\aleph(\mathfrak{g}, \mathfrak{p}) = \mathsf{S}(\mathfrak{g})\Upsilon(\mathfrak{g}) + \mathsf{S}(\mathfrak{p})\Upsilon(\mathfrak{p}),$$
$$\nabla(\mathfrak{g}, \mathfrak{p}) = \mathsf{S}(\mathfrak{g})\Upsilon(\mathfrak{p}) + \mathsf{S}(\mathfrak{p})\Upsilon(\mathfrak{g})$$

and

$$\aleph(\mathfrak{g}, \mathfrak{p}) = [\aleph_*(\mathfrak{g}, \mathfrak{p}), \aleph^*(\mathfrak{g}, \mathfrak{p})],$$
$$\nabla(\mathfrak{g}, \mathfrak{p}) = [\nabla_*(\mathfrak{g}, \mathfrak{p}), \nabla^*(\mathfrak{g}, \mathfrak{p})].$$

Proof. Considering $\mathtt{S}$ and $\mathtt{Y}$ as LR-convex interval valued functions, we have

$$\mathtt{S}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p}) \leqq \mathtt{r}\mathtt{S}_*(\mathfrak{g})+(1-\mathtt{r})\,\mathtt{S}_*(\mathfrak{p}),$$

$$\mathtt{S}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p}) \leqq \mathtt{r}\mathtt{S}^*(\mathfrak{g})+(1-\mathtt{r})\,\mathtt{S}^*(\mathfrak{p}),$$

$$\mathtt{Y}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p}) \leqq \mathtt{r}\mathtt{Y}_*(\mathfrak{g})+(1-\mathtt{r})\,\mathtt{Y}_*(\mathfrak{p}),$$

and

$$\mathtt{Y}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p}) \leqq \mathtt{r}\mathtt{Y}^*(\mathfrak{g})+(1-\mathtt{r})\,\mathtt{Y}^*(\mathfrak{p}).$$

It follows from the definition of LR convex function that $0 \leqq_p \mathtt{S}(\upsilon)$ and $0 \leqq_p \mathtt{Y}(\upsilon)$, which implies

$$\begin{aligned}&\mathtt{S}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\,\mathtt{Y}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\\&\leqq [\mathtt{r}\mathtt{S}_*(\mathfrak{g})+(1-\mathtt{r})\,\mathtt{S}_*(\mathfrak{p})][\mathtt{r}\mathtt{Y}_*(\mathfrak{g})+(1-\mathtt{r})\,\mathtt{Y}_*(\mathfrak{p})]\\&= \mathtt{r}^2\mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{g})+(1-\mathtt{r})^2\,\mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{p})+\mathtt{r}\,(1-\mathtt{r})\,\mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{p})+\mathtt{r}\,(1-\mathtt{r})\,\mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{g}),\end{aligned}$$

and

$$\begin{aligned}&\mathtt{S}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\,\mathtt{Y}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\\&\leqq [\mathtt{r}\mathtt{S}^*(\mathfrak{g})+(1-\mathtt{r})\,\mathtt{S}^*(\mathfrak{p})][\mathtt{r}\mathtt{Y}^*(\mathfrak{g})+(1-\mathtt{r})\,\mathtt{Y}^*(\mathfrak{p})]\\&= \mathtt{r}^2\mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{g})+(1-\mathtt{r})^2\,\mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{p})+\mathtt{r}\,(1-\mathtt{r})\,\mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{p})+\mathtt{r}\,(1-\mathtt{r})\,\mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{g}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\mathtt{S}_*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,\mathtt{Y}_*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\\&\leqq (1-\mathtt{r})^2\,\mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{g})+\mathtt{r}^2\mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{p})+\mathtt{r}\,(1-\mathtt{r})\,\mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{p})+\mathtt{r}\,(1-\mathtt{r})\,\mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{g}),\end{aligned}$$

and

$$\begin{aligned}&\mathtt{S}^*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,\mathtt{Y}^*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\\&\leqq (1-\mathtt{r})^2\,\mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{g})+\mathtt{r}^2\mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{p})+\mathtt{r}\,(1-\mathtt{r})\,\mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{p})+\mathtt{r}\,(1-\mathtt{r})\,\mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{g}).\end{aligned}$$

It follows from the above developments that

$$\begin{aligned}&\mathtt{S}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\,\mathtt{Y}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})+\mathtt{S}_*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,\mathtt{Y}_*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\\&\leqq \left[\mathtt{r}^2+(1-\mathtt{r})^2\right][\mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{g})+\mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{p})]+2\mathtt{r}(1-\mathtt{r})\,[\mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{g})+\mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{p})]\,,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\mathtt{S}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\,\mathtt{Y}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})+\mathtt{S}^*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,\mathtt{Y}^*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\\&\leqq \left[\mathtt{r}^2+(1-\mathtt{r})^2\right][\mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{g})+\mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{p})]+2\mathtt{r}\,(1-\mathtt{r})\,[\mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{g})+\mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{p})]\,.\end{aligned} \tag{3.6}$$

Multiplying both the Eqs (3.5) and (3.6) by $\left(e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\right)$ and then integrating with respect to $\mathtt{r}$ over [0,1], we obtain

$$\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\,\mathtt{Y}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})d\mathtt{r}$$

$$+\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}} \mathtt{S}_*\left((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}\right) \mathtt{Y}_*\left((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}\right) d\mathtt{r}$$
$$\leqq \aleph_*(\mathfrak{g},\mathfrak{p})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\left[\mathtt{r}^2+(1-\mathtt{r})^2\right]d\mathtt{r}+2\nabla_*(\mathfrak{g},\mathfrak{p})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{r}\,(1-\mathtt{r})\,d\mathtt{r}\,,$$

and

$$\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}} \mathtt{S}^*\left(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p}\right) \mathtt{Y}^*\left(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p}\right) d\mathtt{r}$$
$$+\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}} \mathtt{S}^*\left((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}\right) \mathtt{Y}^*\left((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}\right) d\mathtt{r}$$
$$\leqq \aleph^*(\mathfrak{g},\mathfrak{p})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\left[\mathtt{r}^2+(1-\mathtt{r})^2\right]d\mathtt{r}+2\nabla^*(\mathfrak{g},\mathfrak{p})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{r}\,(1-\mathtt{r})\,d\mathtt{r}.$$

In view of the above developments, we have

$$\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{g})\right]$$
$$\leqq \aleph_*(\mathfrak{g},\mathfrak{p})\,\frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{\mu^3}+\nabla_*(\mathfrak{g},\mathfrak{p})\,\frac{2\mu-4+(2\mu+4)e^{-\mu}}{\mu^3},\quad (3.7)$$

and

$$\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{g})\right]$$
$$\leqq \aleph^*(\mathfrak{g},\mathfrak{p})\,\frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{\mu^3}+\nabla^*(\mathfrak{g},\mathfrak{p})\,\frac{2\mu-4+(2\mu+4)e^{-\mu}}{\mu^3}.\quad (3.8)$$

