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Research article

Solvability for a fractional p-Laplacian equation in a bounded domain

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Abstract: In this paper we use the topological degree and the fountain theorem to study the existence of weak solutions for a fractional p-Laplacian equation in a bounded domain. For the nonlinearity f, we consider two situations: (1) the non-resonance case where f is $(p-1)$-asymptotically linear at infinity; (2) the resonance case where f satisfies the Landesman-Lazer type condition.

Keywords: fractional p-Laplacian equation; weak solution; topological degree; fountain theorem; $(p-1)$-asymptotically linear; Landesman-Lazer type condition
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1. Introduction

In this paper we use topological degree and the fountain theorem to study the existence of weak solutions for the fractional p-Laplacian equation

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + f(x,u) - g(x), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $p \geq 2$, $s \in (0,1)$, λ is a parameter, Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$, $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$, and $g : \Omega \to \mathbb{R}$ is a perturbation function. The operator $(-\Delta)_p^s$, defined by

$$(-\Delta)_p^s u(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} dy, \quad x \in \mathbb{R}^N,$$

where $B_\varepsilon(x)$ is the open ε-ball with center x and radius ε, is known as the fractional p-Laplacian and leads to the study of the problem

$$\begin{cases} (-\Delta)_p^s u(x) = h(x,u), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.2}$$

In the literature, there are many papers on the solvability for the fractional p-Laplacian, see for example [2, 4, 5, 7, 8, 13, 15, 17, 18, 22–26]. In [15] the author used the mountain pass theorem and the fractional Moser-Trudinger inequality to study nontrivial solutions for (1.2), where h is superlinear at 0, and is of subcritical or critical exponential growth at ∞, and does not satisfy the Ambrosetti-Rabinowitz condition. In [8] the authors established existence and multiplicity results for weak solutions to (1.2) using Morse Theory. In [4] the authors used truncation and comparison techniques to study the Problem (1.2) with a nonlinearity of the form $\lambda g(x,u) - f(x,u)$.

It has been observed that the Landesman-Lazer type condition is an important tool to study different types of differential equations, see for example [6, 7, 10, 14, 20, 28]. In [28] the authors used the condition

$$\int_{\Omega} f(-\infty)\varphi_1(x)dx < \int_{\Omega} g(x)\varphi_1(x)dx < \int_{\Omega} f(+\infty)\varphi_1(x)dx \tag{1.3}$$

($f(\pm\infty) = \lim_{u\to\pm\infty} f(u)$) to study weak solutions for the fourth-order Navier boundary value problem

$$\begin{cases} \Delta^2 u(x) + c\Delta u(x) = \lambda_1 u(x) + f(u(x)) - g(x), & \text{in } \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where λ_1 is the first eigenvalue of the operator $\Delta^2 + c\Delta$, and φ_1 is an eigenfunction associated with λ_1. In [7] the authors used similar conditions in (1.3) to study weak solutions for (1.1), where $\lambda = \lambda_1(\lambda_1$ is the first eigenvalue of the operator $(-\Delta)_p^s$).

In [10] the authors used minimax methods to study the semilinear elliptic equation:

$$\begin{cases} -\Delta u = \lambda u + g(x,u) - h(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where h, g satisfy the Landesman-Lazer type condition:

$$\int_{\Omega} hudx < \int_{\Omega} \underline{F(x,-\infty)}u^+ dx - \int_{\Omega} \overline{F(x,+\infty)}u^- dx,$$

and

$$\liminf_{t\to-\infty} F(x,t) = \underline{F(x,-\infty)}, \limsup_{t\to+\infty} F(x,t) = \overline{F(x,+\infty)} \text{ uniformly for } x \in \Omega,$$

$$F(x,t) = \begin{cases} \frac{2G(x,t)-g(x,t)t}{t}, & t \neq 0, \\ g(x,0), & t = 0, \end{cases} \quad G(x,t) = \int_0^t g(x,s)ds.$$

Motivated by the above works, in this paper we use topological degree to study weak solutions for (1.1). When $\lambda = 0, g \equiv 0$, we use a $(p-1)$-asymptotically linear growth condition at ∞ to obtain a weak solution result for (1.1). When $\lambda = \lambda_k$, and f, g satisfy the Landesman-Lazer type condition, we obtain a weak solution result, where λ_k are the eigenvalues of the operator $(-\Delta)_p^s$. Our methods are different from those in [7, 14, 20], and the results here generalize and improve the corresponding ones in their works and in [28].

