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Research article

Number of maximal 2-component independent sets in forests

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Abstract: Let $G = (V(G), E(G))$ be a graph. For a positive integer k, we call $S \subseteq V(G)$ a k-component independent set of G if each component of $G[S]$ has order at most k. Moreover, S is maximal if there does not exist a k-component independent set S' of G such that $S \subseteq S'$ and $|S| < |S'|$. A maximal k-component independent set of a graph G is denoted briefly by Mk-CIS. We use $t_k(G)$ to denote the number of Mk-CISs of a graph G. In this paper, we show that for a forest G of order n,

$$t_2(G) \leq \begin{cases} 3^{\frac{n}{3}}, & \text{if} \quad n \equiv 0 \ (mod\ 3) \text{ and } n \geq 3, \\ 4 \cdot 3^{\frac{n-4}{3}}, & \text{if} \quad n \equiv 1 \ (mod\ 3) \text{ and } n \geq 4, \\ 5, & \text{if} \quad n = 5, \\ 4^2 \cdot 3^{\frac{n-8}{3}}, & \text{if} \quad n \equiv 2 \ (mod\ 3) \text{ and } n \geq 8, \end{cases}$$

with equality if and only if $G \cong F_n$, where

$$F_n \cong \begin{cases} \frac{n}{3}P_3, & \text{if} \quad n \equiv 0 \ (mod\ 3) \text{ and } n \geq 3, \\ \frac{n-4}{3}P_3 \cup K_{1,3}, & \text{if} \quad n \equiv 1 \ (mod\ 3) \text{ and } n \geq 4, \\ K_{1,4}, & \text{if} \quad n = 5\ , \\ \frac{n-8}{3}P_3 \cup 2K_{1,3}, & \text{if} \quad n \equiv 2 \ (mod\ 3) \text{ and } n \geq 8. \end{cases}$$

Keywords: tree; forest; independent set; k-component independent set
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1. Introduction

Let $G = (V(G), E(G))$ be a graph. A set $S \subseteq V(G)$ is called an independent set of G if no two vertices of S are adjacent in G. A maximal independent set is an independent set that is not a proper subset of any other independent set. Let k be a positive integer. We call S a k-component independent set of G if each component of $G[S]$ has order at most k. Clearly, the 1-component independent sets are the usual independent sets. A k-component independent set is maximal (maximum) if the set cannot be extended to a larger k-component independent set (if no k-component independent set of G has larger cardinality). A maximal k-component independent set of a graph G is denoted briefly by Mk-CIS. We use $t_k(G)$ to denote the number of Mk-CISs of G.

In 1986, Wilf [12] proved that the maximum number of maximal independent sets for a tree of order n is $2^{\frac{n-1}{2}}$ if n is odd and $2^{\frac{n}{2}-1}+1$ if $n \geq 2$ is even. In 1988, Sagan [9] gave a simple graph-theoretical proof and characterized all extremal trees. In 1991, Zito [15] determined that the maximum number of maximum independent sets for a tree of order n is $2^{\frac{n-3}{2}}$ if $n > 1$ is odd and $2^{\frac{n-2}{2}}+1$ if n is even, and also characterized all extremal trees. In 1993, Hujter and Tuza [4] proved that the maximal number of maximal independent sets in triangle-free graphs is at most $2^{\frac{n}{2}}$ if $n \geq 4$ is even and $5 \cdot 2^{\frac{n-5}{2}}$ if $n \geq 5$ is odd, and characterized the extremal graphs. The number of the maximal independent sets on some classes of graphs were also studied in [5, 6, 10, 13].

In 2021, Tu, Zhang and Shi [11] showed that the maximum number of maximum 2-component independent sets in a tree of order n is $3^{\frac{n}{3}-1}+\frac{n}{3}+1$ if $n \equiv 0 \pmod 3$, $3^{\frac{n-1}{3}-1}+1$ if $n \equiv 1 \pmod 3$, and $3^{\frac{n-2}{3}-1}$ if $n \equiv 2 \pmod 3$, and also characterized the extremal graphs.

In 1981, Yannakakis [14] proved that the problem of computing the number of maximum 2-component independent sets for bipartite graphs is NP-complete. The complexity of the problem on some special families of graphs were studied in [1, 2, 7, 8].

In this paper, we establish a sharp upper bound for $t_2(G)$ of a forest G of order n and characterize all forests achieving the upper bound.

2. The main result

Let G be a graph and v a vertex in G. The neighborhood $N_G(v)$ is the set of vertices adjacent to v and the closed neighborhood $N_G[v]$ is $N_G(v) \cup \{v\}$. In the sequel, we use $t(G)$ to present $t_2(G)$ for simplicity and S denotes an M2-CIS of a tree T under consideration.

Theorem 2.1. *For any forest F of order $n \geq 3$, $t(F) \leq f(n)$, where*

$$f(n) = \begin{cases} 3^{\frac{n}{3}}, & \text{if} \quad n \equiv 0 \ (mod\ 3) \text{ and } n \geq 3, \\ 4 \cdot 3^{\frac{n-4}{3}}, & \text{if} \quad n \equiv 1 \ (mod\ 3) \text{ and } n \geq 4\ , \\ 5, & \text{if} \quad n = 5, \\ 4^2 \cdot 3^{\frac{n-8}{3}}, & \text{if} \quad n \equiv 2 \ (mod\ 3) \text{ and } n \geq 8, \end{cases}$$

with equality if and only if

$$F_n \cong \begin{cases} \frac{n}{3} P_3, & \text{if} \quad n \equiv 0 \ (mod\ 3) \text{ and } n \geq 3, \\ \frac{n-4}{3} P_3 \cup K_{1,3}, & \text{if} \quad n \equiv 1 \ (mod\ 3) \text{ and } n \geq 4, \\ K_{1,4}, & \text{if} \quad n = 5, \\ \frac{n-8}{3} P_3 \cup 2K_{1,3}, & \text{if} \quad n \equiv 2 \ (mod\ 3) \text{ and } n \geq 8. \end{cases}$$

Lemma 2.2. *(Cheng, Wu [3]) Let n and k be two integers with $n \geq k+1 \geq 2$. For any tree T of order n, there exists a vertex v such that $T - v$ has $d(v) - 1$ components, each of which has order at most k, but the sum of their order is at least k. In particular, every nontrivial tree T has a vertex v such that all its neighbors but one are leaves.*

Lemma 2.3. *For any positive integer $n \geq 1$,*

$$t(K_{1,n-1}) = \begin{cases} 1, & \text{if} \quad 1 \leq n \leq 2, \\ n, & \text{if} \quad n \geq 3. \end{cases}$$

We define five special trees, denoted by T_i for each $i \in \{1, \ldots, 5\}$:

T_1 is a tree of order n obtained from $K_{1,3}$ by subdividing an edge of $K_{1,3}$ $n-4$ times, where $5 \leq n \leq 9$.

T_2 is obtained from $2P_4 \cup P_3$ by adding edges connecting a leaf of each copy of P_4 to a leaf x of P_3.

T_3 is obtained from $(K_{1,3} \cup P_4) \cup P_3$ by adding edges connecting a leaf of $K_{1,3}$ and P_4 to a leaf x of P_3.

T_4 is obtained from $aK_{1,3} \cup P_3$ by adding edges connecting a leaf of each copy of $K_{1,3}$ to a leaf x of P_3 for an integer $a \geq 2$.

T_5 is obtained from $bK_{1,3}$ by adding edges connecting a leaf of each copy of $K_{1,3}$ to a fixed vertex x for an integer $b \geq 2$, as shown in Figure 1.

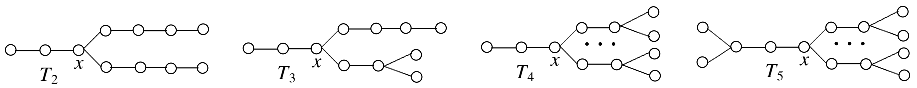

Figure 1. T_i, $i \in \{2, \ldots, 5\}$.

Lemma 2.4. *$t(T_i) \leq t(F_n)$ for each $i \in \{1, \ldots, 5\}$.*

Proof. By a straightforward calculation,

$$t(T_1) = \begin{cases} 4 < 5 = t(K_{1,4}) = t(F_5), & \text{if} \quad n = 5, \\ 6 < 3^2 = t(2P_3) = t(F_6), & \text{if} \quad n = 6, \\ 10 < 4 \cdot 3 = t(P_3 \cup K_{1,3}) = t(F_7), & \text{if} \quad n = 7, \\ 13 < 4^2 = t(2K_{1,3}) = t(F_8), & \text{if} \quad n = 8, \\ 17 < 3^3 = t(3P_3) = t(F_9), & \text{if} \quad n = 9. \end{cases}$$

Obviously, $|V(T_2)| = |V(T_3)| = 11$, $t(T_2) = 28 < 4^2 \cdot 3 = t(2K_{1,3} \cup P_3) = t(F_{11})$, and $t(T_3) = 31 < 4^2 \cdot 3 = t(2K_{1,3} \cup P_3) = t(F_{11})$.

Note that $|V(T_4)| = 4a + 3$. Observe that for an M2-CIS S of T_4, either $x \notin S$ or $x \in S$ with $d_{T[S]}(x) \leq 1$. Let us define $t_x^0 = |\{S : d_{T[S]}(x) = 0\}| = 2^a$, $t_x^1 = |\{S : d_{T[S]}(x) = 1\}| = (a+3) \cdot 3^{a-1}$, $t_{\bar{x}} = |\{S : x \notin S\}| = 4^a$. Thus, $t(T_4) = t_x^0 + t_x^1 + t_{\bar{x}} = 4^a + (a+3) \cdot 3^{a-1} + 2^a$. We consider three cases in terms of the modularity of a (mod 3).

If $a = 3s$, $s \geq 1$, then $|V(T_4)| = 12s + 3$ and $t(F_{12s+3}) = 3^{4s+1}$. Moreover, since $4^{3s} \leq 3^{4s}$ and $(s+1) \cdot 3^{3s} + 2^{3s} \leq 2 \cdot 3^{4s}$ for any $s \geq 1$, it follows that for any $s \geq 1$,

$$\begin{aligned} t(T_4) &= 4^{3s} + (s+1) \cdot 3^{3s} + 2^{3s} \leq 3^{4s} + 2 \cdot 3^{4s} \\ &= 3^{4s+1} = t(F_{12s+3}). \end{aligned}$$

If $a = 3s + 1$, $s \geq 1$, then $|V(T_4)| = 12s + 7$ and $t(F_{12s+7}) = 4 \cdot 3^{4s+1}$. Moreover, since $4^{3s+1} \leq 4 \cdot 3^{4s}$ and $(3s+4) \cdot 3^{3s} + 2^{3s+1} \leq 8 \cdot 3^{4s}$ for any $s \geq 1$, it follows that for any $s \geq 1$,

$$\begin{aligned} t(T_4) &= 4^{3s+1} + (3s+4) \cdot 3^{3s} + 2^{3s+1} \\ &\leq 4 \cdot 3^{4s} + 8 \cdot 3^{4s} = 4 \cdot 3^{4s+1} = t(F_{12s+7}). \end{aligned}$$

If $a = 3s+2$, $s \geq 0$, then $|V(T_4)| = 12s+11$ and $t(F_{12s+11}) = 4^2 \cdot 3^{4s+1}$. Moreover, since $4^{3s+2} \leq 4^2 \cdot 3^{4s}$ and $(3s+5) \cdot 3^{3s+1} + 2^{3s+2} \leq 32 \cdot 3^{4s}$ for any $s \geq 0$, it follows that for any $s \geq 0$,

$$\begin{aligned} t(T_4) &= 4^{3s+2} + (3s+5) \cdot 3^{3s+1} + 2^{3s+2} \\ &\leq 4^2 \cdot 3^{4s} + 32 \cdot 3^{4s} = 4^2 \cdot 3^{4s+1} = t(F_{12s+11}). \end{aligned}$$

Note that $|V(T_5)| = 4b + 1$ and $t(T_5) = 4^b + b \cdot 3^{b-1} - b \cdot 2^{b-1}$. We consider three cases in terms of the modularity of b (mod 3).

If $b = 3s$, $s \geq 1$, then $|V(T_5)| = 12s + 1$ and $t(F_{12s+1}) = 4 \cdot 3^{4s-1}$. Moreover, since $4^{3s} \leq 3^{4s}$ and $s \cdot 3^{3s} \leq 3^{4s-1}$ for any $s \geq 1$, it follows that for any $s \geq 1$,

$$\begin{aligned} t(T_5) &= 4^{3s} + s \cdot 3^{3s} - 3s \cdot 2^{3s-1} \leq 3^{4s} + 3^{4s-1} \\ &= 4 \cdot 3^{4s-1} = t(F_{12s+1}). \end{aligned}$$

If $b = 3s + 1$, $s \geq 1$, then $|V(T_5)| = 12s + 5$ and $t(F_{12s+5}) = 4^2 \cdot 3^{4s-1}$. Moreover, since $4^{3s+1} \leq 4 \cdot 3^{4s}$ and $(3s+1) \cdot 3^{3s} \leq 4 \cdot 3^{4s-1}$ for any $s \geq 1$, it follows that for any $s \geq 1$,

$$\begin{aligned} t(T_5) &= 4^{3s+1} + (3s+1) \cdot 3^{3s} - (3s+1) \cdot 2^{3s} \\ &\leq 4 \cdot 3^{4s} + 4 \cdot 3^{4s-1} = 4^2 \cdot 3^{4s-1} = t(F_{12s+5}). \end{aligned}$$

If $b = 3s + 2$, $s \geq 0$, then $|V(T_5)| = 12s + 9$ and $t(F_{12s+9}) = 3^{4s+3}$. Moreover, since $4^{3s+2} \leq 4^2 \cdot 3^{4s}$ and $(3s+2) \cdot 3^{3s+1} \leq 11 \cdot 3^{4s}$ for any $s \geq 0$, it follows that for any $s \geq 0$,

$$\begin{aligned} t(T_5) &= 4^{3s+2} + (3s+2) \cdot 3^{3s+1} - (3s+2) \cdot 2^{3s+1} \\ &\leq 4^2 \cdot 3^{4s} + 11 \cdot 3^{4s} = 3^{4s+3} = t(F_{12s+9}). \end{aligned}$$

□

3. The proof of Theorem 2.1

In this section, we give the proof of Theorem 2.1.