It follows from the above Eqs (3.7) and (3.8) that

$$\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{g}),\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{g})\right]$$
$$\leqq_p \frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{\mu^3}\left[\aleph_*(\mathfrak{g},\mathfrak{p})\,,\ \aleph^*(\mathfrak{g},\mathfrak{p})\right]$$
$$+\frac{2\mu-4+(2\mu+4)e^{-\mu}}{\mu^3}\left[\nabla_*(\mathfrak{g},\mathfrak{p})\,,\ \nabla^*(\mathfrak{g},\mathfrak{p})\right].$$

Consequently, we have

$$\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathtt{S}(\mathfrak{p})\mathtt{Y}(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\mathtt{S}(\mathfrak{g})\mathtt{Y}(\mathfrak{g})\right]$$
$$\leqq_p \frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{\mu^3}\aleph(\mathfrak{g},\mathfrak{p})+\frac{2\mu-4+(2\mu+4)e^{-\mu}}{\mu^3}\nabla(\mathfrak{g},\mathfrak{p})\,.$$

This concludes the proof of Theorem 3.2. □

Remark 3.2. *If we choose* $\mathtt{S}_*(\upsilon) = \mathtt{S}^*(\upsilon)$ *in Theorem 3.2, we have the following fractional integral inequality for* LR*-convex interval valued functions as given in [26].*

$$\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathtt{S}(\mathfrak{p})\mathtt{Y}(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\mathtt{S}(\mathfrak{g})\mathtt{Y}(\mathfrak{g})\right] \leqq \frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{\mu^3}\aleph(\mathfrak{g},\mathfrak{p}) + \frac{2\mu-4+(2\mu+4)e^{-\mu}}{\mu^3}\nabla(\mathfrak{g},\mathfrak{p}),$$

where $\aleph(\mathfrak{g},\mathfrak{p})$ *and* $\nabla(\mathfrak{g},\mathfrak{p})$ *are defined as Theorem 3.2.*

Example 3.2. *If we let,* $[\mathfrak{g},\mathfrak{p}] = [0, 2\ ], \alpha = \frac{1}{2}$. *Also, let the interval valued functions be given as*

$$\mathtt{S}(\upsilon) = [\upsilon, 2\upsilon] \text{ and } \mathtt{Y}(\upsilon) = \left[\upsilon^2, 2\upsilon^2\right].$$

Since, the left and the right end point functions $\mathtt{S}_*(\upsilon) = \upsilon$, $\mathtt{S}^*(\upsilon) = 2\upsilon$, $\mathtt{Y}_*(\upsilon) = \upsilon^2$ *and* $\mathtt{Y}^*(\upsilon) = 2\upsilon^2$ *are* LR*-convex functions, both* $\mathtt{S}(\upsilon)$ *and* $\mathtt{Y}(\upsilon)$ *are* LR*-convex interval valued functions. Then, we have* $\mathtt{S}(\upsilon)\mathtt{Y}(\upsilon) \in \mathcal{L}\left([\mathfrak{g},\mathfrak{p}], \mathcal{M}^+_C\right)$ *and*

$$\frac{(\alpha)}{\mathfrak{p}-\mathfrak{g}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{g})\right] \approx 1.8346.$$

$$\frac{\alpha}{\mathfrak{p}-\mathfrak{g}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\ \mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\ \mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{g})\right] \approx 7.338.$$

Also, note that

$$\left(\frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{\mu^3}\right)\aleph_*(\mathfrak{g},\mathfrak{p}) = [\mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{g}) + \mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{p})] = 13.556,$$

$$\left(\frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{\mu^3}\right)\aleph^*(\mathfrak{g},\mathfrak{p}) = [\mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{g}) + \mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{p})] = 54.224,$$

$$\left(\frac{2\mu-4+(2\mu+4)e^{-\mu}}{\mu^3}\right)\nabla_*(\mathfrak{g},\mathfrak{p}) = [\mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{p}) + \mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{g})] = 0,$$

$$\left(\frac{2\mu-4+(2\mu+4)e^{-\mu}}{\mu^3}\right)\nabla_*(\mathfrak{g},\mathfrak{p}) = [\mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{p}) + \mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{g})] = 0.$$

Therefore, we have

$$\left(\frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{\mu^3}\right)\aleph(\mathfrak{g},\mathfrak{p}) + \left(\frac{2\mu-4+(2\mu+4)e^{-\mu}}{\mu^3}\right)\nabla(\mathfrak{g},\mathfrak{p}) = [13.556, 54.224]$$

It follows from the above developments that

$$[1.8346, 7.338] \leqq_p [13.556, 54.224],$$

and this evidently verifies Theorem 3.2.

Theorem 3.3. *Let* $\mathtt{S}, \mathtt{Y} : [\mathfrak{g}, \mathfrak{p}] \rightarrow \mathcal{M}_C^+$ *be two* LR*-convex interval valued functions, such that* $\mathtt{S}(\upsilon) = [\mathtt{S}_*(\upsilon), \ \mathtt{S}^*(\upsilon)]$ *and* $\mathtt{Y}(\upsilon) = [\mathtt{Y}_*(\upsilon), \ \mathtt{Y}^*(\upsilon)]$ *for all* $\upsilon \in [\mathfrak{g}, \mathfrak{p}]$. *If* $\mathtt{SY} \in \mathcal{L}\left([\mathfrak{g}, \mathfrak{p}], \mathcal{M}_C^+\right)$, *then*

$$2\mathtt{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\mathtt{Y}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) \leqq_p \frac{(1-\alpha)}{2(1-e^{-\mu})}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathtt{S}(\mathfrak{p})\mathtt{Y}(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathtt{S}(\mathfrak{g})\mathtt{Y}(\mathfrak{g})\right] + \frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{2\mu^2(1-e^{-\mu})}\nabla(\mathfrak{g},\mathfrak{p}) + \frac{\mu-2+(\mu+2)e^{-\mu}}{\mu^2(1-e^{-\mu})}\aleph(\mathfrak{g},\mathfrak{p}), \quad (3.9)$$

where

$$\aleph(\mathfrak{g},\mathfrak{p}) = \mathtt{S}(\mathfrak{g})\mathtt{Y}(\mathfrak{g}) + \mathtt{S}(\mathfrak{p})\mathtt{Y}(\mathfrak{p}),\ \nabla(\mathfrak{g},\mathfrak{p}) = \mathtt{S}(\mathfrak{g})\mathtt{Y}(\mathfrak{p}) + \mathtt{S}(\mathfrak{p})\mathtt{Y}(\mathfrak{g}),$$

and

$$\aleph(\mathfrak{g},\mathfrak{p}) = [\aleph_*(\mathfrak{g},\mathfrak{p}), \aleph^*(\mathfrak{g},\mathfrak{p})],\ \nabla(\mathfrak{g},\mathfrak{p}) = [\nabla_*(\mathfrak{g},\mathfrak{p}),\ \nabla^*(\mathfrak{g},\mathfrak{p})].$$

