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Research article

A fixed point principle in ordered metric spaces and applications to rational type contractions

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Abstract: Fixed points results for rational type contractions in metric spaces have been widely studied in the literature. In the last years, many of these results are obtained in the context of partially ordered metric spaces. In this paper, we introduce a fixed point principle for a class of mappings between partially ordered metric spaces that we call orbitally order continuous. We show that the hypotheses in the statement of such a principle are not redundant and, in addition, that they cannot be weakened in order to guarantee the existence of a fixed point. Moreover, the relationship between this kind of mappings and those that are continuous and orbitally continuous is discussed. As an application, we extend many fixed point theorems for continuous contractions of rational type to the framework of those that are only orbitally order continuous. Furthermore, we get extensions of the aforementioned metric fixed point results to the framework of partial metrics. This is achieved thanks to the fact that each partial metric induces in a natural way a metric in such a way that our new principle is applicable. In both approaches, the metric and the partial metric, we show that there are orbitally order continuous mappings that satisfy all assumptions in our new fixed point principle but that they are not contractions of rational type. The explored theory is illustrated by means of appropriate examples.

Keywords: partial order; metric space; partial metric space; fixed point principle; rational contraction
Mathematics Subject Classification:47H10, 54E50, 54E55, 54F05, 54H25

1. Introduction

Throughout this paper, it is to be expected that the reader is familiar with basics of metric fixed point theory (see, for instance, [30]). Let us recall that, in 1977, D. S. Jaggi proved the following fixed point result for self-mappings in metric spaces known as contractions of rational type (see [16]).

Theorem 1.1. *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a mapping such that there exist real numbers* $\alpha, \beta \in [0, 1[$ *with* $\alpha + \beta < 1$ *and*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$ *with* $x \neq y$*. Then* T *has a unique fixed point in* X*.*

Later on, in 2010, J. Harjani, B. López and K. Sadarangani gave a version of Theorem 1.1 for contractions of rational type defined on partially ordered metric case [13], i.e., a triplet $(X, d, \preceq)$ where (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set. In order to state such a result, let us recall that, given a partially ordered set $(X, \preceq)$, a mapping $T : X \to X$ is said to be $\preceq$-monotone provided that $Tx \preceq Ty$ whenever $x \preceq y$. Moreover, from now on, we will denote by $D_{X,\preceq,\neq}$ the set $\{(x, y) \in X \times X : x \neq y, y \preceq x\}$.

Theorem 1.2. *Let* $(X, d, \preceq)$ *be a partially ordered complete metric space and let* $T : X \to X$ *be a continuous and* $\preceq$*-monotone mapping such that there exist real numbers* $\alpha, \beta \in [0, 1[$ *with* $\alpha + \beta < 1$ *and*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $(x, y) \in D_{X,\preceq,\neq}$*. If there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*, then* T *has a fixed point in* X*.*

In [13] the next version of the preceding result was also proved when $\beta = 0$.

Theorem 1.3. *Let* $(X, d, \preceq)$ *be a partially ordered complete metric space and let* $T : X \to X$ *be a continuous and* $\preceq$*-monotone mapping such that there exists a real number* $\alpha \in [0, 1[$ *such that*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)}$$

for all $(x, y) \in D_{X,\preceq,\neq}$*. If there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*, then* T *has a fixed point in* X*.*

In [25] (see also [4, 7]) the next version of the preceding result was also proved when $\alpha = 0$.

Theorem 1.4. *Let* $(X, d, \preceq)$ *be a partially ordered complete metric space and let* $T : X \to X$ *be a continuous and* $\preceq$*-monotone mapping such that there exist a real number* $\beta \in [0, 1[$ *and*

$$d(Tx, Ty) \leq \beta d(x, y)$$

for all $(x, y) \in D_{X,\preceq,\neq}$*. If there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*, then* T *has a fixed point in* X*.*

It must be stressed that, although the contractive condition Theorem 1.3 is assumed to be hold for all $(x, y) \in D_{X,\preceq} = \{(x, y) \in X \times X : y \preceq x\}$ in the version published in [25], such a contractive condition is only required to be satisfied by the sequence $(T^n x_0)_{n \in \mathbb{N}}$ defined by $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for all

$n \in \mathbb{N}$ and, in addition, it is assumed that $(x_{n+1}, x_n) \in D_{X,\preceq,\neq}$ with $n \geq 1$ because otherwise the existence of a fixed point is evident and the contractive condition becomes unnecessary. Motivated by this fact we have stated the aforesaid theorem in the most restrictive sense regarding the contractive condition. Notice that the original version can be retrieved as a particular case of Theorem 1.4.

From now on, we will denote by $\mathbb{R}^+$ the set of nonnegative real numbers.

In 2013, M. Arshad, E. Karapinar and J. Ahmad extended Theorem 1.1 to new classes of rational type contractions in [8]. On the one hand, one of the aforementioned extensions can be stated as follows:

Theorem 1.5. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a continuous and $\preceq$-monotone mapping such that there exist real numbers $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ with $\alpha + \beta < 1$ and*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta d(x, y) + L \min\{d(x, Ty), d(y, Tx)\}$$

for all $(x, y) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

On the other hand, in [8] a relaxation of the contractivity condition was considered in such a way that elements $\preceq$-related not necessarily different can be considered.

Theorem 1.6. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a continuous and $\preceq$-monotone mapping such that there exist real numbers $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ such that $\alpha + \beta < 1$ and*

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)\,[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + L \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

for all $(y, x) \in D_{X,\preceq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

As in the case of Theorem 1.4 we only consider the version of the preceding result given by Theorem 1.7 below, since in the proof given in [8] such a contractive condition is only required to be satisfied by the sequence $(T^n x_0)_{n\in\mathbb{N}}$ which is assumed, at the same time, to satisfy that $(x_{n+1}, x_n) \in D_{X,\preceq,\neq}$ with $n \geq 1$.

Theorem 1.7. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a continuous and $\preceq$-monotone mapping such that there exist real numbers $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ such that $\beta > 0$, $\alpha + \beta < 1$ and*

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)\,[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + L \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

for all $(y, x) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

When $L = 0$ in the result above, the next consequence was obtained in [8].

Theorem 1.8. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a continuous and $\preceq$-monotone mapping such that there exist real numbers $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ with $\alpha + \beta < 1$ and*

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)\,[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y)$$

for all $(y, x) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

In the light of the research activity that has been developed in this direction, the main purpose of this paper is to provide a fixed point principle which allows us to retrieve as particular cases all the exposed fixed point results. Such a principle is motivated by the appreciation that all proofs of the aforementioned fixed point results can derive from the application of the same technique. Inspired by this fact we introduce the notion of orbitally $\preceq$-continuity which captures the essence of such a technique making continuity unnecessary. Moreover, we show that the hypotheses in the statement of our new principle are not redundant and, in addition, that they cannot be weakened in order to guarantee the existence of a fixed point. Furthermore, the relationship between orbitally $\preceq$-continuity and (orbitally) continuity is discussed. With the help of the new principle, we extend all the exposed fixed point theorems for those contractions of rational type that are only orbitally $\preceq$-continuous. In addition, we show that there are orbitally $\preceq$-continuous mappings that satisfy all assumptions in our new fixed point principle but that they are not contractions of rational type.

Inspired by the preceding facts, we get also extensions of all exposed metric fixed point results to the framework of partial metric spaces, a generalized notion of metric introduced by S. G. Matthews in 1994 [22]. Let us recall that the introduction of such a notion was motivated by the fact that the utility of metric spaces in Computer Science is limited. However, the notion of partial metric has been shown to be an appropriate tool for mathematical modeling in areas like denotational semantics for programming languages, parallel processing, complexity analysis of algorithms and logic programming (see [3, 11, 14, 22, 23, 32, 33]). Recently, an attempt to extent Theorem 1.2 has been established in the framework of partial metric spaces in [15]. Since every partial metric induces in a natural way a metric our new fixed point principle is also applicable. It is worthy to mention that our approach improves that given in [15]. Again we show that there are orbitally $\preceq$-continuous mappings that satisfy all assumptions in our new fixed point principle but that they are not contractions of rational type in the partial metric context. In both approaches, the metric and the partial metric, the explored theory is illustrated by means of appropriate examples.

2. The main result

In this section we present our main result which will be crucial in our subsequent discussion. To this end let us recall that, given a partially ordered set $(X, \preceq)$, a sequence $(x_n)_{n\in\mathbb{N}}$ is increasing when $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ stands for the set of positive integer numbers. The next notion will play a central role later on.