2. Preliminaries

Define the Gagliardo seminorm

$$[u]_{s,p} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{\frac{1}{p}},$$

where $u : \mathbb{R}^N \to \mathbb{R}$ is a measurable function, and consider the fractional Sobolev space as follows:

$$W^{s,p}\left(\mathbb{R}^N\right) = \left\{u \in L^p\left(\mathbb{R}^N\right) : u \text{ is measurable and } [u]_{s,p} < +\infty\right\}$$

endowed with the norm

$$\|u\|_{s,p} = \left(\|u\|_p^p + [u]_{s,p}^p\right)^{\frac{1}{p}},$$

where $\|\cdot\|_p$ is the norm in $L^p(\Omega)$. Let $X(\Omega)$ be the closed linear subspace

$$X(\Omega) = \left\{u \in W^{s,p}\left(\mathbb{R}^N\right) : u(x) = 0 \text{ a.e. } x \in \mathbb{R}^N\backslash\Omega\right\}$$

which can be equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$(see [4, 7, 8]).

Lemma 2.1. *(see [4,7,8]).* $(X(\Omega), \|\cdot\|)$ *is a uniformly convex Banach space. Moreover, the embedding* $X(\Omega)$ *into* $L^q(\Omega)$ *is continuous for all* $1 \le q \le p_s^*$ *and compact for all* $1 \le q < p_s^*$*, where* $p_s^* = \frac{Np}{N-sp}$ *if* $sp < N$ *and* $p_s^* = +\infty$ *if* $sp \ge N$.

Now, we list some basic results on the topological degree of type $(S)_+$.

Definition 2.1. *(see [20]). Let* X *be a reflexive real Banach space and* X^* *its dual. The operator* $T : X \to X^*$ *is said to satisfy the* $(S)_+$ *condition if the assumptions* $u_n \rightharpoonup u_0$ *weakly in* X *and* $\limsup_{n\to\infty} \langle T(u_n), u_n - u_0\rangle \leqslant 0$ *imply* $u_n \to u_0$ *strongly in* X.

Definition 2.2. *(see [11]). The operator* $T : X \to X^*$ *is said to be demicontinuous if* T *maps strongly convergent sequences in* X *to weakly convergent sequences in* X^*

Lemma 2.2. *(see [20]). Let* $T : X \to X^*$ *satisfy the* $(S)_+$ *condition and let* $K : X \to X^*$ *be a compact operator. Then the sum* $T + K : X \to X^*$ *satisfies the* $(S)_+$ *condition.*

Lemma 2.3. *(see [1, 20, 21]). Let* $T : X \to X^*$ *be a bounded and demicontinuous operator satisfying the* $(S)_+$ *condition. Let* $\mathcal{D} \subset X$ *be an open, bounded and nonempty set with the boundary* $\partial\mathcal{D}$ *such that* $T(u) \neq 0$ *for* $u \in \partial\mathcal{D}$*. Then there exists an integer* $\deg(T, \mathcal{D}, 0)$ *such that*

(C1) $\deg(T, \mathcal{D}, 0) \neq 0$ *implies that there exists an element* $u_0 \in \mathcal{D}$ *such that* $T(u_0) = 0$.

(C2) If $\mathcal{D}$ *is symmetric with respect to the origin and* T *satisfies* $T(u) = -T(-u)$ *for any* $u \in \partial\mathcal{D}$*, then* $\deg(T, \mathcal{D}, 0)$ *is an odd number.*

(C3) Let T_λ *be a family of bounded and demicontinuous mappings which satisfy the* $(S)_+$ *condition and which depend continuously on a real parameter* $\lambda \in [0, 1]$*, and let* $T_\lambda(u) \neq 0$ *for any* $u \in \partial\mathcal{D}$ *and* $\lambda \in [0, 1]$*. Then* $\deg(T_\lambda, \mathcal{D}, 0)$ *is constant with respect to* $\lambda \in [0, 1]$.