Proof. Consider $\mathtt{S}, \mathtt{Y} : [\mathfrak{g}, \mathfrak{p}] \rightarrow \mathcal{M}_C^+$ are LR-convex interval valued functions. Then we have

$$\begin{aligned}
&\mathtt{S}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\mathtt{Y}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\\
&\leqq \frac{1}{4}\left[\mathtt{S}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p})\mathtt{Y}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p}) + \mathtt{S}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p})\mathtt{Y}_*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\right]\\
&+ \frac{1}{4}\left[\mathtt{S}_*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\mathtt{Y}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p}) + \mathtt{S}_*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\mathtt{Y}_*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\right]\\
&\leqq \frac{1}{4}\Big(\mathtt{S}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p})\mathtt{Y}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p}) + \mathtt{S}_*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\mathtt{Y}_*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\Big)\\
&+ \frac{1}{4}\Big((\mathtt{r}\mathtt{S}_*(\mathfrak{g})+(1-\mathtt{r})\mathtt{S}_*(\mathfrak{p}))((1-\mathtt{r})\mathtt{Y}_*(\mathfrak{g})+\mathtt{r}\mathtt{Y}_*(\mathfrak{p}))\\
&+ ((1-\mathtt{r})\mathtt{S}_*(\mathfrak{g})+\mathtt{r}\mathtt{S}_*(\mathfrak{p}))(\mathtt{r}\mathtt{Y}_*(\mathfrak{g})+(1-\mathtt{r})\mathtt{Y}_*(\mathfrak{p}))\Big)\\
&= \frac{1}{4}\Big(\mathtt{S}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p})\mathtt{Y}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p}) + \mathtt{S}_*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\mathtt{Y}_*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\Big)\\
&+ \frac{1}{4}\Big(\left\{\mathtt{r}^2+(1-\mathtt{r})^2\right\}\nabla_*(\mathfrak{g},\mathfrak{p}) + \{\mathtt{r}(1-\mathtt{r})+(1-\mathtt{r})\mathtt{r}\}\aleph_*(\mathfrak{g},\mathfrak{p})\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\mathtt{S}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\mathtt{Y}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\\
&\leqq \frac{1}{4}\left[\mathtt{S}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p})\mathtt{Y}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p}) + \mathtt{S}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p})\mathtt{Y}^*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\right]\\
&+ \frac{1}{4}\left[\mathtt{S}^*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\mathtt{Y}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p}) + \mathtt{S}^*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\mathtt{Y}^*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\right]\\
&\leqq \frac{1}{4}\Big(\mathtt{S}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p})\mathtt{Y}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\mathfrak{p}) + \mathtt{S}^*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\mathtt{Y}^*((1-\mathtt{r})\mathfrak{g}+\mathtt{r}\mathfrak{p})\Big)\\
&+ \frac{1}{4}\Big((\mathtt{r}\mathtt{S}^*(\mathfrak{g})+(1-\mathtt{r})\mathtt{S}^*(\mathfrak{p}))((1-\mathtt{r})\mathtt{Y}^*(\mathfrak{g})+\mathtt{r}\mathtt{Y}^*(\mathfrak{p}))\\
&+ ((1-\mathtt{r})\mathtt{S}^*(\mathfrak{g})+\mathtt{r}\mathtt{S}^*(\mathfrak{p}))(\mathtt{r}\mathtt{Y}^*(\mathfrak{g})+(1-\mathtt{r})\mathtt{Y}^*(\mathfrak{p}))\Big)
\end{aligned}$$

$$= \frac{1}{4}\Big(\mathtt{S}^*(\mathtt{r}\mathfrak{g} + (1-\mathtt{r})\,\mathfrak{p})\,\mathtt{Y}^*(\mathtt{r}\mathfrak{g} + (1-\mathtt{r})\,\mathfrak{p}) + \mathtt{S}^*((1-\mathtt{r})\,\mathfrak{g} + \mathtt{r}\mathfrak{p})\,\mathtt{Y}^*((1-\mathtt{r})\,\mathfrak{g} + \mathtt{r}\mathfrak{p})\Big)$$
$$+ \frac{1}{4}\Big(\left\{\mathtt{r}^2 + (1-\mathtt{r})^2\right\}\nabla^*(\mathfrak{g},\mathfrak{p}) + \{\mathtt{r}(1-\mathtt{r}) + (1-\mathtt{r})\,\mathtt{r}\}\,\aleph^*(\mathfrak{g},\mathfrak{p})\Big).$$

Upon multiplying the above developments by $e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}$ and then integrating the obtained result over $[0, 1]$, we get

$$\begin{aligned}&\frac{1-e^{-\mu}}{\mu}\,\mathtt{S}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\mathtt{Y}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\\ &\leqq \frac{\alpha}{4(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\,\mathtt{S}_*(\mathfrak{p})\mathtt{Y}_*(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\,\mathtt{S}_*(\mathfrak{g})\mathtt{Y}_*(\mathfrak{g})\right]\\ &+ \nabla_*(\mathfrak{g},\mathfrak{p})\,\frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{4\mu^3} + \frac{\mu-2+(\mu+2)e^{-\mu}}{2\mu^3}\aleph_*(\mathfrak{g},\mathfrak{p})\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\frac{1-e^{-\mu}}{\mu}\,\mathtt{S}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\mathtt{Y}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\\ &\leqq \frac{\alpha}{4(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\,\mathtt{S}^*(\mathfrak{p})\mathtt{Y}^*(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\,\mathtt{S}^*(\mathfrak{g})\mathtt{Y}^*(\mathfrak{g})\right]\\ &+ \nabla^*(\mathfrak{g},\mathfrak{p})\,\frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{4\mu^3} + \frac{\mu-2+(\mu+2)e^{-\mu}}{2\mu^3}\aleph^*(\mathfrak{g},\mathfrak{p}).\end{aligned} \tag{3.11}$$

In view of the above Eqs (3.10) and (3.11), we have

$$\begin{aligned}2\mathtt{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\mathtt{Y}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) &\leqq_p \frac{(1-\alpha)}{2(1-e^{-\mu})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\,\mathtt{S}(\mathfrak{p})\mathtt{Y}(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\,\mathtt{S}(\mathfrak{g})\mathtt{Y}(\mathfrak{g})\right]\\ &+ \frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{2\mu^2(1-e^{-\mu})}\nabla(\mathfrak{g},\mathfrak{p}) + \frac{\mu-2+(\mu+2)e^{-\mu}}{\mu^2(1-e^{-\mu})}\aleph(\mathfrak{g},\mathfrak{p}).\end{aligned}$$

