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*Research article*

# Function kernels and divisible groupoids

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**Abstract:** In this paper, we introduce the notion of a function kernel which was motivated from the kernel in group theory, and we apply this notion to several algebraic structures, e.g., groups, groupoids, $BCK$-algebras, semigroups, leftoids. Using the notions of left and right cosets in groupoids, we investigate some relations with function kernels. Moreover, the notion of an idenfunction in groupoids is introduced, which is a generalization of an identity axiom in algebras by functions, and we discuss it with function kernels.

**Keywords:** groupoid; $BCK$-algebra; $\varphi$-kernel; coset; divisible; idenfunction
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## 1. Introduction

Bruck [2] discussed the theory of groupoids, loops and quasigroups, and several algebraic structures. Borúvka [1] stated the theory of decompositions of sets and its application to binary systems. Recently, several researchers investigated groupoids, and obtained some interesting results [4,8,10]. The notions of $BCK$-algebras and $BCI$-algebras were formulated in 1966 by Imai and Iséki [5], and were studied by many researchers [3,6,11,13].

Kim and Neggers [7] introduced the notion of $Bin(X)$ of all binary systems (groupoids, algebras) defined on a set $X$, and showed that it becomes a semigroup under suitable operation.

For the construction of the quotient group, we introduce the notion of a kernel by using group homomorphisms, and construct left (right) cosets. In this paper, we introduce the notion of right and left function $\varphi$-kernels, which was motivated from the kernel in groups, and we apply this notion to several algebraic structures, e.g., groupoids, $BCK$-algebras, groups, semigroups, leftoids. Moreover, we apply the left and right $\varphi$-kernels to the semigroup $Bin(X)$ of all binary systems (groupoids, algebras) defined on $X$, and investigate some roles of $R_\varphi(*)$ and $L_\varphi(*)$. We apply the notion of a kernel in groups will be defined in general algebraic structures, i.e., groupoids, and investigate its roles with left and right cosets.

Moreover, we introduce the notion of left and right divisible groupoids, and obtain some relations with function kernels. We show that every subgroup of a group is divisible, but the converse need not be true. The identity axiom in groups, semigroups, $BCK$-algebras and other general algebraic structures plays an important role for developing the theory. Finally, we introduce the notion of an idenfunction by using functions. We investigate some relations between idenfunctions and function kernels. The notions $F\rho(*, \varphi)$ and $FR_\varphi(*)$ will be discussed with function kernels and $KER(*, \varphi)$. The notion of the idenfunction, which is a generalization of an identity axiom in algebras, will be applied to the notion of function kernels, and obtain some useful results.

## 2. Preliminaries

A $d$-algebra [9,12] is a nonempty set $X$ with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A $BCK$-algebra [3,6,11] is a $d$-algebra $X$ satisfying the following additional axioms:

(IV) $(x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

A groupoid $(X, *)$ is said to be a left zero semigroup if $x * y = x$ for any $x, y \in X$, and a groupoid $(X, *)$ is said to be a right zero semigroup if $x * y = y$ for any $x, y \in X$. A groupoid $(X, *)$ is said to be a leftoid for $f : X \to X$ if $x * y = f(x)$ for any $x, y \in X$. Similarly, a groupoid $(X, *)$ is said to be a rightoid for $f : X \to X$ if $x * y = f(y)$ for any $x, y \in X$. Note that a left (right, resp.) zero semigroup is a special case of a leftoid (right, resp.) (see [7]).

Given two groupoids $(X, *)$ and $(X, \bullet)$, we define a new binary operation $\square$ by $x \square y := (x * y) \bullet (y * x)$ for all $x, y \in X$. Then we obtain a new groupoid $(X, \square)$, i.e., $(X, \square) = (X, *)\square(X, \bullet)$. We denote the collection of all binary systems (groupoid, algebras) defined on $X$ by $Bin(X)$ [7].

**Theorem 2.1.** [7] *$(Bin(X), \square)$ is a semigroup and the left zero semigroup is an identity.*

## 3. Function kernels

Given a groupoid $(X, *)$, i.e., $(X, *) \in Bin(X)$, and a function $\varphi : X \to Y$, we define subsets $R_\varphi(*)$ and $L_\varphi(*)$ as follows:

$$R_\varphi(*) := \{t \in X \,|\, \varphi(x * t) = \varphi(x), \forall x \in X\},$$

$$L_\varphi(*) := \{t \in X \,|\, \varphi(t * x) = \varphi(x), \forall x \in X\}.$$

We call $R_\varphi(*)$ the right $\varphi$-kernel of a groupoid $(X, *)$, and $L_\varphi(*)$ the left $\varphi$-kernel of a groupoid $(X, *)$.