Suppose that there exists a nontrivial weak solution $\varphi(x)$, $x \in \Omega$ *of the problem*

$$\begin{cases} (-\Delta)_p^s u = \lambda|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N\backslash\Omega, \end{cases} \tag{2.1}$$

where $\lambda \in \mathbb{R}$ is a parameter. Now, λ is called an eigenvalue of the eigenvalue Problem (2.1)*, and φ an eigenfunction associated with the eigenvalue λ. Moreover, the set of all eigenvalues of* (2.1) *in $X(\Omega)$, denoted by $\sigma(s,p)$, has the following properties.*

Lemma 2.4. *(see [3, 4, 9, 16]). For the problem* (2.1)*, we have*

(D1) $\lambda_1 = \min \sigma(s,p)$ is an isolated point of $\sigma(s,p)$;

(D2) all λ_1-eigenfunctions are proportional, and if u is a λ_1-eigenfunction, then either $u(x) > 0$ a.e. in Ω or $u(x) < 0$ a.e. in Ω;

(D3) if $\lambda \in \sigma(s,p) \backslash \{\lambda_1\}$ and u is a λ-eigenfunction, then u changes sign in Ω;

(D4) all eigenfunctions are in $L^\infty(\Omega)$;

(D5) $\sigma(s,p)$ is a closed set.

For convenience, we use λ_k and $\varphi_k (k = 1, 2, \cdots)$ to stand for the eigenvalues and the corresponding eigenfunctions for the Problem (2.1), respectively. We know that $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \lambda_k \to \infty$ as $k \to +\infty$(see [8, 17]), and $\varphi_1 > 0$, $\varphi_k (k = 2, 3, \cdots)$ are sign-changing functions in Ω. In the following section, we assume

$$\|\varphi_k\| = 1, k = 1, 2, \cdots .$$

On the other hand, let $X_i = \operatorname{span}\{\varphi_i\}$, $Y_k = \bigoplus_{i=1}^{k} X_i$, $Z_k = \overline{\bigoplus_{i=k+1}^{\infty} X_i}$, $k = 1, 2, \ldots$ Then from [30] we have

$$X(\Omega) = \overline{\bigoplus_{i=1}^{\infty}} X_i = Y_k \bigoplus Z_k, \dim Y_k = k. \tag{2.2}$$

Definition 2.3. *Let X be a real Banach space, $J \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We say that J satisfies the $(PS)_c$ condition if any sequence $\{u_m\} \subset X$ such that*

$$J(u_m) \to c \text{ and } J'(u_m) \to 0 \text{ as } m \to \infty$$

has a convergent subsequence.

Lemma 2.5. *(see [12, 27, 29]). Let Y_k, Z_k be defined in* (2.2)*. Suppose that*

(A1) $J \in C^1(X(\Omega), \mathbb{R})$ is an even functional.

If for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

(A2) $a_k := \max_{u \in Y_k, \|u\| = \rho_k} J(u) \leqslant 0$.

(A3) $b_k := \inf_{u \in Z_k, \|u\| = r_k} J(u) \to \infty$ as $k \to \infty$.

(A4) J satisfies the $(\mathrm{PS})_\mathrm{c}$ condition for all $c > 0$.

Then J has an unbounded sequence of critical values.

3. Main results

Define the nonlinear operator $A : X(\Omega) \to X(\Omega)^*$ as follows:

$$\langle A(u), v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} dxdy, \forall u, v \in X(\Omega).$$

Clearly, A is odd, $(p-1)$-homogeneous, and for all $u \in X(\Omega)$ we have

$$\langle A(u), u\rangle = \|u\|^p, \quad \|A(u)\|_* \leq \|u\|^{p-1}.$$

Lemma 3.1. *(see [4]). The operator A satisfies the* $(S)_+$ *condition.*

We first study the problem

$$\begin{cases}(-\Delta)_p^s u = f(x,u), & x \in \Omega,\\ u = 0, & x \in \mathbb{R}^N \backslash \Omega,\end{cases} \tag{3.1}$$

where p, s, N, Ω are as in (1.1), and f satisfies the $(p-1)$-asymptotically linear condition:

(H1) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = b$ uniformly for $x \in \Omega$ with $b \neq \lambda_k$.

Theorem 3.1. *Let (H1) hold. Then* (3.1) *has at least one weak solution.*

Proof. A weak solution of Problem (1.1) is a function $u \in X(\Omega)$ such that

$$\langle A(u), v\rangle = \int_\Omega f(x,u)v dx, \forall v \in X(\Omega).$$

From (H1) there exist $\varepsilon_0 > 0$ and $M_0 > 0$ such that

$$|f(x,t)| \leq (|b| + \varepsilon_0)|t|^{p-1}, \text{ for } x \in \Omega, |t| > M_0.$$

