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Research article

Some fixed point results via auxiliary functions on orthogonal metric spaces and application to homotopy

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Abstract: In 2017, the concepts of orthogonal set and orthogonal metric spaces are presented. And an extension of Banach fixed point theorem is proved in this type metric spaces. Further in 2019, on orthogonal metric spaces, some fixed point theorems via altering distance functions are investigated. In this paper, presence and uniqueness of fixed points of the generalizations of contraction principle via auxiliary functions are investigated. And some consequences and an illustrative example are presented. On the other hand, homotopy theory constitute an important area of algebraic topology, but the application of fixed point results in orthogonal metric spaces to homotopy has not been done until now. As a different application in this field, the homotopy application of the one of the corollaries is given at the end of this paper.

Keywords: fixed point; altering distance functions; orthogonal contraction; orthogonal metric space; homotopy application
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

The field of fixed point theory which mainly cares about the presence and uniqueness of fixed points is one of the most researched areas in the last hundred years. In 1922, Stefan Banach [3] indicated his well known theorem on the presence and uniqueness of a fixed point of exact self maps presented on exact metric spaces for the first time. Particularly, this valuable theorem can be established as below: (K, d) is a complete metric space, h is a self mapping on this complete metric space satisfying the condition

$$d(hk, hl) \leq \lambda d(k, l), \text{ for all } k, l \in K, \lambda \in (0, 1). \tag{1.1}$$

In this case h has a unique fixed point.

This theorem has been used to show the presence and uniqueness of the solution of differential equation

$$y'(x) = F(x, y); y(x_0) = y_0 \tag{1.2}$$

where F is a continuously differentiable function.

A lot of researchers have been studied on fixed point theory and gave some generalization of Banach Contraction Principle on complete metric. (See [7, 8, 14, 16, 24]) Studies in this area have been conducted with two important techiniques; one of them is change the contractive condition of mappings and the other is to replace the existing metric with a more general one.

As one of the results in the first technique, Khan et al. [17] enlarged the research of the metric fixed point theory to a new category by presenting a control function which they called an altering distance in 1984.

Definition 1. *([17]) Let* $\eta : [0, \infty) \to [0, \infty)$ *be a function which satisfies*

(i) $\eta(s)$ *is continuous and nondecreasing,*

(ii) $\eta(s) = 0 \iff s = 0$

properties. Then η *is named altering distance function. And* Δ *is denoted as the set of altering distance functions* η*.*

Theorem 2. *([17]) Let* (K, d) *be a complete metric space, let* η *be an altering distance function and let* $h : K \to K$ *be a self mapping which satisfies the folowing inequality:*

$$\eta(d(hk, hl)) \leq \alpha\eta(d(k, l)) \tag{1.3}$$

for all $k, l \in K$ *and for some* $\alpha \in (0, 1)$*. In this case h has a unique fixed point.*

Altering distance functions have been used in metric fixed point theory in a lot of papers. (See [2, 20, 26, 27]).

Alber and Guerre-Delabriere [1] presented the notion of weak contractions ,which is an another generalization of the contraction principle, in Hilbert Spaces in 1997. Also this notion was enlarged to metric spaces by Rhoades [25] in 2001. Then Doric [9] introduced and studied $(\psi - \phi)$-weak contractions in metric spaces and further developed by Proinov [21].

Definition 3. *([25]) Let* (K, d) *be a metric space, let* η *be an altering distance function and let* $h : K \to K$ *be a self mapping which satisfies the folowing inequality*

$$d(hk, hl) \leq d(k, l) - \eta(d(k, l)) \tag{1.4}$$

where $k, l \in K$*. In this case h is said to be weakly contractive mapping.*

Theorem 4. *([25]) Let* (K, d) *be a complete metric space, let* $h : K \to K$ *be a weakly contractive mapping. After that h has a unique fixed point.*

And as one of the results in the second technique, Gordji et al. [12] presented the concept of an orthogonal set and orthogonal metric spaces in 2017. In their article, extension of Banach fixed point theorem was proved. Also they applied their obtained consequences to indicate the presence of a solution of an ordinary differential equation. Then Gordji and Habibi [10] defined a new concept of generalized orthogonal metric space and they applied the obtained results to show presence and uniqueness of solution of Cauchy problem for the first order differential equation. Recently, some fixed point theorems on various orthogonal metric spaces have been given. (See [4–6, 11, 15, 18, 19, 22, 23, 28–32]).

On the other hand, Bilgili Gungor and Turkoglu [13] presented some fixed point theorems via altering distance functions on orthogonal metric spaces inspired by [12, 17]. In this paper, presence and uniqueness of fixed points of the generalizations of contraction principle via auxiliary functions are proved inspired by [12, 25]. And some consequences and an illustrative example are presented.

Other than, homotopy theory constitute an important area of algebraic topology, but the application of fixed point results in orthogonal metric spaces to homotopy has not been done until now. As a different application in this field, the homotopy application of the one of the corollaries is given at the end of this paper.

In the sequel, respectively, $\mathbb{Z}, \mathbb{R}, \mathbb{N}$ denote integers, real numbers and positive integers.

