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Research article

Hybrid ideals in near-subtraction semigroups

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Abstract: The fuzzy set is highly beneficial for expressing people's hesitations in their everyday lives, and it is a great tool for dealing with uncertainty, which can be described precisely and perfectly from the decision-maker's point of view. Soft set theory has been developed in recent years to address real-world issues. Jun et al. merged fuzzy and soft sets to produce hybrid structures. Hybrid structures are soft set and fuzzy set speculations. The concept of hybrid ideals in near-subtraction semigroups is introduced in this paper, and their equivalent results are obtained. Additionally, we demonstrate the concept of hybrid intersection. Moreover, we define the concept of homomorphism of a hybrid structure in a near-subtraction semigroup.

Keywords: semigroup; ideals; subtraction semigroup; hybrid structure; hybrid ideals; hybrid intersection
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1. Introduction

In [31], Schein investigated systems of the form $(\emptyset, \circ, \backslash)$, which is a collection of closed functions under the composition $\circ$ of functions (and thus $(\emptyset, \circ)$ is a function semigroup) and the set theoretic subtraction $\backslash$ (and thus $(\emptyset, \backslash)$ is a subtraction algebra), and he demonstrated that every subtraction semigroup is isomorphic to an invertible function of a difference semigroup. Zelinka [34] discussed Schein's proposal for the structure of multiplication in a subtraction semigroup. He figured out how to solve a problem involving atomic subtraction algebras, a type of subtraction algebra. Jun et al. [7] studied the characterization of ideals in subtraction algebras and established the concept of ideals. Jun and Kim established the ideals generated by a set and their related outcomes in their paper [8].

Near-rings are one of the generalized structures of rings. In 1930, Zassenhaus and Wielandt studied near-rings in relation to group theory and ring theory. However, Wielandt, Frohlich and Blackett redeveloped near-ring research in 1950. Since then, this field has expanded its applications to include automata theory, formal language theory, non-linear interpolation theory and optimization theory, among others (see [2, 12, 14]). Near-subtraction semigroups and strongly regular near-subtraction semigroups were introduced by Dheena et al. [3], and it was demonstrated that a near subtraction semigroup is strongly regular if and only if it is regular and contains no non-zero nilpotent elements.

Zadeh [33] introduced the notion of a fuzzy subset ν of a set T as a mapping from T into $[0, 1]$, which has since been successfully applied in a variety of applications such as control systems engineering, image processing, industrial automation, robotics and optimization. Rosenfeld [30] first proposed the concept of a fuzzy subgroup of a group in 1971. Jun et al. [9] proposed the definition of fuzzy ideals in a gamma near-ring and investigated the related properties. Lee and Park [11] introduced the concept of a fuzzy ideal in subtraction algebras and outlined several conditions for a fuzzy set to be a fuzzy ideal. In near-subtraction semigroups, Prince Williams [32] proposed the concept of fuzzy ideals and fuzzy intersection, as well as the fuzzy image and preimage of the near-subtraction semigroup under homomorphism.

Molodtsov [15] established the concept of a soft set $(F, \mathbb{Z})$ for an initial universe set $\mathbb{Y}$ and the set of parameters $\mathbb{Z}$ as a mapping from $\mathbb{Z}$ into the power set of $\mathbb{Y}$ in 1999, and he successfully applied it to a wide range of fields. Maji et al. [13] were the first to apply soft sets to decision-making problems. Jun et al. [10] pioneered the concept of hybrid structures, which is similar to fuzzy set theory and soft set theory, and investigated several properties of hybrid structures in a set of parameters over an initial universe set. Based on this concept, they created the notions of a hybrid linear space, a hybrid subalgebra and a hybrid field.

Anis et al. [1] proposed the ideas of hybrid subsemigroups and hybrid left (resp., right) ideals in semigroups and described the notions of hybrid products, which were used to examine the characterizations of hybrid subsemigroups and hybrid left (resp., right) ideals. In [4], Elavarasan and Jun obtained some equivalent characterizations of regular and intra-regular semigroups in terms of their hybrid ideals and hybrid bi-ideals, and they derived major results of semigroup characterization through the properties of their hybrid ideals and hybrid bi-ideals. In [6], Elavarasan et al. discussed the properties of a hybrid generalised bi-ideal, an extension of a hybrid bi-ideal, and characterised the regularity of semigroups in terms of hybrid generalised bi-ideals.

In [5], Elavarasan et al. defined the concept of hybrid ideals in near-rings and investigated their properties, as well as the relation between hybrid intersections and hybrid products of hybrid left (resp., right) ideals in zero-symmetric near-rings. G. Muhiuddin et al. introduced the concept of hybrid subsemimodules and examined the hybrid ideals over semirings in [16]. They also obtained equivalent conditions for a semiring to be completely idempotent. Hybrid structures have been applied to a wide range of algebraic systems, yielding a variety of results (see [17–21]). More concepts related to this study have been studied in [22–29].

We introduce the concept of hybrid ideals in near-subtraction semigroups and their associated properties in this paper. In addition, we build an example of a hybrid left ideal that is not a hybrid right ideal and vice versa. We also define the hybrid intersection, as well as the hybrid image and preimage of the near-subtraction semigroup under homomorphism.

2. Preliminaries

We collect a few fundamental definitions in this section that are related to near-subtraction semigroups, which we will use in this paper.

Definition 2.1. *[31] A subtraction algebra is defined as a set $\mathbb{Z}$ ($\neq \emptyset$) with a binary operation "$-$" that meets the below criteria: $\forall q_0, l_0, i_0 \in \mathbb{Z}$,*
(i) $q_0 - (l_0 - q_0) = q_0$.
(ii) $q_0 - (q_0 - l_0) = l_0 - (l_0 - q_0)$.
(iii) $(q_0 - l_0) - i_0 = (q_0 - i_0) - l_0$.

The properties of a subtraction algebra are as follows:
(i) $q_0 - 0 = q_0$ and $0 - q_0 = 0$.
(ii) $(q_0 - l_0) - q_0 = 0$.
(iii) $(q_0 - l_0) - l_0 = q_0 - l_0$.
(iv) $(q_0 - l_0) - (l_0 - q_0) = q_0 - l_0$, where $0 = q_0 - q_0$ is a value independent of the choice of $q_0 \in \mathbb{Z}$.

Definition 2.2. *[34] A subtraction semigroup is defined as a set $\mathbb{Z}$ ($\neq \emptyset$) with the binary operations "$-$" and "." that meets the below criteria:*
(i) $(\mathbb{Z}, -)$ is a subtraction algebra.
(ii) $(\mathbb{Z}, .)$ is a semigroup.
(iii) $l_0(l_1 - l_2) = l_0l_1 - l_0l_2$ and $(l_0 - l_1)l_2 = l_0l_2 - l_1l_2$, $\forall l_0, l_1, l_2 \in \mathbb{Z}$.

Definition 2.3. *[3] A near-subtraction semigroup (briefly, NSS) is defined as a set $\mathbb{Z}$ ($\neq \emptyset$) with the binary operations "$-$" and "." that meets the below criteria:*
(i) $(\mathbb{Z}, -)$ is a subtraction algebra.
(ii) $(\mathbb{Z}, .)$ is a semigroup.
(iii) $(l_0 - l_1)l_2 = l_0l_2 - l_1l_2$, $\forall l_0, l_1, l_2 \in \mathbb{Z}$.

It is obvious that $0l_0 = 0$, $\forall l_0 \in \mathbb{Z}$.

Definition 2.4. *A subset L ($\neq \emptyset$) of $\mathbb{Z}$ is described as a subalgebra of $\mathbb{Z}$ if $l_0 - l_1 \in L$ whenever $l_0, l_1 \in L$.*

Definition 2.5. *Let $(\mathbb{Z}, -, .)$ be an NSS. A subset C ($\neq \emptyset$) of $\mathbb{Z}$ is described as follows:*
(i) A left ideal if C is a subalgebra of $(\mathbb{Z}, -)$ and $nc_1 - n(v - c_1) \in C$, $\forall n, v \in \mathbb{Z}$, $c_1 \in C$;
(ii) A right ideal if C is a subalgebra of $(\mathbb{Z}, -)$ and $C\mathbb{Z} \subseteq C$;
(iii) An ideal if C is both a left and right ideal.