When $x \in \Omega, |t| \leq M_0$, $f(x,t)$ is bounded. Hence, there exists $c_0 > 0$ such that

$$|f(x,t)| \leq c_0 + (|b| + \varepsilon_0)|t|^{p-1}, \ x \in \Omega, \ t \in \mathbb{R}. \tag{3.2}$$

Let $B, S : X(\Omega) \to X(\Omega)^*$ be as follows:

$$\langle B(u), v\rangle = \int_\Omega |u|^{p-2}uv dx, \ \langle S(u), v\rangle = \int_\Omega f(x,u)v dx, \forall v \in X(\Omega).$$

Note that $p < p_s^*$, f satisfies the subcritical Condition (3.2), and then $B, S : X(\Omega) \to X(\Omega)^*$ are compact operators.

From the definitions of A, S, we know that if there is an $u_0 \in X(\Omega)$ such that $\langle A(u_0), v\rangle = \langle S(u_0), v\rangle, \forall v \in X(\Omega)$, then u_0 is a weak solution for (1.1). Now, we define a homotopy

$$T_\tau(u) = A(u) - (1-\tau)S(u) - \tau b B(u), \qquad \text{for } \tau \in [0,1], u \in X(\Omega), \tag{3.3}$$

where b is in (H1). From Lemma 3.1 and Lemma 2.2 we have that T_τ satisfies the $(S)_+$ condition. Now we prove that there exists $R > 0$ such that this homotopy (3.3) is admissible within the ball $Q(0,R) \subset X(\Omega)$. On the contrary, for any $n \in \mathbb{N}$, there exist $\tau_n \in [0,1]$ and $u_n \in X(\Omega), \|u_n\| \geqslant n$ such that $T_{\tau_n}(u_n) = 0$, i.e., $A(u_n) - (1-\tau_n)S(u_n) - \tau_n b B(u_n) = 0$, and this can be rewritten in the form

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+sp}} dxdy - (1-\tau_n)\int_\Omega f(x,u_n)v dx - \tau_n b\int_\Omega |u_n|^{p-2}u_n v dx = 0. \tag{3.4}$$

Let $\omega_n = \frac{u_n}{\|u_n\|}$. Then, dividing (3.4) by $\|u_n\|^{p-1}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\omega_n(x)-\omega_n(y)|^{p-2}(\omega_n(x)-\omega_n(y))(v(x)-v(y))}{|x-y|^{N+sp}} dxdy \\ &\quad - (1-\tau_n)\int_\Omega \frac{f(x,u_n)}{|u_n|^{p-2}u_n}|\omega_n|^{p-2}\omega_n v dx - \tau_n b\int_\Omega |\omega_n|^{p-2}\omega_n v dx = 0.\end{aligned} \tag{3.5}$$

Note the definitions of ω_n, τ_n, and we may assume that $\omega_n \rightharpoonup \omega$ weakly in $X(\Omega)$, $\omega_n \to \omega$ strongly in $L^p(\Omega)$, and $\tau_n \to \tau \in [0,1]$. Moreover, we also have a basic result from Lemma 2.2 of [19]:

$$\int_\Omega \frac{f(x,u_n)}{|u_n|^{p-2} u_n} |\omega_n|^{p-2} \omega_n v dx \to \int_\Omega b|\omega|^{p-2}\omega v dx. \tag{3.6}$$

Therefore, let $n \to \infty$ in (3.5) and we have

$$\langle A(\omega), v\rangle - (1-\tau)\int_\Omega b|\omega|^{p-2}\omega v dx - \tau b \int_\Omega |\omega|^{p-2}\omega v dx = 0,$$

i.e., $\langle A(\omega), v\rangle = b\langle B(\omega), v\rangle, \forall v \in X(\Omega)$, and this contradicts $b \neq \lambda_k$. Thus our claim is true, i.e., the homotopy T_τ is admissible within the ball $Q(0,R)$. Hence, from Lemma 2.3(C3) we have

$$\deg(A - S, Q(0,R), 0) = \deg(A - bB, Q(0,R), 0). \tag{3.7}$$

Note that the right-hand side of (3.7) is an odd number by Lemma 2.3(C2) and $b \neq \lambda_k$. Hence $\deg(A - S, Q(0,R), 0) \neq 0$, and Lemma 2.3(C1) implies that (1.1) has a weak solution. This completes the proof. □

In Section 2 we know that for any fixed $k \in \mathbb{N}$, $\dim Y_k = k < \infty$, and all norms in finite dimensional spaces are equivalent, and thus for all $u \in Y_k$ there exists a positive constant μ_1 such that

$$\mu_1\|u\|^p \leq \|u\|_p^p, \ u \in Y_k. \tag{3.8}$$

Condition (H1) is only needed to guarantee that (1.1) has at least one weak solution (see Theorem 3.1). To obtain that (1.1) has infinitely many weak solutions, we need an extra assumption, namely

(H2) $f(x,t) + f(x,-t) = 0$ for $x \in \Omega, t \in \mathbb{R}$.