Definition 5. *([12]) Let K be a non-empty set, $\perp \subseteq K \times K$ be a binary relation. $(K, \perp)$ is called orthogonal set if $\perp$ satisfies the following condition*

$$\exists k_0 \in K; (\forall l \in K, l \perp k_0) \vee (\forall l \in K, k_0 \perp l). \tag{1.5}$$

And also this k_0 element is named orthogonal element.

Example 6. *([10]) Let $K = \mathbb{Z}$ and define $a \perp b$ if there exists $t \in \mathbb{Z}$ such that $a = tb$. It is effortless to see that $0 \perp b$ for all $b \in \mathbb{Z}$. On account of this $(K, \perp)$ is an O-set.*

This k_0 element does not have to be unique. For example;

Example 7. *([10]) Let $K = [0, \infty)$, define $k \perp l$ if $kl \in \{k, l\}$, then by setting $k_0 = 0$ or $k_0 = 1$, $(K, \perp)$ is an O-set.*

Definition 8. *([12]) A sequence $\{k_n\}$ is named orthogonal sequence if*

$$(\forall n \in \mathbb{N}; k_n \perp k_{n+1}) \vee (\forall n \in \mathbb{N}; k_{n+1} \perp k_n). \tag{1.6}$$

In the same way, a Cauchy sequence $\{k_n\}$ is named to be an orthogonally Cauchy sequence if

$$(\forall n \in \mathbb{N}; k_n \perp k_{n+1}) \vee (\forall n \in \mathbb{N}; k_{n+1} \perp k_n). \tag{1.7}$$

Definition 9. *([12]) Let $(K, \perp)$ be an orthogonal set, d be a usual metric on K. Afterwards $(K, \perp, d)$ is named an orthogonal metric space.*

Definition 10. *([12]) An orthogonal metric space $(K, \perp, d)$ is named to be a complete orthogonal metric space if every orthogonally Cauchy sequence converges in K.*

Definition 11. *([12]) Let $(K, \perp, d)$ be an orthogonal metric space and a function $h : K \to K$ is named to be orthogonally continuous at k if for each orthogonal sequence $\{k_n\}$ converging to k implies $hk_n \to hk$ as $n \to \infty$. Also h is orthogonal continuous on K if h is orthogonal continuous in each $k \in K$.*

Definition 12. *([12]) Let* $(K, \perp, d)$ *be an orthogonal metric space and* $\alpha \in \mathbb{R}$, $0 < \alpha < 1$. *A function* $h : K \to K$ *is named to be orthogonal contraction with Lipschitz constant* α *if*

$$d(hk, hl) \leq \alpha d(k, l) \tag{1.8}$$

for all $k, l \in K$ *whenever* $k \perp l$.

Definition 13. *([12]) Let* $(K, \perp, d)$ *be an orthogonal metric space and a function* $h : K \to K$ *is named orthogonal preserving if* $hk \perp hl$ *whenever* $k \perp l$.

Remark 14. *The authors of [10] gave an example which shows the orthogonal continuity and orthogonal contraction are weaker than the classic continuity and classic contraction in classic metric spaces.*

Theorem 15. *([12]) Let* $(K, \perp, d)$ *be an orthogonal complete metric space,* $0 < \alpha < 1$ *and let* $h : K \to K$ *be orthogonal continuous , orthogonal contraction (with Lipschitz constant* α*) and orthogonal preserving. Afterwards h has a unique fixed point* $k^* \in K$ *and* $\lim_{n\to\infty} h^n(k) = k^*$ *for all* $k \in K$.

And in [13], notable fixed point theorems on orthogonal metric spaces via altering distance functions are presented by Bilgili Gungor and Turkoglu.

2. Main results

Theorem 16. *Let* $(K, \perp, d)$ *be an orthogonal complete metric space,* $h : K \to K$ *be a self map ,* $\kappa, \eta \in \Delta$ *and* κ *is a sub-additive function. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$\kappa(d(hk, hl)) \leq \kappa(N(k, l)) - \eta(N(k, l)) \tag{2.1}$$

for all $k, l \in K$ *where* $k \perp l$ *and* $k \neq l$ *and*

$$N(k, l) = \max\{d(k, l), d(k, hk), d(l, hl), \frac{1}{2}\{d(k, hl) + d(l, hk)\}\}. \tag{2.2}$$

At that rate, there exists a point $k^* \in K$ *such that for any orthogonal element* $k_0 \in K$*, the iteration sequence* $\{h^n k_0\}$ *converges to this point. And, if h is orthogonal continuous at* $k^* \in K$*, then* $k^* \in K$ *is a unique fixed point of h.*

Proof. Because of $(K, \perp)$ is an orthogonal set,

$$\exists k_0 \in K; (\forall k \in K, k \perp k_0) \vee (\forall k \in K, k_0 \perp k). \tag{2.3}$$

And from h is a self mapping on K, for any orthogonal element $k_0 \in K$, $k_1 \in K$ can be chosen as $k_1 = h(k_0)$. Thus,

$$\begin{aligned} & k_0 \perp hk_0 \vee hk_0 \perp k_0 \\ \Rightarrow\ & k_0 \perp k_1 \vee k_1 \perp k_0. \end{aligned} \tag{2.4}$$

Then, if we continue in the same way

$$k_1 = hk_0, k_2 = hk_1 = h^2 k_0, ..., k_n = hk_{n-1} = h^n k_0, \tag{2.5}$$

so $\{h^n k_0\}$ is an iteration sequence.