Throughout the paper, $\mathbb{Z}$ denotes a near-subtraction semigroup, and $\mathbb{P}(Q)$ represents the power set of a set Q.

3. Hybrid structures in near-subtraction semigroups

We collect some basic definitions of hybrid structure provided by Jun et al. [10] and define the notion of hybrid ideals in near-subtraction semigroup. We also construct an example of a hybrid left ideal that is not a hybrid right ideal and vice versa.

Definition 3.1. *[10] Let $\mathbb{Y}$ be a universal set. A hybrid structure in $\mathbb{Z}$ over $\mathbb{Y}$ is defined to be a mapping*

$$\tilde{d}_\rho := (\tilde{d}, \rho) : \mathbb{Z} \to \mathbb{P}(\mathbb{Y}) \times [0,1],\ u \mapsto (\tilde{d}(u), \rho(u)),$$

where $\tilde{d} : \mathbb{Z} \to \mathbb{P}(\mathbb{Y})$ and $\rho : \mathbb{Z} \to [0,1]$ are mappings.

Define a relation $\ll$ on the gathering of all hybrid structures, denoted by $\mathscr{H}(\mathbb{Z})$, in $\mathbb{Z}$ over $\mathbb{Y}$ as below:

$$\left(\forall \tilde{d}_\rho, \tilde{b}_\gamma \in \mathscr{H}(\mathbb{Z})\right)\left(\tilde{d}_\rho \ll \tilde{b}_\gamma \iff \tilde{d} \tilde{\subseteq} \tilde{b}, \rho \geq \gamma\right),$$

where $\tilde{d} \tilde{\subseteq} \tilde{b}$ means that $\tilde{d}(u) \subseteq \tilde{b}(u)$ and $\rho \geq \gamma$ means that $\rho(u) \geq \gamma(u)$, $\forall u \in \mathbb{Z}$. Then, the set $(\mathscr{H}(\mathbb{Z}), \ll)$ is partially ordered.

Definition 3.2. *$\tilde{d}_\rho \in \mathscr{H}(\mathbb{Z})$ is described as a hybrid ideal if it meets the below criteria:*

(i) $(\forall l, g \in \mathbb{Z})\begin{pmatrix} \tilde{d}(g-l) \supseteq \tilde{d}(g) \cap \tilde{d}(l) \\ \rho(g-l) \leq \rho(g) \vee \rho(l) \end{pmatrix}$.

(ii) $(\forall s, j, l \in \mathbb{Z})\begin{pmatrix} \tilde{d}(sl - s(j-l)) \supseteq \tilde{d}(l) \\ \rho(sl - s(j-l)) \leq \rho(l) \end{pmatrix}$.

(iii) $(\forall l, q \in \mathbb{Z})\begin{pmatrix} \tilde{d}(lq) \supseteq \tilde{d}(l) \\ \rho(lq) \leq \rho(l) \end{pmatrix}$.

Note that $\tilde{d}_\rho$ is a left hybrid ideal of $\mathbb{Z}$ if it satisfies (i) and (ii), and $\tilde{d}_\rho$ is a right hybrid ideal of $\mathbb{Z}$ if it satisfies (i) and (iii).

Below are some examples of hybrid ideals.

Example 3.1. *Let $\mathbb{Z} = \{0, i, p\}$ be a set in which " $-$ " and "." are defined as below.*

-	0	i	p
0	0	0	0
i	i	0	i
p	p	p	0

.	0	i	p
0	0	0	0
i	0	i	0
p	i	0	p

Then, $(\mathbb{Z}, -, .)$ is an NSS. For $V, K, L \in \mathbb{P}(\mathbb{Y})$ and with $w, r, y \in [0,1]$, define a hybrid structure $\tilde{d}_\rho$ in $\mathbb{Z}$ by $\tilde{d}(0) = L$, $\tilde{d}(i) = K$, $\tilde{d}(p) = V$ and $\rho(0) = w$, $\rho(i) = r$, $\rho(p) = y$.

(i) If $V \subset K = L$, and $y > r = w$, then $\tilde{d}_\rho$ is a hybrid ideal of $\mathbb{Z}$.

(ii) If $K = V \subset L$, and $y = r > w$, then $\tilde{d}_\rho$ is a hybrid right ideal, but it is not a hybrid left ideal, as $\tilde{d}(p.0 - p.(p-0)) = \tilde{d}(i) = K \not\supseteq L = \tilde{d}(0)$, and $\rho(p.0 - p.(p-0)) = \rho(i) = r \nleq w = \rho(0)$.

(iii) If $K \subset V \subset L$, and $r > y > w$, then $\tilde{d}_\rho$ is neither a hybrid right ideal nor a hybrid left ideal, as $\tilde{d}(p.0 - p.(i-0)) = \tilde{d}(i) = K \not\supseteq L = \tilde{d}(0)$, $\rho(p.0 - p.(i-0)) = \rho(i) = r \nleq w = \rho(0)$, and $\tilde{d}(p.0) = \tilde{d}(i) = K \not\supseteq V = \tilde{d}(p)$, $\rho(p.0) = \rho(i) = r \nleq y = \rho(p)$. However, it satisfies condition (i) of Definition 3.2.

Example 3.2. *Let $\mathbb{Z} = \{0, r, a, k\}$ be a set in which " $-$ " and "." are defined as below.*

-	0	r	a	k
0	0	0	0	0
r	r	0	k	a
a	a	0	0	a
k	k	0	k	0

.	0	r	a	k
0	0	0	0	0
r	0	r	a	k
a	0	0	0	0
k	0	r	a	k

Then, $(\mathbb{Z}, -, .)$ *is an NSS. For* $P, W, N \in \mathbb{P}(\mathbb{Y})$ *and* $m, y, s \in [0,1]$*, define a hybrid structure* $\tilde{d}_\rho$ *in* $\mathbb{Z}$ *by* $\tilde{d}(0) = P$*,* $\tilde{d}(r) = W$*,* $\tilde{d}(a) = N = \tilde{d}(k)$ *and* $\rho(0) = m$*,* $\rho(r) = y$*,* $\rho(a) = s = \rho(k)$*.*

If $N \subset W \subset P$*, and* $s > y > m$*, then* $\tilde{d}_\rho$ *is a hybrid left ideal, but it is not a hybrid right ideal, as* $\tilde{d}(r.a) = \tilde{d}(a) = N \not\supseteq W = \tilde{d}(r)$*, and* $\rho(r.a) = \rho(a) = s \nleq y = \rho(r)$*.*

Definition 3.3. *For* $\tilde{d}_\tau \in \mathscr{H}(\mathbb{Z})$ *and* $G \in \mathbb{P}(\mathbb{Z})\backslash\{\emptyset\}$*, the characteristic hybrid structure in* $\mathbb{Z}$ *over* $\mathbb{Y}$ *is denoted by* $\chi_G(\tilde{d}_\tau)$*, and it is described as*

$$\begin{aligned} \chi_G(\tilde{d}_\tau) = (\chi_G(\tilde{d}), \chi_G(\tau)) : \mathbb{Z} \longrightarrow \mathbb{P}(\mathbb{Y}) \times [0,1], \\ u \mapsto \left(\chi_G(\tilde{d})(u), \chi_G(\tau)(u)\right), \end{aligned}$$

where

$$\chi_G(\tilde{d}) : \mathbb{Z} \to \mathbb{P}(\mathbb{Y}), u \mapsto \begin{cases} \mathbb{Y}, & if \;\; u \in G, \\ \emptyset, & otherwise, \end{cases}$$

$$\chi_G(\tau) : \mathbb{Z} \to [0,1], u \mapsto \begin{cases} 0, & if \; u \in G, \\ 1, & otherwise, \end{cases}$$

for any $u \in \mathbb{Z}$*.*

Definition 3.4. *Let* $\tilde{d}_\rho \in \mathscr{H}(\mathbb{Z})$*. For any* $(\Gamma, \omega) \in \mathbb{P}(\mathbb{Y}) \times [0,1]$*, we define*

$$\mathbb{Z}_{\tilde{d}}^{\Gamma} := \{l \in \mathbb{Z} \; : \; \tilde{d}(l) \supseteq \Gamma\} \;\; and \;\; \mathbb{Z}_\rho^\omega := \{l \in \mathbb{Z} \; : \; \rho(l) \leq \omega\}.$$