This concludes the proof of Theorem 3.3. □

Remark 3.3. *If* $\mathtt{S}_*(\upsilon) = \mathtt{S}^*(\upsilon)$, *then, from Theorem 3.3 we obtain the following fractional integral inequality for* LR*-convex I-V-Fs as given in [26].*

$$\begin{aligned}2\mathtt{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\mathtt{Y}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right) &\leqq \frac{(1-\alpha)}{2(1-e^{-\mu})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\,\mathtt{S}(\mathfrak{p})\mathtt{Y}(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\,\mathtt{S}(\mathfrak{g})\mathtt{Y}(\mathfrak{g})\right]\\ &+ \frac{\mu^2-2\mu+4-(\mu^2+2\mu+4)e^{-\mu}}{2\mu^2(1-e^{-\mu})}\nabla(\mathfrak{g},\mathfrak{p}) + \frac{\mu-2+(\mu+2)e^{-\mu}}{\mu^2(1-e^{-\mu})}\aleph(\mathfrak{g},\mathfrak{p}),\end{aligned} \tag{3.12}$$

where $\aleph(\mathfrak{g},\mathfrak{p}) = \mathtt{S}(\mathfrak{g})\mathtt{Y}(\mathfrak{g}) + \mathtt{S}(\mathfrak{p})\mathtt{Y}(\mathfrak{p})$, $\nabla(\mathfrak{g},\mathfrak{p}) = \mathtt{S}(\mathfrak{g})\mathtt{Y}(\mathfrak{p}) + \mathtt{S}(\mathfrak{p})\mathtt{Y}(\mathfrak{g})$.

4. Interval-valued Fejér type fractional inequalities

Theorem 4.1. *Let* $\mathtt{S} : [\mathfrak{g},\mathfrak{p}] \to \mathcal{M}^+_C$ *be an* LR*-convex interval valued function with* $(\mathfrak{g} < \mathfrak{p})$ *and given by* $\mathtt{S}(\upsilon) = [\mathtt{S}_*(\upsilon), \mathtt{S}^*(\upsilon)]$ *for all* $\upsilon \in [\mathfrak{g},\mathfrak{p}]$. *Let* $\mathtt{S} \in \mathcal{L}\left([\mathfrak{g},\mathfrak{p}], \mathcal{M}^+_C\right)$ *and* $\mathtt{Y} : [\mathfrak{g},\mathfrak{p}] \to \mathbb{R}, \mathtt{Y}(\upsilon) \geqq 0$, *symmetric with respect to* $\frac{\mathfrak{g}+\mathfrak{p}}{2}$. *Then*

$$\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\,\mathtt{S}\mathtt{Y}(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\,\mathtt{S}\mathtt{Y}(\mathfrak{g})\right] \leqq_p \frac{\mathtt{S}(\mathfrak{g})+\mathtt{S}(\mathfrak{p})}{2}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\mathtt{Y}(\mathfrak{p}) + \mathcal{I}^{\alpha}_{\mathfrak{p}^-}\mathtt{Y}(\mathfrak{g})\right]. \tag{4.1}$$

Proof. Since $\mathtt{S}$ be an LR-convex interval valued function and

$$e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\Upsilon(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\geqq 0.$$

We have

$$e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\leqq e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}(\mathtt{r}\mathtt{S}_*(\mathfrak{g})+(1-\mathtt{r})\,\mathtt{S}_*(\mathfrak{p}))\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}),$$

$$e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\leqq e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}(\mathtt{r}\mathtt{S}^*(\mathfrak{g})+(1-\mathtt{r})\,\mathtt{S}^*(\mathfrak{p}))\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}),$$

$$e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}_*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\leqq e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}((1-\mathtt{r})\,\mathtt{S}_*(\mathfrak{g})+\mathtt{r}\mathtt{S}_*(\mathfrak{p}))\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}),$$

and

$$e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}^*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\leqq e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}((1-\mathtt{r})\,\mathtt{S}^*(\mathfrak{g})+\mathtt{r}\mathtt{S}^*(\mathfrak{p}))\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}).$$

Consequently, from the above developments, we have

$$\begin{aligned}
&\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}_*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})d\mathtt{r}\\
&+\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}_*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})d\mathtt{r}\\
&\leqq\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}_*(\mathfrak{g})\{\mathtt{r}\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})+(1-\mathtt{r})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\}\\
&+e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}_*(\mathfrak{p})\{(1-\mathtt{r})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})+\mathtt{r}\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\}\,d\mathtt{r}\\
&=\mathtt{S}_*(\mathfrak{g})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,d\mathtt{r}+\mathtt{S}_*(\mathfrak{p})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,d\mathtt{r},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}^*((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})d\mathtt{r}\\
&+\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}^*(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})d\mathtt{r}\\
&\leqq\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}^*(\mathfrak{g})\{\mathtt{r}\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})+(1-\mathtt{r})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\}\\
&+e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}^*(\mathfrak{p})\{(1-\mathtt{r})\,\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})+\mathtt{r}\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\}\,d\mathtt{r}\\
&=\mathtt{S}^*(\mathfrak{g})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,d\mathtt{r}+\mathtt{S}^*(\mathfrak{p})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,d\mathtt{r}.
\end{aligned}$$

Since Υ is symmetric, and using the fact that

$$\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\Upsilon(\mathfrak{p})=\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\Upsilon(\mathfrak{g})=\frac{1}{2}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\Upsilon(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\Upsilon(\mathfrak{g})\right],$$