Proof. Let F be a forest of order n. It is straightforward to check that the result is true if $n \leq 5$. We proceed with the induction on the order n of F. If $F \cong K_{1,n-1}$, then by Lemma 2.3, the result trivially holds. Next we assume that F is not a star. By Lemma 2.2, for a tree T, there exists a vertex x with $d(x) - 1$ neighbors being leaves. Let $N(x) = \{x_1, \ldots, x_{d(x)-1}, y\}$, where y is the neighbor of x which is not a leaf of T, as shown in Figure 2.

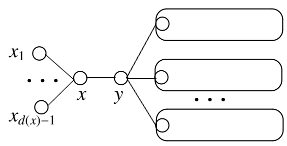

Figure 2. T.

Claim 1. If $d(x) \geq 6$, then $t(T) \leq t(F_n)$.

Proof. Let T_x and T_y be two components of $T - xy$ containing x and y respectively. Then $|V(T_x)| = d(x)$. Observe that for an M2-CIS S of T, either $x \notin S$ or $x \in S$ with $d_{T[S]}(x) = 1$. Let us define

$t_{\bar{x}} = |\{S : x \notin S\}|$,
$t_x^1 = |\{S : d_{T[S]}(x) = 1\}|$
$= |\{S : d_{T[S]}(x) = 1, \{x, y\} \subseteq S\}| + |\{S : d_{T[S]}(x) = 1, \{x, x_i\} \subseteq S,$
$i \in \{1, \ldots, d(x) - 1\}\}|$.

Thus, $t(T) = t_{\bar{x}} + t_x^1$. Since $t_{\bar{x}} = t(T_y)$ and $t_x^1 \leq d(x) \cdot t(T_y)$, we have

$$t(T) \leq (d(x) + 1) \cdot t(T_y). \tag{3.1}$$

Let $V(T_x') = V(T_x)$. We consider three cases in terms of the modularity of $d(x)$ (mod 3).

Case 1. $d(x) = 3s$, $s \geq 2$

Let $T_x' = sP_3$. Then $t(T_x') = 3^s$. By (3.1), it follows that for any $s \geq 2$,

$$t(T) \leq (3s + 1) \cdot t(T_y) \leq 3^s \cdot t(T_y).$$

By the induction hypothesis, $t(T_y) \leq t(F_{n-d(x)})$. Hence, $t(T) \leq t(F_n)$.

Case 2. $d(x) = 3s + 1$, $s \geq 2$

Let $T_x' = (s - 1)P_3 \cup K_{1,3}$. Then $t(T_x') = 4 \cdot 3^{s-1}$. By (3.1), it follows that for any $s \geq 2$,

$$t(T) \leq (3s + 2) \cdot t(T_y) \leq 4 \cdot 3^{s-1} \cdot t(T_y).$$

By the induction hypothesis, $t(T_y) \leq t(F_{n-d(x)})$. Hence, $t(T) \leq t(F_n)$.

Case 3. $d(x) = 3s + 2$, $s \geq 2$

Let $T_x' = (s - 2)P_3 \cup 2K_{1,3}$. Then $t(T_x') = 4^2 \cdot 3^{s-2}$. By (3.1), it follows that for any $s \geq 2$,

$$t(T) \leq (3s + 3) \cdot t(T_y) \leq 4^2 \cdot 3^{s-2} \cdot t(T_y).$$

By the induction hypothesis, $t(T_y) \leq t(F_{n-d(x)})$. Hence, $t(T) \leq t(F_n)$. □

Claim 2. If $d(x) = 4$ or 5, then $t(T) \leq t(F_n)$.

Proof. The meanings of notations here are same as those adopted in Claim 1. Let $F_1 = T - (N[x] \setminus \{y\})$, $F_2 = T - N(x) - N(y)$ and $F_3 = T - N[x]$. Combining these observations with the definition of F_i, we get that $t_{\bar{x}} = t(F_1)$, $t_x^1 = t(F_2) + (d(x) - 1) \cdot t(F_3)$. Since $t(T) = t_{\bar{x}} + t_x^1$, we have

$$t(T) = t(F_1) + t(F_2) + (d(x) - 1) \cdot t(F_3). \tag{3.2}$$

Let n_i be the order of F_i for each $i \in \{1, 2, 3\}$. Then $n_1 = n - d(x)$, $n_2 = n - d(x) - d(y)$ and $n_3 = n - d(x) - 1$. We consider three cases in terms of the modularity of n (mod 3).

Case 1. $n = 3s$, $s \geq 2$

Subcase 1.1. $d(y) = 3l$, $l \geq 1$

If $d(x) = 4$, then $n_1 = 3(s-2)+2$, $n_2 = 3(s-l-2)+2$ and $n_3 = 3(s-2)+1$. By the induction hypothesis, $t(F_1) \leq 4^2 \cdot 3^{s-4}$, $t(F_2) \leq 4^2 \cdot 3^{s-l-4}$ and $t(F_3) \leq 4 \cdot 3^{s-3}$. Moreover, since $\frac{4^2}{3^l} + 52 \leq 3^4$ for any $l \geq 1$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) = (4^2 + \frac{4^2}{3^l} + 4 \cdot 3^2) \cdot 3^{s-4} = (\frac{4^2}{3^l} + 52) \cdot 3^{s-4} \leq 3^s = t(F_n).$$

If $d(x) = 5$, then $n_1 = 3(s-2)+1$, $n_2 = 3(s-l-2)+1$ and $n_3 = 3(s-2)$. By the induction hypothesis, $t(F_1) \leq 4 \cdot 3^{s-3}$, $t(F_2) \leq 4 \cdot 3^{s-l-3}$ and $t(F_3) \leq 3^{s-2}$. Moreover, since $\frac{4}{3^l} + 16 \leq 3^3$ for any $l \geq 1$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) = (4 + \frac{4}{3^l} + 12) \cdot 3^{s-3} = (\frac{4}{3^l} + 16) \cdot 3^{s-3} \leq 3^s = t(F_n).$$

Subcase 1.2. $d(y) = 3l+1, l \geq 1$

If $d(x) = 4$, then $n_2 = 3(s-l-2)+1$. By the induction hypothesis, $t(F_2) \leq 4 \cdot 3^{s-l-3}$. Moreover, since $\frac{12}{3^l} + 52 \leq 3^4$ for any $l \geq 1$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) = (4^2 + \frac{12}{3^l} + 4 \cdot 3^2) \cdot 3^{s-4} = (\frac{12}{3^l} + 52) \cdot 3^{s-4} \leq 3^s = t(F_n).$$

If $d(x) = 5$, then $n_2 = 3(s-l-2)$. By the induction hypothesis, $t(F_2) \leq 3^{s-l-2}$. Moreover, since $\frac{3}{3^l} + 16 \leq 3^3$ for any $l \geq 1$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) = (4 + \frac{3}{3^l} + 12) \cdot 3^{s-3} = (\frac{3}{3^l} + 16) \cdot 3^{s-3} \leq 3^s = t(F_n).$$

Subcase 1.3. $d(y) = 3l+2, l \geq 0$

If $d(x) = 4$, then $n_2 = 3(s-l-2)$. By the induction hypothesis, $t(F_2) \leq 3^{s-l-2}$. Moreover, since $\frac{9}{3^l} + 52 \leq 3^4$ for any $l \geq 0$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 0$,

$$t(T) = (4^2 + \frac{9}{3^l} + 4 \cdot 3^2) \cdot 3^{s-4} = (\frac{9}{3^l} + 52) \cdot 3^{s-4} \leq 3^s = t(F_n).$$

If $d(x) = 5$, then $n_2 = 3(s-l-3)+2$. By the induction hypothesis, $t(F_2) \leq 4^2 \cdot 3^{s-l-5}$. Moreover, since $\frac{4^2}{3^l} + 144 \leq 3^5$ for any $l \geq 0$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 0$,

$$t(T) = (4 \cdot 3^2 + \frac{4^2}{3^l} + 4 \cdot 3^3) \cdot 3^{s-5} = (\frac{4^2}{3^l} + 144) \cdot 3^{s-5} \leq 3^s = t(F_n).$$

Case 2. $n = 3s+1, s \geq 2$

Subcase 2.1. $d(y) = 3l, l \geq 1$

If $d(x) = 4$, then $n_1 = 3(s-1)$, $n_2 = 3(s-l-1)$ and $n_3 = 3(s-2)+2$. By the induction hypothesis, $t(F_1) \leq 3^{s-1}$, $t(F_2) \leq 3^{s-l-1}$ and $t(F_3) \leq 4^2 \cdot 3^{s-4}$. Moreover, since $\frac{9}{3^l} + 25 \leq 4 \cdot 3^2$ for any $l \geq 1$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) = (3^2 + \frac{9}{3^l} + 4^2) \cdot 3^{s-3} = (\frac{9}{3^l} + 25) \cdot 3^{s-3} \leq 4 \cdot 3^{s-1} = t(F_n).$$

If $d(x) = 5$, then $n_1 = 3(s-2)+2$, $n_2 = 3(s-l-2)+2$ and $n_3 = 3(s-2)+1$. By the induction hypothesis, $t(F_1) \le 4^2 \cdot 3^{s-4}$, $t(F_2) \le 4^2 \cdot 3^{s-l-4}$ and $t(F_3) \le 4 \cdot 3^{s-3}$. Moreover, since $\frac{4^2}{3^l} + 64 \le 4 \cdot 3^3$ for any $l \ge 1$, by (3.2), it follows that for any $s \ge 2$ and $l \ge 1$,

$$t(T) = (4^2 + \frac{4^2}{3^l} + 4^2 \cdot 3) \cdot 3^{s-4} = (\frac{4^2}{3^l} + 64) \cdot 3^{s-4} \le 4 \cdot 3^{s-1} = t(F_n).$$

Subcase 2.2. $d(y) = 3l+1, l \ge 1$

If $d(x) = 4$, then $n_2 = 3(s-l-2)+2$. By the induction hypothesis, $t(F_2) \le 4^2 \cdot 3^{s-l-4}$. Moreover, since $\frac{4^2}{3^{l+1}} + 25 \le 4 \cdot 3^2$ for any $l \ge 1$, by (3.2), it follows that for any $s \ge 2$ and $l \ge 1$,

$$t(T) = (3^2 + \frac{4^2}{3^{l+1}} + 4^2) \cdot 3^{s-3} = (\frac{4^2}{3^{l+1}} + 25) \cdot 3^{s-3} \le 4 \cdot 3^{s-1} = t(F_n).$$

If $d(x) = 5$, then $n_2 = 3(s-l-2)+1$. By the induction hypothesis, $t(F_2) \le 4 \cdot 3^{s-l-3}$. Moreover, since $\frac{12}{3^l} + 64 \le 4 \cdot 3^3$ for any $l \ge 1$, by (3.2), it follows that for any $s \ge 2$ and $l \ge 1$,

$$t(T) = (4^2 + \frac{12}{3^l} + 4^2 \cdot 3) \cdot 3^{s-4} = (\frac{12}{3^l} + 64) \cdot 3^{s-4} \le 4 \cdot 3^{s-1} = t(F_n).$$

Subcase 2.3. $d(y) = 3l+2, l \ge 0$

If $d(x) = 4$, then $n_2 = 3(s-l-2)+1$. By the induction hypothesis, $t(F_2) \le 4 \cdot 3^{s-l-3}$. Moreover, since $\frac{4}{3^l} + 25 \le 4 \cdot 3^2$ for any $l \ge 0$, by (3.2), it follows that for any $s \ge 2$ and $l \ge 0$,

$$t(T) = (3^2 + \frac{4}{3^l} + 4^2) \cdot 3^{s-3} = (\frac{4}{3^l} + 25) \cdot 3^{s-3} \le 4 \cdot 3^{s-1} = t(F_n).$$

If $d(x) = 5$, then $n_2 = 3(s-l-2)$. By the induction hypothesis, $t(F_2) \le 3^{s-l-2}$. Moreover, since $\frac{3^2}{3^l} + 64 \le 4 \cdot 3^2$ for any $l \ge 0$, by (3.2), it follows that for any $s \ge 2$ and $l \ge 0$,

$$t(T) = (4^2 + \frac{3^2}{3^l} + 4^2 \cdot 3) \cdot 3^{s-4} = (\frac{3^2}{3^l} + 64) \cdot 3^{s-4} \le 4 \cdot 3^{s-1} = t(F_n).$$

Case 3. $n = 3s+2, s \ge 2$

Subcase 3.1. $d(y) = 3l, l \ge 1$

If $d(x) = 4$, then $n_1 = 3(s-1)+1$, $n_2 = 3(s-l-1)+1$ and $n_3 = 3(s-1)$. By the induction hypothesis, $t(F_1) \le 4 \cdot 3^{s-2}$, $t(F_2) \le 4 \cdot 3^{s-l-2}$ and $t(F_3) \le 3^{s-1}$. Moreover, since $\frac{4}{3^l} + 13 \le 4^2$ for any $l \ge 1$, by (3.2), it follows that for any $s \ge 2$ and $l \ge 1$,

$$t(T) = (4 + \frac{4}{3^l} + 3^2) \cdot 3^{s-2} = (\frac{4}{3^l} + 13) \cdot 3^{s-2} \le 4^2 \cdot 3^{s-2} = t(F_n).$$