If any $n \in \mathbb{N}, k_n = k_{n+1}$ then $k_n = hk_n$ and so h has a fixed point. Assume that $k_n \neq k_{n+1}$ for all $n \in \mathbb{N}$. Since h is orthogonal preserving, $\{h^n k_0\}$ is an orthogonal sequence and by using inequality (2.1)

$$\begin{aligned} \kappa(d(k_{n+1}, k_n)) &= \kappa(d(hk_n, hk_{n-1})) \\ &\leq \kappa(N(k_n, k_{n-1})) - \eta(N(k_n, k_{n-1})) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} N(k_n, k_{n-1}) &= \max\{d(k_n, k_{n-1}), d(k_n, hk_n), d(k_{n-1}, hk_{n-1}), \tfrac{1}{2}\{d(k_n, hk_{n-1}) + d(k_{n-1}, hk_n)\}\} \\ &= \max\{d(k_n, k_{n-1}), d(k_n, k_{n+1}), d(k_{n-1}, k_n), \tfrac{1}{2}\{d(k_n, k_n) + d(k_{n-1}, k_{n+1})\}\} \\ &= \max\{d(k_n, k_{n-1}), d(k_n, k_{n+1})\}. \end{aligned} \tag{2.7}$$

If $N(k_n, k_{n-1}) = d(k_n, k_{n+1})$ then

$$\kappa(d(k_{n+1}, k_n)) \leq \kappa(d(k_n, k_{n+1})) - \eta(d(k_n, k_{n+1})) \tag{2.8}$$

is obtained. It is a contradiction. And so $N(k_n, k_{n-1}) = d(k_n, k_{n-1})$. Thus,

$$\kappa(d(k_{n+1}, k_n)) \leq \kappa(d(k_n, k_{n-1})) - \eta(d(k_n, k_{n-1})). \tag{2.9}$$

Using the monotone property of $\kappa \in \Delta$, $\{d(k_{n+1}, k_n)\}$ is a sequence of decreasing nonnegative real numbers. Thus there is a $m \geq 0$ and $\lim_{n\to\infty} d(k_{n+1}, k_n) = m$. We will show that $m = 0$. Assume, conversely, that $m > 0$. At that rate, by passing to the limit $n \to \infty$ in inequality (2.9) and using κ, η are continuous functions, we obtain

$$\kappa(m) \leq \kappa(m) - \eta(m). \tag{2.10}$$

This is a inconsistency. So we get $m = 0$. Now we prove that $\{k_n\}$ is an orthogonally Cauchy sequence. If $\{k_n\}$ is not an orthogonally Cauchy sequence, there exists $\epsilon > 0$ and corresponding subsequences $\{t(n)\}$ and $\{s(n)\}$ of $\mathbb{N}$ satisfying $t(n) > s(n) > n$ for which

$$d(k_{t(n)}, k_{s(n)}) \geq \epsilon \tag{2.11}$$

and where $t(n)$ is chosen as the smallest integer satisfying (2.11), that is

$$d(k_{t(n)-1}, k_{s(n)}) < \epsilon. \tag{2.12}$$

By (2.11), (2.12) and triangular inequality of d, we easily derive that

$$\varepsilon \leq d(k_{t(n)}, k_{s(n)}) \leq d(k_{t(n)}, k_{t(n)-1}) + d(k_{t(n)-1}, k_{s(n)}) < d(k_{t(n)}, k_{t(n)-1}) + \epsilon. \tag{2.13}$$

Letting $n \to \infty$, by using $\lim_{n\to\infty} d(k_{n+1}, k_n) = m = 0$, we get

$$\lim_{n\to\infty} d(k_{t(n)}, k_{s(n)}) = \epsilon. \tag{2.14}$$

And, for each $n \in \mathbb{N}$, by using the triangular inequality of d,

$$\begin{aligned} d(k_{t(n)}, k_{s(n)}) - d(k_{t(n)}, k_{t(n)+1}) - d(k_{s(n)+1}, k_{s(n)}) &\leq d(k_{t(n)+1}, k_{s(n)+1}) \\ &\leq d(k_{t(n)}, k_{t(n)+1}) + d(k_{t(n)}, k_{s(n)}) \\ &\quad + d(k_{s(n)+1}, k_{s(n)}). \end{aligned} \tag{2.15}$$