Theorem 3.2. *Let (H1) and (H2) hold with $b > \mu_1^{-1}$. Then* (3.1) *has infinitely many weak solutions.*

Proof. We define the energy functional J as follows:

$$J(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(x,u)dx, \tag{3.9}$$

where $F(x,u) = \int_0^u f(x,t)dt$. Note that (3.2) implies that J is well defined on $X(\Omega)$, and of class C^1. We first prove that J satisfies the $(\mathrm{PS})_\mathrm{c}$ condition for all $c > 0$. Assume that $\{u_n\} \subset X(\Omega)$ is a $(\mathrm{PS})_\mathrm{c}$ sequence, i.e.,

$$J(u_n) \to c \text{ and } J'(u_n) \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Now, we claim that $\{u_n\}$ is bounded in $X(\Omega)$. If not, we may assume that there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\|u_n\| \to +\infty$ as $n \to \infty$. Let $\omega_n = \frac{u_n}{\|u_n\|}$. Then $\{\omega_n\}$ is bounded, $\|\omega_n\| = 1$, and there exists $\omega \in X(\Omega)$ such that

$$\omega_n \rightharpoonup \omega \text{ weakly in } X(\Omega),\ \omega_n \to \omega \text{ strongly in } L^q(\Omega) \text{ with } q \in [1, p_s^*),\ \omega_n(x) \to \omega(x),\ \text{a.e. } x \in \Omega. \tag{3.11}$$

If $\omega(x) \equiv 0, x \in \Omega$, note that $J(u_n) \to c,\ \|u_n\| \to +\infty$ as $n \to \infty$, then dividing (3.9) by $\|u_n\|^p$ we have

$$\frac{1}{p} - \int_\Omega \frac{F(x, u_n)}{\|u_n\|^p} dx = \frac{J(u_n)}{\|u_n\|^p} \to 0, \text{ as } n \to \infty. \tag{3.12}$$

From (3.2) we have

$$|F(x,u)| = \left| \int_0^1 f(x, us)u ds \right| \le \int_0^1 (c_0 + (|b| + \varepsilon_0)|us|^{p-1})|u| ds \le c_0|u| + (|b| + \varepsilon_0)|u|^p, \forall x \in \Omega, u \in \mathbb{R}.$$

Together with this and the Lebesgue dominated convergence theorem we find

$$\left| \int_\Omega \frac{F(x, u_n)}{\|u_n\|^p} dx \right| \le \int_\Omega \frac{|F(x, u_n)|}{\|u_n\|^p} dx \le \int_\Omega \frac{c_0|u_n| + (|b| + \varepsilon_0)|u_n|^p}{|u_n|^p} |\omega_n|^p dx \to 0, \text{ as } n \to \infty.$$

This is a contradiction with (3.12), and hence $\omega(x) \not\equiv 0, x \in \Omega$.

Note that J' is

$$\langle J'(u), v \rangle = \langle A(u), v \rangle - \int_\Omega f(x, u) v dx, \ \forall v \in X(\Omega).$$

Then, dividing this equation by $\|u_n\|^{p-1}$ we have

$$\frac{\langle J'(u_n), v \rangle}{\|u_n\|^{p-1}} = \frac{\langle A(u_n), v \rangle}{\|u_n\|^{p-1}} - \int_\Omega \frac{f(x, u_n)}{\|u_n\|^{p-1}} v dx,$$

and from (3.10) we obtain

$$\langle A(\omega_n), v \rangle = \int_\Omega \frac{f(x, u_n)}{|u_n|^{p-2} u_n} |\omega_n|^{p-2} \omega_n v dx + o(1).$$

Let $n \to \infty$. Then from (3.6) and (3.11) we have

$$\langle A(\omega), v \rangle = b \int_\Omega |\omega|^{p-2} \omega v dx.$$

This is a contradiction with $b \neq \lambda_k$.