**Example 3.1.** Let $X := \{0, 1, 2, 3\}$ be a groupoid with the following table:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 |
| 2 | 2 | 2 | 1 | 2 |
| 3 | 3 | 3 | 3 | 0 |

If we define a map $\varphi : X \to X$ by $\varphi(0) = 0$, $\varphi(1) = \varphi(2) = 1$ and $\varphi(3) = 3$, then it it easy to see that $R_\varphi(*) = \{0, 2\}$, but $L_\varphi(*) = \emptyset$.

**Example 3.2.** Consider $BCK$-algebras $(X, *, 0)$ and $(Y, \bullet, e)$ with the following tables [11, p. 245]:

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | 2 |
| 3 | 3 | 3 | 3 | 0 |

| $\bullet$ | e | a | b | c |
|---|---|---|---|---|
| e | e | e | e | e |
| a | a | e | a | e |
| b | b | b | e | e |
| c | c | b | a | e |

If we define a map $\varphi : X \to Y$ by $\varphi(0) = \varphi(1) = \varphi(2) = e$ and $\varphi(3) = a$, then it it easy to see that $\varphi$ is a $BCK$-homomorphism and $R_\varphi(*) = \{0, 1, 2\}$, but $L_\varphi(*) = \emptyset$.

**Proposition 3.1.** *If $(X, *, 0)$ is a $BCK$-algebra, then $0 \in R_\varphi(*)$ for any map $\varphi : X \to Y$.*

*Proof.* If $(X, *, 0)$ is a $BCK$-algebra, then $x * 0 = x$ for all $x \in X$. For any map $\varphi : X \to Y$, we have $\varphi(x * 0) = \varphi(x)$ for all $x \in X$. This shows that $0 \in R_\varphi(*)$. □

**Proposition 3.2.** *If $(X, *, e)$ is a group, then $e \in R_\varphi(*) \cap L_\varphi(*)$ for any map $\varphi : X \to Y$.*

*Proof.* If $(X, *, e)$ is a group, then $x * e = x = e * x$ for all $x \in X$. It follows that, for any map $\varphi : X \to Y$, $\varphi(x * e) = \varphi(e * x) = \varphi(x)$ for all $x \in X$, proving that $e \in R_\varphi(*) \cap L_\varphi(*)$. □

**Example 3.3.** Let $\mathbf{R}$ be the set of all real numbers and let "+" be the usual addition on $\mathbf{R}$. Define a map $\varphi : \mathbf{R} \to \mathbf{R}$ by $\varphi(x) := ax + b$, where $a \neq 0$, $b \in \mathbf{R}$. Then $R_\varphi(+) = L_\varphi(+) = \{0\}$. In fact, if $u \in R_\varphi(+)$, then $\varphi(x + u) = \varphi(x)$ for all $x \in \mathbf{R}$. It follows that $a(x + u) + b = ax + b$, and hence $au = 0$. Since $a \neq 0$, we obtain $u = 0$. Hence $R_\varphi(+) = \{0\}$. Similarly, we have $L_\varphi(+) = \{0\}$.

**Proposition 3.3.** *If $\varphi : (X, *, e) \to (Y, \bullet, e^*)$ is a group homomorphism, then $Ker(\varphi) = R_\varphi(*) = L_\varphi(*)$.*

*Proof.* If $t \in Ker(\varphi)$, then $\varphi(t) = e^*$. Given $x \in X$, we have $\varphi(x * t) = \varphi(x) \bullet \varphi(t) = \varphi(x) \bullet e^* = \varphi(x)$. This shows that $t \in R_\varphi(*)$. Similarly, we obtain $t \in L_\varphi(*)$. If $t \in R_\varphi(*)$, then $\varphi(x * t) = \varphi(x)$ for all $x \in X$. Since $\varphi$ is a homomorphism, we obtain $\varphi(x) = \varphi(x * t) = \varphi(x) \bullet \varphi(t)$, which shows that $\varphi(t) = e^*$. This proves that $t \in Ker(\varphi)$. Similarly, we prove that if $t \in L_\varphi(*)$, then $t \in Ker(\varphi)$. □

Proposition 3.3 shows that the notion of a left (right) $\varphi$-kernel is the same as the notion of the kernel in groups.

**Proposition 3.4.** *If $(X, *)$ is a semigroup, then $(R_\varphi(*), *)$ is also a semigroup.*

*Proof.* Since $R_\varphi(*) \subseteq X$, it is enough to show that $(R_\varphi(*), *)$ is a subgroupoid of $(X, *)$. If $u, t \in R_\varphi(*)$, then $\varphi(x * u) = \varphi(x)$, $\varphi(x * t) = \varphi(x)$ for all $x \in X$. Since $(X, *)$ is a semigroup, we obtain $\varphi(x * (u * t)) = \varphi((x * u) * t) = \varphi(x * u) = \varphi(x)$ for all $x \in X$, and hence $u * t \in R_\varphi(*)$. This proves the proposition. □

Let $(X, *)$ be a groupoid (not necessarily a semigroup). Given $a \in X$, we define a map $r_a : X \to X$ by $r_a(x) := x * a$ for all $x \in X$. The set of all such maps is defined by $R^*_\varphi(*)$, i.e.,

$$R^*_\varphi(*) = \{r_a \,|\, \varphi(r_a(x)) = \varphi(x), \forall x \in X, a \in X\},$$

where $\varphi : X \to Y$ is a map. We obtain the following proposition.