we have,

$$
\begin{aligned}
&\mathrm{S}_*(\mathfrak{g})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}+\mathrm{S}_*(\mathfrak{p})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}\\
&=\left[\mathrm{S}_*(\mathfrak{g})+\mathrm{S}_*(\mathfrak{p})\right]\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}\\
&=\frac{\mathrm{S}_*(\mathfrak{g})+\mathrm{S}_*(\mathfrak{p})}{2}\frac{\alpha}{\mathfrak{p}-\mathfrak{g}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\Upsilon(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\Upsilon(\mathfrak{g})\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{S}^*(\mathfrak{g})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}+\mathrm{S}^*(\mathfrak{p})\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}.\\
&=\left[\mathrm{S}^*(\mathfrak{g})+\mathrm{S}^*(\mathfrak{p})\right]\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}.\\
&=\frac{\mathrm{S}^*(\mathfrak{g})+\mathrm{S}^*(\mathfrak{p})}{2}\frac{\alpha}{\mathfrak{p}-\mathfrak{g}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\Upsilon(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\Upsilon(\mathfrak{g})\right].
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\mathrm{S}_*\left(\mathrm{r}\mathfrak{g}+(1-\mathrm{r})\,\mathfrak{p}\right)\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}\\
&+\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\mathrm{S}_*\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}\\
&=\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}_*\left(\mathfrak{g}+\mathfrak{p}-\upsilon\right)\Upsilon(\upsilon)\,d\upsilon+\frac{1}{\mathfrak{p}-\mathfrak{g})}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}_*(\upsilon)\Upsilon(\upsilon)\,d\upsilon\\
&=\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{\left(-\frac{1-\alpha}{\alpha}(\mathfrak{p}-z)\right)}\mathrm{S}_*(z)\Upsilon\left(\mathfrak{g}+\mathfrak{p}-z\right)dz+\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}_*(\upsilon)\Upsilon(\upsilon)\,d\upsilon\\
&=\frac{\alpha}{\mathfrak{p}-\mathfrak{g}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\mathrm{S}_*\Upsilon(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\mathrm{S}_*\Upsilon(\mathfrak{g})\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\mathrm{S}^*\left(\mathrm{r}\mathfrak{g}+(1-\mathrm{r})\,\mathfrak{p}\right)\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}\\
&+\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\mathrm{S}^*\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}\\
&=\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}^*\left(\mathfrak{g}+\mathfrak{p}-\upsilon\right)\Upsilon(\upsilon)\,d\upsilon+\frac{1}{\mathfrak{p}-\mathfrak{g})}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}^*(\upsilon)\Upsilon(\upsilon)\,d\upsilon\\
&=\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{\left(-\frac{1-\alpha}{\alpha}(\mathfrak{p}-z)\right)}\mathrm{S}^*(z)\Upsilon\left(\mathfrak{g}+\mathfrak{p}-z\right)dz+\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}} e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}^*(\upsilon)\Upsilon(\upsilon)\,d\upsilon\\
&=\frac{\alpha}{\mathfrak{p}-\mathfrak{g}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\,\mathrm{S}^*\Upsilon(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\,\mathrm{S}^*\Upsilon(\mathfrak{g})\right].
\end{aligned}
$$

It follows from the above developments that

$$
\frac{\alpha}{\mathfrak{p}-\mathfrak{g}}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\mathrm{S}_*\Upsilon(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\mathrm{S}_*\Upsilon(\mathfrak{g})\right]\leqq\frac{\mathrm{S}_*(\mathfrak{g})+\mathrm{S}_*(\mathfrak{p})}{2}\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}^{\alpha}_{\mathfrak{g}^+}\Upsilon(\mathfrak{p})+\mathcal{I}^{\alpha}_{\mathfrak{p}^-}\Upsilon(\mathfrak{g})\right],
$$

and

$$\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\mathtt{S}^*\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\mathtt{S}^*\Upsilon(\mathfrak{g})\right]\leqq\frac{\mathtt{S}^*(\mathfrak{g})+\mathtt{S}^*(\mathfrak{p})}{2}\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right].$$

Consequently,

$$\begin{aligned}&\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\mathtt{S}_*\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\mathtt{S}_*\Upsilon(\mathfrak{g})\right],\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\mathtt{S}^*\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathtt{S}^*\Upsilon(\mathfrak{g})\right]\\&\qquad\leqq_p\left[\frac{\mathtt{S}_*(\mathfrak{g})+\mathtt{S}_*(\mathfrak{p})}{2},\ \frac{\mathtt{S}^*(\mathfrak{g})+\mathtt{S}^*(\mathfrak{p})}{2}\right]\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right],\end{aligned}$$

which readily follows

$$\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\mathtt{S}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\mathtt{S}\Upsilon(\mathfrak{g})\right]\leqq_p\frac{\mathtt{S}(\mathfrak{g})+\mathtt{S}(\mathfrak{p})}{2}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right].$$

This concludes the proof of Theorem 4.1. □

Corollary 4.1. *Let* $\mathtt{S}:[\mathfrak{g},\mathfrak{p}]\to\mathcal{M}_C^+$ *be an* LR-*concave function with* $\mathfrak{g}<\mathfrak{p}$, *and* $\mathtt{S}\in\mathcal{L}([\mathfrak{g},\mathfrak{p}])$ *defined by* $\mathtt{S}(\upsilon)=[\mathtt{S}_*(\upsilon),\ \mathtt{S}^*(\upsilon)]$ *for all* $\upsilon\in[\mathfrak{g},\mathfrak{p}]$. *If* $\Upsilon:[\mathfrak{g},\mathfrak{p}]\to\mathbb{R}$, $\Upsilon(\upsilon)\geqq 0$, *is symmetric with respect to* $\frac{\mathfrak{g}+\mathfrak{p}}{2}$. *Then the following inequality holds true.*

$$\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\mathtt{S}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\mathtt{S}\Upsilon(\mathfrak{g})\right]\geqq_p\frac{\mathtt{S}(\mathfrak{g})+\mathtt{S}(\mathfrak{p})}{2}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right].$$

Theorem 4.2. *Let* $\mathtt{S}:[\mathfrak{g},\mathfrak{p}]\to\mathcal{M}_C^+$ *be an* LR-*convex interval valued function with* $\mathfrak{g}<\mathfrak{p}$, *and defined by* $\mathtt{S}(\upsilon)=[\mathtt{S}_*(\upsilon),\ \mathtt{S}^*(\upsilon)]$ *for all* $\upsilon\in[\mathfrak{g},\mathfrak{p}]$. *If* $\mathtt{S}\in\mathcal{L}\left([\mathfrak{g},\mathfrak{p}],\mathcal{M}_C^+\right)$ *and* $\Upsilon:[\mathfrak{g},\mathfrak{p}]\to\mathbb{R}$, $\Upsilon(\upsilon)\geqq 0$, *symmetric with respect to* $\frac{\mathfrak{g}+\mathfrak{p}}{2}$, *then*

$$\mathtt{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right]\leqq_p\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathtt{S}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathtt{S}\Upsilon(\mathfrak{g})\right].\tag{4.2}$$

Proof. If we utilize the definition of LR-convex interval valued function, then we have

$$\mathtt{S}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\leqq\frac{1}{2}\left(\mathtt{S}_*\left(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p}\right)+\mathtt{S}_*\left((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}\right)\right),$$

and

$$\mathtt{S}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\leqq\frac{1}{2}\left(\mathtt{S}^*\left(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p}\right)+\mathtt{S}^*\left((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}\right)\right).\tag{4.3}$$

Since, $\Upsilon(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p})=\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})$, then multiplying the above Eq (4.3) by $e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})$ and integrate it with respect to $\mathtt{r}$ over $[0,\ 1]$, we obtain

$$\begin{aligned}&\mathtt{S}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,d\mathtt{r}\\&\qquad\leqq\frac{1}{2}\Big(\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}_*\left(\mathtt{r}\mathfrak{g}+(1-\mathtt{r})\,\mathfrak{p}\right)\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,d\mathtt{r}\\&\qquad\qquad+\int_0^1 e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathtt{r}}\mathtt{S}_*\left((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p}\right)\Upsilon((1-\mathtt{r})\,\mathfrak{g}+\mathtt{r}\mathfrak{p})\,d\mathtt{r}\Big),\end{aligned}$$

and

$$\mathrm{S}^{*}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\int_{0}^{1}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}$$
$$\leqq\frac{1}{2}\Big(\int_{0}^{1}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\mathrm{S}^{*}\left(\mathrm{r}\mathfrak{g}+(1-\mathrm{r})\,\mathfrak{p}\right)\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}$$
$$+\int_{0}^{1}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\mathrm{S}^{*}\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}\Big).$$