Definition 3.5. *For* $\tilde{d}_\rho \in \mathscr{H}(\mathbb{Z})$*, the set*

$$\tilde{d}_\rho[T, t] := \{l \in \mathbb{Z} \; : \; \tilde{d}(l) \supseteq T \; and \; \rho(l) \leq t\}$$

is known as the $[T, t]$*-hybrid cut of* $\tilde{d}_\rho$*, where* $T \in \mathbb{P}(\mathbb{Y})$*, and* $t \in [0,1]$*.*

Note that $\mathbb{Z}_{\tilde{d}}^{\Gamma} \cap \mathbb{Z}_\rho^\omega = \{l \in \mathbb{Z} \; : \; \tilde{d}(l) \supseteq \Gamma \; and \; \rho(l) \leq \omega\} = \tilde{d}_\rho[\Gamma, \omega]$*.*

Definition 3.6. *Let* $\{\tilde{d}_{i_{\rho_i}}\}$ *be the family of hybrid structures in* $\mathbb{Z}$*. Then, the hybrid intersection* $\left\{\tilde{\Cap}\tilde{d}_{i_{\rho_i}}\right\}$ *of* $(\tilde{d}_{i_{\rho_i}})$ *over* $\mathbb{Y}$ *is defined by* $\left\{\tilde{\Cap}\tilde{d}_{i_{\rho_i}}\right\} := \left(\tilde{\cap}\tilde{d}_i, \tilde{\vee}\rho_i\right)$*, where* $(\tilde{\cap}_i\tilde{d}_i)(l) = \bigcap(\tilde{d}_i(l))$*, and* $(\tilde{\vee}_i\rho_i)(l) = \bigvee(\rho_i(l))$*.*

4. Hybrid ideals in near-subtraction semigroups

Theorem 4.1. *If* $\tilde{d}_\rho$ *is a hybrid left (resp., right) ideal of* $\mathbb{Z}$*, then the sets*

$$\mathbb{Z}_{\tilde{d}} = \{l \in \mathbb{Z} \; : \; \tilde{d}(l) = \tilde{d}(0)\} \; and \; \mathbb{Z}_\rho = \{l \in \mathbb{Z} \; : \; \rho(l) = \rho(0)\}$$

are left (resp., right) ideals of $\mathbb{Z}$*.*

Proof. For $l, w \in \mathbb{Z}_{\tilde{d}_\rho}$, $\tilde{d}(l - w) \supseteq \tilde{d}(l) \cap \tilde{d}(w) = \tilde{d}(0)$, and $\rho(l - w) \leq \rho(l) \vee \rho(w) = \rho(0)$. Thus, $l - w \in \mathbb{Z}_{\tilde{d}_\rho}$. For $s \in \mathbb{Z}$, we have $\tilde{d}(sl - s(w - l)) \supseteq \tilde{d}(l) = \tilde{d}(0)$, and $\rho(sl - s(w - l)) \leq \rho(l) = \rho(0)$. Thus, $sl - s(w - l) \in \mathbb{Z}_{\tilde{d}_\rho}$. So, $\mathbb{Z}_{\tilde{d}_\rho}$ is a left ideal of $\mathbb{Z}$.

Theorem 4.2. *Let* $\tilde{d}_\rho, \tilde{y}_\nu \in \mathscr{H}(\mathbb{Z})$*. If they are hybrid ideals of* $\mathbb{Z}$*, then* $\tilde{d}_\rho \Cap \tilde{y}_\nu$ *is a hybrid ideal of* $\mathbb{Z}$ *.*

Proof. Let $w, g \in \mathbb{Z}$. Then,

$$\begin{aligned}
(\tilde{d} \cap \tilde{y})(g - w) &= \tilde{d}(g - w) \cap \tilde{y}(g - w) \\
&\supseteq \{\tilde{d}(g) \cap \tilde{d}(w)\} \cap \{\tilde{y}(g) \cap \tilde{y}(w)\} \\
&= (\tilde{d} \cap \tilde{y})(g) \cap (\tilde{d} \cap \tilde{y})(w), \\
(\rho \vee \nu)(g - w) &= \rho(g - w) \vee \nu(g - w) \\
&\leq \{\rho(g) \vee \rho(w)\} \vee \{\nu(g) \vee \nu(w)\} \\
&= (\rho \vee \nu)(g) \vee (\rho \vee \nu)(w).
\end{aligned}$$

For $s \in \mathbb{Z}$, we have

$$\begin{aligned}
(\tilde{d} \cap \tilde{y})(sw - s(g - w)) &= \tilde{d}(sw - s(g - w)) \cap \tilde{y}(sw - s(g - w)) \\
&\supseteq \tilde{d}(w) \cap \tilde{y}(w) \\
&= (\tilde{d} \cap \tilde{y})(w), \\
(\rho \vee \nu)(sw - s(g - w)) &= \rho(sw - s(g - w)) \vee \nu(sw - s(g - w)) \\
&\leq \rho(w) \vee \nu(w) \\
&= (\rho \vee \nu)(w).
\end{aligned}$$

Also,

$$\begin{aligned}
(\tilde{d} \cap \tilde{y})(wg) &= \tilde{d}(wg) \cap \tilde{y}(wg) \\
&\supseteq \tilde{d}(w) \cap \tilde{y}(w) \\
&= (\tilde{d} \cap \tilde{y})(w), \\
(\rho \vee \nu)(wg) &= \rho(wg) \vee \nu(wg) \\
&\leq \rho(w) \vee \nu(w) \\
&= (\rho \vee \nu)(w).
\end{aligned}$$

So, $\tilde{d}_\rho \Cap \tilde{y}_\nu$ is a hybrid ideal of $\mathbb{Z}$.

Note that for a family of hybrid ideals $\{\tilde{d}_{i_{\rho_i}}\}$ of $\mathbb{Z}$, $\widehat{\Cap}\tilde{d}_{i_{\rho_i}}$ of $\mathbb{Z}$ is a hybrid ideal.

Theorem 4.3. *For $G \in \mathbb{P}(\mathbb{Z}) \setminus \{\emptyset\}$, define a hybrid structure $\tilde{d}_\rho$ in $\mathbb{Z}$ by*

$$\tilde{d}(l) := \begin{cases} Q_0, & if \;\; l \in G, \\ Q_1, & otherwise, \end{cases} \quad and \;\; \rho(l) := \begin{cases} q_0, & if \;\; l \in G, \\ q_1, & otherwise, \end{cases}$$

where $Q_0, Q_1 \in \mathbb{P}(\mathbb{Y})$ with $Q_0 \supset Q_1$ and $q_0, q_1 \in [0, 1]$ with $q_0 < q_1$. The conditions mentioned below are equivalent.

(i) $\tilde{d}_\rho$ is a hybrid ideal of $\mathbb{Z}$.

(ii) G of $\mathbb{Z}$ is an ideal. Moreover, $\mathbb{Z}_{\tilde{d}_\rho} = G$.

Proof. $(i) \Rightarrow (ii)$ Let $l, b \in G$. Then, $\tilde{d}(l - b) \supseteq \tilde{d}(l) \cap \tilde{d}(b) = Q_0$, and $\rho(l - b) \leq \rho(l) \vee \rho(b) = q_0$. So, $l - b \in G$. For $s \in \mathbb{Z}$, we have $\tilde{d}(sl - s(b - l)) \supseteq \tilde{d}(l) = Q_0$, and $\rho(sl - s(b - l)) \leq \rho(l) = q_0$, implying that $sl - s(b - l) \in G$. Also, $\tilde{d}(lb) \supseteq \tilde{d}(l) = Q_0$, and $\rho(lb) \leq \rho(l) = q_0$. We get $lb \in G$. Therefore, G of $\mathbb{Z}$ is an ideal.

$(ii) \Rightarrow (i)$ Let $l, b, s \in \mathbb{Z}$. If $l \notin G$ or $b \notin G$, then $\tilde{d}(l-b) \supseteq Q_1 = \tilde{d}(l) \cap \tilde{d}(b)$, and $\rho(l-b) \leq q_1 = \rho(l) \vee \rho(b)$. Otherwise, $l, b \in G$, and then $l - b \in G$, which implies $\tilde{d}(l-b) \supseteq Q_0 = \tilde{d}(l) \cap \tilde{d}(b)$, and $\rho(l-b) \leq q_0 = \rho(l) \vee \rho(g)$.