**Proposition 3.5.** *$(R^*_\varphi(*), \circ)$ is a semigroup, where "$\circ$" is the composition of functions.*

*Proof.* Given $a, b \in R^*_\varphi(*)$ and $x \in X$, we have $(r_a \circ r_b)(x) = r_a(r_b(x)) = r_a(x * b) = (x * b) * a$. Since $a, b \in R^*_\varphi(*)$, we obtain $\varphi(x * a) = \varphi(x)$, $\varphi(x * b) = \varphi(x)$ for all $x \in X$. It follows that $\varphi((x * b) * a) = \varphi(x * b) = \varphi(x)$ for all $x \in X$. Hence, $\varphi(x) = \varphi((x * b) * a) = \varphi((r_a \circ r_b)(x))$, which shows that $r_a \circ r_b \in R^*_\varphi(*)$. Thus, $(R^*_\varphi(*), \circ)$ is a semigroup. □

Similarly, given $a \in X$, we define a map $l_a : X \to X$ by $l_a(x) := a * x$ for all $x \in X$. The set of all such maps is defined by $L^*_\varphi(*)$, i.e.,

$$L^*_\varphi(*) = \{l_a \,|\, \varphi(l_a(x)) = \varphi(x), \forall x \in X, a \in X\},$$

where $\varphi : X \to Y$ is a map. We obtain the following proposition.

**Corollary 3.1.** *$(L^*_\varphi(*), \circ)$ is a semigroup, where "$\circ$" is the composition of functions.*

**Proposition 3.6.** *Let $\varphi : (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids. If $a \in R_\varphi(*)$ and $\varphi$ is injective, then $r_a$ is an identity function on $(X, *)$.*

*Proof.* Assume that $r_a$ is not an identity function on $(X, *)$. Then there exists $x_0 \in X$ such that $r_a(x_0) \neq x_0$, i.e., $x_0 * a \neq x_0$. Since $\varphi$ is injective, we obtain $\varphi(x_0 * a) \neq \varphi(x_0)$, which means that $a \notin R_\varphi(*)$, which is a contradiction. □

**Corollary 3.2.** *Let $\varphi : (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids. If $a \in L_\varphi(*)$ and $\varphi$ is injective, then $l_a$ is an identity function on $(X, *)$.*

*Proof.* The proof is similar to Proposition 3.6. □

**Remark 3.1.** *It is necessary to give a condition: $\varphi$ is injective, in Proposition 3.6. The mapping $\varphi$ in Example 3.2 is a BCK-homomorphism, but not injective, and $R_\varphi(*) = \{0, 1, 2\}$. Since $r_2(1) = 1 * 2 = 0 \neq 1$ and $r_1(2) = 2 * 1 = 1 \neq 2$, i.e., $r_1$ and $r_2$ are not an identity function.*

**Example 3.4.** Let $+$ be the usual addition on $\mathbf{R}$. If we define a map $\varphi : \mathbf{R} \to \mathbf{R}$ by $\varphi(x) := \sin x$, then $R_\varphi(+) = \{2n\pi \,|\, n \in \mathbf{Z}\}$. In fact, if $t \in R_\varphi(+)$, then $\varphi(x + t) = \varphi(x)$ for all $x \in \mathbf{R}$, which shows that $\sin(x + t) = \sin x$ for all $x \in \mathbf{R}$. Hence, $t$ is a period of $\sin x$, i.e., there exists $n \in \mathbf{Z}$ such that $t = 2n\pi$. Since $\sin(x + 2(m \pm n)\pi) = \sin x$, $(R_\varphi(+), +)$ is a normal subgroup of $(\mathbf{R}, +)$, whence it is a kernel of the map $\nu : (\mathbf{R}, +) \to (\mathbf{R}, +)/(R_\varphi(+), +)$.