Taking limit when $n \to \infty$ in the last inequality we obtain

$$d(k_{t(n)+1}, k_{s(n)+1}) = \epsilon. \tag{2.16}$$

Using the inequality (2.1),

$$\begin{aligned} \kappa(d(k_{t(n)+1}, k_{s(n)+1})) &= \kappa(d(hk_{t(n)}, hk_{s(n)})) \\ &\leq \kappa(N(k_{t(n)}, k_{s(n)})) - \eta(N(k_{t(n)}, k_{s(n)})) \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} N(k_{t(n)}, k_{s(n)}) &= \max\{d(k_{t(n)}, k_{s(n)}), d(k_{t(n)}, hk_{t(n)}), d(k_{s(n)}, hk_{s(n)}), \\ &\quad \tfrac{1}{2}\{d(k_{t(n)}, hk_{s(n)}) + d(k_{s(n)}, hk_{t(n)})\}\} \\ &= \max\{d(k_{t(n)}, k_{s(n)}), d(k_{t(n)}, k_{t(n)+1}), d(k_{s(n)}, k_{s(n)+1}), \\ &\quad \tfrac{1}{2}\{d(k_{t(n)}, k_{s(n)+1}) + d(k_{s(n)}, k_{t(n)+1})\}\}. \end{aligned} \tag{2.18}$$

Taking limit when $n \to \infty$ in the last inequality we obtain

$$\kappa(\epsilon) \leq \kappa(\epsilon) - \eta(\epsilon). \tag{2.19}$$

It is a inconsistency. Thus $\{k_n\}$ is a orthogonally Cauchy sequence. By the orthogonally completeness of K, there exists $k^* \in K$ such that $\{k_n\} = \{h^n k_0\}$ converges to this point.

Now it can be shown that k^* is a fixed point of h when h is orthogonal continuous at $k^* \in K$. Suppose that h is orthogonal continuous at $k^* \in K$. Therefore,

$$k^* = \lim_{n\to\infty} k_{n+1} = \lim_{n\to\infty} hk_n = hk^*. \tag{2.20}$$

thus $k^* \in K$ is a fixed point of h.

Currently we can be show the uniqueness of the fixed point. Assume that there exist two distinct fixed points k^* and l^*. Then,

(i) If $k^* \perp l^* \vee l^* \perp k^*$, by using the inequality (2.1)

$$\begin{aligned} \kappa(d(k^*, l^*)) &= \kappa(d(hk^*, hl^*)) \\ &\leq \kappa(N(k^*, l^*)) - \eta(N(k^*, l^*)) \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} N(k^*, l^*) &= \max\{d(k^*, l^*), d(k^*, hk^*), d(l^*, hl^*), \tfrac{1}{2}\{d(k^*, hl^*) + d(l^*, hk^*)\}\} \\ &= d(k^*, l^*). \end{aligned} \tag{2.22}$$

And so

$$\kappa(d(k^*, l^*)) \leq \kappa(d(k^*, l^*)) - \eta(d(k^*, l^*)) \tag{2.23}$$

is obtained. This is a inconsistency and $k^* \in K$ is an unique fixed point of h.

(ii) If not, for the chosen orthogonal element $k_0 \in K$,

$$[(k_0 \perp k^*) \wedge (k_0 \perp l^*)] \vee [(k^* \perp k_0) \wedge (l^* \perp k_0)] \tag{2.24}$$

and since h is orthogonal preserving,

$$[(hk_n \perp k^*) \wedge (hk_n \perp l^*)] \vee [(k^* \perp hk_n) \wedge (l^* \perp hk_n)] \tag{2.25}$$

is obtained. And by using the triangular inequality of d, κ is nondecreasing sub-additive function and the inequality (2.1)

$$\begin{aligned}\kappa(d(k^*, l^*)) &= \kappa(d(hk^*, hl^*)) \\ &\leq \kappa(d(hk^*, hk_{n+1}) + d(hk_{n+1}, hl^*)) \\ &\leq \kappa(d(hk^*, h(hk_n))) + \kappa(d(h(hk_n), hl^*)) \\ &\leq \kappa(N(k^*, hk_n)) - \eta(N(k^*, hk_n)) + \kappa(N(hk_n, l^*)) - \eta(N(hk_n, l^*)),\end{aligned} \tag{2.26}$$

where

$$\begin{aligned}N(k^*, hk_n) &= \max\{d(k^*, hk_n), d(k^*, hk^*), d(k_n, h(hk_n)), \\ &\tfrac{1}{2}\{d(k^*, h(hk_n)) + d(hk_n, hk^*)\}\}\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}N(hk_n, l^*) &= \max\{d(hk_n, l^*), d(hk_n, h(hk_n)), d(l^*, hl^*), \\ &\tfrac{1}{2}\{d(hk_n, h(l^*)) + d(l^*, h(hk_n))\}\},\end{aligned} \tag{2.28}$$

in the last inequality taking limit $n \to \infty$, we obtain $k^* = l^*$. Thus, $k^* \in K$ is a unique fixed point of h. □

Setting $\kappa = I$ in Theorem 16, we conclude the following corollary.

Corollary 17. *Let $(K, \perp, d)$ be an orthogonal complete metric space, $h : K \to K$ be a self map , $\eta \in \Delta$. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$d(hk, hl) \leq N(k, l) - \eta(N(k, l)) \tag{2.29}$$

for all $k, l \in K$ where $k \perp l, k \neq l$ and

$$N(k, l) = \max\{d(k, l), d(k, hk), d(l, hl), \frac{1}{2}\{d(k, hl) + d(l, hk)\}\}. \tag{2.30}$$

At that rate, there exists a point $k^* \in K$ such that for any orthogonal element $k_0 \in K$, the iteration sequence $\{h^n k_0\}$ converges to this point. And if h is orthogonal continuous at $k^* \in K$, then $k^* \in K$ is a unique fixed point of h.