If $d(x) = 5$, then $n_1 = 3(s-1)$, $n_2 = 3(s-l-1)$ and $n_3 = 3(s-2)+2$. By the induction hypothesis, $t(F_1) \le 3^{s-1}$, $t(F_2) \le 3^{s-l-1}$ and $t(F_3) \le 4^2 \cdot 3^{s-4}$. Moreover, since $\frac{3^3}{3^l} + 91 \le 4^2 \cdot 3^2$ for any $l \ge 1$, by (3.2), it follows that for any $s \ge 2$ and $l \ge 1$,

$$t(T) = (3^3 + \frac{3^3}{3^l} + 4^3) \cdot 3^{s-4} = (\frac{3^3}{3^l} + 91) \cdot 3^{s-4} \le 4^2 \cdot 3^{s-2} = t(F_n).$$

Subcase 3.2. $d(y) = 3l + 1, l \geq 1$

If $d(x) = 4$, then $n_2 = 3(s - l - 1)$. By the induction hypothesis, $t(F_2) \leq 3^{s-l-1}$. Moreover, since $\frac{3}{3^l} + 13 \leq 4^2$ for any $l \geq 1$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) = (4 + \frac{3}{3^l} + 3^2) \cdot 3^{s-2} = (\frac{3}{3^l} + 13) \cdot 3^{s-2} \leq 4^2 \cdot 3^{s-2} = t(F_n).$$

If $d(x) = 5$, then $n_2 = 3(s - l - 2) + 2$. By the induction hypothesis, $t(F_2) \leq 4^2 \cdot 3^{s-l-4}$. Moreover, since $\frac{4^2}{3^l} + 91 \leq 4^2 \cdot 3^2$ for any $l \geq 1$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) = (3^3 + \frac{4^2}{3^l} + 4^3) \cdot 3^{s-4} = (\frac{4^2}{3^l} + 91) \cdot 3^{s-4} \leq 4^2 \cdot 3^{s-2} = t(F_n).$$

Subcase 3.3. $d(y) = 3l + 2, l \geq 0$

If $d(x) = 4$, then $n_2 = 3(s - l - 2) + 2$. By the induction hypothesis, $t(F_2) \leq 4^2 \cdot 3^{s-l-4}$. Moreover, since $\frac{4^2}{3^{l+2}} + 13 \leq 4^2$ for any $l \geq 0$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 0$,

$$t(T) = (4 + \frac{4^2}{3^{l+2}} + 3^2) \cdot 3^{s-2} = (\frac{4^2}{3^{l+2}} + 13) \cdot 3^{s-2} \leq 4^2 \cdot 3^{s-2} = t(F_n).$$

If $d(x) = 5$, then $n_2 = 3(s - l - 2) + 1$. By the induction hypothesis, $t(F_2) \leq 4 \cdot 3^{s-l-3}$. Moreover, since $\frac{12}{3^l} + 91 \leq 4^2 \cdot 3^2$ for any $l \geq 0$, by (3.2), it follows that for any $s \geq 2$ and $l \geq 0$,

$$t(T) = (3^3 + \frac{12}{3^l} + 4^3) \cdot 3^{s-4} = (\frac{12}{3^l} + 91) \cdot 3^{s-4} \leq 4^2 \cdot 3^{s-2} = t(F_n).$$

□

Claim 3. $t(T) \leq t(F_n)$ if one of the following conditions holds:

(1) $n \geq 6$, $n \equiv 0$ or $1 \pmod 3$, $d(x) = 3$;

(2) $n \geq 8$, $n \equiv 2 \pmod 3$, $d(x) = 3$, $d(y) \geq 3$, where $y \in N(x)$.

Proof. The meanings of notations here are same as those adopted in Claims 1 and 2. By (3.2), we have

$$t(T) = t(F_1) + t(F_2) + 2t(F_3). \tag{3.3}$$

Let n_i be the order of F_i for each $i \in \{1, 2, 3\}$. We consider three cases in terms of the modularity of $n \pmod 3$.

Case 1. $n = 3s, s \geq 2$.

$n_1 = 3(s - 1)$, $n_3 = 3(s - 2) + 2$. By the induction hypothesis, $t(F_1) \leq 3^{s-1}$ and $t(F_3) \leq 4^2 \cdot 3^{s-4}$.

Subcase 1.1. $d(y) = 3l, l \geq 1$

Since $n_2 = 3(s - l - 1)$, by the induction hypothesis, $t(F_2) \leq 3^{s-l-1}$. Moreover, since $\frac{3^3}{3^l} + 59 \leq 3^4$ for any $l \geq 1$, by (3.3), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (3^3 + \frac{3^3}{3^l} + 2 \cdot 4^2) \cdot 3^{s-4} = (\frac{3^3}{3^l} + 59) \cdot 3^{s-4} \leq 3^s = t(F_n).$$

Subcase 1.2. $d(y) = 3l + 1, l \geq 1$

Since $n_2 = 3(s-l-2)+2$, by the induction hypothesis, $t(F_2) \leq 4^2 \cdot 3^{s-l-4}$. Moreover, since $\frac{4^2}{3^l} + 59 \leq 3^4$ for any $l \geq 1$, by (3.3), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (3^3 + \frac{4^2}{3^l} + 2 \cdot 4^2) \cdot 3^{s-4} = (\frac{4^2}{3^l} + 59) \cdot 3^{s-4} \leq 3^s = t(F_n).$$

Subcase 1.3. $d(y) = 3l+2$, $l \geq 0$

Since $n_2 = 3(s-l-2)+1$, by the induction hypothesis, $t(F_2) \leq 4 \cdot 3^{s-l-3}$. Moreover, since $\frac{12}{3^l} + 59 \leq 3^4$ for any $l \geq 0$, by (3.3), it follows that for any $s \geq 2$ and $l \geq 0$,

$$t(T) \leq (3^3 + \frac{12}{3^l} + 2 \cdot 4^2) \cdot 3^{s-4} = (\frac{12}{3^l} + 59) \cdot 3^{s-4} \leq 3^s = t(F_n).$$

Case 2. $n = 3s+1$, $s \geq 2$

By the definition of n_i, $n_1 = 3(s-1)+1$, $n_3 = 3(s-1)$. By the induction hypothesis, $t(F_1) \leq 4 \cdot 3^{s-2}$ and $t(F_3) \leq 3^{s-1}$.

Subcase 2.1. $d(y) = 3l$, $l \geq 1$

Since $n_2 = 3(s-l-1)+1$, by the induction hypothesis, $t(F_2) \leq 4 \cdot 3^{s-l-2}$. Moreover, since $\frac{4}{3^l} + 10 \leq 4 \cdot 3$ for any $l \geq 1$, by (3.3), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (4 + \frac{4}{3^l} + 2 \cdot 3) \cdot 3^{s-2} = (\frac{4}{3^l} + 10) \cdot 3^{s-2} \leq 4 \cdot 3^{s-1} = t(F_n).$$

Subcase 2.2. $d(y) = 3l+1$, $l \geq 1$

Since $n_2 = 3(s-l-1)$, by the induction hypothesis, $t(F_2) \leq 3^{s-l-1}$. Moreover, since $\frac{3}{3^l} + 10 \leq 4 \cdot 3$ for any $l \geq 1$, by (3.3), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (4 + \frac{3}{3^l} + 2 \cdot 3) \cdot 3^{s-2} = (\frac{3}{3^l} + 10) \cdot 3^{s-2} \leq 4 \cdot 3^{s-1} = t(F_n).$$

Subcase 2.3. $d(y) = 3l+2$, $l \geq 0$

Since $n_2 = 3(s-l-2)+2$, by the induction hypothesis, $t(F_2) \leq 4^2 \cdot 3^{s-l-4}$. Moreover, since $\frac{4^2}{3^l} + 90 \leq 4 \cdot 3^3$ for any $l \geq 0$, by (3.3), it follows that for any $s \geq 2$ and $l \geq 0$,

$$t(T) \leq (4 \cdot 3^2 + \frac{4^2}{3^l} + 2 \cdot 3^3) \cdot 3^{s-4} = (\frac{4^2}{3^l} + 90) \cdot 3^{s-4} \leq 4 \cdot 3^{s-1} = t(F_n).$$

Case 3. $n = 3s+2$, $s \geq 2$

By the definition of n_i, $n_1 = 3(s-1)+2$, $n_3 = 3(s-1)+1$. By the induction hypothesis, $t(F_1) \leq 4^2 \cdot 3^{s-3}$ and $t(F_3) \leq 4 \cdot 3^{s-2}$.

Subcase 3.1. $d(y) = 3l$, $l \geq 1$

Since $n_2 = 3(s-l-1)+2$, by the induction hypothesis, $t(F_2) \leq 4^2 \cdot 3^{s-l-3}$. Moreover, since $\frac{4^2}{3^l} + 40 \leq 4^2 \cdot 3$ for any $l \geq 1$, by (3.3), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (4^2 + \frac{4^2}{3^l} + 8 \cdot 3) \cdot 3^{s-3} = (\frac{4^2}{3^l} + 40) \cdot 3^{s-3} \leq 4^2 \cdot 3^{s-2} = t(F_n).$$

Subcase 3.2. $d(y) = 3l + 1, l \geq 1$

Since $n_2 = 3(s - l - 1) + 1$, by the induction hypothesis, $t(F_2) \leq 4 \cdot 3^{s-l-2}$. Moreover, since $\frac{12}{3^l} + 40 \leq 4^2 \cdot 3$ for any $l \geq 1$, by (3.3), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (4^2 + \frac{12}{3^l} + 8 \cdot 3) \cdot 3^{s-3} = (\frac{12}{3^l} + 40) \cdot 3^{s-3} \leq 4^2 \cdot 3^{s-2} = t(F_n).$$

Subcase 3.3. $d(y) = 3l + 2, l \geq 1$

Since $n_2 = 3(s - l - 1)$, by the induction hypothesis, $t(F_2) \leq 3^{s-l-1}$. Moreover, since $\frac{3^2}{3^l} + 40 \leq 4^2 \cdot 3$ for any $l \geq 1$, by (3.3), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (4^2 + \frac{3^2}{3^l} + 8 \cdot 3) \cdot 3^{s-3} = (\frac{3^2}{3^l} + 40) \cdot 3^{s-3} \leq 4^2 \cdot 3^{s-2} = t(F_n).$$

□

In view of Claim 3, we consider the case that $d(y) = 2$ and $n = 3s + 2$ where $s \geq 2$.

Claim 4. Assume that $d(y) = 2$ for the remaining neighbor y of x and $d(z) \geq 1$ for the neighbor of y other than x, as shown in Figure 3. If $T - z$ has an isolated vertex or an isolated edge, then $t(T) \leq t(F_n)$.

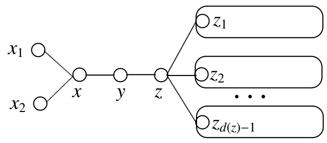

Figure 3. T.

Proof. Let T_x and T_y be two components of $T - xy$ containing x and y respectively.

Case 1. $T - z$ has exactly an isolated vertex.

By Lemma 2.4, we distinguish two subcases in terms of $d(z) \geq 3$.

Subcse 1.1. $d(z) = 3$

Observe that for an M2-CIS S' of T_y, either $y \notin S'$ or $y \in S'$ with $d_{T[S']}(y) \leq 1$. Let us define $\tilde{t}_y^0 = |\{S' : d_{T[S']}(y) = 0\}|$, $\tilde{t}_y^1 = |\{S' : d_{T[S']}(y) = 1\}|$, $\tilde{t}_{\bar{y}} = |\{S' : y \notin S'\}|$. Thus, $t(T_y) = \tilde{t}_y^0 + \tilde{t}_y^1 + \tilde{t}_{\bar{y}}$.

Observe that for an M2-CIS S of T, either $y \notin S$ or $y \in S$ with $d_{T[S]}(y) \leq 1$. Let

$t_y^0 = |\{S : d_{T[S]}(y) = 0\}| = \tilde{t}_y^0,$

$t_y^1 = |\{S : d_{T[S]}(y) = 1\}|$

$= |\{S : d_{T[S]}(y) = 1, \{x, y\} \subseteq S\}| + |\{S : d_{T[S]}(y) = 1, \{y, z\} \subseteq S\}|$

$= \tilde{t}_y^0 + \tilde{t}_y^1.$

$t_{\bar{y}} = |\{S : y \notin S\}| = |\{S : y \notin S, x \in S\}| + |\{S : y \notin S, x \notin S\}|$

$\leq (\tilde{t}_y^0 + 2\tilde{t}_y^1 + 2\tilde{t}_{\bar{y}}) + \tilde{t}_{\bar{y}} = \tilde{t}_y^0 + 2\tilde{t}_y^1 + 3\tilde{t}_{\bar{y}}.$

Clearly, $t(T) = t_y^0 + t_y^1 + t_{\bar{y}} \leq 3\tilde{t}_y^0 + 3\tilde{t}_y^1 + 3\tilde{t}_{\bar{y}}$. Since $t(T_y) = \tilde{t}_y^0 + \tilde{t}_y^1 + \tilde{t}_{\bar{y}}$, $t(T) \leq 3t(T_y) = t(T_x) \cdot t(T_y)$. By the induction hypothesis, $t(T_y) \leq t(F_{n-3})$. Hence, $t(T) \leq t(T_x) \cdot t(F_{n-3}) \leq t(F_n)$.

Subcase 1.2. $d(z) \geq 4$

Let $T - yz = T_y \cup T_z$ where $z \in T_z$. Observe that for an M2-CIS S' of T_z, either $z \notin S'$ or $z \in S'$ with $d_{T[S']}(z) = 1$. Let us define $t_{\bar{z}} = |\{S' : z \notin S'\}|$ and $t_z^1 = |\{S' : d_{T[S']}(z) = 1\}|$. Thus, $t(T_z) = t_{\bar{z}} + t_z^1$.

The meanings of notations here are same as those adopted in Subcase 1.1. Note that $t_y^0 + t_y^1 \leq t_z^1 + 2t_{\bar{z}}$ and $t_{\bar{y}} = 2t(T_z) + t_z^1$. Thus, $t(T) \leq 2t(T_z) + 2(t_z^1 + t_{\bar{z}})$. Since $t(T_z) = t_{\bar{z}} + t_z^1$, $t(T) \leq 4t(T_z) = t(T_y) \cdot t(T_z)$. By the induction hypothesis, $t(T_z) \leq t(F_{n-4})$. Hence, $t(T) \leq t(T_y) \cdot t(F_{n-4}) \leq t(F_n)$.