**Proposition 3.7.** *Let $(X, *)$ be a left zero semigroup.*

(i) *If $\varphi : X \to Y$ is any map, then $R_\varphi(*) = X$.*
(ii) *If $\varphi : X \to Y$ is not a constant map, then $L_\varphi(*) = \emptyset$.*

*Proof.* (i) Given $a \in X$, since $(X, *)$ is a left zero semigroup, we have $x * a = x$ for all $x \in X$. It follows that $\varphi(x * a) = \varphi(x)$ for all $x \in X$. This shows that $a \in R_\varphi(*)$.
(ii) Assume that $L_\varphi(*) \neq \emptyset$. Then there exists $a \in L_\varphi(*)$, which means that $\varphi(a * x) = \varphi(x)$ for all $x \in X$. Since $\varphi$ is not a constant function, there exists $b \in X$ such that $a \neq b$ and $\varphi(a) \neq \varphi(b)$. Since $(X, *)$ is a left zero semigroup, we have $\varphi(a) = \varphi(a * b) = \varphi(b) \neq \varphi(a)$, which is a contradiction. □

**Remark 3.2.** *Proposition 3.7 (i) does not hold for non-left zero semigroup in general. In fact, we see that* $(\mathbf{R}, +)$ *is not a left zero semigroup and* $R_\varphi(+) = \{2n\pi \mid n \in \mathbf{Z}\} \neq \mathbf{R}$ *in Example 3.4.*

**Theorem 3.1.** *Let* $(X, *)$ *and* $(X, \bullet)$ *be groupoids and let* $(X, \square) := (X, *)\square(X, \bullet)$*, i.e.,* $x\square y = (x*y)\bullet(y*x)$ *for all* $x, y \in X$*. If* $\varphi : X \to Y$ *is a mapping, then*

(i) $t * x \in R_\varphi(\bullet),\ t \in R_\varphi(*)$ *implies* $t \in R_\varphi(\square)$*;*
(ii) $t * x \in L_\varphi(\bullet),\ t \in R_\varphi(*)$ *implies* $t \in R_\varphi(\square)$*;*
(iii) $x * t \in L_\varphi(\bullet),\ t \in L_\varphi(*)$ *implies* $t \in L_\varphi(\square)$*;*
(iv) $x * t \in R_\varphi(\bullet),\ t \in L_\varphi(*)$ *implies* $t \in L_\varphi(\square)$*.*

*Proof.* (i) If $t * x \in R_\varphi(\bullet),\ t \in R_\varphi(*)$, then $\varphi(x\square t) = \varphi((x * t) \bullet (t * x)) = \varphi(x * t) = \varphi(x)$, which shows that $t \in R_\varphi(\square)$.
(ii) If $t * x \in L_\varphi(\bullet),\ t \in R_\varphi(*)$, then $\varphi(t\square x) = \varphi((t * x) \bullet (x * t)) = \varphi(x * t) = \varphi(x)$, which shows that $t \in R_\varphi(\square)$.
(iii) If $x * t \in L_\varphi(\bullet),\ t \in L_\varphi(*)$, then $\varphi(x\square t) = \varphi((x * t) \bullet (t * x)) = \varphi(t * x) = \varphi(x)$, which shows that $t \in L_\varphi(\square)$.
(iv) If $x * t \in R_\varphi(\bullet),\ t \in L_\varphi(*)$, then $\varphi(t\square x) = \varphi((t * x) \bullet (x * t)) = \varphi(t * x) = \varphi(x)$, which shows that $t \in L_\varphi(\square)$. □

Let $\varphi : (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids and let $t \in X$. Define a set $KER(*, \varphi)$ by

$$KER(*, \varphi) := \{(x, y) | \varphi(x) = \varphi(y)\}.$$

Given $t \in (X, *)$, we define the right coset $\rho(*, t)$ and the left coset $\lambda(*, t)$ respectively by

$$\rho(*, t) := \{(x * t, x) \mid x \in X\}$$

and

$$\lambda(*, t) := \{(t * x, x) \mid x \in X\}.$$

We define two sets $RK(*, \varphi)$ and $LK(*, \varphi)$ as follows:

$$RK(*, \varphi) := \{t \in X \mid \rho(*, t) \subseteq KER(*, \varphi)\}$$

and

$$LK(*, \varphi) := \{t \in X \mid \lambda(*, t) \subseteq KER(*, \varphi)\}.$$

**Proposition 3.8.** *Given a groupoid* $(X, *)$*, i.e.,* $(X, *) \in Bin(X)$*, and a function* $\varphi : X \to Y$*, we have*

$$\text{(i) } R_\varphi(*) = RK(*, \varphi), \quad \text{(ii) } L_\varphi(*) = LK(*, \varphi).$$

*Proof.* (i) Let $t \in R_\varphi(*)$. If $(x*t, x) \in \rho(*, t)$, then $\varphi(x*t) = \varphi(x)$, i.e., $(x*t, x) \in KER(*, \varphi)$. This shows that $\rho(*, t) \subseteq KER(*, \varphi)$, which means that $t \in RK(*, \varphi)$. If $t \in RK(*, \varphi)$, then $\rho(*, t) \subseteq KER(*, \varphi)$. It follows that $(x * t, x) \in KER(*, \varphi)$ for all $x \in X$, and hence $\varphi(x * t) = \varphi(x)$. Hence, $t \in R_\varphi(*)$. This proves that $RK(*, \varphi) \subseteq R_\varphi(*)$.
(ii) The proof is similar to (i). □