If $l \in G$, then $sl - s(b-l) \in G$, which implies $\tilde{d}(sl - s(b-l)) \supseteq Q_0 = \tilde{d}(l)$, and $\rho(sl - s(b-l)) \leq q_0 = \rho(l)$. Otherwise, $l \notin G$, and then $\tilde{d}(sl - s(b-l)) \supseteq Q_1 = \tilde{d}(l)$, and $\rho(sl - s(b-l)) \leq q_1 = \rho(l)$.

If $l \in G$, then $lb \in G$, which implies $\tilde{d}(lb) \supseteq Q_0 = \tilde{d}(l)$, and $\rho(lb) \leq q_0 = \rho(l)$. Otherwise, $l \notin G$, and then $\tilde{d}(lb) \supseteq Q_1 = \tilde{d}(l)$, and $\rho(lb) \leq q_1 = \rho(l)$.

Therefore, $\tilde{d}_\rho$ is a hybrid ideal of $\mathbb{Z}$.

Moreover,

$$\begin{aligned}
\mathbb{Z}_{\tilde{d}} &= \{l \in \mathbb{Z} \ : \ \tilde{d}(l) = \tilde{d}(0)\} \\
&= \{l \in \mathbb{Z} \ : \ \tilde{d}(l) = Q_0\} \\
&= \{l \in \mathbb{Z} \ : \ l \in G\} = G, \\
\mathbb{Z}_{\rho} &= \{l \in \mathbb{Z} \ : \ \rho(l) = \rho(0)\} \\
&= \{l \in \mathbb{Z} \ : \ \rho(l) = q_0\} \\
&= \{l \in \mathbb{Z} \ : \ l \in G\} = G.
\end{aligned}$$

Hence, $\mathbb{Z}_{\tilde{d}_\rho} = G$.

Corollary 4.1. *For $G \in \mathbb{P}(\mathbb{Z}) \setminus \{\emptyset\}$ and $\tilde{d}_\rho \in \mathscr{H}(\mathbb{Z})$, the below conditions are equivalent:*

(i) $\chi_G(\tilde{d}_\rho)$ of $\mathbb{Z}$ is a hybrid left (resp., right) ideal.

(ii) G of $\mathbb{Z}$ is a left (resp., right) ideal.

Theorem 4.4. *Let $\tilde{d}_\rho \in \mathscr{H}(\mathbb{Z})$. Then, the assertions mentioned below are equivalent:*

(i) $\tilde{d}_\rho$ is a hybrid ideal in $\mathbb{Z}$.

(ii) $\forall (\Gamma, \omega) \in \mathbb{P}(\mathbb{Y}) \times [0,1]$, the sets $\mathbb{Z}_{\tilde{d}}^{\Gamma} \neq \emptyset$ and $\mathbb{Z}_{\rho}^{\omega} \neq \emptyset$ are ideals of $\mathbb{Z}$.

Proof. $(i) \Rightarrow (ii)$ Let $\Gamma \in \mathbb{P}(\mathbb{Y})$ and $\omega \in [0,1]$ be such that $\mathbb{Z}_{\tilde{d}}^{\Gamma} \neq \emptyset$, and $\mathbb{Z}_{\rho}^{\omega} \neq \emptyset$. Let $l, g, s \in \mathbb{Z}$.

If $l, g \in \mathbb{Z}_{\tilde{d}}^{\Gamma} \cap \mathbb{Z}_{\rho}^{\omega}$, then $\tilde{d}(l-g) \supseteq \tilde{d}(l) \cap \tilde{d}(g) \supseteq \Gamma$, $\rho(l-g) \leq \rho(l) \vee \rho(g) \leq \omega$, and so $l - g \in \mathbb{Z}_{\tilde{d}}^{\Gamma} \cap \mathbb{Z}_{\rho}^{\omega}$.

Also, if $l \in \mathbb{Z}_{\tilde{d}}^{\Gamma} \cap \mathbb{Z}_{\rho}^{\omega}$, then $\tilde{d}(sl - s(g-l)) \supseteq \tilde{d}(l) \supseteq \Gamma$, $\rho(sl - s(g-l)) \leq \rho(l) \leq \omega$, and $\tilde{d}(lg) \supseteq \tilde{d}(l) \supseteq \Gamma$, $\rho(lg) \leq \rho(l) \leq \omega$. Thus, $sl - s(g-l)$, $lg \in \mathbb{Z}_{\tilde{d}}^{\Gamma} \cap \mathbb{Z}_{\rho}^{\omega}$, and hence $\mathbb{Z}_{\tilde{d}}^{\Gamma}$ and $\mathbb{Z}_{\rho}^{\omega}$ are ideals in $\mathbb{Z}$.

$(ii) \Rightarrow (i)$ Let $l, g \in \mathbb{Z}$ be such that $\tilde{d}(l) = \Gamma_l$, and $\tilde{d}(g) = \Gamma_g$ for some $\Gamma_l, \Gamma_g \in \mathbb{P}(\mathbb{Y})$. If we put $\Gamma := \Gamma_l \cap \Gamma_g$, then $l, g \in \mathbb{Z}_{\tilde{d}}^{\Gamma}$, and $l - g \in \mathbb{Z}_{\tilde{d}}^{\Gamma}$, so $\tilde{d}(l-g) \supseteq \Gamma = \Gamma_l \cap \Gamma_g = \tilde{d}(l) \cap \tilde{d}(g)$. Let $s \in \mathbb{Z}$. Then, $sl - s(g-l) \in \mathbb{Z}_{\tilde{d}}^{\Gamma}$, and $ls \in \mathbb{Z}_{\tilde{d}}^{\Gamma}$, implying that $\tilde{d}(sl - s(g-l)) \supseteq \Gamma_l = \tilde{d}(l)$, and $\tilde{d}(ls) \supseteq \Gamma_l = \tilde{d}(l)$.

Also, let $\rho(l) = \omega_l$ and $\rho(g) = \omega_g$ for some $\omega_l, \omega_g \in [0,1]$. Then, by taking $\omega := \omega_l \vee \omega_g$, we get $l, g \in \mathbb{Z}_{\rho}^{\omega}$ and $l - g \in \mathbb{Z}_{\rho}^{\omega}$, so $\rho(l-g) \leq \omega = \omega_l \vee \omega_g = \rho(l) \vee \rho(g)$. Let $s \in \mathbb{Z}$. Then, $sl - s(g-l)$, and $ls \in \mathbb{Z}_{\rho}^{\omega}$ imply $\rho(sl - s(g-l)) \leq \omega_l = \rho(l)$, and $\rho(ls) \leq \omega_l = \rho(l)$.

So, $\tilde{d}_\rho$ in $\mathbb{Z}$ over $\mathbb{Y}$ is a hybrid ideal.

Theorem 4.5. *For $\tilde{d}_\rho \in \mathscr{H}(\mathbb{Z})$, let $\{\mathbb{Z}_{\tilde{d}}^{\Gamma} \mid \Gamma \in \mathbb{P}(\mathbb{Y})\}$ and $\{\mathbb{Z}_{\rho}^{\sigma} \ : \ \sigma \in [0,1]\}$ be the gathering of ideals of $\mathbb{Z}$ such that*

(i) $\mathbb{Z} = \bigcup_{\Gamma \in \mathbb{P}(\mathbb{Y})} \mathbb{Z}_{\tilde{d}}^{\Gamma} = \bigcup_{\sigma \in [0,1]} \mathbb{Z}_{\rho}^{\sigma}$,

(ii) $Z \supset H$ if and only if $\mathbb{Z}_{\tilde{d}}^{Z} \subset \mathbb{Z}_{\tilde{d}}^{H}$, $\forall Z, H \in \mathbb{P}(\mathbb{Y})$,

(iii) $z < h$ if and only if $\mathbb{Z}_{\rho}^{z} \subset \mathbb{Z}_{\rho}^{h}$, $\forall z, h \in [0,1]$.

Construct a hybrid structure $\tilde{b}_\delta$ in $\mathbb{Z}$ over $\mathbb{Y}$ by $\tilde{b}(c) = \bigcup\{\Gamma \mid \Gamma \in \mathbb{P}(\mathbb{Y})$ and $c \in \mathbb{Z}_{\tilde{d}}^{\Gamma}\}$, and $\delta(c) = \bigwedge\{\sigma \mid \sigma \in [0,1]$ and $c \in \mathbb{Z}_\rho^\sigma\}$, $\forall c \in \mathbb{Z}$. Then, $\tilde{b}_\delta$ is a hybrid ideal in $\mathbb{Z}$ over $\mathbb{Y}$.