Case 2. $T - z$ has two isolated vertices.

The meanings of notations here are same as those adopted in Subcases 1.1 and 1.2. Note that $t(T_z) = t_z^1 + t_{\bar{z}}$, $t_y^0 = t_{\bar{z}}$, $t_y^1 \leq t_{\bar{z}} + t_z^1$, $t_{\bar{y}} = 2t(T_z) + t_z^1$. Thus, $t(T) \leq 2t(T_z) + 2(t_z^1 + t_{\bar{z}}) = 4t(T_z) = t(T_y) \cdot t(T_z)$. By the induction hypothesis, $t(T_z) \leq t(F_{n-4})$. Hence, $t(T) \leq t(T_y) \cdot t(F_{n-4}) \leq t(F_n)$.

Case 3. $T - z$ has an isolated edge.

Note that $t_z^1 + t_{\bar{z}} \leq t(T_z)$. By a similar argument as in the proof of Case 2, we show that $t(T) \leq t(F_n)$. □

In view of Claim 4, we consider the case that $d(z) = 2$.

Claim 5. Assume that there exists a path $P := xyzw$ in T with $d(x) = 3, d(y) = d(z) = 2$, as shown in Figure 4. We have $t(T) \leq t(F_n)$ if one of the following conditions holds:

(1) $T - w$ has an isolated vertex or an isolated edge;

(2) $T - w$ has no isolated vertex or isolated edge, where $d(w) \neq 2$.

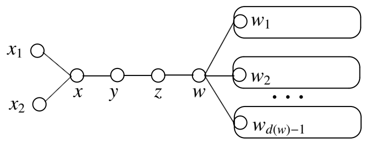

Figure 4. T.

Proof. Let $N(w) = \{w_1, \ldots, w_{d(w)-1}, z\}$. We consider two cases in the following.

Case 1. $T - w$ has an isolated vertex or an isolated edge.

Subcase 1.1. $T - w$ has an isolated vertex.

Let $T - yz = T_y \cup T_z$ where $z \in T_z$. Observe that for an M2-CIS S' of T_z, either $z \notin S'$ or $z \in S'$ with $d_{T[S']}(z) \leq 1$. Let us define $\tilde{t}_z^0 = |\{S' : d_{T[S']}(z) = 0\}|$, $\tilde{t}_{\bar{z}} = |\{S' : z \notin S'\}|$, $\tilde{t}_z^1 = |\{S' : d_{T[S']}(z) = 1\}|$. Thus, $t(T_z) = \tilde{t}_z^0 + \tilde{t}_z^1 + \tilde{t}_{\bar{z}}$.

Observe that for an M2-CIS S of T, either $z \notin S$ or $z \in S$ with $d_{T[S]}(z) \leq 1$. Let

$$
\begin{aligned}
t_z^0 &= |\{S : d_{T[S]}(z) = 0\}| = 2\tilde{t}_z^0, \\
t_z^1 &= |\{S : d_{T[S]}(z) = 1\}| \\
&= |\{S : d_{T[S]}(z) = 1, \{y, z\} \subseteq S\}| + |\{S : d_{T[S]}(z) = 1, \{z, w\} \subseteq S\}| \\
&= \tilde{t}_z^0 + 3\tilde{t}_z^1, \\
t_{\bar{z}} &= |\{S : z \notin S\}| \\
&= |\{S : z \notin S, y \in S, d_{T[S]}(y) = 0\}| + |\{S : z \notin S, y \in S, d_{T[S]}(y) = 1\}| \\
&\quad + |\{S : z \notin S, y \notin S\}| \\
&= \tilde{t}_{\bar{z}} + (|\{S : z \notin S, y \in S, d_{T[S]}(y) = 1, w \notin S\}| \\
&\quad + |\{S : z \notin S, y \in S, d_{T[S]}(y) = 1, w \in S\}|) + 2\tilde{t}_{\bar{z}} \\
&\leq 3\tilde{t}_{\bar{z}} + (\tilde{t}_z^0 + \tilde{t}_{\bar{z}}) = \tilde{t}_z^0 + 4\tilde{t}_{\bar{z}}.
\end{aligned}
$$

Obviously, $t(T) = t_z^0 + t_z^1 + t_{\bar{z}} \leq 4\tilde{t}_z^0 + 3\tilde{t}_z^1 + 4\tilde{t}_{\bar{z}}$. Since $t(T_z) = \tilde{t}_z^0 + \tilde{t}_z^1 + \tilde{t}_{\bar{z}}$, $t(T) \leq 4t(T_z) = t(T_y) \cdot t(T_z)$. By the induction hypothesis, $t(T_z) \leq t(F_{n-4})$. Hence, $t(T) \leq t(T_y) \cdot t(F_{n-4}) \leq t(F_n)$.

Subcase 1.2. $T - w$ has an isolated edge.

By Lemma 2.4, $d(w) \geq 3$. Let $T - zw = T_z \cup T_w$ where $w \in T_w$ and $T'_z = K_{1,4}$ where $V(T'_z) = V(T_z)$. Then $t(T'_z) = 5$. Observe that for an M2-CIS S' of T_w, either $w \notin S'$ or $w \in S'$ with $d_{T[S']}(w) \leq 1$. Let us define $t^0_w = |\{S' : d_{T[S']}(w) = 0\}|$, $t_{\bar{w}} = |\{S' : w \notin S'\}|$, $t^1_w = |\{S' : d_{T[S']}(w) = 1\}|$. Thus, $t(T_w) = t^0_w + t^1_w + t_{\bar{w}}$.

The meanings of notations here are same as those adopted in Subcase 1.1. Note that $t^0_z = 2t_{\bar{w}}$, $t^1_z \leq 2t^0_w + t^1_w + 2t_{\bar{w}}$ and $t_{\bar{z}} = t(T_w) + t^0_w + 3t^1_w$.

We obtain that $t(T) = t^0_z + t^1_z + t_{\bar{z}} \leq t(T_w) + 3t^0_w + 4t^1_w + 4t_{\bar{w}}$. Since $t(T_w) = t^0_w + t^1_w + t_{\bar{w}}$, $t(T) \leq 5t(T_w) = t(T'_z) \cdot t(T_w)$. By the induction hypothesis, $t(T_w) \leq t(F_{n-5})$. Hence, $t(T) \leq t(T'_z) \cdot t(F_{n-5}) \leq t(F_n)$.

Case 2. $T - w$ has no isolated vertex or isolated edge, where $d(w) \neq 2$.

The meanings of notations here are same as those adopted in Subcase 1.1. Let $F_1 = T - (N[x] \cup \{z\})$, $F_2 = T - N[x] - V(P)$, $F_3 = T - N[x] - V(P) - N(w)$ and $F_4 = T - N[x] - V(P) - N(w) - N(w_i)$. Combining these observations with the definition of F_i, we get that $t^0_z = 2t(F_2)$, $t^1_z = t(F_2) + 3t(F_3)$, $t_{\bar{z}} = t(F_1) + t(F_3) + 3(d(w) - 1) \cdot t(F_4)$. Since $t(T) = t^0_z + t^1_z + t_{\bar{z}}$, we have

$$t(T) = t(F_1) + 3t(F_2) + 4t(F_3) + 3(d(w) - 1) \cdot t(F_4). \tag{3.4}$$

Let n_i be the order of F_i for each $i \in \{1, 2, 3, 4\}$. Then $n_1 = 3(s - 1)$, $n_2 = 3(s - 2) + 2$. By the induction hypothesis, $t(F_1) \leq 3^{s-1}$ and $t(F_2) \leq 4^2 \cdot 3^{s-4}$. We consider three cases in terms of the modularity of $d(w)$ (mod 3).

Subcase 2.1. $d(w) = 3l, l \geq 1$

Since $n_3 = 3(s - l - 1)$, $n_4 \leq 3(s - l - 2) + 2$, by the induction hypothesis, $t(F_3) \leq 3^{s-l-1}$ and $t(F_4) \leq 4^2 \cdot 3^{s-l-4}$. Moreover, since $\frac{48l+20}{3^l} + 25 \leq 4^2 \cdot 3$ for any $l \geq 1$, by (3.4), it follows that for any $s \geq 2$ and $l \geq 1$,

$$\begin{aligned} t(T) &\leq (3^2 + 4^2 + \tfrac{4 \cdot 3^2}{3^l} + \tfrac{4^2(3l-1)}{3^l}) \cdot 3^{s-3} \\ &= (\tfrac{48l+20}{3^l} + 25) \cdot 3^{s-3} \leq 4^2 \cdot 3^{s-2} = t(F_n). \end{aligned}$$

Subcase 2.2. $d(w) = 3l + 1, l \geq 1$

Since $n_3 = 3(s - l - 2) + 2$, $n_4 \leq 3(s - l - 2) + 1$, by the induction hypothesis, $t(F_3) \leq 4^2 \cdot 3^{s-l-4}$ and $t(F_4) \leq 4 \cdot 3^{s-l-3}$. Moreover, since $\frac{108l+64}{3^l} + 75 \leq 4^2 \cdot 3^2$ for any $l \geq 1$, by (3.4), it follows that for any $s \geq 2$ and $l \geq 1$,

$$\begin{aligned} t(T) &\leq (3^3 + 4^2 \cdot 3 + \tfrac{4^3}{3^l} + \tfrac{4l \cdot 3^3}{3^l}) \cdot 3^{s-4} \\ &= (\tfrac{108l+64}{3^l} + 75) \cdot 3^{s-4} \leq 4^2 \cdot 3^{s-2} = t(F_n). \end{aligned}$$

Subcase 2.3. $d(w) = 3l + 2, l \geq 1$

Since $n_3 = 3(s - l - 2) + 1$, $n_4 \leq 3(s - l - 2)$, by the induction hypothesis, $t(F_3) \leq 4 \cdot 3^{s-l-3}$ and $t(F_4) \leq 3^{s-l-2}$. Moreover, since $\frac{27l+25}{3^l} + 25 \leq 4^2 \cdot 3$ for any $l \geq 1$, by (3.4), it follows that for any $s \geq 2$ and $l \geq 1$,

$$\begin{aligned} t(T) &\leq (3^2 + 4^2 + \tfrac{4^2}{3^l} + \tfrac{3^2(3l+1)}{3^l}) \cdot 3^{s-3} \\ &= (\tfrac{27l+25}{3^l} + 25) \cdot 3^{s-3} \leq 4^2 \cdot 3^{s-2} = t(F_n). \end{aligned}$$

□

In view of Claim 5, we proceed to consider the case that $d(w) = 2$.

Claim 6. Assume that there exists a path $P := xyzwu$ in T with $d(x) = 3, d(y) = d(z) = d(w) = 2$, as shown in Figure 5. We have $t(T) \leq t(F_n)$ if one of the following conditions holds:

(1) $T - u$ has an isolated vertex or an isolated edge;

(2) $T - u$ has no isolated vertex or isolated edge, where $d(u) \neq 2$.

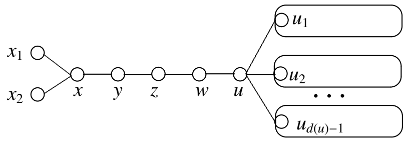

Figure 5. T.

Proof. Let $T - wu = T_w \cup T_u$ where $u \in T_u$ and $N(u) = \{u_1, \ldots, u_{d(u)-1}, w\}$.

Case 1. $T - u$ has an isolated vertex or an isolated edge.

Subcase 1.1. $T - u$ has an isolated vertex.

By Lemma 2.4, $d(u) \geq 3$. Let $T'_w = 2P_3$ where $V(T'_w) = V(T_w)$. Then $t(T'_w) = 9$. Observe that for an M2-CIS S' of T_u, either $u \notin S'$ or $u \in S'$ with $d_{T[S']}(u) = 1$. Let us define $t_u^1 = |\{S' : d_{T[S']}(u) = 1\}|$, $t_{\bar{u}} = |\{S' : u \notin S'\}|$. Thus, $t(T_u) = t_u^1 + t_{\bar{u}}$.

Observe that for an M2-CIS S of T, either $w \notin S$ or $w \in S$ with $d_{T[S]}(w) \leq 1$. Let

$$
\begin{aligned}
t_w^0 &= |\{S : d_{T[S]}(w) = 0\}| \leq 4t_{\bar{u}},\\
t_w^1 &= |\{S : d_{T[S]}(w) = 1\}|\\
&= |\{S : d_{T[S]}(w) = 1, \{w, u\} \subseteq S\}| + |\{S : d_{T[S]}(w) = 1, \{w, z\} \subseteq S\}|\\
&\leq 4t_u^1 + (t_u^1 + 4t_{\bar{u}}) = 5t_u^1 + 4t_{\bar{u}}.\\
t_{\bar{w}} &= |\{S : w \notin S\}|\\
&= |\{S : w \notin S, z \in S, d_{T[S]}(z) = 0\}| + |\{S : w \notin S, z \in S, d_{T[S]}(z) = 1\}|\\
&\quad + |\{S : w \notin S, z \notin S\}|\\
&= 2t_u^1 + t(T_u) + t_u^1 = 3t_u^1 + t(T_u).
\end{aligned}
$$

Clearly, $t(T) = t_w^0 + t_w^1 + t_{\bar{w}} \leq t(T_u) + 8(t_u^1 + t_{\bar{u}})$. Since $t(T_u) = t_u^1 + t_{\bar{u}}$, $t(T) \leq 9t(T_u) = t(T'_w) \cdot t(T_u)$. By the induction hypothesis, $t(T_u) \leq t(F_{n-6})$. Hence, $t(T) \leq t(T'_w) \cdot t(F_{n-6}) \leq t(F_n)$.

Subcase 1.2. $T - u$ has an isolated edge.