**Theorem 3.2.** *Let* $\mathbf{R}$ *be the set of all real numbers and let "+" be the usual addition on* $\mathbf{R}$ *and let* $\varphi : \mathbf{R} \to \mathbf{R}$ *be a map. Then*

$$\cup_{t\in R_\varphi(+)} \rho(+, t) \subseteq KER(+, \varphi) \subseteq \cup_{t\in \mathbf{R}} \rho(+, t).$$

*Proof.* If $(\alpha,\beta) \in \cup_{t\in R_\varphi(+)} \rho(+, t)$, then there exists $t \in R_\varphi(+)$ such that $(\alpha,\beta) \in \rho(+, t)$, and hence $\alpha = \beta + t$. Since $t \in R_\varphi(+)$, we have $\varphi(x + t) = \varphi(x)$ for all $x \in \mathbf{R}$. It follows that $\varphi(\alpha) = \varphi(\beta + t) = \varphi(\beta)$, i.e., $(\alpha,\beta) \in KER(+, \varphi)$. Hence, $\cup_{t\in R_\varphi(+)} \rho(+, t) \subseteq KER(+, \varphi)$. If $(x, y) \in KER(+, \varphi)$, then $\varphi(x) = \varphi(y)$. Let $t \in \mathbf{R}$ such that $x = y + t$. It follows that $(x, y) = (y + t, y) \in \rho(+, t)$. Hence, $(x, y) \in \cup_{t\in \mathbf{R}} \rho(+, t)$, proving the theorem. □

## 4. Divisible groupoids

A groupoid $(X, *)$ is said to be right divisible (resp., left divisible ) if for any $x, y \in X$, there exists $t \in X$ such that $x = y * t$ (resp., $x = t * y$). Note that $(\mathbf{Z}, +)$ is divisible. We take $t := x - y$ for any $x, y \in \mathbf{Z}$. It follows that $x = (x - y) + y = t + y$.

**Proposition 4.1.** *If* $(X, *)$ *is a right divisible groupoid, then*

$$X \times X = \cup_{t\in X}\ \rho(*, t).$$

*Proof.* For any $(x, y) \in X \times X$, there exists $t \in X$ such that $x = y * t$, since $(X, *)$ is right divisible. It follows that $(x, y) = (y * t, y) \in \rho(*, t)$, which proves that $(x, y) \in \cup_{t\in X}\ \rho(*, t)$. □

Given a groupoid $(X, *)$, we define a set $\mathbb{T}_\varphi(*)$ by

$$\mathbb{T}_\varphi(*) := \{t \in X \,|\, \rho(*, t) \cap KER(*, \varphi) \neq \emptyset\}.$$

**Theorem 4.1.** *Let* $(X, *)$ *be a right divisible groupoid. If* $\varphi : (X, *) \to (Y, \bullet)$ *is a homomorphism of groupoids, then*

$$KER(*, \varphi) \subseteq \cup_{t\in \mathbb{T}_\varphi(*)}\ \rho(*, t).$$

*Proof.* If $(x, y) \in KER(*, \varphi)$, then $\varphi(x) = \varphi(y)$. Since $(X, *)$ is right divisible, there exists $t \in X$ such that $x = y * t$. It follows that $(x, y) = (y * t, y) \in \rho(*, t)$ and $\varphi(y * t) = \varphi(x) = \varphi(y)$. We claim that $t \in \mathbb{T}_\varphi(*)$. In fact,

$$\begin{aligned} t \in \mathbb{T}_\varphi(*) &\iff \rho(*, t) \cap KER(*, \varphi) \neq \emptyset \\ &\iff \exists\ y \in X \text{ such that } (y * t, y) \in \rho(*, t) \cap KER(*, \varphi) \\ &\iff \varphi(y * t) = \varphi(y) \text{ holds.} \end{aligned}$$

Hence, $(x, y) \in \cup_{t\in \mathbb{T}_\varphi(*)}\ \rho(*, t)$. □

**Corollary 4.1.** *If* $(X, *)$ *is a left divisible groupoid, then*

(i) $X \times X = \cup_{t\in X}\ \lambda(*, t)$,
(ii) $KER(*, \varphi) \subseteq \cup_{t\in\mathbb{T}_{\varphi}(*)}\ \lambda(*, t)$.