Setting $\kappa = I$ and $\eta(s) = (1 - \delta)s$ $(\forall s \in (0, \infty), \delta \in (0, 1))$ in Theorem 16, we conclude the following corollary.

Corollary 18. *Let $(K, \perp, d)$ be an orthogonal complete metric space, $h : K \to K$ be a self map, $\delta \in \mathbb{R}$ where $0 < \delta < 1$. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$d(hk, hl) \leq \delta N(k, l) \tag{2.31}$$

for all $k, l \in K$ where $k \perp l, k \neq l$ and

$$N(k, l) = \max\{d(k, l), d(k, hk), d(l, hl), \frac{1}{2}\{d(k, hl) + d(l, hk)\}\}. \tag{2.32}$$

In this case, there exists a point $k^* \in K$ such that for any orthogonal element $k_0 \in K$, the iteration sequence $\{h^n k_0\}$ converges to this point. And if h is orthogonal continuous at $h^* \in K$, then $k^* \in K$ is a unique fixed point of h.

Corollary 19. *Let $(K,\perp,d)$ be an orthogonal complete metric space, $h: K \to K$ be a self map, $\delta \in \mathbb{R}$ where $0<\delta<1$. Assume that h is orthogonal preserving self mapping satisfying the inequality*

$$\int_0^{d(hk,hl)} \vartheta(\zeta)d(\zeta) \leq \delta \int_0^{\max\{d(k,l),d(k,hk),d(l,h)l,\frac{1}{2}\{d(k,hl)+d(l,hk)\}} \vartheta(\zeta)d(\zeta) \tag{2.33}$$

for all $k,l \in K$ where $k \perp l, k \neq l$ and $\vartheta: \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable and $\int_0^{\epsilon} \vartheta(\zeta)d(\zeta) > 0$ (for each $\epsilon > 0$). In this case, there exists a point $k^* \in K$ such that for any orthogonal element $k_0 \in K$, the iteration sequence $\{h^n k_0\}$ converges to this point. Also, if h is orthogonal continuous at $k^* \in K$, then $k^* \in K$ is a unique fixed point of h.

Proof. Choose $\kappa(s) = \int_0^s \vartheta(\zeta)d(\zeta)$ and $\eta(t) = (1-\delta)\int_0^s \theta(\zeta)d(\zeta)$ (for all $s \in [0,\infty)$). Clearly $\kappa,\eta \in \Delta$ and κ is a sub-additive function. Now, in view of Theorem 16, result follows. □

Remark 20. *The main theorem in [12] is the result of Theorem 16. (In Theorem 16, let $N(k,l) = d(k,l)$, $\kappa = I$ and $\eta(s) = (1-\delta)s$ ($\forall s \in (0,\infty), \delta \in (0,1)$).)*

Example 21. *Let $K = [0,1)$ be a set and define the $d: K \times K \to K$ such that $d(k,l) =\mid k-l \mid$. Also, let the binary relation $\perp$ on K such that $k \perp l \iff kl \leq \max\{\frac{k}{3},\frac{l}{3}\}$. Then, $(K,\perp)$ is an orthogonal set and d is a metric on K. So $(K,\perp,d)$ is an orthogonal metric space. In this space, any orthogonally Cauchy sequence is convergent. Indeed, suppose that (k_n) is an arbitrary orthogonal Cauchy sequence in K. Then*

$$\begin{aligned} & k_n.k_{n+1} \leq \tfrac{k_n}{3} \text{ or } k_n.k_{n+1} \leq \tfrac{k_{n+1}}{3} \\ \Rightarrow\ & k_n(k_{n+1} - \tfrac{1}{3}) \leq 0 \text{ or } k_{n+1}(k_n - \tfrac{1}{3}) \leq 0 \\ \Rightarrow\ & (k_n = 0 \text{ or } k_{n+1} \leq \tfrac{1}{3}) \text{ or } (k_{n+1} = 0 \text{ or } k_n \leq \tfrac{1}{3}) \end{aligned} \tag{2.34}$$

and for any $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$, for all $n \in \mathbb{N}$ that is $n \geq n_0$,

$$\mid k_n - k_{n+1} \mid < \epsilon \tag{2.35}$$

is provided. So, for any $\epsilon > 0$ and for all $n \in \mathbb{N}$, that is $n \geq n_0$, $\mid k_n - 0 \mid < \epsilon$ that is $\{k_n\}$ is convergent to $0 \in K$. Thus $(K,\perp,d)$ is a complete orthogonal metric space. Remark that, (K,d) is not a complete sub-metric space of $(\mathbb{R},d)$ because of K is not a closed subset of $(\mathbb{R},d)$.