As a result, $\{u_n\}$ is a bounded sequence. Thus, there exist a subsequence(still denoted by $\{u_n\}$) and $u \in X(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in } X(\Omega),\ u_n \to u \text{ strongly in } L^q(\Omega) \text{ with } q \in [1, p_s^*),\ u_n(x) \to u(x), \text{ a.e. } x \in \Omega. \tag{3.13}$$

Consequently, from (3.2) we have

$$\begin{aligned}\left| \int_\Omega (f(x, u_n) - f(x, u))(u_n - u) dx \right| &\le \int_\Omega |2c_0 + (|b| + \varepsilon_0)(|u_n|^{p-1} + |u|^{p-1})||u_n - u| dx \\ &\le 2c_0 [\text{meas}(\Omega)]^{\frac{p-1}{p}} \|u_n - u\|_p + (|b| + \varepsilon_0)(\|u_n\|_p^{p-1} + \|u\|_p^{p-1})\|u_n - u\|_p \\ &\to 0.\end{aligned}$$

Obviously, $\langle J'(u_n) - J'(u), u_n - u \rangle = o(1)$. Hence, we have

$$\langle A(u_n) - A(u), u_n - u \rangle = 0, \text{ as } n \to \infty. \tag{3.14}$$

Using the Hölder inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u(x)-u(y))}{|x-y|^{N+sp}}dxdy\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u(x)-u(y))}{(|x-y|^{N+sp})^{\frac{p-1}{p}}(|x-y|^{N+sp})^{\frac{1}{p}}}dxdy\\
&\leq\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{(|x-y|^{N+sp})^{\frac{p-1}{p}}}\right|^{\frac{p}{p-1}}dxdy\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{u(x)-u(y)}{(|x-y|^{N+sp})^{\frac{1}{p}}}\right|^{p}dxdy\right)^{\frac{1}{p}}\\
&=\|u_n\|^{p-1}\|u\|.
\end{aligned}$$

Combining this with (3.14) we have

$$\begin{aligned}
o(1)&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^{p}}{|x-y|^{N+sp}}dxdy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u(x)-u(y))}{|x-y|^{N+sp}}dxdy\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u_n(x)-u_n(y))}{|x-y|^{N+sp}}dxdy+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p}}{|x-y|^{N+sp}}dxdy\\
&\geq\|u_n\|^p-\|u\|\|u_n\|^{p-1}-\|u_n\|\|u\|^{p-1}+\|u\|^p\\
&=(\|u_n\|-\|u\|)(\|u_n\|^{p-1}-\|u\|^{p-1})\\
&\geq 0.
\end{aligned}$$

This implies that $\|u_n\|\to\|u\|$, i.e., $u_n\to u$ strongly in $X(\Omega)$. Therefore, J satisfies the $(\mathrm{PS})_{\mathrm{c}}$ condition for all $c>0$, as required.

From (H1) we have $\lim_{|t|\to\infty}\frac{f(x,t)}{|t|^{p-2}t}>\mu_1^{-1}$ uniformly for $x\in\Omega$, and there exits $\varepsilon_1>0, c_1>0$ such that

$$f(x,t)\geq(\mu_1^{-1}+\varepsilon_1)|t|^{p-2}t-c_1,\ \text{for } t\in\mathbb{R}, x\in\Omega.$$

This implies that

$$F(x,t)\geq\frac{\mu_1^{-1}+\varepsilon_1}{p}|t|^p-c_1|t|,\ \text{for } t\in\mathbb{R}, x\in\Omega.$$

For any $u\in Y_k$, and from (3.8) we obtain

$$J(u)\leq\frac{1}{p}\|u\|^p-\int_\Omega\left[\frac{\mu_1^{-1}+\varepsilon_1}{p}|u|^p-c_1|u|\right]dx\leq\frac{1}{p}\|u\|^p\left(1-(\mu_1^{-1}+\varepsilon_1)\mu_1\right)+c_1\|u\|_1.$$

Note that $1-(\mu_1^{-1}+\varepsilon_1)\mu_1<0$ and $p\geq 2$, there exist positive constants d_k such that

$$J(u)\leq 0,\ \text{for } u\in Y_k, \|u\|\geq d_k. \tag{3.15}$$

Note that $\lambda_k\to+\infty$ as $k\to\infty$, and thus there exists $k\in\mathbb{N}$ such that $b<\lambda_{k+1}$. Then using the limit $\lim_{|t|\to\infty}\frac{f(x,t)}{|t|^{p-2}t}<\lambda_{k+1}$ uniformly for $x\in\Omega$, and there exits $\varepsilon_2>0, c_2>0$ such that

$$f(x,t)\leq(\lambda_{k+1}-\varepsilon_2)|t|^{p-2}t+c_2,\ \text{for } t\in\mathbb{R}, x\in\Omega.$$