Definition 2.1. *Let $(X, d, \preceq)$ be a partially ordered metric space. A mapping $T : X \to X$ will be said to be orbitally $\preceq$-continuous at $x_0 \in X$ provided that $(T^{n+1}x_0)_{n\in\mathbb{N}}$ has a subsequence which converges to Tx_0 with respect to $\mathcal{T}(d)$ whenever the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is increasing in $(X, \preceq)$ and converges to x_0 with respect to $\mathcal{T}(d)$.*

The next example gives an instance of a mapping that is not orbitally $\preceq$-continuous at any point.

Example 2.1. *Consider the metric space $([0, 1], d_E)$, where d_E stands for the Euclidean metric. Consider the partial order $\preceq_{[0,1]}$ on $[0, 1]$ given by $x \preceq_{[0,1]} y \Leftrightarrow x, y \in]0, 1]$ and $x \geq y$ or $x = y$. Next define the mapping $T : [0, 1] \to [0, 1]$ by $Tx = \frac{x}{2}$ for all $x \in]0, 1]$ and $T0 = \frac{1}{2}$. It is clear that there is no point $x_0 \in [0, 1]$ such that T is orbitally $\preceq_{[0,1]}$-continuous at x_0. Indeed, consider $x_0 \in [0, 1]$. Then $(T^n x_0)_{n\in\mathbb{N}}$ is increasing in $([0, 1], \preceq_{[0,1]})$ and always converges to 0 with respect to $\mathcal{T}(d_E)$. However,* *the sequence* $(T^{n+1}x_0)_{n\in\mathbb{N}}$ *does not have any subsequence which converges to* $T0 = \frac{1}{2}$ *with respecto to* $\mathcal{T}(d_E)$.

Observe that a $\mathcal{T}(d)$-continuous mapping is clearly an orbitally $\preceq$-continuous one. However, the converse is not true in general such as Example 2.2 shows.

Example 2.2. *Consider the metric space* $([0,1], d_E)$. *Define the mapping* $T : [0,1] \to [0,1]$ *by* $Tx = x$ *for all* $x \in]0,1]$ *and* $T0 = \frac{1}{2}$. *Now endow* $[0,1]$ *with the partial order* $\preceq_=$ *given by* $x \preceq_= y \iff x = y$. *Then it is clear that* T *is orbitally* $\preceq_=$*-continuous at every* $x_0 \in]0,1]$, *since the unique increasing sequences in* $([0,1], \preceq_=)$ *are the constant sequences. It is not hard to see that* T *is not* $\mathcal{T}(d_E)$*-continuous at* 0.

According to [12], given a metric space (X, d), a mapping $T : X \to X$ is called orbitally continuous at $x_0 \in X$ provided that the sequence $(T^{n+1}x_0)_{n\in\mathbb{N}}$ converges to Tx_0 with respect to $\mathcal{T}(d)$ whenever the sequence $(T^n x_0)_{n\in\mathbb{N}}$ converges to x_0 with respect to $\mathcal{T}(d)$. Obviously, every orbitally continuous mapping is orbitally $\preceq$-continuous. The next example shows that the converse does not hold.

Example 2.3. *Consider the partially ordered metric space* $([0,1], d_E, \leq)$, *where* $\leq$ *denotes the usual partial order on* $[0,1]$. *Let* $T : [0,1] \to [0,1]$ *be the mapping introduced in Example 2.2. It is a simple matter to check that* T *is orbitally* $\leq$*-continuous at* $x_0 \in [0,1]$ *but* T *is not orbitally continuous at* 0.

Since Definition 2.1 involves increasing sequences it seems natural to ask for the relationship between$\preceq$-monotone mappings and those that are orbitally $\preceq$-continuous. Example 2.1 provided an instance of $\preceq$-monotone mapping which is not orbitally $\preceq$-continuous. Indeed, it is not hard to check that the mapping introduced in the aforesaid example is $\preceq_{[0,1]}$-monotone but it is not orbitally $\preceq_{[0,1]}$-continuous at any $x_0 \in [0,1]$.

The next example shows that there are mappings that are orbitally $\preceq$-continuous but they are not $\preceq$-monotone.

Example 2.4. *Let* (X, d_E) *with* $X = \{0,1\}$. *Endow* X *with the partial order* $\preceq$ *defined by* $1 \preceq 1$, $0 \preceq 0$ *and* $1 \preceq 0$. *Now, define the mapping* $T : X \to X$ *by* $T0 = 1$ *and* $T1 = 0$. *It follows easily that* T *is* $\mathcal{T}(d_E)$*-continuous and, thus, orbitally* $\preceq$*-continuous. Clearly* T *is not* $\preceq$*-monotone, since* $1 \preceq 0$ *and* $0 = T1 \npreceq T0 = 1$.

Our main result, which yields the promised fixed point principle, can be stated as follows.

Theorem 2.1. *Let* $(X, d, \preceq)$ *be a partially ordered metric space and* $T : X \to X$ *be a* $\preceq$*-monotone mapping. If there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$, T *is orbitally* $\preceq$*-continuous at* x_0 *and* $(T^n x_0)_{n\in\mathbf{N}}$ *is convergent to* x *with respect to* $\mathcal{T}(d)$, *then* x *is a fixed point of* T.

Proof. Let $x_0 \in X$ such that $x_0 \preceq Tx_0$. Since T is $\preceq$-monotone, we have that

$$x_0 \preceq Tx_0 \preceq T^2x_0 \preceq \ldots \preceq T^n x_0 \preceq T^{n+1}x_0 \preceq \ldots$$

Put $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Suppose that $x_{n+1} \neq x_n$ for $n \geq 0$, since otherwise we have that T has a fixed point. Clearly the sequence $(T^n x_0)_{n\in\mathbf{N}}$ is increasing. Moreover, it is convergent to x with respect to $\mathcal{T}(d)$. Then we obtain that there exists a subsequence $(T^{n_k+1}x_0)_{k\in\mathbf{N}}$ of $(T^{n+1}x_0)_{n\in\mathbf{N}}$ such that $(T^{n_k+1}x_0)_{k\in\mathbf{N}}$ converges to Tx with respect to $\mathcal{T}(d)$, since T is orbitally continuous at x_0. Moreover, we have that

$$d(Tx,x) \leq d(Tx, T^{n_k+1}x_0) + d(T^{n_k+1}x_0, x)$$

for all $k \in \mathbb{N}$. Whence we deduce that $d(Tx,x) = 0$ and, hence, that x is a fixed point of T. □

In the light of the preceding result it seems natural to wonder whether the assumptions in the statement of Theorem 2.1 are redundant. The above exposed examples evidence that $\preceq$-monotony and orbitally $\preceq$-continuity are independent. So it remains to discuss if every $\preceq$-monotone orbitally $\preceq$-continuous mapping T at $x_0 \in X$ must fulfill that $x_0 \preceq Tx_0$. The next example gives a negative answer to the posed question .

Example 2.5. *Let (X, d_E) be the metric space such that $X = \{0, 1, 2\}$. Moreover, consider the partial order $\preceq_X$ on X defined by $x \preceq y \iff y = 2x$ or $y = x$. Thus $0 \preceq_X 0$, $1 \preceq_X 1$ and $1 \preceq_X 2$. Define the $\preceq$-monotone mapping $T : X \to X$ by $T1 = T2 = 0$ and $T0 = 1$. Clearly, T is orbitally continuous at any $x_0 \in X$, since increasing and convergent sequences must be eventually constant. Observe that $x \npreceq Tx$ for any $x \in X$. Note that T has no fixed points.*

From the preceding example we derive that assumptions in the statement of Theorem 2.1 are not redundant. In addition, the example above allows us to state that the condition "$x_0 \preceq Tx_0$" cannot be deleted from the aforesaid theorem. Moreover, Example 2.4 gives that the $\preceq$-monotony of the mapping cannot be deleted in the statement of Theorem 2.1. Notice that the mapping T introduced in such an example has no fixed points and it is orbitally $\preceq$-continuous at 1, $1 \preceq T(1)$ but it is not $\preceq$-monotone. Furthermore, Example 2.1 provides that the orbitally $\preceq$-continuity cannot be removed from the statement of Theorem 2.1. Indeed, the mapping T introduced in the aforementioned example has no fixed point and it is not orbitally $\preceq_{[0,1]}$-continuous at any $x_0 \in [0, 1]$ but it is $\preceq_{[0,1]}$-monotone and $x \preceq_{[0,1]} T(x)$ for all $x \in]0, 1]$.

It must be stressed that Example 2.2 guarantees that Theorem 2.1 does not provide uniqueness of fixed points in general.

3. A few consequences in metric fixed point theory for rational type contractions

This section is devoted to extend Theorems 1.2, 1.3, 1.5, 1.7 and 1.8 to the context of orbital $\preceq$-continuity.

The next result extends Theorem 1.5 to our new context.

Corollary 3.1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a $\preceq$-monotone mapping such that there exist real numbers $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ with $\alpha + \beta < 1$ and*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta d(x, y) + L d(y, Tx)$$

for all $(x, y) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is orbitally $\preceq$-continuous at x_0, then T has a fixed point in X.