Let $\kappa: [0,\infty) \to [0,\infty)$ be defined as $\kappa(s) = \frac{s}{2}$ and let $\eta: [0,\infty) \to [0,\infty)$ be defined as $\eta(s) = \frac{s}{3}$. Also let $h: K \to K$ be defined as

$$h(k) = \begin{cases} \frac{k}{3}, & k \leq \frac{1}{3}, \\ 0, & k > \frac{1}{3}. \end{cases} \tag{2.36}$$

In this case, one can see that $\kappa,\eta \in \Delta$, κ is a sub-additive function. Also h is orthogonal preserving mapping. Indeed,

$$k \perp l \Rightarrow (kl \leq \frac{k}{3}) \text{ or } (kl \leq \frac{l}{3}). \tag{2.37}$$

Without loss of generality, suppose that $kl \leq \frac{k}{3}$.

So $k = 0$ or $l \leq \frac{1}{3}$. Then, there exists the following cases:
Case I: $k = 0$ and $l \leq \frac{1}{3}$. Then $h(k) = 0$ and $h(l) = \frac{l}{3}$.
Case II: $k = 0$ and $l > \frac{1}{3}$. Then $h(k) = h(l) = 0$.
Case III: $l \leq \frac{1}{3}$ and $k \leq \frac{1}{3}$. Then $h(l) = \frac{l}{3}$ and $h(k) = \frac{k}{3}$.

Case IV: $l \leq \frac{1}{3}$ *and* $k > \frac{1}{3}$. *Then* $h(l) = \frac{l}{3}$ *and* $h(k) = 0$.
These cases implies that $h(k)h(l) \leq \frac{h(k)}{3}$.

On the other hand, h is orthogonal continuous at $0 \in K$. *Indeed, assume that* $\{k_n\}$ *is an orthogonal sequence and* $k_n \to 0$. *In this case,*

$$\begin{aligned} & (k_n.k_{n+1} \leq \tfrac{k_n}{3}) \ or \ (k_n.k_{n+1} \leq \tfrac{k_{n+1}}{3}) \\ \Rightarrow \ & (k_n = 0 \ or \ k_{n+1} \leq \tfrac{1}{3}) \ or \ (k_{n+1} = 0 \ or \ k_n \leq \tfrac{1}{3}) \end{aligned} \tag{2.38}$$

and also because of $k_n \to 0$, *for any* $\varepsilon > 0$ *there exists a* $n_0 \in \mathbb{N}$, *for all* $n \in \mathbb{N}$ *that is* $n > n_0$, $k_n < \epsilon$ *is obtained. So, for all* $n \in \mathbb{N}$ *that is* $n > n_0$, $k_n \in [0, \frac{1}{3}]$. *Therefore, from the definition of h, for the same* $n_0 \in \mathbb{N}$ *that is* $n > n_0$, $\mid h(x_n) - h(0) \mid < \epsilon$ *that is* $h(x_n) \to h(0) = 0$.

Now, it can be shown that h is a self mapping satisfying the inequality (2.1) for all $k, l \in K$ *where* $k \perp l$ *and* $k \neq l$ *and*

$$N(k,l) = \max\{d(k,l), d(k,hk), d(l,hl), \frac{1}{2}\{d(k,hl) + d(l,hk)\}\}. \tag{2.39}$$

Assume that $k, l \in K$ *two element of K,* $k \perp l$ *and* $k \neq l$. *In this case*

$$(kl \leq \frac{k}{3}) \ or \ (kl \leq \frac{l}{3}). \tag{2.40}$$

Without loss of generality, suppose that $kl \leq \frac{k}{3}$.

So $k = 0$ *or* $l \leq \frac{1}{3}$. *Then there exist the following cases:*
Case I: If $k = 0$ *and* $l \leq \frac{1}{3}$. *Then* $h(k) = 0$ *and* $h(l) = \frac{l}{3}$.

$$\begin{aligned} \kappa(d(hk,hl)) &= \tfrac{|0-\frac{l}{3}|}{2} = \tfrac{l}{6}, \\ N(k,l) &= \max\{d(k,l), d(k,hk), d(l,hl), \tfrac{1}{2}\{d(k,hl) + d(l,hk)\}\} = l, \\ \kappa(N(k,l)) &= \tfrac{l}{2}, \\ \eta(N(k,l)) &= \tfrac{l}{3}. \end{aligned} \tag{2.41}$$

So $\kappa(d(hk,hl)) = \frac{l}{6} \leq \frac{l}{2} - \frac{l}{3} = \kappa(N(k,l)) - \eta(N(k,l))$.
Case II: If $k = 0$ *and* $l > \frac{1}{3}$. *Then* $hk = hl = 0$.

$$\begin{aligned} \kappa(d(hk,hl)) &= 0, \\ N(k,l) &= \max\{d(k,l), d(k,hk), d(l,hl), \tfrac{1}{2}\{d(k,hl) + d(l,h)k\}\} = l, \\ \kappa(N(k,l)) &= \tfrac{l}{2}, \\ \eta(N(k,l)) &= \tfrac{l}{3}. \end{aligned} \tag{2.42}$$

So $\kappa(d(hk,hl)) = 0 \leq \frac{l}{2} - \frac{l}{3} = \kappa(N(k,l)) - \eta(N(k,l))$.
Case III: If $l \leq \frac{1}{3}$ *and* $k \leq \frac{1}{3}$. *Then* $hl = \frac{l}{3}$ *and* $hk = \frac{k}{3}$.