Proof. By Theorem 4.4, it is enough to show that $\mathbb{Z}_{\tilde{b}}^{\Gamma}$ and $\mathbb{Z}_\delta^\sigma$ are ideals of $\mathbb{Z}$ for all $(\Gamma, \sigma) \in \mathbb{P}(\mathbb{Y}) \times [0,1]$.

Now, we prove that $\mathbb{Z}_{\tilde{b}}^{\Gamma}$ is an ideal of $\mathbb{Z}$ for $\Gamma \in \mathbb{P}(\mathbb{Y})$.

Consider the below two cases:

(*i*) $\Gamma = \bigcup\{\Gamma_0 \mid \Gamma_0 \in \mathbb{P}(\mathbb{Y})$ and $c \in \mathbb{Z}_{\tilde{d}}^{\Gamma_0}\}$.

(*ii*) $\Gamma \neq \bigcup\{\Gamma_0 \mid \Gamma_0 \in \mathbb{P}(\mathbb{Y})$ and $c \in \mathbb{Z}_{\tilde{d}}^{\Gamma_0}\}$.

Case (*i*) gives that $c \in \mathbb{Z}_{\tilde{b}}^{\Gamma}$ if and only if $c \in \mathbb{Z}_{\tilde{d}}^{\Gamma_1}$, $\forall \Gamma_1 \subset \Gamma$ if and only if $c \in \bigcap_{\Gamma_1 \subset \Gamma} \mathbb{Z}_{\tilde{d}}^{\Gamma_1}$. Thus, $\mathbb{Z}_{\tilde{b}}^{\Gamma} = \bigcap_{\Gamma_1 \subset \Gamma} \mathbb{Z}_{\tilde{d}}^{\Gamma_1}$, and hence $\mathbb{Z}_{\tilde{b}}^{\Gamma}$ is an ideal of $\mathbb{Z}$.

For case (*ii*), we claim that $\mathbb{Z}_{\tilde{b}}^{\Gamma} = \bigcup_{\Gamma_1 \supseteq \Gamma} \mathbb{Z}_{\tilde{d}}^{\Gamma_1}$. If $c_1 \in \bigcup_{\Gamma_1 \supseteq \Gamma} \mathbb{Z}_{\tilde{d}}^{\Gamma_1}$, then $c_1 \in \mathbb{Z}_{\tilde{d}}^{\Gamma_1}$ for some $\Gamma_1 \supseteq \Gamma$. By assumption, we have $c_1 \in \mathbb{Z}_{\tilde{d}}^{\Gamma}$, so $\bigcup_{\Gamma_1 \supseteq \Gamma} \mathbb{Z}_{\tilde{d}}^{\Gamma_1} \subseteq \mathbb{Z}_{\tilde{b}}^{\Gamma}$. For the converse, if $c_1 \notin \bigcup_{\Gamma_1 \supseteq \Gamma} \mathbb{Z}_{\tilde{d}}^{\Gamma_1}$, then $c_1 \notin \mathbb{Z}_{\tilde{d}}^{\Gamma_1}$, $\forall \Gamma_1 \supseteq \Gamma$. In particular, $c_1 \notin \mathbb{Z}_{\tilde{b}}^{\Gamma}$, so $\mathbb{Z}_{\tilde{b}}^{\Gamma} = \bigcup_{\Gamma_1 \supseteq \Gamma} \mathbb{Z}_{\tilde{d}}^{\Gamma_1}$, and $\mathbb{Z}_{\tilde{b}}^{\Gamma}$ is an ideal of $\mathbb{Z}$.

Next, we prove that $\mathbb{Z}_\rho^{\sigma_0}$ is an ideal of $\mathbb{Z}$ for $\sigma_0 \in [0,1]$.

Consider the two cases:

(*i*) $\sigma_0 = \bigwedge\{\sigma \mid \sigma \in [0,1]$ and $\sigma > \sigma_0\}$.

(*ii*) $\sigma_0 \neq \bigwedge\{\sigma \mid \sigma \in [0,1]$ and $\sigma > \sigma_0\}$.

Case (*i*) gives that $c \in \mathbb{Z}_\delta^{\sigma_0}$ if and only if $c \in \mathbb{Z}_\rho^\sigma$, $\forall \sigma > \sigma_0$ if and only if $c \in \bigcap_{\sigma > \sigma_0} \mathbb{Z}_\rho^\sigma$. Thus, $\mathbb{Z}_\delta^{\sigma_0} = \bigcap_{\sigma > \sigma_0} \mathbb{Z}_\rho^\sigma$, and $\mathbb{Z}_\delta^{\sigma_0}$ is an ideal of $\mathbb{Z}$.

For case (*ii*), we claim that $\mathbb{Z}_\delta^{\sigma_0} = \bigcup_{\sigma \geq \sigma_0} \mathbb{Z}_\rho^\sigma$. If $c \in \bigcup_{\sigma \geq \sigma_0} \mathbb{Z}_\rho^\sigma$, then $c \in \mathbb{Z}_\rho^\sigma$ for some $\sigma \leq \sigma_0$. By assumption, we have $c \in \mathbb{Z}_\delta^\sigma$, so $\bigcup_{\sigma \geq \sigma_0} \mathbb{Z}_\rho^\sigma \subseteq \mathbb{Z}_\delta^{\sigma_0}$. For the converse, if $c_1 \notin \bigcup_{\sigma \geq \sigma_0} \mathbb{Z}_\rho^\sigma$, then $c_1 \notin \mathbb{Z}_\rho^\sigma$, $\forall \sigma \geq \sigma_0$. In particular, $c_1 \notin \mathbb{Z}_\delta^{\sigma_0}$. So, $\mathbb{Z}_\delta^{\sigma_0} = \bigcup_{\sigma \geq \sigma_0} \mathbb{Z}_\rho^\sigma$, and $\mathbb{Z}_\delta^{\sigma_0}$ is an ideal of $\mathbb{Z}$. □

Theorem 4.6. *For a family of ideals $\{G_m$: m is a positive integer and $\mathbb{Z} = G_1 \supset G_2 \supset ...\}$ in $\mathbb{Z}$, define a hybrid structure $\tilde{d}_\rho$ in $\mathbb{Z}$ over $\mathbb{Y} = [0,1]$ by*

$$\tilde{d}(l) := \begin{cases} \left[0, \frac{m}{m+1}\right], & if \;\; l \in G_m \setminus G_{m+1}, \\ [0,1], & if \; l \in \bigcap_{m=1}^{\infty} G_m, \end{cases} \quad and \; \rho(l) := \begin{cases} \frac{1}{m+1}, & if \;\; l \in G_m \setminus G_{m+1}, \\ 0, & if \; l \in \bigcap_{m=1}^{\infty} G_m, \end{cases}$$

$\forall l \in \mathbb{Z}$. Then, $\tilde{d}_\rho$ in $\mathbb{Z}$ is a hybrid ideal over $\mathbb{Y} = [0,1]$.

Proof. Let $l, g \in \mathbb{Z}$.

(i) Consider that $l \in G_p \setminus G_{p+1}$ and $g \in G_a \setminus G_{a+1}$ for some $p, a \in \{1, 2, ...\}$. Without loss of generality, consider that $p \leq a$. Then, $l - g \in G_p$ implies $\tilde{d}(l-g) \supseteq \left[0, \frac{p}{p+1}\right] = \tilde{d}(l) \cap \tilde{d}(g)$, and $\rho(l-g) \leq \frac{1}{p+1} = \rho(l) \vee \rho(g)$.

If $l, g \in \bigcap_{m=1}^{\infty} G_m$, then $l - g \in \bigcap_{m=1}^{\infty} G_m$, and thus $\tilde{d}(l-g) = [0,1] = \tilde{d}(l) \cap \tilde{d}(g)$, and $\rho(l-g) = 0 = \rho(l) \vee \rho(g)$.

If $l \in \bigcap_{m=1}^{\infty} G_m$, and $g \notin \bigcap_{m=1}^{\infty} G_m$, then $\exists\, q \in \mathbb{N} : g \in G_q \setminus G_{q+1}$. It follows that $l - g \in G_q$, $\tilde{d}(l-g) \supseteq \left[0, \frac{q}{q+1}\right] = \tilde{d}(l) \cap \tilde{d}(g)$, and $\rho(l-g) \leq \frac{1}{q+1} = \rho(l) \vee \rho(g)$.