Let $\upsilon=(1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}$. Then we have

$$\int_{0}^{1}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\mathrm{S}_{*}\left(\mathrm{r}\mathfrak{g}+(1-\mathrm{r})\,\mathfrak{p}\right)\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}$$
$$+\int_{0}^{1}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\mathrm{S}_{*}\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}$$
$$=\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}}e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}_{*}\left(\mathfrak{g}+\mathfrak{p}-\upsilon\right)\Upsilon\left(\upsilon\right)d\upsilon+\frac{1}{\mathfrak{p}-\mathfrak{g})}\int_{\mathfrak{g}}^{\mathfrak{p}}e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}_{*}\left(\upsilon\right)\Upsilon\left(\upsilon\right)d\upsilon$$
$$=\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}}e^{\left(-\frac{1-\alpha}{\alpha}(\mathfrak{p}-z)\right)}\mathrm{S}_{*}\left(z\right)\Upsilon\left(\mathfrak{g}+\mathfrak{p}-z\right)dz+\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}}e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}_{*}\left(\upsilon\right)\Upsilon\left(\upsilon\right)d\upsilon$$
$$=\frac{\alpha}{\mathfrak{p}-\mathfrak{g}}\left[\mathcal{I}_{\mathfrak{g}^{+}}^{\alpha}\ \mathrm{S}_{*}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^{-}}^{\alpha}\ \mathrm{S}_{*}\Upsilon(\mathfrak{g})\right],$$

and

$$\int_{0}^{1}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\mathrm{S}^{*}\left(\mathrm{r}\mathfrak{g}+(1-\mathrm{r})\,\mathfrak{p}\right)\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}$$
$$+\int_{0}^{1}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\mathrm{S}^{*}\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}$$
$$=\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}}e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}^{*}\left(\mathfrak{g}+\mathfrak{p}-\upsilon\right)\Upsilon\left(\upsilon\right)d\upsilon+\frac{1}{\mathfrak{p}-\mathfrak{g})}\int_{\mathfrak{g}}^{\mathfrak{p}}e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}^{*}\left(\upsilon\right)\Upsilon\left(\upsilon\right)d\upsilon$$
$$=\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}}e^{\left(-\frac{1-\alpha}{\alpha}(\mathfrak{p}-z)\right)}\mathrm{S}^{*}\left(z\right)\Upsilon\left(\mathfrak{g}+\mathfrak{p}-z\right)dz+\frac{1}{\mathfrak{p}-\mathfrak{g}}\int_{\mathfrak{g}}^{\mathfrak{p}}e^{\left(-\frac{1-\alpha}{\alpha}(\upsilon-\mathfrak{g})\right)}\mathrm{S}^{*}\left(\upsilon\right)\Upsilon\left(\upsilon\right)d\upsilon$$
$$=\frac{\alpha}{\mathfrak{p}-\mathfrak{g}}\left[\mathcal{I}_{\mathfrak{g}^{+}}^{\alpha}\ \mathrm{S}^{*}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^{-}}^{\alpha}\ \mathrm{S}^{*}\Upsilon(\mathfrak{g})\right].$$

Also,

$$\mathrm{S}_{*}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\int_{0}^{1}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}=\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\mathrm{S}_{*}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\left[\mathcal{I}_{\mathfrak{g}^{+}}^{\alpha}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^{-}}^{\alpha}\Upsilon(\mathfrak{g})\right],$$
$$\text{and }\ \mathrm{S}^{*}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\int_{0}^{1}e^{-\frac{1-\alpha}{\alpha}(\mathfrak{p}-\mathfrak{g})\mathrm{r}}\Upsilon\left((1-\mathrm{r})\,\mathfrak{g}+\mathrm{r}\mathfrak{p}\right)d\mathrm{r}=\frac{\alpha}{(\mathfrak{p}-\mathfrak{g})}\mathrm{S}_{*}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\left[\mathcal{I}_{\mathfrak{g}^{+}}^{\alpha}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^{-}}^{\alpha}\Upsilon(\mathfrak{g})\right].$$

In view of the above developments, we have

$$\left[\mathrm{S}_{*}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right),\ \ \mathrm{S}^{*}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\right]\left[\mathcal{I}_{\mathfrak{g}^{+}}^{\alpha}\Upsilon(\mathfrak{p})+\mathcal{I}_{\mathfrak{p}^{-}}^{\alpha}\Upsilon(\mathfrak{g})\right]$$

$$\leqq_p \left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}_*\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}_*\Upsilon(\mathfrak{g}),\ \ \mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}^*\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}^*\Upsilon(\mathfrak{g})\right].$$

It readily follows

$$\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right] \leqq_p \left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}\Upsilon(\mathfrak{g})\upsilon\right].$$

This concludes the proof of Theorem 4.2. □

Corollary 4.2. *Let* $\mathsf{S} : [\mathfrak{g}, \mathfrak{p}] \to \mathcal{M}_C^+$ *be an* LR*-concave function with* $\mathfrak{g} < \mathfrak{p}$*, and* $\mathsf{S} \in \mathcal{L}([\mathfrak{g}, \mathfrak{p}])$ *defined by* $\mathsf{S}(\upsilon) = [\mathsf{S}_*(\upsilon),\ \mathsf{S}^*(\upsilon)]$ *for all* $\upsilon \in [\mathfrak{g}, \mathfrak{p}]$*. If* $\Upsilon : [\mathfrak{g}, \mathfrak{p}] \to \mathbb{R}$, $\Upsilon(\upsilon) \geqq 0$, *is symmetric with respect to* $\frac{\mathfrak{g}+\mathfrak{p}}{2}$*. Then the following inequalities hold true:*

$$\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right] \geqq_p \left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}\Upsilon(\mathfrak{g})\right]. \tag{4.4}$$

Remark 4.1. *If we choose* $\mathsf{S}_*(\upsilon) = \mathsf{S}^*(\upsilon)$ *in the above Theorem 4.1 and Theorem 4.2, we obtain the following fractional H-H-F inequality (*see [26]).

$$\mathsf{S}\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right] \leqq_p \left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}\Upsilon(\mathfrak{g})\right] \leqq_p \frac{\mathsf{S}(\mathfrak{g}) + \mathsf{S}(\mathfrak{p})}{2}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right].$$