*Proof.* The proof is similar to Proposition 4.1 and Theorem 4.1. □

**Proposition 4.2.** *If* $(X, *)$ *is a left zero semigroup, then it is left divisible, but not right divisible.*

*Proof.* Given $x, y \in X$, if we let $u := x$, then $u*y = x*y = x$, since $(X, *)$ is a left zero semigroup. Hence, $(X, *)$ is left divisible. We claim that $(X, *)$ is not a right divisible. Assume $(X, *)$ is right divisible. Let $x \neq y$ in $X$. Then there exists $z \in X$ such that $y = x * z$. Since $(X, *)$ is a left-zero semigroup, we obtain $y = x * z = x$, which is a contradiction. □

**Proposition 4.3.** *If* $(X, *)$ *is a right zero semigroup, then it is right divisible, but not left divisible.*

**Proposition 4.4.** *Let* $(X, *)$ *be a leftoid for* $\varphi : X \to X$. *If* $\varphi$ *is onto, then* $(X, *)$ *is left divisible, but not right divisible.*

*Proof.* Given $x, y \in X$, since $\varphi$ is onto, there exists $u \in X$ such that $y = \varphi(u)$. It follows that $y = \varphi(u) = u * x$. Hence, $(X, *)$ is left divisible. Assume $(X, *)$ is right divisible. Let $y_1 \neq y_2$ in $X$. Since $\varphi$ is onto, there exist $u_1, u_2 \in X$ such that $y_1 = x * u_1, y_2 = x * u_2$. Since $(X, *)$ is a leftoid for $\varphi$, we obtain $y_1 = \varphi(x) = y_2$, which is a contradiction. □

**Remark 4.1.** *It is necessary to add the condition,* $\varphi$ *is onto, in Proposition 4.4. See the following example.*

**Example 4.1.** Let $\mathbf{R}$ be the set of all real numbers. If we define a map $\varphi : \mathbf{R} \to \mathbf{R}$ by $\varphi(x) := x + 1$, then it is a bijective function. Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := \varphi(x)$ for all $x, y \in \mathbf{R}$. Then $(\mathbf{R}, *)$ is left divisible over $\varphi$. In fact, for any $y \in \mathbf{R}$, if we let $u := y - 1$, then $u * x = \varphi(u) = \varphi(y - 1) = y$ for any $x \in \mathbf{R}$. Define a map $\psi : \mathbf{R} \to \mathbf{R}$ by $\psi(x) := x^2$. Given $-3, 5 \in \mathbf{R}$, there exists no real number $t \in \mathbf{R}$ such that $-3 = t * 5 = \psi(t) = t^2 \geq 0$. This means that $(\mathbf{R}, *)$ is not left divisible over $\psi$.

**Proposition 4.5.** *Let* $(X, *)$ *be a rightoid for* $\varphi : X \to X$. *If* $\varphi$ *is onto, then* $(X, *)$ *is right divisible, but not left divisible.*

*Proof.* The proof is similar to Proposition 4.4. □

Let $(X, *)$ and $(Y, \bullet)$ be groupoids. Define maps $\varphi : (X, *) \to (X, *)$ and $\psi : (Y, \bullet) \to (Y, \bullet)$. We define a binary operation "$\nabla$" on $X \times Y$ by $(x, y)\nabla(u, v) := (x * u, y \bullet v)$ and a map $\xi : (X \times Y, \nabla) \to (X \times Y, \nabla)$ by $\xi(x, y) := (\varphi(x), \psi(y))$. We call $(X \times Y, \nabla)$ a direct product of $(X, *)$ and $(Y, \bullet)$ with respect to $(\varphi, \psi)$.

**Theorem 4.2.** *Let* $(X \times Y, \nabla)$ *be a direct product of* $(X, *)$ *and* $(Y, \bullet)$ *with respect to* $(\varphi, \psi)$. *If we define a map* $\xi : (X \times Y, \nabla) \to (X \times Y, \nabla)$ *by* $\xi(x, y) := (\varphi(x), \psi(y))$, *then* $R_{\varphi}(*) \times R_{\psi}(\bullet) = R_{\xi}(\nabla)$.

*Proof.* If $(\alpha, \beta) \in R_{\varphi}(*) \times R_{\psi}(\bullet)$, then $\alpha \in R_{\varphi}(*)$ and $\beta \in R_{\psi}(\bullet)$, i.e., $\varphi(x * \alpha) = \varphi(x)$ and $\psi(y \bullet \beta) = \psi(y)$ for all $x \in X$ and $y \in Y$. It follows that

$$\begin{aligned} & \xi((x, y)\nabla(\alpha, \beta)) \\ = \ & \xi(x * \alpha, y \bullet \beta) = (\varphi(x * \alpha), \psi(y \bullet \beta)) \\ = \ & (\varphi(x), \psi(y)) = \xi(x, y), \end{aligned}$$

for all $(x, y) \in X \times Y$. This shows that $(\alpha, \beta) \in R_{\xi}(\nabla)$. The converse is similar, and we omit it. □

**Corollary 4.2.** *Let $(X \times Y, \nabla)$ be a direct product of $(X, *)$ and $(Y, \bullet)$ with respect to $(\varphi, \psi)$. If we define a map $\xi : (X \times Y, \nabla) \to (X \times Y, \nabla)$ by $\xi(x, y) := (\varphi(x), \psi(y))$, then $L_\varphi(*) \times L_\psi(\bullet) = L_\xi(\nabla)$.*