Similarly, $\tilde{d}(l-g) \supseteq \tilde{d}(l) \cap \tilde{d}(g)$, and $\rho(l-g) \leq \rho(l) \vee \rho(g)$ whenever $l \notin \bigcap_{m=1}^{\infty} G_m$ and $g \in \bigcap_{m=1}^{\infty} G_m$.

(ii) If $s, g \in \mathbb{Z}$ and $l \in G_p \setminus G_{p+1}$ for some $p = 1, 2, ...$, then $sl - s(g-l) \in G_p$. Thus, $\tilde{d}(sl - s(g-l)) \supseteq \left[0, \frac{p}{p+1}\right] = \tilde{d}(l)$, and $\rho(sl - s(g-l)) \leq \frac{1}{p+1} = \rho(l)$.

If $l \in \bigcap_{m=1}^{\infty} G_m$, then $sl - s(g-l) \in \bigcap_{m=1}^{\infty} G_m, \forall s, g \in \mathbb{Z}$. Thus, $\tilde{d}(sl - s(g-l)) = [0,1] = \tilde{d}(l)$, and $\rho(sl - s(g-l)) = 0 = \rho(l)$.

If $s \in G_a \setminus G_{a+1}$ for some $a = 1, 2, ...$, and $g \in \bigcap_{m=1}^{\infty} G_m$ (or, $s \in \bigcap_{m=1}^{\infty} G_m$, and $g \in G_a \setminus G_{a+1}$ for some $a = 1, 2, ...$), then $l \in G_a$. Thus,

$$\tilde{d}(sl - s(g-l)) \supseteq \left[0, \frac{a}{a+1}\right] = \tilde{d}(l), \text{ and } \rho(sl - s(g-l)) \leq \frac{1}{a+1} = \rho(l).$$

(iii) Now, if $l, g \in G_p \setminus G_{p+1}$ for some $p = 1, 2, ...$, then $lg \in G_a$, as G_a is an ideal of $\mathbb{Z}$. Thus,

$$\tilde{d}(lg) \supseteq \left[0, \frac{a}{a+1}\right] = \tilde{d}(l), \text{ and } \rho(lg) \leq \frac{1}{a+1} = \rho(l).$$

If $l, g \in \bigcap_{m=1}^{\infty} G_m$, then $g \in \bigcap_{m=1}^{\infty} G_m$. Thus, $\tilde{d}(lg) = [0,1] = \tilde{d}(l)$, and $\rho(lg) = 0 = \rho(l)$.

Therefore, $\tilde{d}_\rho$ of $\mathbb{Z}$ is a hybrid ideal over $\mathbb{Y} = [0,1]$.

5. Homomorphism of a hybrid structure

In this section, we present some characteristics related to homomorphism of hybrid structures in a near-subtraction semigroup. Throughout this section, $\mathbb{Z}$ and $\mathbb{Z}'$ denote the near-subtraction semigroups.

Definition 5.1. *A homomorphism of $\mathbb{Z}$ into $\mathbb{Z}'$ such that $\vartheta(w-a) = \vartheta(w) - \vartheta(a)$ and $\vartheta(wa) = \vartheta(w)\vartheta(a)$, $\forall w, a \in \mathbb{Z}$ is defined.*

Definition 5.2. *Let $\vartheta : \mathbb{Z} \to \mathbb{Z}'$ be a mapping, where $\mathbb{Z}, \mathbb{Z}' \neq \{\emptyset\}$ and $\tilde{d}_\rho \in \mathscr{H}(\mathbb{Z}')$. The preimage of $\tilde{d}_\rho$ under ϑ, denoted as $\vartheta^{-1}(\tilde{d}_\rho)$, is a hybrid structure of $\mathbb{Z}$ defined by $\vartheta^{-1}(\tilde{d}_\rho) := (\vartheta^{-1}(\tilde{d}), \vartheta^{-1}(\rho))$, where $\vartheta^{-1}(\tilde{d}) = \tilde{d}(\vartheta(l))$ and $\vartheta^{-1}(\rho) = \rho(\vartheta(l))$, $\forall l \in \mathbb{Z}$.*

Theorem 5.1. *Let $\vartheta : \mathbb{Z} \to \mathbb{Z}'$ be a homomorphism of an NSS. If $\tilde{d}_\rho$ of $\mathbb{Z}'$ is a hybrid ideal, then $\vartheta^{-1}(\tilde{d}_\rho)$ of $\mathbb{Z}$ is a hybrid ideal.*

Proof. Assume that $\tilde{d}_\rho$ of $\mathbb{Z}'$ is a hybrid ideal. Let $a, g \in \mathbb{Z}$. Then,

$$\begin{aligned}
\vartheta^{-1}(\tilde{d})(a-g) &= \tilde{d}(\vartheta(a-g)) = \tilde{d}(\vartheta(a) - \vartheta(g)) \\
&\supseteq \tilde{d}(\vartheta(a)) \cap \tilde{d}(\vartheta(g)) \\
&= \vartheta^{-1}(\tilde{d})(a) \cap \vartheta^{-1}(\tilde{d})(g), \\
\vartheta^{-1}(\rho)(a-g) &= \rho(\vartheta(a-g)) \\
&= \rho(\vartheta(a) - \vartheta(g)) \\
&\leq \rho(\vartheta(a)) \vee \rho(\vartheta(g)) \\
&= \vartheta^{-1}(\rho)(a) \vee \vartheta^{-1}(\rho)(g).
\end{aligned}$$

Let $q \in \mathbb{Z}$. Then,

$$\begin{aligned}
\vartheta^{-1}(\tilde{d})(qa - q(g-a)) &= \tilde{d}(\vartheta(qa - q(g-a))) \\
&= \tilde{d}(\vartheta(qa) - \vartheta(q(g-a))) \\
&= \tilde{d}(\vartheta(q)\vartheta(a) - \vartheta(q)(\vartheta(g) - \vartheta(a))) \\
&\supseteq \tilde{d}(\vartheta(a)) \\
&= \vartheta^{-1}(\tilde{d})(a), \\
\vartheta^{-1}(\rho)(qa - q(g-a)) &= \rho(\vartheta(qa - q(g-a))) \\
&= \rho(\vartheta(qa) - \vartheta(q(g-a))) \\
&= \rho(\vartheta(q)\vartheta(a) - \vartheta(q)(\vartheta(g) - \vartheta(a))) \\
&\leq \rho(\vartheta(a)) \\
&= \vartheta^{-1}(\rho)(a).
\end{aligned}$$

Also,

$$\vartheta^{-1}(\tilde{d})(ag) = \tilde{d}(\vartheta(ag)) = \tilde{d}(\vartheta(a)\vartheta(g)) \supseteq \tilde{d}(\vartheta(a)) = \vartheta^{-1}(\tilde{d})(a).$$
$$\vartheta^{-1}(\rho)(ag) = \rho(\vartheta(ag)) = \rho(\vartheta(a)\vartheta(g)) \leq \rho(\vartheta(a)) = \vartheta^{-1}(\rho)(a).$$

So, $\vartheta^{-1}(\tilde{d}_\rho)$ of $\mathbb{Z}$ is a hybrid ideal.