Example 4.1. *If let* $\alpha = \frac{1}{2}$ *and The* LR*-convex I-V-Fs* $\mathsf{S} : [0,\ 2] \to \mathcal{M}_C^+$ *be given by* $\mathsf{S}(\upsilon) = [\upsilon^2, 2\upsilon^2]$*. If*

$$\Upsilon(\upsilon) = \begin{cases} \upsilon, & \upsilon \in [0, 1], \\ 2-\upsilon, & \upsilon \in (1,\ 2], \end{cases}$$

then $\Upsilon(2-\upsilon) = \Upsilon(\upsilon) \geqq 0$*, for all* $\upsilon \in [0,\ 2]$*. We clearly observe that:*

$$\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}\Upsilon(\mathfrak{g})\right] \leqq_p \frac{\mathsf{S}(\mathfrak{g}) + \mathsf{S}(\mathfrak{p})}{2}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right].$$

$$\frac{\mathsf{S}_*(\mathfrak{g}) + \mathsf{S}_*(\mathfrak{p})}{2}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right] \approx 3.1966,$$
$$\frac{\mathsf{S}^*(\mathfrak{g}) + \mathsf{S}^*(\mathfrak{p})}{2}\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right] \approx 6.3932,$$

$$\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}_*\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}_*\Upsilon(\mathfrak{g})\right] \approx 1.8948,$$
$$\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\ \mathsf{S}^*\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\ \mathsf{S}^*\Upsilon(\mathfrak{g})\right] \approx 3.7896.$$

It follows from the above developments that $[1.8948, 3.7896] \leqq_p [3.1966, 6.3932]$.
This evidently verifies Theorem 4.1.
Next, for Theorem 4.2, we have the following computations:

$$\mathsf{S}_*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right] = 1.5983,$$

and

$$\mathsf{S}^*\left(\frac{\mathfrak{g}+\mathfrak{p}}{2}\right)\left[\mathcal{I}_{\mathfrak{g}^+}^{\alpha}\Upsilon(\mathfrak{p}) + \mathcal{I}_{\mathfrak{p}^-}^{\alpha}\Upsilon(\mathfrak{g})\right] = 3.1966,$$

which readily follows

$$[1.5983, 3.1966] \leqq_p [1.8948, 3.7896].$$

Hence, Theorem 4.2 is also verified.

5. Conclusions

In this study, we have obtained the Hermite-Hadamard type fractional inclusions for LR-convex interval-valued functions. Following that, we have shown fractional-order Pachpatte type inclusions for the product of two LR-convex interval-valued functions, as well as the Hermite-Hadamard-Fejér type fractional-order inclusions for symmetric functions. We can look into the LR-convex interval-valued functions on coordinates and the quantum (or q-) calculus in the future. The new work is intended to inspire scholars in fractional calculus, interval analysis, and other relevant fields.

We choose to conclude our current investigation by commenting that, in numerous recent scientific articles, fractional-order analogues of many different well-known integral inequalities have been regularly investigated by utilizing some trivial or redundant parametric variations of some widely- and extensively-studied fractional integral and fractional derivative operators (see [9] for detailed observation about the triviality and inconsequential aspect of the so-called "post quantum" calculus involving a redundant or superficial forced-in parameter).

Acknowledgments

This Research was supported by Taif University Researchers Supporting Project Number (TURSP-2020/217), Taif University, Taif, Saudi Arabia and the National Science, Research, and Innovation Fund (NSRF), Thailand.

Conflicts of interest

The authors declare they have no conflicts of interest.

References

1. M. A. El Shaed, *Fractional Calculus Model of Semilunar Heart Valve Vibrations*, International Mathematica Symposium, London, UK, 2003.
2. A. Atangana, Application of fractional calculus to epidemiology, *Fractional Dynamics*, **2015** (2015), 174–190.
3. V. V. Kulish, J. L. Lage, Application of fractional calculus to fluid mechanics, *J. Fluids Engrg.*, **124** (2002), 803–806. https://doi.org/10.1115/1.1478062
4. D. Baleanu, Z. B. Güvenç, J. A. T. Machado, Eds., *New Trends in Nanotechnology and Fractional Calculus Applications,* New York: Springer, 2010.
5. M. Caputo, *Modeling social and economic cycles*, In: Alternative Public Economics, F. Forte, P. Navarra, R. Mudambi, Eds., Elgar, Cheltenham, UK, 2014.
6. R. L. Magin, *Fractional Calculus in Bio-Engineering,* Begell House Inc. Publishers, Danbury, USA, 2006.
7. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* North-Holland Mathematical Studies, Vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

8. H. M. Srivastava, An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions, *J. Adv. Engrg. Comput.*, **5** (2021), 135–166.

9. H. M. Srivastava, Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations, *J. Nonlinear Convex Anal.,* **22** (2021), 1501–1520.

10. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, **57** (2013), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048

11. I. Işcan, Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals, *Studia Univ. Babeş-Bolyai Sect. A Math.*, **60** (2015), 355–366.

12. F. Chen, A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals, *Ital. J. Pure Appl. Math.*, **33** (2014), 299–306.

13. A. Guessab, Generalized barycentric coordinates and approximations of convex functions on arbitrary convex polytopes, *Comput. Math. Appl.*, **66** (2013), 1120–1136. https://doi.org/10.1016/j.camwa.2013.07.014

14. A. Guessab, G. Schmeisser, Two Korovkin-type theorems in multivariate approximation, *Banach J. Math. Anal.*, **2** (2008), 121–128. https://doi.org/10.15352/bjma/1240336298

15. O. Alabdali, A. Guessab, G. Schmeisser, Characterizations of uniform convexity for differentiable functions, *Appl. Anal. Discret. Math.*, **13** (2019), 721–732. https://doi.org/10.2298/AADM190322029A

16. A. Guessab, O. Nouisser, G. Schmeisser, Enhancement of the algebraic precision of a linear operator and consequences under positivity, *Positivity*, **13** (2009), 693–707. https://doi.org/10.1007/s11117-008-2253-4

17. A. Fernandez, P. O. Mohammed, Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels, *Math. Meth. Appl. Sci.*, **44** (2021), 8414–8431.

18. H. Ogulmus, M. Z. Sarikaya, Hermite-Hadamard-Mercer type inequalities for fractional integrals, *Filomat*, **35** (2021), 2425–2436. https://doi.org/10.2298/FIL2107425O

19. M. Andrić, J. Pečarič, I. Perić, A multiple Opial type inequality for the Riemann-Liouville fractional derivatives, *J. Math. Inequal.*, **7** (2013), 139–150.

20. H. Ahmad, M. Tariq, S.K. Sahoo, J. Baili, C. Cesarano, New estimations of Hermite-Hadamard type integral inequalities for special functions. *Fractal Fract.* **5** (2021), 144. https://doi.org/10.3390/fractalfract5040144

21. S. K. Sahoo, M. Tariq, H. Ahmad, B. Kodamasingh, A. A. Shaikh, T. Botmart, et al., Some novel fractional integral inequalities over a new class of generalized convex function, *Fractal Fract.*, **6** (2022), article ID 42, 1–22. https://doi.org/10.3390/fractalfract6010042

22. S. K. Sahoo, P. O. Mohammed, B. Kodamasingh, M. Tariq, Y. S. Hamed, New fractional integral inequalities for convex functions pertaining to Caputo-Fabrizio operator, *Fractal Fract.*, **6** (2022), 171. https://doi.org/10.3390/fractalfract6030171

23. M. B. Khan, M. A. Noor, K. I. Noor, Y. M. Chu, New Hermite-Hadamard type inequalities for (h_1, h_2)-convex fuzzy-interval-valued functions, *Adv. Differ. Equ.*, **2021** (2021), Article ID 149, 1–21.