*Proof.* The proof is similar to Theorem 4.2. □

**Proposition 4.6.** *Let $(X \times Y, \nabla)$ be a direct product of $(X, *)$ and $(Y, \bullet)$. If $(X, *)$ and $(Y, \bullet)$ are right (resp., left) divisible, then $(X \times Y, \nabla)$ is also right (resp., left) divisible.*

*Proof.* The proof is straightforward. □

**Proposition 4.7.** *Every homomorphic image of a right (resp., left) divisible groupoid is right (resp., left) divisible.*

*Proof.* Let $\varphi : (X, *) \to (Y, \bullet)$ be an epimorphism of groupoids and let $(X, *)$ be a right (resp., left) divisible groupoid. Given $x, y \in Y$, since $\varphi$ is onto, there exist $a, b \in X$ such that $x = \varphi(a), y = \varphi(b)$. Since $(X, *)$ is right divisible, there exists $c \in X$ such that $a = b * c$. Hence, $\varphi(a) = \varphi(b * c) = \varphi(b) \bullet \varphi(c)$, i.e., $x = y \bullet \varphi(c)$, which shows that $(Y, \bullet)$ is right divisible. □

**Proposition 4.8.** *Every subgroup of a group is divisible.*

*Proof.* Let $(X, *, e)$ be a group and let $H$ be a subgroup of $X$. Given $x, y \in H$, we let $t := x^{-1} * y$. Then $x * t = x * (x^{-1} * y) = (x * x^{-1}) * y = e * y = y$. Hence, $(H, *, e)$ is right divisible. Similarly, we proves that $(H, *, e)$ is left divisible. □

In Proposition 4.8, we showed that every subgroup of a group is divisible. But it does not hold for subgroupoids which are not subgroups. Consider a set $\mathbb{U}_k = \{k, k + 1, k + 2, \cdots\}$. It is a subgroupoid of an abelian group $\mathbb{Z}$, but it is not a subgroup of $\mathbb{Z}$. Clearly, it is not divisible, since there exists no element $t$ in $\mathbb{Z}$ such that $(k + 1) + t = k$.

## 5. Idenfunctions

Let $(X, *)$ be a groupoid. A map $\xi : X \to X$ is said to be a right (resp., left) idenfunction of $(X, *)$ if $x * \xi(x) = x$ (resp., $\xi(x) * x = x$) for all $x \in X$. If $\xi$ is an identity map, i.e., $\xi(x) := e$ for all $x \in X$, then $e$ is an right identity of a groupoid $(X, *)$. The notion of an idenfunction is a generalization of an identity axiom in groupoids.

**Example 5.1.** Consider a group $(\mathbb{Z}/(5), \cdot)$. For any $x \in \mathbb{Z}/(5)$, we have $x^5 = x$. If we take $\xi(x) := x^4$, then $x \cdot \xi(x) = x \cdot x^4 = x^5 = x$ for all $x \in \mathbb{Z}/(5)$. Now, $1^4 = 1, 2^4 = 16 = 1, 3^4 = 81 = 1, 4^4 = (-1)^4 = 1$, $0^4 = 0$, so that $\xi$ is an idenfunction which is not a constant function. Of course, $\xi(x) \equiv 1$ yields an identity element.

**Example 5.2.** Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := 2x + 3y$ for all $x, y \in \mathbf{R}$. Define a map $\xi : \mathbf{R} \to \mathbf{R}$ satisfying $x * \xi(x) = x$ for all $x \in \mathbf{R}$. Then $x = x * \xi(x) = 2x + 3\xi(x)$. It follows that $\xi(x) = -\frac{1}{3}x$ is the right idenfunction of $(\mathbf{R}, *)$. Assume $\psi : \mathbf{R} \to \mathbf{R}$ is a map satisfying $\psi(x) * x = x$ for all $x \in \mathbf{R}$. Then $2\psi(x) + 3x = x$, and hence $\psi(x) = -x$ is the left idenfunction of $(\mathbf{R}, *)$.

Given a map $\varphi : (X, *) \to (Y, \bullet)$, we define sets $F\rho(*, \varphi)$ and $FR_\varphi(*)$ by

$$F\rho(*, \varphi) := \{(x * \varphi(x), x) \mid x \in X\}$$

and

$$FR_\varphi(*) := \{\psi : X \to X \,|\, \varphi(x * \psi(x)) = \varphi(x), \forall x \in X\}.$$

We may define $F\lambda(*, \varphi)$ and $FL_\varphi(*)$ for a map $\varphi : X \to Y$.

Let $(X, *)$ be a groupoid and let $a \in X$. If we define a map $\delta_a : X \to X$ by $\delta_a(x) := a$ for all $x \in X$, then we may identify the map $\delta_a$ with $a$, i.e., $\delta_a \equiv a$. If a groupoid $(X, *)$ has a right identity $e$, then $e \in F\rho(*, \delta_e) = \rho(*, e)$.