Without loss of generality, suppose that $0 \leq l \leq k \leq \frac{1}{3}$. *Then,*

$$\begin{aligned} \kappa(d(hk,hl)) &= \tfrac{|\frac{k}{3}-\frac{l}{3}|}{2} = \tfrac{k-l}{6}, \\ N(k,l) &= \max\{d(k,l), d(k,hk), d(l,hl), \tfrac{1}{2}\{d(k,hl) + d(l,hk)\}\} \\ &= \max\{k-l, \tfrac{2k}{3}, \tfrac{2l}{3}, \tfrac{1}{2}\{k - \tfrac{l}{3} + \mid l - \tfrac{k}{3} \mid\}\}. \end{aligned} \tag{2.43}$$

Now there are two cases:
(i) If $\frac{k}{3} \leq l$*, then* $-l \leq -\frac{k}{3}$ *and so* $k - l \leq k - \frac{k}{3} = \frac{2k}{3}$*. Also because of* $l \leq k$ *we get* $\frac{2l}{3} \leq \frac{2k}{3}$ *and* $\frac{k+l}{3} \leq \frac{2k}{3}$*. Thus* $N(k,l) = \max\{k - l, \frac{2k}{3}, \frac{2l}{3}, \frac{k+l}{3}\} = \frac{2k}{3}$ *and so* $\kappa(N(k,l)) = \frac{k}{3}$ *and* $\eta(N(k,l)) = \frac{2k}{9}$*.*
(ii) If $l \leq \frac{k}{3}$*, then* $-\frac{k}{3} \leq -l$ *and so* $k - \frac{k}{3} = \frac{2k}{3} \leq k - l$*. Also because of* $l \leq k$ *we get* $\frac{2l}{3} \leq \frac{2k}{3}$ *and* $\frac{2k-2l}{3} \leq \frac{2k}{3}$*. Thus* $N(k,l) = \max\{k - l, \frac{2k}{3}, \frac{2l}{3}, \frac{2k-2l}{3}\} = k - l$ *and so* $\kappa(N(k,l)) = \frac{k-l}{2}$ *and* $\eta(N(k,l)) = \frac{k-l}{3}$*.*

In both cases it can be easily seen that $\kappa(d(hk, hl)) \leq \kappa(N(k,l)) - \eta(N(k,l))$*.*
Case IV: If $l \leq \frac{1}{3}$ *and* $k > \frac{1}{3}$*. Then* $hl = \frac{l}{3}$ *and* $hk = 0$*.*

$$\begin{aligned} \kappa(d(hk,hl)) &= \frac{|0-\frac{l}{3}|}{2} = \frac{l}{6}, \\ N(k,l) &= \max\{d(k,l), d(k,hk), d(l,hl), \tfrac{1}{2}\{d(k,h)l + d(l,hk)\}\} = k, \\ \kappa(N(k,l)) &= \frac{k}{2}, \\ \eta(N(k,l)) &= \frac{k}{3}. \end{aligned} \tag{2.44}$$

So $\kappa(d(hk,hl)) = \frac{l}{6} \leq \frac{k}{2} - \frac{k}{3} = \kappa(N(k,l)) - \eta(N(k,l))$*.*

Consequently, h is a self mapping satisfying the inequality (2.1) for all $k, l \in K$ *whenever* $k \perp l$ *and* $k \neq l$*. Thus, all hypothesis of Theorem 16 satisfy and so, it is evident that h has a unique fixed point* $0 \in K$*.*

3. An application to homotopy

Theorem 22. *Let* $(K, \perp, d)$ *be an orthogonal complete metric space. Let W be a nonempty open subset of orthogonal elements of K. Assume that* $H : \overline{W} \times [0,1] \to K$ *with the following properties:*
(1) $k \neq H(k,s)$ *for every* $k \in \partial W$ *and* $s \in [0,1]$ *(here* ∂W *denotes the boundary of W in K.)*
(2) *For all* $k, l \in \overline{W}$ *where* $k \perp l$ *and* $k \neq l$ *and* $s \in [0,1], \lambda \in [0,1)$ *such that*

$$d(H(k,s), H(l,s)) \leq \lambda d(k,l). \tag{3.1}$$

(3) *There exists* $L \geq 0$*, such that*

$$d(H(k,s), H(k,r)) \leq L \mid s - r \mid \tag{3.2}$$

for every $k \in \overline{W}$ *and* $s, r \in [0,1]$*.*

In this case $H(.,0)$ *has a fixed point in W if and only if* $H(.,1)$ *has a fixed point in W.*

Proof. Determine the set

$$M = \{s \in [0,1] : k = H(k,s) \text{ for some } k \in W\}. \tag{3.3}$$

($\Rightarrow$:) Because of $H(.,0)$ has a fixed point in W, then M is nonempty, that is $0 \in M$. If it is shown that the M is both closed and open in $[0,1]$, then from the connectedness of $[0,1]$, it is obtained $M = [0,1]$. Therefore $H(.,1)$ has a fixed point in W.