Theorem 5.2. *Let $\vartheta : \mathbb{Z} \to \mathbb{Z}'$ be an onto homomorphism of an NSS, and $\tilde{d}_\rho \in \mathscr{H}(\mathbb{Z}')$. If $\vartheta^{-1}(\tilde{d}_\rho)$ of $\mathbb{Z}$ is a hybrid ideal, then $\tilde{d}_\rho$ of $\mathbb{Z}'$ is a hybrid ideal.*

Proof. Let $\vartheta^{-1}(\tilde{d}_\rho)$ in $\mathbb{Z}$ be a hybrid ideal, and $l', r' \in \mathbb{Z}'$. Then, $\exists\ l, r \in \mathbb{Z}$ such that $\vartheta(l) = l'$ and $\vartheta(r) = r'$. Now,

$$\begin{aligned}\tilde{d}(l'-r') &= \tilde{d}(\vartheta(l)-\vartheta(r)) = \tilde{d}(\vartheta(l-r)) = \vartheta^{-1}(\tilde{d})(l-r)\\ &\supseteq \vartheta^{-1}(\tilde{d})(l)\cap\vartheta^{-1}(\tilde{d})(r)\\ &= \tilde{d}(\vartheta(l))\cap\tilde{d}(\vartheta(r))\\ &= \tilde{d}(l')\cap\tilde{d}(r'),\\ \rho(l'-r') &= \rho(\vartheta(l)-\vartheta(r))\\ &= \rho(\vartheta(l-r))\\ &= \vartheta^{-1}(\rho)(l-r)\\ &\leq \vartheta^{-1}(\rho)(l)\vee\vartheta^{-1}(\rho)(r)\\ &= \rho(\vartheta(l))\vee\rho(\vartheta(r))\\ &= \rho(l')\vee\rho(r').\end{aligned}$$

Let $s'\in\mathbb{Z}'$. Then, $\exists s\in\mathbb{Z}$ such that $\vartheta(s)=s'$. Now,

$$\begin{aligned}\tilde{d}(s'l'-s'(r'-l')) &= \tilde{d}(\vartheta(s)\vartheta(l)-\vartheta(s)(\vartheta(r)-\vartheta(l)))\\ &= \tilde{d}(\vartheta(sl)-\vartheta(s)\vartheta(r-l))\\ &= \tilde{d}(\vartheta(sl)-\vartheta(s(r-l)))\\ &= \tilde{d}(\vartheta(sl-s(r-l)))\\ &= \vartheta^{-1}(\tilde{d})(sl-s(r-l))\\ &\supseteq \vartheta^{-1}(\tilde{d})(l)=\tilde{d}(\vartheta(l))=\tilde{d}(l'),\\ \rho(s'l'-s'(r'-l')) &= \rho(\vartheta(s)\vartheta(l)-\vartheta(s)(\vartheta(r)-\vartheta(l)))\\ &= \rho(\vartheta(sl)-\vartheta(s)\vartheta(r-l))\\ &= \rho(\vartheta(sl)-\vartheta(s(r-l)))\\ &= \rho(\vartheta(sl-s(r-l)))\\ &= \vartheta^{-1}(\rho)(sl-s(r-l))\\ &\leq \vartheta^{-1}(\rho)(l)=\rho(\vartheta(l))=\rho(l').\end{aligned}$$

Also,

$$\begin{aligned}&\tilde{d}(l'r')=\tilde{d}(\vartheta(l)\vartheta(r))=\tilde{d}(\vartheta(lr))=\vartheta^{-1}(\tilde{d})(lr)\supseteq\vartheta^{-1}(\tilde{d})(l)=\tilde{d}(\vartheta(l))=\tilde{d}(l').\\ &\rho(l'r')=\rho(\vartheta(l)\vartheta(r))=\rho(\vartheta(lr))=\vartheta^{-1}(\rho)(lr)\leq\vartheta^{-1}(\rho)(l)=\rho(\vartheta(l))=\rho(l').\end{aligned}$$

So, $\tilde{d}_\rho$ in $\mathbb{Z}'$ is a hybrid ideal.

Definition 5.3. *Let $\tilde{d}_\rho\in\mathscr{H}(\mathbb{Z})$ and $\Lambda:\mathbb{Z}\to\mathbb{Z}'$ be a mapping. Then, the hybrid structure $\Lambda(\tilde{d}_\rho):=(\Lambda(\tilde{d}),\Lambda(\rho))$ in $\Lambda(\mathbb{Z})$ defined by*

$$\Lambda(\tilde{d}(v))=\begin{cases}\bigcup\limits_{n\in\Lambda^{-1}(v)}\tilde{d}(n), & if\ \Lambda^{-1}(v)\neq\emptyset,\\ \emptyset, & otherwise,\end{cases}$$

$$\Lambda(\rho(v)) = \begin{cases} \bigwedge_{n\in\Lambda^{-1}(v)} \rho(n), & if\ \Lambda^{-1}(v) \neq \emptyset, \\ 1, & otherwise, \end{cases}$$

$\forall v \in \mathbb{Z}'$, is said to be the image of $\tilde{d}_\rho$ under Λ.

A hybrid structure $\tilde{d}_\rho$ in $\mathbb{Z}$ is said to satisfy the sup property if for every subset $G \subseteq \mathbb{Z}$, $\exists\ k_0 \in G$ such that

$$\tilde{d}(k_0) = \bigcup_{k\in G} \tilde{d}(k) \ \ and \ \ \rho(k_0) = \bigwedge_{k\in G} \rho(k).$$

Proposition 5.1. *A homomorphic image of a hybrid ideal with the sup property is a hybrid ideal.*

Proof. Consider a homomorphism of an NSS $\varsigma : \mathbb{Z} \to \mathbb{Z}'$, and let $\tilde{d}_\rho$ of $\mathbb{Z}$ be a hybrid ideal with the sup property.

Given $\varsigma(p), \varsigma(w) \in \mathbb{Z}'$, let $p_0 \in \varsigma^{-1}(\varsigma(p))$ and $w_0 \in \varsigma^{-1}(\varsigma(w))$ be such that

$$\tilde{d}(p_0) = \bigcup_{k\in\varsigma^{-1}(\varsigma(p))} \tilde{d}(k), \quad \rho(p_0) = \bigwedge_{k\in\varsigma^{-1}(\varsigma(p))} \rho(k),$$
$$\tilde{d}(w_0) = \bigcup_{k\in\varsigma^{-1}(\varsigma(w))} \tilde{d}(k), \quad \rho(w_0) = \bigwedge_{k\in\varsigma^{-1}(\varsigma(w))} \rho(k).$$

Then,

$$\begin{aligned} \varsigma(\tilde{d})(\varsigma(p) - \varsigma(w)) &= \bigcup_{z\in\varsigma^{-1}(\varsigma(p)-\varsigma(w))} \tilde{d}(z) \\ &\supseteq \tilde{d}(p_0) \cap \tilde{d}(w_0) \\ &= \left(\bigcup_{k\in\varsigma^{-1}(\varsigma(p))} \tilde{d}(k) \right) \cap \left(\bigcup_{k\in\varsigma^{-1}(\varsigma(w))} \tilde{d}(k) \right) \\ &= \varsigma(\tilde{d})(\varsigma(p)) \cap \varsigma(\tilde{d})(\varsigma(w)), \\ \varsigma(\rho)(\varsigma(p) - \varsigma(w)) &= \bigwedge_{z\in\varsigma^{-1}(\varsigma(p)-\varsigma(w))} \rho(z) \\ &\leq \rho(p_0) \vee \rho(w_0) \\ &= \left(\bigwedge_{k\in\varsigma^{-1}(\varsigma(p))} \rho(k) \right) \vee \left(\bigwedge_{k\in\varsigma^{-1}(\varsigma(w))} \rho(k) \right) \\ &= \varsigma(\rho)(\varsigma(p)) \vee \varsigma(\rho)(\varsigma(w)). \end{aligned}$$

Given $\varsigma(s) \in \mathbb{Z}'$, let $s_0 \in \varsigma^{-1}(\varsigma(s))$.

$$\begin{aligned} \varsigma(\tilde{d})(\varsigma(s)\varsigma(p) - \varsigma(s)(\varsigma(w) - \varsigma(p))) &= \bigcup_{z\in\varsigma^{-1}(\varsigma(s)\varsigma(p)-\varsigma(s)(\varsigma(w)-\varsigma(p)))} \tilde{d}(z) \\ &\supseteq \tilde{d}(p_0) \\ &= \bigcup_{k\in\varsigma^{-1}(\varsigma(p))} \tilde{d}(k) \\ &= \varsigma(\tilde{d})(\varsigma(p)), \end{aligned}$$

$$\begin{aligned}\varsigma(\rho)(\varsigma(s)\varsigma(p)-\varsigma(s)(\varsigma(w)-\varsigma(p))) &= \bigwedge_{z\in\varsigma^{-1}(\varsigma(s)\varsigma(p)-\varsigma(s)(\varsigma(w)-\varsigma(p)))}\rho(z)\\ &\le \rho(p_0)\\ &= \bigwedge_{k\in\varsigma^{-1}(\varsigma(p))}\rho(k)\\ &= \varsigma(\rho)(\varsigma(p)).\end{aligned}$$

Also,

$$\begin{aligned}\varsigma(\tilde{d})(\varsigma(p)\varsigma(w)) &= \bigcup_{z\in\varsigma^{-1}(\varsigma(p)\varsigma(w))}\tilde{d}(z)\\ &\supseteq \tilde{d}(p_0)\\ &= \bigcup_{k\in\varsigma^{-1}(\varsigma(p))}\tilde{d}(k)\\ &= \varsigma(\tilde{d})(\varsigma(p)),\\ \varsigma(\rho)(\varsigma(p)\varsigma(w)) &= \bigwedge_{z\in\varsigma^{-1}(\varsigma(p)\varsigma(w))}\rho(z)\\ &\le \rho(p_0)\\ &= \bigwedge_{k\in\varsigma^{-1}(\varsigma(p))}\rho(k)\\ &= \varsigma(\rho)(\varsigma(p)).\end{aligned}$$

Hence, $\varsigma(\tilde{d}_\rho)$ is a hybrid ideal in $\varsigma(\mathbb{Z})$.