The meanings of notations here are same as those adopted in Subcase 1.1, with exception that adding the definition of t_u^0. More precisely, let $t_u^0 = |\{S' : d_{T[S']}(u) = 0\}|$. Then $t(T_u) = t_u^0 + t_u^1 + t_{\bar{u}}$.

Note that $t_w^0 = 2t_{\bar{u}}$, $t_w^1 \leq 4t_u^1 + 3t_{\bar{u}}$ and $t_{\bar{w}} \leq 2t_u^0 + 3t_u^1 + t(T_u)$. Thus, $t(T) = t_w^0 + t_w^1 + t_{\bar{w}} \leq t(T_u) + 2t_u^0 + 7t_u^1 + 5t_{\bar{u}}$. Moreover, $t(T_u) = t_u^0 + t_u^1 + t_{\bar{u}}$, $t(T) \leq 8t(T_u) < 9t(T_u) = t(T'_w) \cdot t(T_u)$. By the induction hypothesis, $t(T_u) \leq t(F_{n-6})$. Hence, $t(T) \leq t(T'_w) \cdot t(F_{n-6}) \leq t(F_n)$.

Case 2. $T - u$ has no isolated vertex or isolated edge, where $d(u) \neq 2$.

The meanings of notations here are same as those adopted in Subcase 1.1. Let $F_1 = T - N(x) - V(P)$, $F_2 = T - N(x) - (V(P) \backslash \{u\})$, $F_3 = T - N(x) - V(P) - N(u)$ and $F_4 = T - N(x) - V(P) - N(u) - N(u_i)$. Combining these observations with the definition of F_i, we get that $t_w^0 = 2t(F_1)$, $t_w^1 = 3t(F_1) + 4t(F_3)$, $t_{\bar{w}} = t(F_2) + 2t(F_3) + 3(d(u) - 1) \cdot t(F_4)$. Since $t(T) = t_w^0 + t_w^1 + t_{\bar{w}}$, we have

$$t(T) = 5t(F_1) + t(F_2) + 6t(F_3) + 3(d(u) - 1) \cdot t(F_4). \tag{3.5}$$

Let n_i be the order of F_i for $i \in \{1, 2, 3, 4\}$. Then $n_1 = 3(s-2)+1$ and $n_2 = 3(s-2)+2$. By the induction hypothesis, $t(F_1) \le 4 \cdot 3^{s-3}$ and $t(F_2) \le 4^2 \cdot 3^{s-4}$. Now we consider three subcases in terms of $d(u)$ (mod 3).

Subcase 2.1. $d(u) = 3l, l \ge 1$

By the definition of n_i, $n_3 = 3(s-l-2)+2$ and $n_4 \le 3(s-l-2)+1$. By the induction hypothesis, $t(F_3) \le 4^2 \cdot 3^{s-l-4}$ and $t(F_4) \le 4 \cdot 3^{s-l-3}$. Moreover, since $\frac{108l+60}{3^l} + 76 \le 4^2 \cdot 3^2$ for any $l \ge 1$, by (3.5), it follows that for any $s \ge 2$ and $l \ge 1$,

$$\begin{aligned} t(T) &= (5 \cdot 4 \cdot 3 + 4^2 + \tfrac{6 \cdot 4^2}{3^l} + \tfrac{4 \cdot 3^2 \cdot (3l-1)}{3^l}) \cdot 3^{s-4} \\ &= (\tfrac{108l+60}{3^l} + 76) \cdot 3^{s-4} \\ &\le 4^2 \cdot 3^{s-2} = t(F_n). \end{aligned}$$

Subcase 2.2. $d(u) = 3l+1, l \ge 1$

By the definition of n_i, $n_3 = 3(s-l-2)+1$ and $n_4 = 3(s-l-2)$. By the induction hypothesis, $t(F_3) \le 4 \cdot 3^{s-l-3}$ and $t(F_4) \le 3^{s-l-2}$. Moreover, since $\frac{81l+72}{3^l} + 76 \le 4^2 \cdot 3^2$ for any $l \ge 1$, by (3.5), it follows that for any $s \ge 2$ and $l \ge 1$,

$$\begin{aligned} t(T) &= (5 \cdot 4 \cdot 3 + 4^2 + \tfrac{6 \cdot 12}{3^l} + \tfrac{3^4 l}{3^l}) \cdot 3^{s-4} \\ &= (\tfrac{81l+72}{3^l} + 76) \cdot 3^{s-4} \\ &\le 4^2 \cdot 3^{s-2} = t(F_n). \end{aligned}$$

Subcase 2.3. $d(u) = 3l+2, l \ge 1$

By the definition of n_i, $n_3 = 3(s-l-2)$ and $n_4 = 3(s-l-3)+2$. By the induction hypothesis, $t(F_3) \le 3^{s-l-2}$ and $t(F_4) \le 4^2 \cdot 3^{s-l-5}$. Moreover, since $\frac{48l+70}{3^l} + 76 \le 4^2 \cdot 3^2$ for any $l \ge 1$, by (3.5), it follows that for any $s \ge 2$ and $l \ge 1$,

$$\begin{aligned} t(T) &= (5 \cdot 4 \cdot 3 + 4^2 + \tfrac{6 \cdot 3^2}{3^l} + \tfrac{4^2(3l+1)}{3^l}) \cdot 3^{s-4} \\ &\le (\tfrac{48l+70}{3^l} + 76) \cdot 3^{s-4} \\ &\le 4^2 \cdot 3^{s-2} = t(F_n). \end{aligned}$$

□

In view of Claim 6, we proceed to consider the case that $d(u) = 2$.

Claim 7. Assume that there exists a path $P := xyzwuq$ in T with $d(x) = 3, d(y) = d(z) = d(w) = d(u) = 2$, as shown in Figure 6. We have $t(T) \le t(F_n)$.

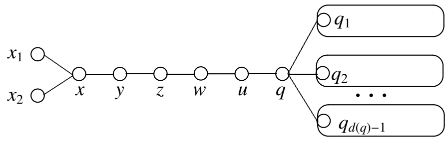

Figure 6. T.

Proof. Let $T - uq = T_u \cup T_q$ where $q \in T_q$ and $N(q) = \{q_1, \dots, q_{d(q)-1}, u\}$.

Case 1. $T - q$ has an isolated vertex.

By Lemma 2.4, $d(q) \geq 3$. Let $T'_u = P_3 \cup K_{1,3}$ where $V(T'_u) = V(T_u)$. Then $t(T'_u) = 12$. Observe that for an M2-CIS S' of T_q, either $q \notin S'$ or $q \in S'$ with $d_{T[S']}(q) = 1$. Let us define $t_q^1 = |\{S' : d_{T[S']}(q) = 1\}|$, $t_{\bar{q}} = |\{S' : q \notin S'\}|$. Thus, $t(T_q) = t_q^1 + t_{\bar{q}}$.

Observe that for an M2-CIS S of T, either $u \notin S$ or $u \in S$ with $d_{T[S]}(u) \leq 1$. Let

$$\begin{aligned} t_u^0 &= |\{S : d_{T[S]}(u) = 0\}| \\ &= |\{S : d_{T[S]}(u) = 0, z \in S, d_{T[S]}(z) = 1\}| + |\{S : d_{T[S]}(u) = 0, z \in S, \\ &\quad d_{T[S]}(z) = 0\}| \\ &\leq 2t_{\bar{q}} + (t_q^1 + t_{\bar{q}}) = t_q^1 + 3t_{\bar{q}}. \\ t_u^1 &= |\{S : d_{T[S]}(u) = 1\}| \\ &= |\{S : d_{T[S]}(u) = 1, \{u, q\} \subseteq S\}| + |\{S : d_{T[S]}(u) = 1, \{w, u\} \subseteq S\}| \\ &\leq 4t_q^1 + (6t_{\bar{q}} + t_q^1) = 5t_q^1 + 6t_{\bar{q}}, \\ t_{\bar{u}} &= |\{S : u \notin S\}| \\ &= |\{S : u \notin S, w \in S, d_{T[S]}(w) = 0\}| + |\{S : u \notin S, w \in S, d_{T[S]}(w) = 1\}| \\ &\quad + |\{S : u \notin S, w \notin S\}| \\ &= 2t_q^1 + 3t(T_q) + t_q^1 = 3t_q^1 + 3t(T_q). \end{aligned}$$

Clearly, $t(T) = t_u^0 + t_u^1 + t_{\bar{u}} \leq 3t(T_q) + 9(t_q^1 + t_{\bar{q}})$. Since $t(T_q) = t_q^1 + t_{\bar{q}}$, $t(T) \leq 12t(T_q) = t(T'_u) \cdot t(T_q)$. By the induction hypothesis, $t(T_q) \leq t(F_{n-7})$. Hence, $t(T) \leq t(T'_u) \cdot t(F_{n-7}) \leq t(F_n)$.

Case 2. $T - q$ has no isolated vertex.

The meanings of notations here are same as those adopted in Case 1. Let $F_1 = T - N(x) - V(P) - N(q)$, $F_2 = T - N(x) - V(P) - N(q) - N(q_i)$, $F_3 = T - N(x) - V(P)$ and $F_4 = T - N(x) - (V(P) \backslash \{q\})$. Combining these observations with the definition of F_i, we get that $t_u^0 = 3t(F_3)$, $t_u^1 = 4t(F_1) + 4t(F_3)$, $t_{\bar{u}} = 2t(F_1) + 3(d(q) - 1) \cdot t(F_2) + 3t(F_4)$. Since $t(T) = t_u^0 + t_u^1 + t_{\bar{u}}$, we have

$$t(T) = 6t(F_1) + 3(d(q) - 1) \cdot t(F_2) + 7t(F_3) + 3t(F_4). \tag{3.6}$$

Let n_i be the order of F_i for $i \in \{1, 2, 3, 4\}$. Then $n_3 = 3(s-2)$ and $n_4 = 3(s-2)+1$. By the induction hypothesis, $t(F_1) \leq 3^{s-2}$ and $t(F_4) \leq 4 \cdot 3^{s-3}$. We consider three subcases in terms of $d(q)$ (mod 3).

Subcase 2.1. $d(q) = 3l, l \geq 1$

By the definition of n_i, $n_1 = 3(s - l - 2) + 1$ and $n_2 \leq 3(s - l - 2)$. By the induction hypothesis, $t(F_1) \leq 4 \cdot 3^{s-l-3}$ and $t(F_2) \leq 3^{s-l-2}$. Moreover, since $\frac{9l+5}{3^l} + 11 \leq 4^2$ for any $l \geq 1$, by (3.6), it follows that for any $s \geq 2$ and $l \geq 1$,

$$\begin{aligned} t(T) &\leq \left(\tfrac{8}{3^l} + \tfrac{3(3l-1)}{3^l} + 11\right) \cdot 3^{s-2} \leq \left(\tfrac{9l+5}{3^l} + 11\right) \cdot 3^{s-2} \\ &\leq 4^2 \cdot 3^{s-2} = t(F_n). \end{aligned}$$

Subcase 2.2. $d(q) = 3l + 1, l \geq 1$

By the definition of n_i, $n_1 = 3(s - l - 2)$ and $n_2 \leq 3(s - l - 3) + 2$. By the induction hypothesis, $t(F_1) \leq 3^{s-l-2}$ and $t(F_2) \leq 4^2 \cdot 3^{s-l-5}$. Moreover, since $\frac{16l+18}{3^l} + 33 \leq 4^2 \cdot 3$ for any $l \geq 1$, by (3.6), it follows that for any $s \geq 2$ and $l \geq 1$,

$$\begin{aligned} t(T) &\leq \left(\tfrac{18}{3^l} + \tfrac{4^2 l}{3^l} + 33\right) \cdot 3^{s-3} \leq \left(\tfrac{16l+18}{3^l} + 33\right) \cdot 3^{s-3} \\ &\leq 4^2 \cdot 3^{s-2} = t(F_n). \end{aligned}$$

Subcase 2.3. $d(q) = 3l + 2, l \geq 0$

By the definition of n_i, $n_1 = 3(s-l-3)+2$ and $n_2 \leq 3(s-l-3)+1$. By the induction hypothesis, $t(F_1) \leq 4^2 \cdot 3^{s-l-5}$ and $t(F_2) \leq 4 \cdot 3^{s-l-4}$. Moreover, since $\frac{36l+44}{3^l} + 99 \leq 4^2 \cdot 3^2$ for any $l \geq 0$, by (3.6), it follows that for any $s \geq 2$ and $l \geq 0$,

$$\begin{aligned} t(T) &\leq (\tfrac{2\cdot 4^2}{3^l} + \tfrac{12(3l+1)}{3^l} + 99) \cdot 3^{s-4} \leq (\tfrac{36l+44}{3^l} + 99) \cdot 3^{s-4} \\ &\leq 4^2 \cdot 3^{s-2} = t(F_n). \end{aligned}$$

□

By Lemma 2.2, we consider the case that there exists a vertex with one neighbor being leaf.

Claim 8. Assume that there exists a path $P := xyz$ in T with $d(x) = 1, d(y) = 2$, as shown in Figure 7. We have $t(T) \leq t(F_n)$ if one of the following conditions holds:

(1) $T - z$ has an isolated vertex or an isolated edge other than the component xy;

(2) $T - z$ has no isolated vertex or isolated edge, where $d(z) \geq 3$.

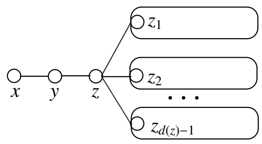

Figure 7. T.

Proof. Let $T - yz = T_y \cup T_z$ where $z \in T_z$ and $N(z) = \{z_1, \ldots, z_{d(z)-1}, y\}$.

Case 1. $T - z$ has an isolated vertex.

Let $T'_y = P_3$ where $V(T'_y) = \{x, y, z_1\}$. Then $t(T'_y) = 3$.

Subcase 1.1. $T - z$ has exactly an isolated vertex, say z_1.