Proof. Since $x_0 \preceq Tx_0$ and T is $\preceq$-monotone, we have that the sequence $(T^n x_0)_{n \in \mathbb{N}}$ is increasing. Next put $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Of course without loss of generality we can assume that $x_{n+1} \neq x_n$ for all $n \geq 0$, since otherwise the existence of a fixed point is guaranteed. With the aim of proving that T has a fixed point, we only show that the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is convergent with respect to $\mathcal{T}(d)$. To this end, we distinguish two possible cases:

Case 1. Assume that $\beta \neq 0$. According to [8], we have that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha \frac{d(x_n, Tx_n) \cdot d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1})} + \beta d(x_n, x_{n-1}) + Ld(x_n, x_n)$$
$$= d(x_n, x_{n+1}) + \beta d(x_n, x_{n-1})$$

for all $n \geq 1$, since $(x_{n+1}, x_n) \in D_{X,\leq,\neq}$ for all $n \geq 1$. It follows, as in the proof of Theorem 2 given in [8], that $d(x_{n+1}, x_n) \leq \left(\frac{\beta}{1-\alpha}\right)^n d(x_1, x_0)$ for all $n \geq 1$ and, thus, that $d(x_m, x_n) \leq (\frac{t^n}{1-t}) \cdot d(x_1, x_0)$, where $t = \frac{\beta}{(1-\alpha)} < 1$ and $m, n \in \mathbb{N}$.

In the light of the last inequalities we deduce that $\lim_{m,n\to\infty} d(x_m, x_n) = 0$ and, thus, that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space there exists $x \in X$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to x with respect to $\mathcal{T}(d)$. Therefore we have obtained that $(T^n x_0)_{n\in\mathbb{N}}$ converges to x with respect to $\mathcal{T}(d_p)$. Thus Theorem 2.1 gives us that x is a fixed point of T.

Case 2. Assume that $\beta = 0$. Then we show that there exists $n_1 \in \mathbb{N}$ such that $T^{n_1+1}x_0 = T^{n_1}x_0$. Indeed, suppose that $T^{n+1}x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$. Then, we proceed analogously to the proof of the preceding case obtaining that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha \frac{d(x_n, Tx_n) \cdot d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1})} = d(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. Whence we deduce that $1 \leq \alpha$. So $T^{n_1+1}x_0 = T^{1^n}x_0$ and, thus, we get that $T^{n+1}x_0 = T^n x_0$ for all $n \in \mathbb{N}$ with $n \geq n_1$. Therefore $(T^n x_0)_{n\in\mathbb{N}}$ is eventually constant and, hence, it converges to x with respect to $\mathcal{T}(d_p)$ and Theorem 2.1 gives us that x is a fixed point of T.

Note that if in addition to the fact that $\beta = 0$ we have that $\alpha = 0$, then we deduce that $d(x_{n+1}, x_n) = 0$ for all $n \in \mathbb{N}$ which provides immediately that T has x_0 as fixed point. □

Notice that the preceding result is more general than Theorem 1.5, since

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta d(x, y) + L \min\{d(x, Ty), d(y, Tx)\}$$
$$\leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta d(x, y) + Ld(y, Tx).$$

for all $(x, y) \in D_{X,\leq,\neq}$.

The next corollaries can be obtained following similar arguments to those applied to the proof of the preceding corollary.

Taking $L = 0$ in the statement of Corollary 3.1 the next extension of Theorem 1.2 can be obtained whose proof is exactly the proof of Corollary 3.1.

Corollary 3.2. *Let $(X, d, \leq)$ be a partially ordered complete metric space and let $T : X \to X$ be a $\leq$-monotone mapping such that there exist real numbers $\alpha, \beta \in [0, 1[$ with $\alpha + \beta < 1$ and*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $(x, y) \in D_{X,\leq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$ and T is orbitally $\leq$-continuous at x_0, then T has a fixed point in X.

Taking $L = 0$ and $\beta = 0$ in the statement of Corollary 3.1 the next extension of Theorem 1.3 can be obtained whose proof is given by those arguments applied to Case 2 of the proof of Corollary 3.1.

Corollary 3.3. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a $\preceq$-monotone mapping such that there exists a real number $\alpha \in [0, 1[$ such that*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx) \cdot d(y, Ty)}{d(x, y)}$$

for all $(x, y) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is orbitally $\preceq$-continuous at x_0, then T has a fixed point in X.

Taking $L = 0$ and $\alpha = 0$ in the statement of Corollary 3.1 we get the next extension of Theorem 1.4.

Corollary 3.4. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a $\preceq$-monotone mapping such that there exist a real number $\beta \in [0, 1[$ and*

$$d(Tx, Ty) \leq \beta d(x, y)$$

for all $(x, y) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is orbitally $\preceq$-continuous at x_0, then T has a fixed point in X.

Since every continuous mapping is orbitally $\preceq$-continuous we immediately obtain that the preceding results recover Theorems 1.2–1.4 as a particular case. Notice that the same results remain true when orbitally continuous mappings are considered, since every orbitally continuous mapping is orbitally $\preceq$-continuous.

Observe that the above exposed corollaries do not warranty the uniqueness of fixed point such as the example below shows.

Example 3.1. *Consider the metric space $([0, 1], d_E)$ and the mapping $T : [0, 1] \to [0, 1]$ given by $Tx = x$ for all $x \in [0, 1]$. Consider $[0, 1]$ endowed with the partial order $\preceq_=$ introduced in Example 2.2, i.e., $x \preceq_= y \iff x = y$. Then T is orbitally $\preceq_=$-continuous at every $x_0 \in [0, 1]$, $\preceq_=$-monotone, $x \preceq_= T(x)$ for all $x \in [0, 1]$ and T satisfies all contractive conditions of rational type assumed in the statements of Corollaries 3.1–3.3, since $D_{[0,1],\preceq_=,\neq} = \emptyset$. Clearly T has more than one fixed point.*

Next we prove a version of Theorems 1.7 and 1.8.

Corollary 3.5. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a $\preceq$-monotone mapping such that there exist real numbers $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ such that $\alpha + \beta < 1$ and*

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)\,[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + L d(y, Tx)$$

for all $(y, x) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is orbitally $\preceq$-continuous at x_0, then T has a fixed point in X.

Proof. As pointed out before, since $x_0 \preceq Tx_0$ and T is $\preceq$-monotone, we have that the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is increasing. Next put $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ and assume without loss of generality that $x_{n+1} \neq x_n$ for all $n \geq 0$. Next we show that the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is convergent with respect to $\mathcal{T}(d)$. To this end, we distinguish two possible cases:

Case 1. Assume that $\beta \neq 0$. According to [8], we have that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha \frac{d(x_n, Tx_n)\,[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) + Ld(x_n, Tx_{n-1})$$
$$= \alpha \frac{d(x_{n+1}, x_n)\,[1 + d(x_n, x_{n-1})]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) + Ld(x_n, x_n)$$
$$\alpha d(x_{n+1}, x_n) + \beta d(x_{n-1}, x_n)$$

for all $n \geq 1$. Whence we obtain that

$$d(x_{n+1}, x_n) \leq \frac{\beta}{1-\alpha} d(x_n, x_{n-1})$$

for all $n \geq 1$, since $(x_{n+1}, x_n) \in D_{X,\leq,\neq}$ for all $n \geq 1$. It follows that $d(x_{n+1}, x_n) \leq (\frac{\beta}{1-\alpha})^n d(x_1, x_0)$ for all $n \geq 1$ and, hence, that $d(x_m, x_n) \leq (\frac{t^n}{1-t}) \cdot d(x_1, x_0)$, where $t = \frac{\beta}{(1-\alpha)} < 1$ and $m, n \in \mathbb{N}$.

The same reasoning to that applied in the proof of Corollary 3.1 gives the existence of $x \in X$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to x with respect to $\mathcal{T}(d)$. Therefore we have obtained that $(T^n x_0)_{n\in\mathbb{N}}$ converges to x with respect to $\mathcal{T}(d)$. Thus Theorem 2.1 provides that x is a fixed point of T.

Case 2. Assume that $\beta = 0$. Then we show that there exists $n_1 \in \mathbb{N}$ such that $T^{n_1+1}x_0 = T^{n_1}x_0$. Indeed, suppose that $T^{n+1}x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$. Then, we proceed analogously to the proof of the preceding case obtaining that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha \frac{d(x_n, Tx_n)\,[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} = \alpha d(x_n, Tx_n) = \alpha d(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. Whence we deduce that $1 \leq \alpha$. So $T^{n_1+1}x_0 = T^{1^n}x_0$ and, thus, we get that $T^{n+1}x_0 = T^n x_0$ for all $n \in \mathbb{N}$ with $n \geq n_1$. Therefore $(T^n x_0)_{n\in\mathbb{N}}$ is eventually constant and, hence, it converges to x with respect to $\mathcal{T}(d_p)$ and Theorem 2.1 gives us that x is a fixed point of T.