6. Conclusions

In this paper, we have defined and investigated hybrid ideals in near-subtraction semigroups. In a near-subtraction semigroup, we formed ideals for a hybrid ideal, and various properties of the hybrid image and hybrid preimage of a near-subtraction semigroup's hybrid ideal under homomorphism mapping were also discussed. Using the ideas and results presented in this paper, it is intended to demonstrate the concept of a hybrid prime (resp., semi) ideal and its related properties for a hybrid ideal to be a hybrid prime (resp., semi) ideal in near-subtraction semigroups.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. S. Anis, M. Khan, Y. B. Jun, Hybrid ideals in semigroups, *Cogent Math.*, **4** (2017), 1–12. https://doi.org/10.1080/23311835.2017.1352117
2. J. R. Clay, The near-rings on groups of low order, *Math. Z.*, **104** (1968), 364–371. https://doi.org/10.1007/BF01110428
3. P. Dheena, G. S. Kumar, On strongly regular near-subtraction semigroups, *Commun. Korean Math. Soc.*, **22** (2007), 323–330. https://doi.org/10.4134/CKMS.2007.22.3.323
4. B. Elavarasan, Y. B. Jun, Regularity of semigroups in terms of hybrid ideals and hybrid bi-ideals, *Kragujev. J. Math.*, **46** (2022), 857–864.
5. B. Elavarasan, G. Muhiuddin, K. Porselvi, Y. B. Jun, Hybrid structures applied to ideals in near-rings, *Complex Intell. Syst.*, **7** (2021), 1489–1498. https://doi.org/10.1007/s40747-021-00271-7
6. B. Elavarasan, K. Porselvi, Y. B. Jun, Hybrid generalized bi-ideals in semigroups, *Int. J. Math. Comput. Sci.*, **14** (2019), 601–612.
7. Y. B. Jun, H. S. Kim, E. H. Roh, Ideal theory of subtraction algebras, *Sci. Math. Jpn.*, **61** (2005), 459–464.
8. Y. B. Jun, H. S. Kim, On ideals in subtraction algebras, *Sci. Math. Jpn.*, **65** (2007), 129–134. https://doi.org/10.32219/isms.65.1_129
9. Y. B. Jun, M. Sapanci, M. A. Öztürk, Fuzzy ideals in Gamma near-rings, *Tr. J. Math.*, **22** (1998), 449–459.
10. Y. B. Jun, S. Z. Song, G. Muhiuddin, Hybrid structures and applications, *Ann. Commun. Math.*, **1** (2018), 11–25.
11. K. J. Lee, C. H. Park, Some questions on fuzzifications of ideals in subtraction algebras, *Commun. Korean Math. Soc.*, **22** (2007), 359–363. https://doi.org/10.4134/CKMS.2007.22.3.359
12. G. Mason, Strongly regular near-rings, *Proc. Edinb. Math. Soc.*, **23** (1980), 27–35. https://doi.org/10.1017/S0013091500003564
13. P. K. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl.*, **44** (2002), 1077–1083. https://doi.org/10.1016/S0898-1221(02)00216-X
14. J. D. P. Meldrum, Varieties and d. g. near-rings, *Proc. Edinb. Math. Soc.*, **17** (1971), 271–274. https://doi.org/10.1017/S0013091500027000
15. D. Molodtsov, Soft set theory–first results, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
16. G. Muhiuddin, J. C. G. John, B. Elavarasan, Y. B. Jun, K. Porselvi, Hybrid structures applied to modules over semirings, *J. Intell. Fuzzy Syst.*, **42** (2022), 2521–2531. https://doi.org/10.3233/JIFS-211751
17. G. Muhiuddin, J. C. G. John, B. Elavarasan, K. Porselvi, D. Al-Kadi, Properties of *k*-hybrid ideals in ternary semiring, *J. Intell. Fuzzy Syst.*, **42** (2022), 5799–5807. https://doi.org/10.3233/JIFS-212311

18. G. Muhiuddin, D. Al-Kadi, W. A. Khan, C. Jana, Hybrid structures applied to subalgebras of BCH-algebras, *Secur. Commun. Netw.*, **2021** (2021), 1–8. https://doi.org/10.1155/2021/8960437
19. G. Muhiuddin, D. Al-Kadi, A. Mahboob, Hybrid structures applied to ideals in BCI-algebras, *J. Math.*, **2020** (2020), 1–7. https://doi.org/10.1155/2020/2365078
20. K. Porselvi, B. Elavarasan, On hybrid interior ideals in semigroups, *Probl. Anal. Issues Anal.*, **8** (2019), 137–146. https://doi.org/10.15393/j3.art.2019.6150
21. K. Porselvi, B. Elavarasan, Y. B. Jun, Hybrid interior ideals in ordered semigroups, *New Math. Nat. Comput.*, **18** (2022), 1–8. https://doi.org/10.1142/S1793005722500016
22. G. Muhiuddin, H. Harizavi, Y. B. Jun, Bipolar-valued fuzzy soft hyper BCK ideals in hyper BCK algebras, *Discrete Math. Algorithms Appl.*, **12** (2020), 2050018. https://doi.org/10.1142/S1793830920500184
23. G. Muhiuddin, Bipolar fuzzy KU-subalgebras/ideals of KU-algebras, *Ann. Fuzzy Math. Inform.*, **8** (2014), 409–418.
24. G. Muhiuddin, D. Al-Kadi, A. Mahboob, A. Albjedi, Interval-valued m-polar fuzzy positive implicative ideals in BCK-algebras, *Math. Probl. Eng.*, **2021** (2021), 1–9. https://doi.org/10.1155/2021/1042091
25. G. Muhiuddin, D. Al-Kadi, Interval valued m-polar fuzzy BCK/BCI-algebras, *Int. J. Comput. Intell. Syst.*, **14** (2021), 1014–1021. https://doi.org/10.2991/ijcis.d.210223.003
26. G. Muhiuddin, D. Al-Kadi, A. Mahboob, A. Aljohani, Generalized fuzzy ideals of BCI-algebras based on interval valued m-polar fuzzy structures, *Int. J. Comput. Intell. Syst.*, **14** (2021), 1–9. https://doi.org/10.1007/s44196-021-00006-z
27. G. Muhiuddin, K. P. Shum, New types of (α,β)-fuzzy subalgebras of BCK/BCI-algebras, *Int. J. Math. Comput. Sci.*, **14** (2019), 449–464.
28. G. Muhiuddin, A. M. Al-Roqi, Subalgebras of BCK/BCI-algebras based on (α,β)-type fuzzy sets, *J. Comput. Anal. Appl.*, **18** (2015), 1057–1064.
29. D. Al-Kadi, G. Muhiuddin, Bipolar fuzzy BCI-implicative ideals of BCI-algebras, *Ann. Commun. Math.*, **3** (2020), 88–96.
30. A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.*, **35** (1971), 512–517.
31. B. M. Schein, Difference semigroups, *Commun. Algebra*, **20** (1992), 2153–2169. https://doi.org/10.1080/00927879208824453
32. D. R. P. Williams, Fuzzy ideals in near-subtraction semigroups, *Int. Scholarly Sci. Res. Innov.*, **2** (2008), 625–632.
33. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353.
34. B. Zelinka, Subtraction semigroups, *Math. Bohem.*, **120** (1995), 445–447. https://doi.org/10.21136/MB.1995.126093

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