Observe that for an M2-CIS S' of $T_z - z_1$, either $z \notin S'$ or $z \in S'$ with $d_{T[S']}(z) \leq 1$. Let us define $t_z^0 = |\{S' : d_{T[S']}(z) = 0\}|$, $t_{\bar{z}} = |\{S' : z \notin S'\}|$, $t_z^1 = |\{S' : d_{T[S']}(z) = 1\}|$. Thus, $t(T_z - z_1) = t_z^0 + t_z^1 + t_{\bar{z}}$.

Observe that for an M2-CIS S of T, either $y \notin S$ or $y \in S$ with $d_{T[S]}(y) = 1$. Let

$$\begin{aligned} t_{\bar{y}} &= |\{S : y \notin S\}| \leq t(T_z - z_1) + t_z^1, \\ t_y^1 &= |\{S : d_{T[S]}(y) = 1\}| \\ &= |\{S : d_{T[S]}(y) = 1, \{x, y\} \subseteq S\}| + |\{S : d_{T[S]}(y) = 1, \{y, z\} \subseteq S\}| \\ &\leq t(T_z - z_1) + t_z^0 + t_{\bar{z}}, \end{aligned}$$

Clearly, $t(T) = t_{\bar{y}} + t_y^1 \leq 2t(T_z - z_1) + t_z^0 + t_z^1 + t_{\bar{z}}$. Since $t(T_z - z_1) = t_z^0 + t_z^1 + t_{\bar{z}}$, $t(T) \leq 3t(T_z - z_1) = t(T'_y) \cdot t(T_z - z_1)$. By the induction hypothesis, $t(T_z - z_1) \leq t(F_{n-3})$. Hence, $t(T) \leq t(T'_y) \cdot t(F_{n-3}) \leq t(F_n)$.

Subcase 1.2. $T - z$ has two isolated vertices.

Observe that for an M2-CIS S' of $T_z - z_1$, either $z \notin S'$ or $z \in S'$ with $d_{T[S']}(z) = 1$. Let $t_{\bar{z}} = |\{S' : z \notin S'\}|$, $t_z^1 = |\{S' : d_{T[S']}(z) = 1\}|$. Then $t(T_z - z_1) = t_{\bar{z}} + t_z^1$.

The meanings of notations here are same as those adopted in Subcase 1.1. Note that $t_{\bar{y}} \leq 2t_z^1$ and $t_y^1 \leq 2t_{\bar{z}} + t(T_z - z_1)$. We have $t(T) = t_{\bar{y}} + t_y^1 \leq t(T_z - z_1) + 2(t_{\bar{z}} + t_z^1)$. Moreover, $t(T_z - z_1) = t_{\bar{z}} + t_z^1$, $t(T) \leq 3t(T_z - z_1) = t(T'_y) \cdot t(T_z - z_1)$. By the induction hypothesis, $t(T_z - z_1) \leq t(F_{n-3})$. Hence, $t(T) \leq t(T'_y) \cdot t(F_{n-3}) \leq t(F_n)$.

Case 2. $T - z$ has an isolated edge, say $z_1z_1^1$.

Let $T'_y = K_{1,3}$ where $V(T'_y) = \{x, y, z_1, z_1^1\}$. Then $t(T'_y) = 4$. The meanings of nations here are same as those adopted in Subcase 1.1. It is sufficient to note that $t_z^0 + t_z^1 \leq t(T_z - z_1z_1^1)$, $t_{\bar{y}} \leq t(T_z - z_1z_1^1) + t_z^0 + t_z^1$ and $t_y^1 \leq 2t(T_z - z_1z_1^1)$. Thus, $t(T) = t_{\bar{y}} + t_y^1 \leq 4t(T_z - z_1z_1^1) = t(T'_y) \cdot t(T_z - z_1z_1^1)$. By the induction hypothesis, $t(T_z - z_1z_1^1) \leq t(F_{n-4})$. Hence, $t(T) \leq t(T'_y) \cdot t(F_{n-4}) \leq t(F_n)$. □

In view of Claim 8, we consider the case that $d(z) = 2$.

Claim 9. Assume that there exists a path $P := xyzw$ in T with $d(x) = 1, d(y) = d(z) = 2$, as shown in Figure 8. We have $t(T) \leq t(F_n)$ if one of the following conditions holds:

(1) $n \geq 6$, $n \equiv 0$ or 1 (mod 3);

(2) $n \geq 8$, $n \equiv 2$ (mod 3), $d(w) \geq 3$.

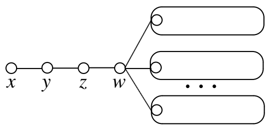

Figure 8. T.

Proof. Let $F_1 = T - N[y]$, $F_2 = T - V(P)$, and $F_3 = T - V(P) - N(w)$. Observe that for an M2-CIS S of T, either $z \notin S$ or $z \in S$ with $d_{T[S]}(z) \leq 1$. Let us define

$t_{\bar{z}} = |\{S : z \notin S\}| = t(F_1)$, $t_z^0 = |\{S : d_{T[S]}(z) = 0\}| = t(F_2)$,

$t_z^1 = |\{S : d_{T[S]}(z) = 1\}|$

$= |\{S : d_{T[S]}(z) = 1, \{y, z\} \subseteq S\}| + |\{S : d_{T[S]}(z) = 1, \{z, w\} \subseteq S\}|$

$= t(F_2) + t(F_3)$.

Since $t(T) = t_{\bar{z}} + t_z^0 + t_z^1$, we have

$$t(T) = t(F_1) + 2t(F_2) + t(F_3) \tag{3.7}$$

Let n_i be the order of F_i for each $i \in \{1, 2, 3\}$. We consider three cases in terms of the modularity of n (mod 3).

Case 1. $n = 3s$, $s \geq 2$.

By the definition of n_i, $n_1 = 3(s-1)$, $n_2 = 3(s-2)+2$. By the induction hypothesis, $t(F_1) \leq 3^{s-1}$ and $t(F_2) \leq 4^2 \cdot 3^{s-4}$. We distinguish three subcases according to $d(w)$ (mod 3).

Subcase 1.1. $d(w) = 3l$, $l \geq 1$,

Since $n_3 = 3(s-l-1)$, by the induction hypothesis, $t(F_3) \leq 3^{s-l-1}$. Moreover, since $\frac{3^3}{3^l} + 59 \leq 3^4$ for any $l \geq 1$, by (3.7), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (3^3 + 2 \cdot 4^2 + \frac{3^3}{3^l}) \cdot 3^{s-4} = (\frac{3^3}{3^l} + 59) \cdot 3^{s-4} \leq 3^s = t(F_n).$$

Subcase 1.2. $d(w) = 3l + 1$, $l \geq 1$

Since $n_3 = 3(s-l-2)+2$, by the induction hypothesis, $t(F_3) \leq 4^2 \cdot 3^{s-l-4}$. Moreover, since $\frac{4^2}{3^l} + 59 \leq 3^4$ for any $l \geq 1$, by (3.7), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (3^3 + 2 \cdot 4^2 + \frac{4^2}{3^l}) \cdot 3^{s-4} = (\frac{4^2}{3^l} + 59) \cdot 3^{s-4} \leq 3^s = t(F_n).$$

Subcase 1.3. $d(w) = 3l + 2$, $l \geq 0$

Since $n_3 = 3(s-l-2)+1$, by the induction hypothesis, $t(F_3) \leq 4\cdot 3^{s-l-3}$. Moreover, since $\frac{12}{3^l}+59 \leq 3^4$ for any $l \geq 0$, by (3.7), it follows that for any $s \geq 2$ and $l \geq 0$,

$$t(T) \leq (3^3 + 2\cdot 4^2 + \frac{12}{3^l})\cdot 3^{s-4} = (\frac{12}{3^l} + 59)\cdot 3^{s-4} \leq 3^s = t(F_n).$$

Case 2. $n = 3s + 1$, $s \geq 2$

By the definition of n_i, $n_1 = 3(s-1)+1$, $n_2 = 3(s-1)$, by the induction hypothesis, $t(F_1) \leq 4\cdot 3^{s-2}$ and $t(F_2) \leq 3^{s-1}$.

Subcase 2.1. $d(w) = 3l$, $l \geq 1$

Since $n_3 = 3(s-l-1)+1$, by the induction hypothesis, $t(F_3) \leq 4\cdot 3^{s-l-2}$. Moreover, since $\frac{4}{3^l}+10 \leq 4\cdot 3$ for any $l \geq 1$, by (3.7), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (4 + 2\cdot 3 + \frac{4}{3^l})\cdot 3^{s-2} = (\frac{4}{3^l} + 10)\cdot 3^{s-2} \leq 4\cdot 3^{s-1} = t(F_n).$$

Subcase 2.2. $d(w) = 3l + 1$, $l \geq 1$

Since $n_3 = 3(s-l-1)$, by the induction hypothesis, $t(F_3) \leq 3^{s-l-1}$. Moreover, since $\frac{3}{3^l} + 10 \leq 4\cdot 3$ for any $l \geq 1$, by (3.7), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (4 + 2\cdot 3 + \frac{3}{3^l})\cdot 3^{s-2} = (\frac{3}{3^l} + 10)\cdot 3^{s-2} \leq 4\cdot 3^{s-1} = t(F_n).$$

Subcase 2.3. $d(w) = 3l + 2$, $l \geq 0$

Since $n_3 = 3(s-l-2)+2$, by the induction hypothesis, $t(F_3) \leq 4^2\cdot 3^{s-l-4}$. Moreover, since $\frac{4^2}{3^l} + 90 \leq 4\cdot 3^3$ for any $l \geq 0$, by (3.7), it follows that for any $s \geq 2$ and $l \geq 0$,

$$t(T) \leq (4\cdot 3^2 + 2\cdot 3^3 + \frac{4^2}{3^l})\cdot 3^{s-4} = (\frac{4^2}{3^l} + 90)\cdot 3^{s-4} \leq 4\cdot 3^{s-1} = t(F_n).$$

Case 3. $n = 3s + 2$, $s \geq 2$

By the definition of n_i, $n_1 = 3(s-1)+2$, $n_2 = 3(s-1)+1$. By the induction hypothesis, we have $t(F_1) \leq 4^2\cdot 3^{s-3}$ and $t(F_2) \leq 4\cdot 3^{s-2}$.

Subcase 3.1. $d(w) = 3l$, $l \geq 1$

Since $n_3 = 3(s-l-1)+2$, by the induction hypothesis, $t(F_3) \leq 4^2\cdot 3^{s-l-3}$. Moreover, since $\frac{4^2}{3^l} + 40 \leq 4^2\cdot 3$ for any $l \geq 1$, by (3.7), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (4^2 + 8\cdot 3 + \frac{4^2}{3^l})\cdot 3^{s-3} = (\frac{4^2}{3^l} + 40)\cdot 3^{s-3} \leq 4^2\cdot 3^{s-2} = t(F_n).$$

Subcase 3.2. $d(w) = 3l + 1$, $l \geq 1$

Since $n_3 = 3(s-l-1)+1$, by the induction hypothesis, $t(F_3) \leq 4\cdot 3^{s-l-2}$. Moreover, since $\frac{12}{3^l} + 40 \leq 4^2\cdot 3$ for any $l \geq 1$, by (3.7), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (4^2 + 8\cdot 3 + \frac{12}{3^l})\cdot 3^{s-3} = (\frac{12}{3^l} + 40)\cdot 3^{s-3} \leq 4^2\cdot 3^{s-2} = t(F_n).$$

Subcase 3.3. $d(w) = 3l + 2$, $l \geq 1$

Since $n_3 = 3(s - l - 1)$, by the induction hypothesis, $t(F_3) \leq 3^{s-l-1}$. Moreover, since $\frac{3^2}{3^l} + 40 \leq 4^2 \cdot 3$ for any $l \geq 1$, by (3.7), it follows that for any $s \geq 2$ and $l \geq 1$,

$$t(T) \leq (4^2 + 8 \cdot 3 + \frac{3^2}{3^l}) \cdot 3^{s-3} = (\frac{3^2}{3^l} + 40) \cdot 3^{s-3} \leq 4^2 \cdot 3^{s-2} = t(F_n).$$

□

In view of Claim 9, we proceed to consider the case that $d(w) = 2$ and $n = 3s + 2$ where $s \geq 2$.

Claim 10. Assume that there exists a path $P := xyzwu$ in T with $d(x) = 1, d(y) = d(z) = d(w) = 2$, as shown in Figure 9. We have $t(T) \leq t(F_n)$ if one of the following conditions holds:

(1) $T - u$ has an isolated vertex or an isolated edge;

(2) $T - u$ has no isolated vertex or isolated edge, where $d(u) \geq 4$.

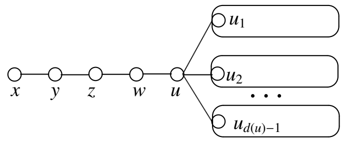

Figure 9. T.

Proof. Let $N(u) = \{u_1, \ldots, u_{d(u)-1}, w\}$.

Case 1. $T - u$ has an isolated vertex or an isolated edge.

Subcase 1.1. $T - u$ has an isolated vertex.

Let $T - zw = T_z \cup T_w$ where $w \in T_w$. Observe that for an M2-CIS S' of T_w, either $w \notin S'$ or $w \in S'$ with $d_{T[S']}(w) \leq 1$. Let us define $\tilde{t}^0_w = |\{S' : d_{T[S']}(w) = 0\}|$, $\tilde{t}^1_w = |\{S' : d_{T[S']}(w) = 1\}|$, $\tilde{t}_{\bar{w}} = |\{S' : w \notin S'\}|$. Then $t(T_w) = \tilde{t}^0_w + \tilde{t}^1_w + \tilde{t}_{\bar{w}}$.