Again, if in addition to the fact that $\beta = 0$ we have that $\alpha = 0$, then we deduce that $d(x_{n+1}, x_n) = 0$ for all $n \in \mathbb{N}$ which provides immediately that T has x_0 as fixed point. □

Observe that the preceding result provides an improved version of Theorems 1.7, since

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)\,[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + L \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$
$$\leq \alpha \frac{d(y, Ty)\,[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + Ld(y, Tx)$$

for all $(y, x) \in D_{X,\leq,\neq}$.

Taking $L = 0$ in the proof of Corollary 3.5 we get the next result.

Corollary 3.6. *Let $(X, d, \leq)$ be a partially ordered complete metric space and let $T : X \to X$ be a $\leq$-monotone mapping such that there exist real numbers α,$\beta \in [0, 1[$ such that $\beta > 0$, $\alpha + \beta < 1$ and*

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)\,[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y)$$

for all $(y, x) \in D_{X,\leq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$ and T is orbitally $\leq$-continuous at x_0, then T has a fixed point in X.

Notice that taking either $\alpha = 0$ or $\beta = 0$ in the statement of the preceding result we get appropriate extensions of Theorem 1.7 to the context of orbital $\preceq$-continuity.

Since every continuous mapping is orbitally $\preceq$-continuous, we get that the preceding results retrieve Theorems 1.7 and 1.8 as a particular case. Observe that the same results remain true when orbitally continuous mappings are under consideration because of every orbitally continuous mapping is orbitally $\preceq$-continuous.

The example below shows that the both preceding corollaries do not guarantee the uniqueness of fixed point.

Example 3.2. *Consider the metric space* $([0,1], d_E)$ *and the mapping* $T : [0,1] \to [0,1]$ *given by* $Tx = x$ *for all* $x \in [0,1]$. *Consider* $[0,1]$ *endowed with the partial order* $\preceq_=$ *introduced in Example 2.2, i.e.,* $x \preceq_= y \iff x = y$. *Then* T *is orbitally* $\preceq_=$*-continuous at every* $x_0 \in [0,1]$, $\preceq_=$*-monotone,* $x \preceq_= T(x)$ *for all* $x \in [0,1]$ *and* T *satisfies all contractive conditions of rational type assumed in the statements of Corollaries 3.5 and 3.6, since* $D_{\preceq_=,\neq} = \emptyset$. *Obviously* T *has more than one fixed point.*

In Section 2 we have shown that the assertions in Theorem 2.1 are not redundant, it seems natural to wonder whether a mapping satisfying all assumptions in the aforementioned theorem needs to be a contraction of rational type. The next example provides a negative answer to the posed question. In fact, it gives an instance of a mapping that satisfies all assumptions in Theorem 2.1 and that does not satisfy any contraction inequality assumed in the statements of Corollaries 3.1–3.6.

Example 3.3. *Consider the complete metric space* $([0,1], d_D)$ *where* d_D *is the discrete metric space. Endow* $[0,1]$ *with the partial order* $\preceq$ *given by* $x \preceq y \Leftrightarrow y \leq x$. *Define the mapping* $T : X \to X$ *by* $Tx = \frac{x}{2}$ *for all* $x \in [0,1]$. *Clearly,* T *is orbitally* $\preceq$*-continuous at every* $x_0 \in X$. *Moreover, it is* $\preceq$*-monotone and* $x \preceq Tx$ *for all* $x \in [0,1]$. *However,* T *is not a rational contraction in any the sense of Corollaries 3.1–3.4. Indeed, there do not exist* $L \in \mathbb{R}^+$ *and* $\alpha, \beta \in [0,1[$ *with* $\alpha + \beta < 1$ *such that*

$$\begin{aligned} 1 = d_D(T1, T\frac{1}{2}) &\leq \alpha \cdot \frac{d_D(1, T1) \cdot d_D(\frac{1}{2}, T\frac{1}{2})}{d_D(1, \frac{1}{2})} + \beta d_D(1, \frac{1}{2}) + L d_D(\frac{1}{2}, T1) \\ &= \alpha + \beta < 1. \end{aligned}$$

Notice that the preceding inequality is not satisfied even if we consider that any of the values $\{L, \alpha, \beta\}$ *is exactly zero.*

Furthermore, T *is not a rational contraction in any the sense of Corollaries 3.5 and 3.6. Indeed, there do not exist* $L \in \mathbb{R}^+$ *and* $\alpha, \beta \in [0,1[$ *with* $\alpha + \beta < 1$ *such that*

$$\begin{aligned} 1 = d_D(T1, T\frac{1}{2}) &\leq \alpha \frac{d_D(\frac{1}{2}, T\frac{1}{2})\,[1 + d_D(1, T1)]}{1 + d_D(1, \frac{1}{2})} + \beta d_D(1, \frac{1}{2}) + L d_D(\frac{1}{2}, T1) \\ &= \alpha + \beta < 1. \end{aligned}$$

Observe again that the preceding inequality is not satisfied even if any of the values $\{L, \alpha, \beta\}$ *is exactly zero.*

4. A few consequences in partial metric fixed point theory for rational type contractions

This section is devoted to show that Theorem 2.1 allows us to deduce a collection of results which involve contractions of rational type and, hence, extend those theorems exposed in Section 1 to the partial metric context.

To this end, let us recall that, following [22], a partial metric space is a pair (X, p), where X is a non-empty set and p is a real-valued function on $X \times X$ such that, for all $x, y, z \in X$, the following axioms are fulfilled:

i) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
ii) $0 \leq p(x, x) \leq p(x, y)$;
iii) $p(x, y) = p(y, x)$;
iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Of course, a partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that a metric space (X, d) is a partial metric space that satisfies in addition the following condition: $d(x, x) = 0$ for all $x \in X$.

According to [22], each partial metric p on X induces a T_0 topology $\mathcal{T}(p)$ on X which has as a base the family of open balls $\{B_p(x; \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x; \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$. Moreover, a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X, p) converges to a point $x \in X$ with respect to $\mathcal{T}(p) \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n)$. Furthermore, a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite. Taking this into account, a partial metric space (X, p) is said to be complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}(p)$, to a point $x \in X$ such that $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$.

On account of [22], every partial metric induces in a natural way a metric. Indeed, given a partial metric space (X, p), then the function $d_p : X \times X \to \mathbb{R}^+$ defined by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

for all $x, y \in X$, is a metric on X.

Observe that, a sequence $(x_n)_{n\in\mathbb{N}}$ in (X, p) converges to $x \in X$ with respect to $\mathcal{T}(d_p) \Leftrightarrow \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n) = p(x, x)$. One can deduce easily that a sequence in (X, p) is Cauchy if and only it is Cauchy in the metric space (X, d_p) and, in addition, that (X, p) is complete if and only if (X, d_p) is complete (see [26]).

Since in a partial metric space (X, p) we can induce two topologies, namely $\mathcal{T}(p)$ and $\mathcal{T}(d_p)$, in the following we will specify what topology in under consideration when continuous mappings are involved. Thus, a mapping continuous from $\mathcal{T}_1$ into $\mathcal{T}_2$ will be said to be $\mathcal{T}_1$-$\mathcal{T}_2$-continuous, where $\mathcal{T}_1, \mathcal{T}_2$ can be one of the topologies $\mathcal{T}(p), \mathcal{T}(d_p)$. In case that $\mathcal{T}_1 = \mathcal{T}_2$, then we will only say that T is $\mathcal{T}_1$-continuous.

In the last years, the interest and the research activity in fixed point theory in partial metric spaces has grown greatly. The large number of published works in this direction evidence that fact, some of them can be found in [1–3, 5, 6, 10, 11, 17–21, 24, 27–33].

Recently, in [15], the following extension of Theorem 1.2 has been tried to be established in the framework of partial metric spaces, where a partially ordered partial metric space is a triplet $(X, p, \preceq)$ where (X, p) is a partial metric space and $(X, \preceq)$ is a partially ordered set:

"Let $(X, p, \preceq)$ be a partially ordered complete partial metric space and let $T : X \to X$ be a $\mathcal{T}(p)$-continuous and $\preceq$-monotone mapping such that there exist real numbers $\alpha, \beta \in [0, 1[$ with $\alpha + \beta < 1$ and $p(Tx, Ty) \leq \alpha \frac{p(x,Tx)\cdot p(y,Ty)}{p(x,y)} + \beta p(x, y)$ for all $(x, y) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X."

However, the next example shows that the preceding attempt of extending Theorem 1.2 is not necessarily true.