24. R. E. Moore, *Interval Analysis*, Prentice Hall: Englewood Cliffs, NJ, USA, **1966**.

25. H. Budak, T. Tunç, M. Z. Sarikaya, Fractional Hermite-Hadamard-type inequalities for interval-valued functions, *Proc. Amer. Math. Soc.*, **148** (2020), 705–718. https://doi.org/10.1090/proc/14741

26. B. Ahmad, A. Alsaedi, M. Kirane, B. T. Torebek, Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals, *J. Comput. Appl. Math.*, **353** (2019), 120–129.

27. D. Zhang, C. Guo, D. Chen, G. Wang, Jensen's inequalities for set-valued and fuzzy set-valued functions, *Fuzzy Sets Syst.*, **404** (2021), 178–204. https://doi.org/10.1016/j.fss.2020.06.003

28. T. M. Costa, H. Román-Flores, Y. Chalco-Cano, Opial-type inequalities for interval-valued functions, *Fuzzy Set. Syst.*, **358** (2019), 48–63. https://doi.org/10.1016/j.fss.2018.04.012

29. Y. Chalco-Cano, W. Lodwick, W. Condori-Equice, Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative, *Comput. Appl. Math.*, **31** (2012), 475–472.

30. H. Román-Flores, Y. Chalco-Cano, W. Lodwick, Some integral inequalities for interval-valued functions, *Comput. Appl. Math.*, **37** (2016), 1306–1318. https://doi.org/10.1007/s40314-016-0396-7

31. D. Zhao, M. A. Ali, G. Murtaza, Z. Zhang, On the Hermite-Hadamard inequalities for interval-valued coordinated convex functions, *Adv. Differ. Equ.*, **2020** (2020), Article ID 570, 1–14.

32. E. R. Nwaeze, M. A. Khan, Y. M. Chu, Fractional inclusions of the Hermite-Hadamard type for m-polynomial convex interval-valued functions, *Adv. Differ. Equ.*, **2020** (2020), 1–17.

33. H. Kara, H. Budak, M. A. Ali, M. Z. Sarikaya, Y. M. Chu, Weighted Hermite-Hadamard type inclusions for products of co-ordinated convex interval-valued functions, *Adv. Differ. Equ.*, **2021** (2021), 1–16.

34. H. Budak, H. Kara, M. A. Ali, S. Khan, Y. M. Chu, Fractional Hermite-Hadamard-type inequalities for interval-valued co-ordinated convex functions, *Open Math.*, **19** (2021), 1081–1097.

35. H. M. Srivastava, S. K. Sahoo, P. O. Mohammed, D. Baleanu, B. Kodamasingh, Hermite-Hadamard type inequalities for interval-valued preinvex functions via fractional integral operators, *Int. J. Comput. Intel. Syst.*, **15** (2022), Article ID 8, 1–12. https://doi.org/10.1007/s44196-021-00061-6

36. N. Sharma, S. K. Singh, S. K. Mishra, A. Hamdi, Hermite-Hadamard-type inequalities for interval-valued preinvex functions via Riemann-Liouville fractional integrals, *J. Inequal. Appl.*, **98** (2021).

37. H. Zhou, M. S. Saleem, W. Nazeer, A. F. Shah, Hermite-Hadamard type inequalities for interval-valued exponential type pre-invex functions via Riemann-Liouville fractional integrals, *AIMS Math.*, **7** (2022), 2602–2617. https://doi.org/10.3934/math.2022146

38. K. Lai, S. K. Mishra, J. Bisht, M. Hassan, Hermite-Hadamard type inclusions for interval-valued coordinated preinvex functions, *Symmetry*, **14** (2022), 771. https://doi.org/10.3390/sym14040771

39. H. Kalsoom, M. A. Latif, Z. A. Khan, M. Vivas-Cortez, Some new Hermite-Hadamard-Fejér fractional type inequalities for h-Convex and Harmonically h-Convex interval-valued functions, *Mathematics*, **10** (2022), 74. https://doi.org/10.3390/math10010074

40. F. Shi, G. Ye, D. Zhao, W. Liu, Some integral inequalities for coordinated log-h-convex interval-valued functions, *AIMS Math.*, **7** (2022), 156–170. https://doi.org/10.3934/math.2022009

41. M. B. Khan, M. A. Noor, M. Al-Shomrani, L. Abdullah, Some novel inequalities for LR-h-convex interval-valued functions by means of pseudo-order relation, *Math. Meth. App. Sci.*, **2022** (2022).

42. M. B. Khan, H. G. Zaini, S. Treanță, M. S. Soliman, K. Nonlaopon, Riemann-Liouville fractional integral inequalities for generalized pre-invex functions of interval-valued settings based upon pseudo order relation, *Mathematics*, **10** (2022), Article ID 204, 1–17.

43. M. B. Khan, M. A. Noor, K. I. Noor, K. S. Nisar, K. A. Ismail, A. Elfasakhany, Some inequalities for LR-(h_1, h_2) convex interval-valued functions by means of pseudo order relation, *Int. J. Comput. Intel. Syst.,* **14** (2021), Article ID 180, 1–15.

44. M. B. Khan, H. M. Srivastava, P. O. Mohammed, J. E. Macías-Diaz, Y. S. Hamed, Some new versions of integral inequalities for log-preinvex fuzzy-interval-valued functions through fuzzy order relation, *Alexandria Engrg. J.*, **61** (2022), 7089–7101. https://doi.org/10.1016/j.aej.2021.12.052

45. M. B. Khan, H. M. Srivastava, P. O. Mohammed, L. L. G. Guirao, T. M. Jawa, Fuzzy-interval inequalities for generalized preinvex fuzzy interval valued functions, *Math. Biosci. Engrg.*, **19** (2022), 812–835. http://doi.org/10.3934/mbe.2022037

46. M. B. Khan, P. O. Mohammed, K. Nonlaopon, Y. S. Hamed, Some new Jensen, Schur and Hermite-Hadamard inequalities for log convex fuzzy interval-valued functions, *AIMS Math.,* **7** (2022), 4338–4358. https://doi.org/10.3934/math.2022241

47. M. B. Khan, S. Treanţă, M. S. Soliman, K. Nonlaopon, H. G. Zaini, Some Hadamard-Fejér type inequalities for LR-convex interval-valued functions, *Fractal Fract.,* **6** (2022), Article ID 6, 1–16. https://doi.org/10.3390/fractalfract6010006

AIMS Press

© 2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)