Observe that for an M2-CIS S of T, either $w \notin S$ or $w \in S$ with $d_{T[S]}(w) \leq 1$. Let

$$\begin{aligned}
t^0_w &= |\{S : d_{T[S]}(w) = 0\}| = \tilde{t}^0_w,\\
t^1_w &= |\{S : d_{T[S]}(w) = 1\}|\\
&= |\{S : d_{T[S]}(w) = 1, \{z, w\} \subseteq S\}| + |\{S : d_{T[S]}(w) = 1, \{w, u\} \subseteq S\}|\\
&= \tilde{t}^0_w + \tilde{t}^1_w\\
t_{\bar{w}} &= |\{S : w \notin S\}|\\
&= |\{S : w \notin S, u \in S, d_{T[S]}(u) = 1\}| + |\{S : w \notin S, u \notin S\}|\\
&\leq 3\tilde{t}_{\bar{w}} + \tilde{t}^0_w.
\end{aligned}$$

It is easy to see that $t(T) = t^0_w + t^1_w + t_{\bar{w}} \leq 3\tilde{t}^0_w + \tilde{t}^1_w + 3\tilde{t}_{\bar{w}}$. Since $t(T_w) = \tilde{t}^0_w + \tilde{t}^1_w + \tilde{t}_{\bar{w}}$, $t(T) \leq 3t(T_w) = t(T_z) \cdot t(T_w)$. By the induction hypothesis, $t(T_w) \leq t(F_{n-3})$. Hence, $t(T) \leq t(T_z) \cdot t(F_{n-3}) \leq t(F_n)$.

Subcase 1.2. $T - u$ has an isolated edge.

Let $T - wu = T_w \cup T_u$ where $u \in T_u$ and $T'_w = K_{1,3}$ where $V(T'_w) = V(T_w)$. Then $t(T'_w) = 4$. Observe that for an M2-CIS S' of T_u, either $u \notin S'$ or $u \in S'$ $d_{T[S']}(u) \leq 1$. Let us define $t^0_u = |\{S' : d_{T[S']}(u) = 0\}|$, $t_{\bar{u}} = |\{S' : u \notin S'\}|$, $t^1_u = |\{S' : d_{T[S']}(u) = 1\}|$. Then $t(T_u) = t^0_u + t^1_u + t_{\bar{u}}$.

The meanings of notations here are same as those adopted in Subcase 1.1. Note that $t^0_w = t_{\bar{u}}$, $t^1_w \leq t^1_u + t_{\bar{u}}$ and $t_{\bar{w}} = t(T_u) + t^0_u + 2t^1_u$.

Thus, $t(T) = t_w^0 + t_w^1 + t_{\bar{w}} \leq t(T_u) + t_u^0 + 3t_u^1 + 2t_{\bar{u}}$. Moreover, $t(T_u) = t_u^0 + t_u^1 + t_{\bar{u}}$, $t(T) \leq 4t(T_u) = t(T'_w) \cdot t(T_u)$. By the induction hypothesis, $t(T_u) \leq t(F_{n-4})$. Hence, $t(T) \leq t(T'_w) \cdot t(F_{n-4}) \leq t(F_n)$.

Case 2. $T - u$ has no isolated vertex or isolated edge, where $d(u) \geq 4$.

The meanings of notations here are same as those adopted in Subcase 1.1. Let $F_1 = T - (V(P) \setminus \{u\})$, $F_2 = T - V(P)$, $F_3 = T - V(P) - N(u)$ and $F_4 = T - V(P) - N(u) - N(u_i)$. Combining these observations with the definition of F_i, we get that $t_w^0 = t(F_2)$, $t_w^1 = t(F_2) + t(F_3)$ and $t_{\bar{w}} = t(F_1) + t(F_3) + 2(d(u)-1) \cdot t(F_4)$. Since $t(T) = t_w^0 + t_w^1 + t_{\bar{w}}$, we have

$$t(T) = t(F_1) + 2t(F_2) + 2t(F_3) + 2(d(u) - 1) \cdot t(F_4). \tag{3.8}$$

Let n_i be the order of F_i for each $i \in \{1, 2, 3, 4\}$. Then $n_1 = 3(s-1) + 1$, $n_2 = 3(s-1)$. By the induction hypothesis, $t(F_1) \leq 4 \cdot 3^{s-2}$ and $t(F_2) \leq 3^{s-1}$. We want to prove that $t(T) \leq 4^2 \cdot 3^{s-2} = t(F_n)$ for any $s \geq 2$. By (3.8), we complete the proof by showing that for any $s \geq 2$,

$$t(F_3) + (d(u) - 1) \cdot t(F_4) \leq 3^{s-1}. \tag{3.9}$$

Subcase 2.1. $d(u) = 3l, l \geq 2$

Since $n_3 = 3(s-l-1) + 1$, $n_4 \leq 3(s-l-1)$, by the induction hypothesis, $t(F_3) \leq 4 \cdot 3^{s-l-2}$ and $t(F_4) \leq 3^{s-l-1}$. Moreover, since $\frac{9l+1}{3^l} \leq 3$ for any $l \geq 2$, it follows that for any $s \geq 2$ and $l \geq 2$,

$$4 \cdot 3^{s-l-2} + (3l - 1) \cdot 3^{s-l-1} = \frac{9l + 1}{3^l} \cdot 3^{s-2} \leq 3^{s-1}.$$

i.e., (3.9) holds.

Subcase 2.2. $d(u) = 3l + 1, l \geq 0$

Since $n_3 = 3(s-l-1)$, $n_4 \leq 3(s-l-2) + 2$, by the induction hypothesis, $t(F_3) \leq 3^{s-l-1}$ and $t(F_4) \leq 4^2 \cdot 3^{s-l-4}$. Moreover, since $\frac{16l+9}{3^l} \leq 3^2$ for any $l \geq 0$, it follows that for any $s \geq 2$ and $l \geq 0$,

$$3^{s-l-1} + 4^2 l \cdot 3^{s-l-3} = \frac{16l + 9}{3^l} \cdot 3^{s-3} \leq 3^{s-1}.$$

i.e., (3.9) holds.

Subcase 2.3. $d(u) = 3l + 2, l \geq 1$

Since $n_3 = 3(s-l-2) + 2$, $n_4 \leq 3(s-l-2) + 1$, by the induction hypothesis, $t(F_3) \leq 4^2 \cdot 3^{s-l-4}$ and $t(F_4) \leq 4 \cdot 3^{s-l-3}$. Moreover, since $\frac{36l+28}{3^l} \leq 3^3$ for any $l \geq 1$, it follows that for any $s \geq 2$ and $l \geq 1$,

$$4^2 \cdot 3^{s-l-4} + 4(3l + 1) \cdot 3^{s-l-3} = \frac{36l + 28}{3^l} \cdot 3^{s-4} \leq 3^{s-1}.$$

i.e., (3.9) holds. □

In view of Claim 10, we proceed to consider the case that $d(u) = 2$ and $d(u) = 3$ respectively.

Claim 11. Assume that there exists a path $P := xyzwuq$ in T with $d(x) = 1, d(y) = d(z) = d(w) = d(u) = 2$, as shown in Figure 10. We have $t(T) \leq t(F)$.

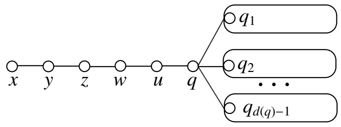

Figure 10. T.

Proof. Let $N(q) = \{q_1, \ldots, q_{d(q)-1}, u\}$ and $T - wu = T_w \cup T_u$ where $u \in T_u$.

Case 1. $T - q$ has an isolated vertex.

Let $T'_w = K_{1,3}$ where $V(T'_w) = V(T_w)$. Then $t(T'_w) = 4$. Observe that for an M2-CIS S' of T_u, either $u \notin S'$ or $u \in S'$ with $d_{T[S']}(u) \leq 1$. Let us define $\tilde{t}^0_u = |\{S' : d_{T[S']}(u) = 0\}|$, $\tilde{t}_{\bar{u}} = |\{S' : u \notin S'\}|$, $\tilde{t}^1_u = |\{S' : d_{T[S']}(u) = 1\}|$. Thus, $t(T_u) = \tilde{t}^0_u + \tilde{t}^1_u + \tilde{t}_{\bar{u}}$.

Observe that for an M2-CIS S of T, either $u \notin S$ or $u \in S$ with $d_{T[S]}(u) \leq 1$. Let

$t^0_u = |\{S : d_{T[S]}(u) = 0\}| = 2\tilde{t}^0_u,$

$t^1_u = |\{S : d_{T[S]}(u) = 1\}|$

$= |\{S : d_{T[S]}(u) = 1, \{w, u\} \subseteq S\}| + |\{S : d_{T[S]}(u) = 1, \{u, q\} \subseteq S\}|$

$= \tilde{t}^0_u + 3\tilde{t}^1_u.$

$t_{\bar{u}} = |\{S : u \notin S\}|$

$= |\{S : u \notin S, w \notin S\}| + |\{S : u \notin S, w \in S, d_{T[S]}(w) = 1\}| + |\{S : u \notin S,$
$w \in S, d_{T[S]}(w) = 0\}|$

$= \tilde{t}_{\bar{u}} + (\tilde{t}^0_u + \tilde{t}_{\bar{u}}) + \tilde{t}_{\bar{u}} = \tilde{t}^0_u + 3\tilde{t}_{\bar{u}}.$

Then $t(T) = t^0_u + t^1_u + t_{\bar{u}} = 4\tilde{t}^0_u + 3\tilde{t}^1_u + 3\tilde{t}_{\bar{u}}$. Moreover, $t(T_u) = \tilde{t}^0_u + \tilde{t}^1_u + \tilde{t}_{\bar{u}}$, $t(T) \leq 4t(T_u) = t(T'_w) \cdot t(T_u)$. By the induction hypothesis, $t(T_u) \leq t(F_{n-4})$. Hence, $t(T) \leq t(T'_w) \cdot t(T_u) \leq t(F_{n-4})$.

Case 2. $T - q$ has no isolated vertex.

The meanings of notations here are same as those adopted in Case 1. Let $F_1 = T - V(P) - N(q)$, $F_2 = T - V(P) - N(q) - N(q_i)$, $F_3 = T - V(P)$ and $F_4 = T - (V(P) \backslash \{q\})$. Combining these observations with the definition of F_i, we get that $t^0_u = 2t(F_3)$, $t^1_u = 3t(F_1) + t(F_3)$ and $t_{\bar{u}} = t(F_1) + 2(d(q) - 1) \cdot t(F_2) + t(F_4)$. Since $t(T) = t^0_u + t^1_u + t_{\bar{u}}$, we have

$$t(T) = 4t(F_1) + 2(d(u) - 1) \cdot t(F_2) + 3t(F_3) + t(F_4). \tag{3.10}$$

Let n_i be the order of F_i for $i \in \{1, 2, 3, 4\}$. Then $n_3 = 3(s - 2) + 2$, $n_4 = 3(s - 1)$. By the induction hypothesis, $t(F_3) \leq 4^2 \cdot 3^{s-4}$ and $t(F_4) \leq 3^{s-1}$. We want to prove that $t(T) \leq 4^2 \cdot 3^{s-2} = t(F_n)$ for any $s \geq 2$. By (3.10), we complete the proof by showing that for any $s \geq 2$,

$$4t(F_1) + 2(d(q) - 1) \cdot t(F_2) \leq 23 \cdot 3^{s-3}. \tag{3.11}$$

Subcase 2.1. $d(q) = 3l$, $l \geq 1$

$n_1 = 3(s-l-1)$, $n_2 \leq 3(s-l-2)+2$, by the induction hypothesis, $t(F_1) \leq 3^{s-l-1}$ and $t(F_2) \leq 4^2 \cdot 3^{s-l-4}$. Moreover, since $\frac{96l+76}{3^l} \leq 23 \cdot 3$ for any $l \geq 1$, it follows that for any $s \geq 2$ and $l \geq 1$,

$$4 \cdot 3^{s-l-1} + 2 \cdot 4^2(3l - 1) \cdot 3^{s-l-4} = \frac{96l + 76}{3^l} \cdot 3^{s-4} \leq 23 \cdot 3^{s-3}.$$

i.e., (3.11) holds.

Subcase 2.2. $d(q) = 3l + 1, l \geq 0$

$n_1 = 3(s - l - 2) + 2$, $n_2 \leq 3(s - l - 2) + 1$, by the induction hypothesis, $t(F_1) \leq 4^2 \cdot 3^{s-l-4}$ and $t(F_2) \leq 4 \cdot 3^{s-l-3}$. Moreover, since $\frac{72l+64}{3^l} \leq 23 \cdot 3$ for any $l \geq 0$, it follows that for any $s \geq 2$ and $l \geq 0$,

$$4^3 \cdot 3^{s-l-4} + 8l \cdot 3^{s-l-2} = \frac{72l + 64}{3^l} \cdot 3^{s-4} \leq 23 \cdot 3^{s-3}.$$

i.e., (3.11) holds.

Subcase 2.3. $d(q) = 3l + 2, l \geq 0$

$n_1 = 3(s-l-2)+1$, $n_2 \leq 3(s-l-2)$, by the induction hypothesis, $t(F_1) \leq 4 \cdot 3^{s-l-3}$ and $t(F_2) \leq 3^{s-l-2}$. Moreover, since $\frac{18l+22}{3^l} \leq 23$ for any $l \geq 0$, it follows that for any $s \geq 2$ and $l \geq 0$,

$$4^2 \cdot 3^{s-l-3} + 2(3l + 1) \cdot 3^{s-l-2} = \frac{18l + 22}{3^l} \cdot 3^{s-3} \leq 23 \cdot 3^{s-3}.$$

i.e., (3.11) holds. □

Claim 12. Assume that there exists a vertex x with $d(x) \geq 3$ such all components of $T - x$ are isomorphic to $K_{1,3}$, but one, denoted by T_y where y is the neighbor of x lying in T_y, is not isomorphic to $K_{1,3}$, as shown in Figure 11. We have $t(T) \leq t(F_n)$.

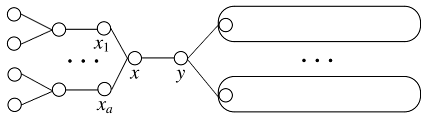

Figure 11. T.