Example 4.1. *Consider the partial metric $p_{\max}$ defined on $[0,1]$ by $p_{\max}(x,y) = \max\{x,y\}$. A straightforward computation shows that the partial metric space $([0,1], p_{\max})$ is complete. Consider the partial order $\preceq_{[0,1]}$ on $[0,1]$ given by $x \preceq_{[0,1]} y \Leftrightarrow x, y \in]0,1]$ and $x \geq y$ or $x = y$. Next define the mapping $T : [0,1] \to [0,1]$ by $Tx = \frac{x}{2}$ for all $x \in]0,1]$ and $T0 = \frac{1}{2}$. Then it is not hard to check that T is $\preceq_{[0,1]}$-monotone and $\mathcal{T}(p_{\max})$-continuous. Moreover, $1 \preceq_{[0,1]} T1 = \frac{1}{2}$. Furthermore, for all $(x,y) \in D_{[0,1],\preceq_{[0,1]},\neq}$ we have that*

$$\frac{y}{2} = p(Tx, Ty) \leq \alpha \frac{p(x,Tx) \cdot p(y,Ty)}{p(x,y)} + \beta p(x,y) = \alpha \frac{x \cdot y}{y} + \beta y = \alpha x + \beta y,$$

where $\beta = \frac{1}{2}$ and $\alpha \in [0, \frac{1}{2}[$. Clearly, T satisfies all assumptions in the aforesaid attempt but it is fixed point free.

In the light of the preceding fact, the main purpose of this section is to provide a real extension of Theorem 1.2 in the framework of partial metric spaces by means of the use of Theorem 2.1 in such a way that the aforementioned theorem can be retrieved as a particular case of our new result. Such extension will show the reason for which the above attempt does not work. Moreover, extensions of Theorems 1.3–1.5, 1.7 and 1.8 will be also provided. Furthermore, we show that the hypothesis in the statements of our new results cannot be weakened.

In order to introduce the promised extensions we need to adapt the notion of orbitally $\preceq$-continuous mapping to the context of partially ordered partial metric spaces.

Definition 4.1. *Let $(X, p, \preceq)$ be an partially ordered partial metric space. A mapping $T : X \to X$ will be said to be orbitally $\preceq$-continuous at $x_0 \in X$ provided that $(T^{n+1}x_0)_{n\in\mathbb{N}}$ has a subsequence which converges to Tx_0 with respect to $\mathcal{T}(d_p)$ whenever the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is increasing in $(X, \preceq)$ and converges to x_0 with respect to $\mathcal{T}(d_p)$.*

In the light of the above notion we extend Corollary 3.1.

Corollary 4.1. *Let $(X, p, \preceq)$ be a partially ordered complete partial metric space and let $T : X \to X$ be a $\preceq$-monotone mapping such that there exist real numbers $L \in \mathbb{R}^+$, $\alpha, \beta \in [0,1[$ with $\alpha + \beta < 1$ and*

$$p(Tx, Ty) \leq \alpha \frac{p(x,Tx) \cdot p(y,Ty)}{p(x,y)} + \beta p(x,y) + L\left(p(x,Ty) - p(x,x)\right)$$

for all $(x,y) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is orbitally $\preceq$-continuous at x_0, then T has a fixed point $x \in X$. Moreover, if $D_{X,\preceq,\neq} \neq \emptyset$ and $\beta \neq 0$ then $p(x,x) = 0$.

Proof. Following the same arguments given in the proof of Corollary 3.1 we have that the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is increasing, where $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ and $x_{n+1} \neq x_n$ for all $n \geq 0$. Next we show that the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is convergent with respect to $\mathcal{T}(d_p)$. To this end, we distinguish two possible cases:

Case 1. Assume that $\beta \neq 0$. Since $(x_{n+1}, x_n) \in D_{X,\preceq,\neq}$ for all $n \geq 1$ we get that

$$\begin{aligned} p(x_{n+1}, x_n) = p(Tx_n, Tx_{n-1}) &\leq \alpha \cdot \tfrac{p(x_n,Tx_n)\cdot p(x_{n-1},Tx_{n-1})}{p(x_n,x_{n-1})} + \beta \cdot p(x_n, x_{n-1}) \\ &\quad + L\,(p(x_n, x_n) - p(x_n, x_n)) \\ &= \alpha \cdot \tfrac{p(x_n,x_{n+1})\cdot p(x_{n-1},x_n)}{p(x_n,x_{n-1})} + \beta \cdot p(x_n, x_{n-1}) \\ &= \alpha \cdot p(x_n, x_{n+1}) + \beta \cdot p(x_n, x_{n-1}). \end{aligned}$$

From the last inequality, we deduce that $p(x_{n+1}, x_n) \leq \frac{\beta}{1-\alpha} p(x_n, x_{n-1})$ for all $n \geq 1$. It follows that $p(x_{n+1}, x_n) \leq (\frac{\beta}{1-\alpha})^n p(x_1, x_0)$ for all $n \geq 1$.

Next we set $t = \frac{\beta}{(1-\alpha)} < 1$ and we show that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, p). Indeed, let $m, n \in \mathbb{N}$ and suppose that $m \geq n$. Then we have

$$\begin{aligned} p(x_m, x_n) &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_n) - p(x_{m-1}, x_{m-1}) \\ &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_n) \\ &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + p(x_{m-2}, x_n) - p(x_{m-2}, x_{m-2}) \\ &\vdots \\ &\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \ldots + p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) \\ &\leq (t^{m-1} + t^{m-2} + \ldots + t^{n+1} + t^n) \cdot p(x_1, x_0) \\ &\leq (\tfrac{t^n}{1-t}) \cdot p(x_1, x_0). \end{aligned}$$

Whence we derive that $\lim_{m,n\to\infty} p(x_m, x_n) = 0$ and, thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, p). So $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d_p). Since (X, p) is a complete partial metric space we have that the metric space (X, d_p) is complete. Hence there exists $x \in X$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to x with respect to $\mathcal{T}(d_p)$. Consequently $0 = \lim_{m,n\to\infty} p(x_m, x_n) = \lim_{n\to\infty} p(x, x_n) = p(x, x)$. Therefore we have obtained that $(T^n x_0)_{n\in\mathbb{N}}$ converges to x with respect to $\mathcal{T}(d_p)$. Thus Theorem 2.1 gives us that x is a fixed point of T.

Case 2. Assume that $\beta = 0$. Then we show that there exists $n_1 \in \mathbb{N}$ such that $T^{n_1+1}x_0 = T^{n_1}x_0$. Indeed, suppose that $T^{n+1}x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$. Then, $p(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}$ and we proceed analogously to the proof of the preceding case obtaining that

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \alpha \cdot \frac{p(x_n, x_{n+1}) \cdot p(x_{n-1}, x_n)}{p(x_n, x_{n-1})} = \alpha p(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$. If $\alpha \in]0, 1[$ then, the preceding inequality provides that $1 \leq \alpha$ which is impossible.

If in addition to the fact that $\beta = 0$ we have that $\alpha = 0$, then we deduce that $p(x_{n+1}, x_n) = 0$ for all $n \in \mathbb{N}$ which contradicts our hypothesis. So $T^{n_1+1}x_0 = T^{1^n}x_0$ and, thus, we get that $T^{n+1}x_0 = T^n x_0$ for all $n \in \mathbb{N}$ with $n \geq n_1$. Therefore $(T^n x_0)_{n\in\mathbb{N}}$ is eventually constant and, hence, it converges to x with respect to $\mathcal{T}(d_p)$ and Theorem 2.1 gives us that x is a fixed point of T □

The example below shows that the zero self-distance condition "$p(x, x) = 0$" is not warranted in general when $\beta = 0$.

Example 4.2. *Endow $X = \left\{0, \frac{1}{2}\right\}$ with the partial order $\preceq$ defined by $\frac{1}{2} \preceq \frac{1}{2}$, $0 \preceq 0$ and $\frac{1}{2} \preceq 0$. Now, consider the partial metric $p_{\max}$ on X and define the mapping $T : X \to X$ by $Tx = \frac{1}{2}$ for all $x \in X$. Observe that $D_{X,\preceq,\neq} \neq \emptyset$. It follows easily that T is orbitally $\preceq$-continuous at every $x_0 \in X$ and it is $\preceq$-monotone. Furthermore, $\frac{1}{2} \preceq T\frac{1}{2}$ and*

$$\frac{1}{2} = p_{\max}(0, \frac{1}{2}) \leq \alpha \frac{p_{\max}(0, T0) \cdot p_{\max}(\frac{1}{2}, T\frac{1}{2})}{p_{\max}(0, \frac{1}{2})} + Lp_{\max}(0, T\frac{1}{2}) - p_{\max}(0, 0),$$

where $L = 2$ and $\alpha \in [0, 1[$. However, T has $\frac{1}{2}$ as unique fixed point with $p_{\max}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$.