**Proposition 5.1.** *If a groupoid $(X, *)$ has a right identity $e$, then $e \in R_\varphi(*)$.*

*Proof.* If $e$ is a right identity of $(X, *)$, then $x * e = x$ for all $x \in X$. It follows that $\varphi(x * e) = \varphi(x)$ for all $x \in X$. This proves that $e \in R_\varphi(*)$. □

**Proposition 5.2.** *Let $(X, *)$ be a groupoid. If $\psi : (X, *) \to (X, *)$ is a right idenfunction of $(X, *)$, then $\psi \in FR_\varphi(*)$ for any map $\varphi : X \to Y$.*

*Proof.* The proof is straightforward. □

**Example 5.3.** (a) Let $\mathbf{R}$ be the set of all real numbers and let "$\cdot$" be the usual multiplication on $\mathbf{R}$. Define a map $\varphi : \mathbf{R} \to \mathbf{R}$ by $\varphi(x) := \sin x$. Assume $\psi : \mathbf{R} \to \mathbf{R}$ satisfies the condition: $\varphi(x \cdot \psi(x)) = \varphi(x)$. Then $\sin(x \cdot \psi(x)) = \sin x$ and hence $x \cdot \psi(x) = x + 2n(x)\pi = x(1 + \frac{1}{x}2n(x)\pi)$ for some $n(x) \in \mathbb{Z}$ with $x \neq 0$. It follows that $\psi(x) = 1 + \frac{1}{x}2n(x)\pi \in FR_\varphi(\cdot)$ for $x \neq 0$.
(b) Define $\varphi : (\mathbf{R}, +) \to [-1, 1]$ by $\varphi(x) := \sin x$. Assume $\delta : \mathbf{R} \to \mathbf{R}$ satisfies the condition: $\varphi(x + \delta(x)) = \varphi(x)$. Then $\sin(x + \delta(x)) = \sin x$, and hence we obtain $x + \delta(x) = x + 2n(x)\pi$, i.e., $\delta(x) = 2n(x)\pi \in FR_\varphi(+)$ for some $n(x) \in \mathbb{Z}$.

**Theorem 5.1.** *Let $(X, *)$ be a groupoid and let $\varphi : X \to Y$ be a map. Then*

$$R_\varphi(*) \subseteq FR_\varphi(*).$$

*Proof.* If $a \in R_\varphi(*)$, then $\varphi(x * a) = \varphi(x)$ for all $x \in X$. If we define a constant map $\delta_a : X \to X$ by $\delta_a(x) := a$, then we may identify $\delta_a \equiv a$. It follows that $\varphi(x) = \varphi(x * a) = \varphi(x * \delta_a(x))$ for all $x \in X$, and hence $a \equiv \delta_a \in FR_\varphi(*)$. □

**Proposition 5.3.** *Let $(X, *)$ be a groupoid and let $\varphi : X \to Y$ be a map. If $\xi \in FR_\varphi(*)$, then $F\rho(*, \xi) \subseteq KER(*, \varphi)$.*

*Proof.* Since $\xi \in FR_\varphi(*)$, if $(x * \xi(x), x) \in F\rho(*, \xi)$, then $\varphi(x * \xi(x)) = \varphi(x)$ for all $x \in X$, which proves that $(x * \xi(x), x) \in KER(*, \varphi)$. □

**Example 5.4.** In Example 5.2, we define a map $\varphi(x) := x^2$ for all $x \in \mathbf{R}$. We find all functions $\xi(x)$ in $FR_\varphi(*)$. If $\xi(x) \in FR_\varphi(*)$, then $\varphi(x*\xi(x)) = \varphi(x)$, and hence $x^2 = (2x+3\xi(x))^2 = 4x^2+12x\xi(x)+9\xi(x)^2$. It follows that $\xi(x) = -\frac{1}{3}x$ or $\xi(x) = -x$, i.e., $FR_\varphi(*) = \{-\frac{1}{3}x, -x\}$. Similarly, we obtain $FL_\varphi(*) = \{-2x, -x\}$.

## 6. Conclusions

In this paper, we introduced the notion of a function kernel, in which the idea came from the kernel in group theory. We applied this concept to several algebraic structures, e.g., groupoids, $BCK$-algebras, semigroups etc. By introducing the notions of left and right divisible groupoids, we discussed some relations between function kernels and divisible groupoids. Finally, we introduced the notion of an idenfunction, which is a generalized identity axiom in several algebraic structures. The notion can be applied to several algebraic structures, e.g., groups, rings, fields and vector spaces in the sequel, since these algebraic structures contain the identity axiom. This approach may open the new door of several algebras in future.

## Conflict of interest

The authors declare no conflicts of interest.

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