Proof. For an integer $a \geq 2$, assume that $N(x) = \{x_1, \ldots, x_a, y\}$. Let $T - xy = T_x \cup T_y$ where $y \in T_y$. Then $|V(T_x)| = 4a + 1$.

Case 1. $T - y$ has no isolated vertex.

Observe that for an M2-CIS S' of T_y, either $y \notin S'$ or $y \in S'$ with $d_{T[S']}(y) \leq 1$. Let us define $t_y^0 = |\{S' : d_{T[S']}(y) = 0\}|$, $t_{\bar{y}} = |\{S' : y \notin S'\}|$, $t_y^1 = |\{S' : d_{T[S']}(y) = 1\}|$. Thus, $t(T_y) = t_y^0 + t_y^1 + t_{\bar{y}}$.

Observe that for an M2-CIS S of T, either $x \notin S$ or $x \in S$ with $d_{T[S]}(x) \leq 1$. Let

$t_x^0 = |\{S : d_{T[S]}(x) = 0\}| \leq 2^a \cdot (t_y^1 + t_{\bar{y}})$,

$t_x^1 = |\{S : d_{T[S]}(x) = 1\}|$

$= |\{S : d_{T[S]}(x) = 1, y \in S\}| + |\{S : d_{T[S]}(x) = 1, y \notin S\}|$

$\leq 3^a \cdot (t_y^0 + t_{\bar{y}}) + a \cdot 3^{a-1} \cdot t(T_y)$,

$t_{\bar{x}y} = |\{S : x \notin S, d_{T[S]}(x_i) = 1 \text{ } or \text{ } d_{T[S]}(x_i) = d_{T[S]}(x_j) = 0, i, j \in \{1, \ldots, a\}\}|$

$= (4^a - 2^a - a \cdot 2^{a-1}) \cdot t(T_y)$,

$t_{\bar{x}y}^1 = |\{S : x \notin S, d_{T[S]}(y) = 1, d_{T[S]}(x_i) \neq 1, \exists \text{ } exactly \text{ } one \text{ } x_i, d_{T[S]}(x_i) =$
$0 \text{ } or \text{ } x_i \notin S, i \in \{1, \ldots, a\}\}|$

$= (2^a + a \cdot 2^{a-1}) \cdot t_y^1$,

$t_{\bar{x}y}^0 = |\{S : x \notin S, d_{T[S]}(y) = 0, d_{T[S]}(x_i) \neq 1, x_i \notin S, i \in \{1, \ldots, a\}\}|$

$= a \cdot 2^{a-1} \cdot t_y^0$,

$t_{\bar{x}} = |\{S : x \notin S\}| = t_{\bar{x}y} + t_{\bar{x}y}^1 + t_{\bar{x}y}^0$

$= (4^a - 2^a - a \cdot 2^{a-1}) \cdot t(T_y) + (2^a + a \cdot 2^{a-1}) \cdot t_y^1 + a \cdot 2^{a-1} \cdot t_y^0.$

Since $t(T_y) = t_y^0 + t_y^1 + t_{\bar{y}}$, $(2^{a+1} + a \cdot 2^{a-1}) \cdot t_y^1 + (3^a + a \cdot 2^{a-1}) \cdot t_y^0 + (3^a + 2^a) \cdot t_{\bar{y}} \leq (3^a + a \cdot 2^{a-1}) \cdot t(T_y)$. Moreover, $t(T) = t_x^0 + t_x^1 + t_{\bar{x}}$. We get that

$$t(T) \leq [4^a + (a+3)3^{a-1} - 2^a] \cdot t(T_y).$$

Let $V(T'_x) = V(T_x)$. Then $t(T'_x \cup T_y) = t(T'_x) \cdot t(T_y)$. By the induction hypothesis, $t(T_y) \leq t(F_{n-(4a+1)})$. We want to prove $t(T) \leq t(T'_x) \cdot t(F_{n-(4a+1)}) \leq t(F_n)$ for any $a \geq 2$. Therefore, we need to show that for any $a \geq 2$,

$$4^a + (a+3) \cdot 3^{a-1} - 2^a \leq t(T'_x). \tag{3.12}$$

We distinguish into three cases based on the modularity of a (mod 3).

Subcase 1.1. $a = 3s$, $s \geq 1$

Let $T'_x = (4s-1)P_3 \cup K_{1,3}$ where $|V(T'_x)| = 12s + 1$. Then $t(T'_x) = 4 \cdot 3^{4s-1}$. Since $4^{3s} \leq 3^{4s}$ and $(s+1)3^{3s} \leq 3^{4s-1}$ for any $s \geq 2$, by (3.12), it follows that for any $s \geq 2$,

$$4^{3s} + (s+1)3^{3s} - 2^{3s} \leq 3^{4s} + 3^{4s-1} = 4 \cdot 3^{4s-1} = t(T'_x).$$

If $s = 1$, then $a = 3$. Note that $t_x^0 \leq 8(t_y^1 + t_{\bar{y}})$, $t_x^1 \leq 23t_y^0 + 25t_{\bar{y}} + 29t(T_y)$ and $t_{\bar{x}} = 44t(T_y) + 20t_y^1 + 12t_y^0$. Meanwhile, $t(T_y) = t_y^0 + t_y^1 + t_{\bar{y}}$, $t(T) \leq 108t(T_y) = t(T'_x) \cdot t(T_y)$. By the induction hypothesis, $t(T_y) \leq t(F_{n-(4a+1)})$. Hence, $t(T) \leq t(T'_x) \cdot t(F_{n-(4a+1)}) \leq t(F_n)$.

Subcase 1.2. $a = 3s + 1$, $s \geq 1$

Let $T'_x = (4s-1)P_3 \cup 2K_{1,3}$ where $|V(T'_x)| = 3(4s+1)+2$. Then $t(T'_x) = 4^2 \cdot 3^{4s-1}$. Since $4^{3s+1} \leq 4 \cdot 3^{4s}$ and $(3s+4)3^{3s} \leq 4 \cdot 3^{4s-1}$ for any $s \geq 2$, by (3.12), it follows that for any $s \geq 2$,

$$4^{3s+1} + (3s+4)3^{3s} - 2^{3s+1} \leq 4 \cdot 3^{4s} + 4 \cdot 3^{4s-1} = 4^2 \cdot 3^{4s-1} = t(T'_x).$$

The result is true for $s = 1$.

Subcase 1.3. $a = 3s + 2$, $s \geq 0$

Let $T'_x = (4s+3)P_3$ where $|V(T'_x)| = 3(4s+3)$. Then $t(T'_x) = 3^{4s+3}$. Since $4^{3s+2} \leq 4^2 \cdot 3^{4s}$ and $(3s+5) \cdot 3^{3s+1} \leq 11 \cdot 3^{4s}$ for any $s \geq 1$, by (3.12), it follows that for any $s \geq 1$,

$$4^{3s+2} + (3s+5) \cdot 3^{3s+1} - 2^{3s+2} \leq 4^2 \cdot 3^{4s} + 11 \cdot 3^{4s} = 3^{4s+3} = t(T'_x).$$

The result is true for $s = 0$.

Case 2. $T - y$ has an isolated vertex.

By Lemma 2.4, $d(y) \geq 3$. The meanings of notations here are same as those adopted in Case 1. Thus $t(T_y) = t_y^1 + t_{\bar{y}}$, $t_x^0 \leq 2^{a+1} \cdot t_{\bar{y}}$, $t_x^1 \leq a \cdot 3^{a-1} \cdot t(T_y)$, $t_{\bar{x}} \leq (4^a - 2^a - a \cdot 2^{a-1}) \cdot t(T_y) + (2^a + a \cdot 2^{a-1}) \cdot t_y^1$.

Furthmore, $2^{a+1} \cdot t_{\bar{y}} + (2^a + a \cdot 2^{a-1}) \cdot t_y^1 \leq (3^a + a \cdot 2^{a-1}) \cdot t(T_y)$, $t(T) \leq [4^a + (a+3)3^{a-1} - 2^a] \cdot t(T_y)$. By a similar argument as in the proof of Case 1, we show that $t(T) \leq t(F_n)$. □

In view of Claim 11, it remains to consider the case that $d(u) = 3$.

Claim 13. Assume that there exists a path $P := xyzwu$ in T with $d(x) = 1, d(y) = d(z) = d(w) = 2, d(u) = 3$. If $T - u$ has no isolated vertex or isolated edge, then $t(T) \leq t(F_n)$.

Proof. By Claims 1–12, it is not difficult to observe that there exists a vertex x such that $T - x$ has two components, one is isomorphic to P_4, the other one is isomorphic to P_4 or $K_{1,3}$, as shown in Figure 12. The meanings of notations adopted here are same in Case 1 of Claim 12. Let $T - xy = T_x \cup T_y$ where $y \in T_y$ and $T'_x = 3P_3$ where $V(T'_x) = V(T_x)$. Then $t(T'_x) = 3^3$.

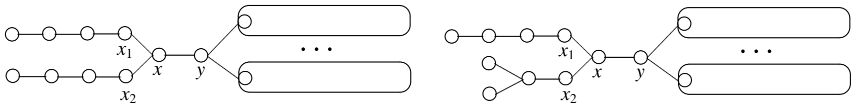

Figure 12. $T - x$ has two components, one is isomorphic to P_4, the other one is isomorphic to P_4 or $K_{1,3}$.

Case 1. $T - x$ has two components which are isomorphic to P_4.

Subcase 1.1. $T - y$ has no isolated vertex.

Note that $t^0_x \leq 4(t^0_y + t_{\bar{y}})$, $t^1_x \leq 6t(T_y) + 9(t^0_y + t^1_y)$, $t_{\bar{x}} \leq 6t(T_y) + 3t^1_y + 2t^0_y$. Meanwhile, $t(T) = t^0_x + t^1_x + t_{\bar{x}}$. This implies that $t(T) \leq 12t(T_y) + 15t^0_y + 12t^1_y + 4t_{\bar{y}}$. Since $t(T_y) = t^0_y + t^1_y + t_{\bar{y}}$, $t(T) \leq 3^3 t(T_y) = t(T'_x) \cdot t(T_y)$. By the induction hypothesis, $t(T_y) \leq t(F_{n-9})$. Hence, $t(T) \leq t(T'_x) \cdot t(F_{n-9}) \leq t(F_n)$.

Subcase 1.2. $T - y$ has an isolated vertex.

By Lemma 2.4, $d(y) \geq 3$. Note that $t^0_x \leq 4t(T_y)$, $t^1_x \leq 9t^1_y + 6t_{\bar{y}}$ and $t_{\bar{x}} \leq 6t(T_y) + 3t^1_y$. Since $t(T) = t^0_x + t^1_x + t_{\bar{x}}$, $t(T) \leq 10t(T_y) + 12t^1_y + 6t_{\bar{y}}$. Moreover, $t(T_y) = t^1_y + t_{\bar{y}}$, $t(T) \leq 22t(T_y) \leq 3^3 t(T_y) = t(T'_x) \cdot t(T_y)$. By the induction hypothesis, $t(T_y) \leq t(F_{n-9})$. Hence, $t(T) \leq t(T'_x) \cdot t(F_{n-9}) \leq t(F_n)$.

Case 2. $T - x$ has two components which are isomorphic to P_4 and $K_{1,3}$, respectively.

Subcase 2.1. $T - y$ has no isolated vertex.

Note that $t^1_x \leq 6t(T_y) + 9(t^0_y + t_{\bar{y}})$, $t^0_x \leq 4(t^1_y + t_{\bar{y}})$, $t_{\bar{x}} \leq 7t(T_y) + 5t^1_y + 3t^0_y$. Since $t(T) = t^0_x + t^1_x + t_{\bar{x}}$, $t(T) \leq 13t(T_y) + 12t^0_y + 9t^1_y + 13t_{\bar{y}}$. Moreover, $t(T_y) = t^0_y + t^1_y + t_{\bar{y}}$ and $t(T) \leq 26t(T_y) \leq 3^3 t(T_y) = t(T'_x) \cdot t(T_y)$. By the induction hypothesis, $t(T_y) \leq t(F_{n-9})$. Hence, $t(T) \leq t(T'_x) \cdot t(F_{n-9}) \leq t(F_n)$.

Subcase 2.2. $T - y$ has an isolated vertex.

By Lemma 2.4, $d(y) \geq 3$. Note that $t^1_x \leq 9(t^1_y + t_{\bar{y}})$, $t^0_x \leq 4t(T_y)$ and $t_{\bar{x}} \leq 7t(T_y) + 5t^1_y$. Since $t(T) = t^0_x + t^1_x + t_{\bar{x}}$, $t(T) \leq 11t(T_y) + 14t^1_y + 9t_{\bar{y}}$. Moreover, $t(T_y) = t^1_y + t_{\bar{y}}$ and $t(T) \leq 25t(T_y) \leq 3^3 t(T_y) = t(T'_x) \cdot t(T_y)$. By the induction hypothesis, $t(T_y) \leq t(F_{n-9})$. Hence, $t(T) \leq t(T'_x) \cdot t(F_{n-9}) \leq t(F_n)$. □

From the above discussion, we proceed to consider the following.

Claim 14. Assume there exists a path $P := xyz$ in T with $d(x) = 1$ or $d(x) = 3$ such that two neighbors of x distinct from y being leaves, $d(y) = 2$ and $d(z) \geq 3$. If $T - z$ has no isolated vertex or isolated edge, then $t(T) \leq t(F_n)$.

Proof. By Claims 1–11 and 13, it remains to consider the case that there exists a vertex w such that $T - w$ has at least two components which are isomorphic to $K_{1,3}$. By Lemma 2.4 and Claim 12, we have $t(T) \leq t(F_n)$. □

This completes the proof of theorem. □

4. Conclusions

In this paper, we determine the maximum number of maximal 2-component independent sets of a forest of order n. It is an interesting problem to determine the maximum number of maximal 2-component independent sets of graphs of order n over some other families, such as trees, bipartite graphs, triangle-free graphs, all connected graphs.

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Conflict of interest

The authors declare no conflict of interests.

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