In the next example we show that the rational contractive condition assumed in Corollary 4.1 cannot be changed by this other one:

$$p(Tx, Ty) \leq \alpha \frac{p(x, Tx) \cdot p(y, Ty)}{p(x, y)} + \beta p(x, y) + L\left(p(Ty, x) - p(x, x)\right).$$

Example 4.3. *Consider the complete partial metric space $(\mathbb{R}^+, p_{\max})$ endowed with the partial order $\preceq$ defined as follows: $x \preceq y \Leftrightarrow x = y$ or there exists $k \in \mathbb{N}$ such that $y = 2^k x$. Consider, in addition, the mapping $T : \mathbb{R}^+ \to \mathbb{R}^+$ by $T(x) = 2x$ for all $x \in]0, \infty[$ and $T(0) = 1$. It is a simple matter to check that T is $\preceq$-monotone, $x \preceq T(x)$ for all $x \in]0, \infty[$ and T is orbitally $\preceq$-continuous at x for all $x \in \mathbb{R}^+$. Moreover, for all $(x, y) \in D_{X,\preceq,\neq}$ we have that*

$$\begin{aligned} 2x = p_{\max}(Tx, Ty) \quad &\leq \quad \frac{1}{2} \frac{p_{\max}(x, Tx) \cdot p_{\max}(y, Ty)}{p_{\max}(x, y)} + \frac{1}{4} p_{\max}(x, y) \\ &\quad\quad +2\left(p_{\max}(Ty, x) - p_{\max}(x, x)\right) \\ &= \quad \frac{2x \cdot 2y}{2x} + \frac{1}{4}x + 2 \cdot (2x - x) = 2y + \frac{1}{4}x + 2x. \end{aligned}$$

Notice that T has no fixed points.

Taking $L = 0$ in the statement of Corollary 4.1 we get the following result which extends Corollary 3.2 to the partial metric context and whose proof is exactly the proof of Corollary 4.1.

Corollary 4.2. *Let $(X, p, \preceq)$ be a partially ordered complete partial metric space and let $T : X \to X$ be a $\preceq$-monotone mapping such that there exist real numbers $\alpha, \beta \in [0, 1[$ with $\alpha + \beta < 1$ and*

$$p(Tx, Ty) \leq \alpha \frac{p(x, Tx) \cdot p(y, Ty)}{p(x, y)} + \beta p(x, y)$$

for all $(x, y) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is orbitally $\preceq$-continuous at x_0, then T has a fixed point $x \in X$. Moreover, if $D_{X,\preceq,\neq} \neq \emptyset$ and $\beta \neq 0$ then $p(x, x) = 0$.

Taking $\beta = 0$ in the statement of Corollary 4.2 we get the following result which extends Corollary 3.3 to the partial metric context whose proof is given by those arguments applied to Case 2 of the proof of Corollary 4.1.

Corollary 4.3. *Let $(X, p, \preceq)$ be a partially ordered complete partial metric space and let $T : X \to X$ be a $\preceq$-monotone mapping such that there exists a real number $\alpha \in [0, 1[$ and*

$$p(Tx, Ty) \leq \alpha \frac{p(x, Tx) \cdot p(y, Ty)}{p(x, y)}$$

for all $(x, y) \in D_{X,\preceq,\neq}$. *If there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$ *and* T *is orbitally* $\preceq$*-continuous at* x_0, *then* T *has a fixed point* $x \in X$.

The next example shows that the preceding result does not guarantee in general that the fixed point satisfies the zero self-distance condition.

Example 4.4. *Endow* $X = \left\{\frac{1}{2}, 1\right\}$ *with the partial order* $\preceq_=$ *defined by* $x \preceq_= y \iff x = y$. *Consider the partial metric* $p_{\max}$ *on* X *and the mapping* $T : X \to X$ *by* $Tx = x$ *for all* $x \in X$. *It follows easily that* T *is orbitally* $\preceq_=$*-continuous at every* $x_0 \in X$ *and it is* $\preceq_=$*-monotone. Furthermore,* $x \preceq_= Tx$ *for all* $x \in X$ *and it satisfies the rational contractive condition given in the statement of Corollary 4.3 because* $D_{X,\preceq_=,\neq} = \emptyset$. *However, all fixed point of* T *verifies that* $p_{\max}(x, x) \neq 0$.

Taking $\alpha = 0$ in the statement of Corollary 4.2 we get the following result which extends Corollary 3.4 to the partial metric context.

Corollary 4.4. *Let* $(X, p, \preceq)$ *be a partially ordered complete partial metric space and let* $T : X \to X$ *be a* $\preceq$*-monotone mapping such that there exists a real number* $\beta \in [0, 1[$ *and*

$$p(Tx, Ty) \leq \beta p(x, y)$$

for all $(x, y) \in D_{X,\preceq,\neq}$. *If there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$ *and* T *is orbitally* $\preceq$*-continuous at* x_0, *then* T *has a fixed point* $x \in X$ *with* $p(x, x) = 0$.

Proof. If $\beta \in]0, 1[$, then the proof runs following similar arguments to those applied to Case 1 of the proof of Corollary 4.1. When $\beta = 0$ then there exists $n_1 \in \mathbb{N}$ such that $T^{n_1+1}x_0 = T^{n_1}x_0$. Indeed, suppose that $T^{n+1}x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$. Then $p(T^{n+1}x_0, T^n x_0) = 0$ which is a contradiction. □

The next example shows that there are mappings which satisfy all assumptions in Theorem 2.1 which are not contractions of rational type in the partial metric context. In fact, the next example shows that the three preceding contractive conditions of rational type are not redundant in Corollaries 4.1–4.4.

Example 4.5. *Consider the complete partial metric space* $([0, 1], p_{\max})$. *Endow* $[0, 1]$ *with the usual partial order* $\leq$. *Define the mapping* $T : X \to X$ *by* $Tx = x$ *for all* $x \in [0, 1]$. *Clearly,* T *is orbitally* $\leq$*-continuous at every* $x_0 \in [0, 1]$. *Moreover, it is* $\leq$*-monotone and* $x \leq Tx$ *for all* $x \in [0, 1]$. *However,* T *is not a contraction of rational type. Indeed, there do not exist* $L \in \mathbb{R}^+$ *and* $\alpha, \beta \in [0, 1[$ *with* $\alpha + \beta < 1$ *such that*

$$\begin{aligned} 1 = p_{\max}(T1, T0) \quad \leq \quad & \alpha \frac{p_{\max}(1, T1) \cdot p_{\max}(0, T0)}{p_{\max}(1, 0)} + \beta p_{\max}(1, 0) \\ & + L\left(p_{\max}(1, T0) - p_{\max}(1, 1)\right) = \beta. \end{aligned}$$

Notice that the preceding inequality is not satisfied even if we consider that any of the values $\{L, \alpha, \beta\}$ *is exactly zero.*

In the following, Corollaries 3.5 and 3.6 are extended to the partial metric framework.

Corollary 4.5. *Let* $(X, p, \preceq)$ *be a partially ordered complete partial metric space and let* $T : X \to X$ *be a* $\preceq$*-monotone mapping such that there exist real numbers* $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ *with* $\alpha + \beta < 1$ *and*

$$p(Tx, Ty) \leq \alpha \frac{p(y, Ty)\left[1 + p(x, Tx)\right]}{1 + p(x, y)} + \beta p(x, y) + L\left(p(y, Tx) - p(y, y)\right).$$

for all $(y, x) \in D_{X,\preceq,\neq}$. *If there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$ *and* T *is orbitally* $\preceq$*-continuous at* x_0, *then* T *has a fixed point* $x \in X$. *Moreover, if* $D_{X,\preceq,\neq} \neq \emptyset$ *and* $\beta \neq 0$ *then* $p(x, x) = 0$.

Proof. Consider the sequence $(T^n x_0)_{n\in\mathbb{N}}$. Clearly it is increasing, where $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ and $x_{n+1} \neq x_n$ for all $n \geq 0$. Next we show that the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is convergent with respect to $\mathcal{T}(d_p)$.