Firstly, we show that M is closed in $[0,1]$. Let $\{s_n\}$ be a sequence in M where $s_n \to s^* \in [0,1]$ as $n \to \infty$. It must be shown that $s^* \in M$. Since $s_n \in M$ for $n \in \mathbb{N}$, there exists $k_n \in W$ with $k_n = H(k_n, s_n)$. Also for $n, m \in \mathbb{N}$,

$$\begin{aligned} d(k_n,k_m) &= d(H(k_n,s_n), H(k_m,s_m)) \\ &\leq d(H(k_n,s_n), H(k_n,s_m)) + d(H(k_n,s_m), H(k_m,s_m)) \\ &\leq L \mid s_n - s_m \mid + \lambda d(k_n,k_m), \end{aligned} \tag{3.4}$$

that is,

$$d(k_n, k_m) \leq (\frac{L}{1-\lambda}) \mid s_n - s_m \mid . \tag{3.5}$$

Since every convergent sequence in metric spaces is a Cauchy sequence, $\{s_n\}$ is a Cauchy sequence. Thus we obtain $\lim_{n,m\to\infty} d(k_n, k_m) = 0$, that is $\{k_n\}$ is an orthogonally Cauchy sequence in K. Since K is an orthogonal complete there exists $k^* \in \overline{W}$ with $\lim_{n\to\infty} d(k_n, k^*) = 0$. Letting $n \to \infty$ in the following inequality,

$$\begin{aligned} d(k_n, H(k^*, s^*)) &= d(H(k_n, s_n), H(k^*, s^*)) \\ &\leq d(H(k_n, s_n), H(k_n, s^*)) + d(H(k_n, s^*), H(k^*, s^*)) \\ &\leq L \mid s_n - s^* \mid + \lambda d(k_n, k^*), \end{aligned} \tag{3.6}$$

we get $\lim_{n\to\infty} d(k_n, H(k^*, s^*)) = 0$ and hence

$$\lim_{n\to\infty} d(k_n, H(k^*, s^*)) = d(k^*, H(k^*, s^*)) = 0 \tag{3.7}$$

that is $H(k^*, s^*) = k^*$. Thus $s^* \in M$ is gotten and so M is closed in $[0, 1]$.

Now, we continue with proving M is open in $[0, 1]$. Let $s_0 \in M$ and $k_0 \in W$ with $k_0 = H(k_0, s_0)$. There exists $r_0 > 0$ such that $B_d(k_0, r_0) \subseteq W$ as W is open in K. Considering $\epsilon > 0$ with $\epsilon < \frac{(1-\lambda)r_0}{L}$.

Let $s \in (s_0 - \epsilon, s_0 + \epsilon)$, then for $k \in \overline{B}_d(k_0, r_0) = \{k \in K : d(k, k_0) \leq r_0\}$,

$$\begin{aligned} d(H(k, s), k_0) &= d(H(k, s), H(k_0, s_0)) \\ &\leq d(H(k, s), H(k, s_0)) + d(H(k, s_0), H(k_0, s_0)) \\ &\leq L \mid s - s_0 \mid + \lambda d(k, k_0) \\ &\leq (1-\lambda) r_0 + \lambda r_0 \\ &= r_0. \end{aligned} \tag{3.8}$$

Thus for each fixed $s \in (s_0 - \epsilon, s_0 + \epsilon)$, $H(., s) : \overline{B}_d(k_0, r_0) \to \overline{B}_d(k_0, r_0)$.

Since all hypothesis of Corollary 18 hold, $H(., s)$ has a fixed point in $\overline{W}$. However it must be in W as (1) obtains. Therefore for any $s_0 \in M$, there exists an $\epsilon > 0$ and $(s_0 - \epsilon, s_0 + \epsilon) \subseteq M$. And so we obtain that M is open in $[0, 1]$.

($\Leftarrow$:) It can be shown similarly same argument in above. □

4. Conclusions

In the first part of this study, as a result of a comprehensive literature review, the developments related to the existence of fixed points for mappings that provide the appropriate contraction conditions from the beginning of the fixed point theory studies are mentioned, and then the general subject of this study is emphasized.

In this paper, presence and uniqueness of fixed points of the generalizations of contraction principle via auxiliary functions are proved inspired by [12, 25]. And some consequences and an illustrative example are presented.

Other than, homotopy theory constitute an important area of algebraic topology, but the application of fixed point results in orthogonal metric spaces to homotopy has not been done until now. As a different application in this field, the homotopy application of the one of the corollaries is given at the end of this paper.

The results of this paper, not only generalize the analogous fixed point theorems but are relatively simpler and more natural than the related ones. The results of this paper are actually three-fold: a relatively more general contraction condition is used, the continuity of the involved mapping is weakened to orthogonal continuity, the comparability conditions used by previous authors between elements are replaced by orthogonal relatedness.

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Conflict of interest

The author declares that there is no conflict interest regarding the publication of this article.

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