This implies that

$$F(x,t) \leq \frac{\lambda_{k+1} - \varepsilon_2}{p}|t|^p + c_2|t|, \text{ for } t \in \mathbb{R}, x \in \Omega.$$

For all $u \in Z_k$, let $\beta_k = \sup_{u\in Z_k,\|u\|=1} \|u\|_p$. From Lemma 2.1 and Lemma 4.1 in [30], $\beta_k \to 0$ as $k \to \infty$. Note that from (2.2) and [8, Lemma 4.5] we have

$$\|u\|^p \geq \lambda_{k+1}\|u\|_p^p, \text{ for } u \in Z_k.$$

Therefore, we obtain

$$J(u) \geq \frac{1}{p}\|u\|^p - \int_\Omega \left[\frac{\lambda_{k+1} - \varepsilon_2}{p}|u|^p + c_2|u|\right]dx \geq \frac{1}{p}\|u\|^p\left(1 - \frac{\lambda_{k+1} - \varepsilon_2}{\lambda_{k+1}}\right) - c_2[\text{meas}(\Omega)]^{\frac{p-1}{p}}\beta_k\|u\|.$$

Let $r_k = \beta_k^{-1}$. Then $r_k \to +\infty$ as $k \to \infty$, and

$$J(u) \geq \frac{1}{p}\left(1 - \frac{\lambda_{k+1} - \varepsilon_2}{\lambda_{k+1}}\right)r_k^p - c_2[\text{meas}(\Omega)]^{\frac{p-1}{p}} \to +\infty, \text{ as } k \to \infty.$$

Hence,

$$b_k = \inf_{u\in Z_k,\|u\|=r_k} J(u) \to +\infty \text{ as } k \to \infty,$$

and let $\rho_k = \max\{d_k, r_k + 1\}$ from (3.15) we have

$$a_k = \max_{u\in Y_k,\|u\|=\rho_k} J(u) \leq 0.$$

Note that (H2) implies that J is an even functional, and we obtain all the conditions in Lemma 2.5 are satisfied, J has an unbounded sequence of critical values, i.e., (3.1) has infinitely many weak solutions. This completes the proof. □

Next we consider the following problem

$$\begin{cases} (-\Delta)_p^s u = \lambda_k |u|^{p-2}u + f(u) - g(x), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.16}$$

where p, s, N, Ω, g are as in (1.1), and $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the condition:

(H3) f is bounded and has finite limits $f(+\infty), f(-\infty)$ such that

$$f(+\infty) = \lim_{u\to+\infty} f(u), f(-\infty) = \lim_{u\to-\infty} f(u).$$

Theorem 3.3. *Suppose that (H3) holds. Let $g \in L^{p'}(\Omega)(p' = \frac{p}{p-1})$ with $g(x) \not\equiv 0$ in Ω be a given function, and satisfies either*

$$\int_\Omega f(+\infty)\varphi_k^+(x)dx - \int_\Omega f(-\infty)\varphi_k^-(x)dx < \int_\Omega g(x)\varphi_k(x)dx < \int_\Omega f(-\infty)\varphi_k^+(x)dx - \int_\Omega f(+\infty)\varphi_k^-(x)dx, \tag{3.17}$$

or

$$\int_\Omega f(-\infty)\varphi_k^+(x)dx - \int_\Omega f(+\infty)\varphi_k^-(x)dx < \int_\Omega g(x)\varphi_k(x)dx < \int_\Omega f(+\infty)\varphi_k^+(x)dx - \int_\Omega f(-\infty)\varphi_k^-(x)dx, \tag{3.18}$$

where φ_k^+ and φ_k^- respectively denote the positive and negative parts of φ_k for $k = 1, 2, \ldots$. Then (3.16) *has at least one weak solution.*