To this end, we distinguish two possible cases:

Case 1. Assume that $\beta \neq 0$. Since $(x_{n+1}, x_n) \in D_{X,\neq}$ for all $n \geq 1$ we get that

$$\begin{aligned} p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) &\leq \alpha\frac{p(x_n, x_{n+1})\left[1 + p(x_{n-1}, x_n)\right]}{1 + p(x_{n-1}, x_n)} + \beta p(x_{n-1}, x_n) \\ &\quad + L\left(p(x_n, x_n) - p(x_n, x_n)\right) \\ &= \alpha p(x_n, x_{n+1}) + \beta p(x_{n-1}, x_n). \end{aligned}$$

It follows that $p(x_{n+1}, x_n) \leq \frac{\beta}{1-\alpha} p(x_n, x_{n-1})$ for all $n \geq 1$ and, thus, that $p(x_{n+1}, x_n) \leq (\frac{\beta}{1-\alpha})^n p(x_1, x_0)$ for all $n \geq 1$. So $\lim_{m,n\to\infty} p(x_m, x_n) = 0$ and, thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, p). So $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d_p). Since (X, p) is a complete partial metric space we have that the metric space (X, d_p) is complete. Hence there exists $x \in X$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to x with respect to $\mathcal{T}(d_p)$. Consequently $0 = \lim_{m,n\to\infty} p(x_m, x_n) = \lim_{n\to\infty} p(x, x_n) = p(x, x)$. Therefore we have obtained that $(T^n x_0)_{n\in\mathbb{N}}$ converges to x with respect to $\mathcal{T}(d_p)$. Thus Theorem 2.1 gives us that x is a fixed point of T.

Case 2. Assume that $\beta = 0$ and $\alpha \neq 0$. Then we show that there exists $n_1 \in \mathbb{N}$ such that $T^{n_1+1}x_0 = T^{n_1}x_0$. Indeed, suppose that $T^{n+1}x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$. Then, we proceed analogously to the proof of the preceding case obtaining that

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \alpha\frac{p(x_n, Tx_n)\left[1 + p(x_{n-1}, Tx_{n-1})\right]}{1 + p(x_{n-1}, x_n)} = \alpha p(x_n, x_{n+1})$$

for all $n \in \mathbb{N}$.

If $\alpha \in]0, 1[$ then, the preceding inequality provides that $1 \leq \alpha$ which is impossible. If in addition to the fact that $\beta = 0$ we have that $\alpha = 0$, then we deduce that $p(x_{n+1}, x_n) = 0$ for all $n \in \mathbb{N}$ which contradicts our hypothesis. So $T^{n_1+1}x_0 = T^{1^n}x_0$ and, thus, we get that $T^{n+1}x_0 = T^n x_0$ for all $n \in \mathbb{N}$ with $n \geq n_1$. Therefore $(T^n x_0)_{n\in\mathbb{N}}$ is eventually constant and, hence, it converges to x with respect to $\mathcal{T}(d_p)$ and Theorem 2.1 gives us that x is a fixed point of T. □

The next example shows that the preceding result does not guarantee that the fixed point satisfies the zero self-distance condition when $\beta = 0$.

Example 4.6. *Consider the partially ordered partial metric space* $(X, \preceq, p_{\max})$ *and the mapping* $T : X \to X$ *introduced in Example 4.2. Clearly* T *is orbitally* $\preceq$*-continuous at every* $x_0 \in X$ *and it is* $\preceq$*-monotone. Observe that* $D_{X,\preceq,\neq} \neq \emptyset$. *Furthermore,* $\frac{1}{2} \preceq T\frac{1}{2}$ *and*

$$\begin{aligned} \frac{1}{2} = p_{\max}(T\frac{1}{2}, T0) &\leq \alpha\frac{p_{\max}(0, T0)\left[1 + p_{\max}(\frac{1}{2}, T\frac{1}{2})\right]}{1 + p_{\max}(0, \frac{1}{2})} + L\left(p_{\max}(0, T\frac{1}{2}) - p_{\max}(0, 0)\right) \\ &= \frac{\alpha}{2} + \frac{L}{2}, \end{aligned}$$

where $L = 1$ *and* $\alpha \in [0, 1[$. *However,* T *has* $\frac{1}{2}$ *as unique fixed point with* $p_{\max}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$.

From Corollary 4.5, we get immediately the next one.

Corollary 4.6. *Let $(X, p, \preceq)$ be a partially ordered complete partial metric space and let $T : X \to X$ be a $\preceq$-monotone mapping such that there exist real numbers $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ such that $\alpha + \beta < 1$ with*

$$\begin{aligned} p(Tx, Ty) \;\; \leq \;\; & \alpha \frac{p(y, Ty)\,[1 + p(x, Tx)]}{1 + p(x, y)} + \beta p(x, y) \\ & + L \min\{p(x, Tx) - p(x, x), p(x, Ty) - p(x, x), p(y, Tx) - p(y, y)\} \end{aligned}$$

for all $(y, x) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is orbitally $\preceq$-continuous at x_0, then T has a fixed point $x \in X$.Moreover, if $D_{X,\preceq,\neq} \neq \emptyset$ and $\beta \neq 0$ then $p(x, x) = 0$.

Proof. Since $\min\{p(x, Tx) - p(x, x), p(x, Ty) - p(x, x), p(y, Tx) - p(y, y)\} \leq p(y, Tx) - p(y, y)$ for all $(y, x) \in D_{X,\preceq,\neq}$ we have, by Corollary 4.5, the desired conclusions.

□

If we force $L = 0$ in the statement of Corollary 4.6 we obtain the result below.

Corollary 4.7. *Let $(X, p, \preceq)$ be a partially ordered complete partial metric space and let $T : X \to X$ be a $\preceq$-monotone mapping such that there exist real numbers $\alpha, \beta \in [0, 1[$ such that $\alpha + \beta < 1$ and*

$$p(Tx, Ty) \leq \alpha \frac{p(y, Ty)\,[1 + p(x, Tx)]}{1 + p(x, y)} + \beta p(x, y)$$

for all $(y, x) \in D_{X,\preceq,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is orbitally $\preceq$-continuous at x_0, then T has a fixed point in X. Moreover, if $D_{X,\preceq,\neq} \neq \emptyset$ and $\beta \neq 0$ then $p(x, x) = 0$.

When $\beta = 0$ in the statement of the preceding result we show that get the corollary below.

Corollary 4.8. *Let $(X, p, \preceq)$ be a partially ordered complete partial metric space and let $T : X \to X$ be a $\preceq$-monotone mapping such that there exists a real number $\alpha \in [0, 1[$ and*

$$p(Tx, Ty) \leq \alpha \frac{p(y, Ty)\,[1 + p(x, Tx)]}{1 + p(x, y)}$$

for all $(y, x) \in D_{X,\preceq}$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is orbitally $\preceq$-continuous at x_0, then T has a fixed point in X.

Observe that Example 4.4 shows that preceding result does not guarantee in general that the fixed point satisfies the zero self-distance condition.

In next example we show that there are mappings which satisfy all assumptions in Theorem 2.1 which are not contractions of rational type in the partial metric context in any the sense of Corollaries 4.5–4.8.

Example 4.7. *Consider the complete partial metric space $([0, 1], p_{\max})$ endowed with the usual partial order $\leq$. Define the mapping $T : X \to X$ by $Tx = x$ for all $x \in [0, 1]$. Clearly, T is orbitally $\leq$-continuous at every $x_0 \in [0, 1]$. Moreover, it is $\leq$-monotone and $x \leq Tx$ for all $x \in [0, 1]$. However, T is not a contraction of rational type. Indeed, there do not exist $(y, x) \in D_{[0,1],\leq,\neq}$ and $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ with $\alpha + \beta < 1$ such that*

$$
\begin{aligned}
y = p_{\max}(Tx, Ty) &\leq \alpha \frac{p_{\max}(y, Ty)\left[1 + p_{\max}(x, Tx)\right]}{1 + p_{\max}(x, y)} + \beta p_{\max}(x, y) \\
&\quad + L\left(p_{\max}(y, Tx) - p(y, y)\right) \leq \\
&= \alpha \frac{y(1+x)}{1+y} + \beta y \\
&\leq (\alpha + \beta)\, y < y.
\end{aligned}
$$

Notice that if $(y, x) \in D_{[0,1],\preceq,\neq}$*, then* $x, y \in]0, 1]$ *and* $x \neq y$*.*

Observe that all exposed results remain true if we interchange the orbitally $\preceq$-continuity by either orbitally continuity or $\mathcal{T}(d_p)$ continuity in their statement because the last notion implies orbitally $\preceq$-continuity. Of course we are assuming that, given a partial metric space (X, p), a mapping $f : X \to X$ is orbitally continuous at $x_0 \in X$ provided that it is orbitally continuous at x_0 with respect to $\mathcal{T}(d_p)$. Observe that Example 4.1 shows that we can consider neither orbitally continuity with respect to $\mathcal{T}(p)$ nor continuity with respect to $\mathcal{T}(p)$.

According to [22], every partial metric p induces a partial order $\preceq_p$ on a non-empty set X, which is known as specialization order and is given by $x \preceq_p y \Leftrightarrow p(x, y) = p(x, x)$. It must be stressed that in the particular case that the partial metric p is exactly a metric d, then the specialization order $\preceq_d$ is exactly the "flat" one, i.e., $x \preceq_d y \Leftrightarrow x = y$, which cannot be considered in general for fixed point theory.

We end the paper obtaining the next results when we consider that the partial order in the statement of all exposed corollaries is exactly the specialization order.

Corollary 4.9. *Let* (X, p) *be a complete partial metric space and let* $T : X \to X$ *be a* $\preceq_p$*-monotone mapping such that there exist real numbers* $L \in \mathbb{R}^+$*,* $\alpha, \beta \in [0, 1[$ *with* $\alpha + \beta < 1$ *and*

$$
p(Tx, Tx) \leq \alpha \frac{p(x, Tx) \cdot p(y, Ty)}{p(x, x)} + \beta p(x, x) + L\left(p(y, Tx) - p(y, y)\right)
$$

for all $(y, x) \in D_{X,\preceq_p,\neq}$*. If there exists* $x_0 \in X$ *such that* $x_0 \preceq_p Tx_0$ *and* T *is orbitally* $\preceq_p$*-continuous at* x_0*, then* T *has a fixed point* $x \in X$*. Moreover, if* $D_{X,\preceq,\neq} \neq \emptyset$ *and* $\beta \neq 0$ *then* $p(x, x) = 0$*.*

Proof. We observe that when the partial order $\preceq_p$ is under consideration, the contractive condition of rational type of a mapping T in statement of Corollary 4.1 is equivalent to the condition $p(Tx, Tx) \leq \alpha \frac{p(x,Tx) \cdot p(y,Ty)}{p(x,x)} + \beta p(x, x) + L\left(p(y, Tx) - p(y, y)\right)$ for all $(y, x) \in D_{X,\preceq_p,\neq}$ because $p(Tx, Ty) = p(Tx, Tx)$ and $p(x, y) = p(x, x)$ when $x \preceq_p y$. □

Notice that when any value of $\{L, \alpha, \beta\}$ is zero we get versions of Corollaries 4.2–4.4.

We get the following result from Corollary 4.5 and 4.6.

Corollary 4.10. *Let* (X, p) *be a complete partial metric space and let* $T : X \to X$ *be a* $\preceq_p$*-monotone mapping such that there exist real numbers* $L \in \mathbb{R}^+$*,* $\alpha, \beta \in [0, 1[$ *with* $\beta > 0$*,* $\alpha + \beta < 1$ *and one of the following contractive conditions is satisfied for all* $(y, x) \in D_{X,\preceq_p,\neq}$*:*

1) $p(Tx, Tx) \leq \alpha \frac{p(y,Ty)\left[1+p(x,Tx)\right]}{1+p(x,x)} + \beta p(x, x) + L\left(p(y, Tx) - p(y, y)\right)$.
2) $p(Tx, Tx) \leq \alpha \frac{p(y,Ty)\left[1+p(x,Tx)\right]}{1+p(x,x)} + \beta p(x, x) + L \min\{p(x, Tx) - p(x, x), p(x, Ty) - p(x, x), p(y, Tx) - p(y, y)\}$.

If there exists $x_0 \in X$ such that $x_0 \preceq_p Tx_0$ and T is orbitally $\preceq_p$-continuous at x_0, then T has a fixed point $x \in X$. Moreover, if $D_{X,\preceq,\neq} \neq \emptyset$ and $\beta \neq 0$ then $p(x,x) = 0$.

Observe that taking either $L = 0$ and $\beta = 0$ we get versions of Corollaries 4.7 and 4.8.

In order to state the next corollaries, let us recall that, given a partially ordered set $(X, \preceq)$, a mapping $T : X \to X$ is said to be $\preceq$-continuous provided that the least upper bound of $(Tx_n)_{n\in\mathbb{N}}$ is Tx for every increasing sequence $(x_n)_{n\in\mathbb{N}}$ whose least upper bound exists and is x (see [9]).

In the light of the preceding notion we can state the following fixed point results for contractions of rational type assuming continuity with respect to $\mathcal{T}(p)$.

Corollary 4.11. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a $\preceq_p$-monotone mapping such that there exist real numbers $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ with $\alpha + \beta < 1$ and*

$$p(Tx, Tx) \leq \alpha \frac{p(x, Tx) \cdot p(y, Ty)}{p(x, x)} + \beta p(x, x) + L\,(p(y, Tx) - p(y, y))$$

for all $(y, x) \in D_{X,\preceq_p,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq_p Tx_0$ and T is $\mathcal{T}(p)$-continuous at x_0, then T has a fixed point $x \in X$. Moreover, if $D_{X,\preceq,\neq} \neq \emptyset$ and $\beta \neq 0$ then $p(x, x) = 0$.

Proof. Following [22], when (X, p) is complete every increasing sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \preceq_p)$ has a least upper bound and, in addition, it is convergent to its least upper bound with respect to $\mathcal{T}(d_p)$ (see remarks in [22, page 194] and [34, Proposition 2.1] for a detailed proof). We only need to show that T is orbitally $\preceq_p$-continuous at x_0. By Remark 7 in [3] we obtain that every $\mathcal{T}(p)$-continuous and $\preceq_p$-monotone mapping is $\preceq_p$-continuous. Since $x_0 \preceq_p Tx_0$ and T is $\preceq_p$-monotone we have that the sequence $(T^n x_0)_{n\in\mathbb{N}}$ is increasing and it is convergent to its least upper bound $x \in X$ with respect to $\mathcal{T}(d_p)$. Since T is $\preceq_p$-continuous we have that the sequence $(T^{n+1}x_0)_{n\in\mathbb{N}}$ has Tx as least upper bound. The fact that $(T^{n+1}x_0)_{n\in\mathbb{N}}$ is increasing yields that it converges to its least upper bound with respect to $\mathcal{T}(d_p)$. So T is orbitally $\preceq_p$-continuous at x_0. Now Corollary 4.9 yields the desired conclusion. □

Notice that the proof of Corollary 4.11 gives that every $\preceq_p$-monotone mapping which satisfies that $x_0 \preceq_p Tx_0$ always has a fixed point even if it is not a contraction of rational type. In fact Corollary 4.11 is a consequence of Corollary 8 in [3].

Applying similar arguments we get the following results.

Corollary 4.12. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a $\preceq_p$-monotone mapping such that there exist real numbers $\alpha, \beta \in [0, 1[$ with $\alpha + \beta < 1$ and*

$$p(Tx, Tx) \leq \alpha \frac{p(x, Tx) \cdot p(y, Ty)}{p(x, x)} + \beta p(x, x)$$

for all $(y, x) \in D_{X,\preceq_p,\neq}$. If there exists $x_0 \in X$ such that $x_0 \preceq_p Tx_0$ and T is $\mathcal{T}(p)$-continuous at x_0, then T has a fixed point $x \in X$. Moreover, if $D_{X,\preceq,\neq} \neq \emptyset$ and $\beta \neq 0$ then $p(x, x) = 0$.

Corollary 4.13. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a $\preceq_p$-monotone mapping such that there exist real numbers $L \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1[$ with $\alpha + \beta < 1$ and one of the following contractive conditions is satisfied for all $(y, x) \in D_{X,\preceq_p,\neq}$:*

1) $p(Tx, Tx) \leq \alpha \frac{p(y,Ty)\left[1+p(x,Tx)\right]}{1+p(x,x)} + \beta p(x, x) + L\,(p(y, Tx) - p(y, y))$.

2) $p(Tx, Tx) \leq \alpha \frac{p(y,Ty)\left[1+p(x,Tx)\right]}{1+p(x,x)} + \beta p(x, x) + L \min\{p(x, Tx) - p(x, x), p(x, Ty) - p(x, x), p(y, Tx) - p(y, y)\}$.

If there exists $x_0 \in X$ such that $x_0 \preceq_p Tx_0$ and T is $\mathcal{T}(p)$-continuous at x_0, then T has a fixed point $x \in X$. Moreover, if $D_{X,\leq,\neq} \neq \emptyset$ and $\beta \neq 0$ then $p(x, x) = 0$.

5. Conclusions

We have introduced a fixed point principle for a class of mappings between partially ordered metric spaces that we have called orbitally order continuous. All the hypotheses in the statement of such a principle cannot be weakened in order to guarantee the existence of a fixed point and they are shown to be not redundant. Moreover, the relationship between orbitally order continuous mappings and those that are continuous and orbitally continuous has been also discussed. As an application, we have extended many fixed point theorems for continuous contractions of rational type to the framework of those that are only orbitally order continuous. Extensions of the aforementioned metric fixed point results to the framework of partial metrics have been obtained. This goal has been achieved thanks to the fact that each partial metric induces in a natural way a metric in such a way that our new principle was applicable. In both approaches, the metric and the partial metric, we have shown that there are orbitally order continuous mappings that satisfy all assumptions in our new fixed point principle but that they are not contractions of rational type. Illustrative examples have been provided in order to support the exposed theory.

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Conflict of interest

All authors declare no conflicts of interest in this paper

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