Proof. Let $\langle g^*, v\rangle = \int_\Omega g(x)v dx, \forall v \in X(\Omega)$. Note that $g \in L^{p'}(\Omega)$ with $p' = \frac{p}{p-1}$, and we have g^* is a compact operator. Note that f is bounded in $\mathbb{R}$, we obtain $\lim_{|t|\to\infty} \frac{f(t)}{|t|^{p-2}t} = 0$. This implies that (3.2) holds, and the operator $\langle S(u), v\rangle = \int_\Omega f(u)v dx, \forall v \in X(\Omega)$ is also compact on $X(\Omega)$. Choosing $\delta_k \in (0, \lambda_{k+1} - \lambda_k)$, and we can define a homotopy as follows:

$$T_\tau(u) = A(u) - \lambda_k B(u) - (1-\tau)\delta_k B(u) - \tau S(u) + \tau g^*, u \in X(\Omega), \tau \in [0,1].$$

We now prove that there exists $R > 0$ such that this homotopy is admissible within the ball $Q(0, R) \subset X(\Omega)$. We argue by contradiction. For any $n \in \mathbb{N}$, assume there exist $\tau_n \in [0,1]$ and $u_n \in X(\Omega), \|u_n\| \geq n$ such that $T_{\tau_n}(u_n) = 0$, and this can be expressed by

$$\langle A(u_n), v\rangle - \lambda_k\langle B(u_n), v\rangle - (1-\tau_n)\delta_k\langle B(u_n), v\rangle - \tau_n\langle S(u_n), v\rangle + \tau_n\langle g^*, v\rangle = 0, \forall v \in X(\Omega). \quad (3.19)$$

Let $\omega_n = \frac{u_n}{\|u_n\|}$. Then, dividing (3.19) by $\|u_n\|^{p-1}$, we obtain

$$\langle A(\omega_n), v\rangle - \lambda_k\langle B(\omega_n), v\rangle - (1-\tau_n)\delta_k\langle B(\omega_n), v\rangle - \tau_n\frac{\langle S(u_n), v\rangle}{\|u_n\|^{p-1}} + \tau_n\frac{\langle g^*, v\rangle}{\|u_n\|^{p-1}} = 0, \forall v \in X(\Omega). \quad (3.20)$$

Using the conditions for f, g we have

$$\lim_{n\to\infty}\frac{|\langle g^*, v\rangle|}{\|u_n\|^{p-1}} = \lim_{n\to\infty}\frac{1}{\|u_n\|^{p-1}}\left|\int_\Omega g(x)v dx\right| \leq \lim_{n\to\infty}\frac{\|g\|_{p'}\|v\|_p}{\|u_n\|^{p-1}} = 0,$$

and note that f is bounded on $\mathbb{R}$, so we have

$$\lim_{n\to\infty}\frac{|\langle S(u_n), v\rangle|}{\|u_n\|^{p-1}} = \lim_{n\to\infty}\frac{1}{\|u_n\|^{p-1}}\left|\int_\Omega f(u_n)v dx\right| \leq \lim_{n\to\infty}\frac{1}{\|u_n\|^{p-1}}\int_\Omega M|v|\,dx = 0,$$

where M is a positive constant. Let $\omega_n \rightharpoonup \omega$ weakly in $X(\Omega)$, $\omega_n \to \omega$ strongly in $L^p(\Omega)$, and $\tau_n \to \tau \in [0,1]$. Then from (3.20) we have

$$\langle A(\omega), v\rangle - \lambda_k\langle B(\omega), v\rangle - (1-\tau)\delta_k\langle B(\omega), v\rangle = 0, \forall v \in X(\Omega).$$

Note that $\lambda_k + (1-\tau)\delta_k, \tau \in [0,1)$ is not eigenvalues for (2.1), so we only consider $\tau = 1$. Hence, we have

$$\langle A(\omega), v\rangle - \lambda_k\langle B(\omega), v\rangle = 0, \forall v \in X(\Omega).$$

As a result, $\omega = \pm\varphi_k$, and $u_n = \omega_n\|u_n\| \to \pm\varphi_k\|u_n\|$. Thus, we have

$$\langle A(u_n), v\rangle - \lambda_k\langle B(u_n), v\rangle \to 0, \forall v \in X(\Omega).$$

In what follows, we consider two cases:

Case 3.1. *$\omega = \varphi_k$, i.e., $\omega_n \to \varphi_k$. Hence, $u_n \to +\infty$ in $\Omega \cap \{x : \varphi_k(x) \geq 0\}$, $u_n \to -\infty$ in $\Omega \cap \{x : \varphi_k(x) \leq 0\}$. Let $n \to \infty$ and $v = \varphi_k$, and from* (3.19) *we have*

$$\begin{aligned}\langle g^*, \varphi_k\rangle &= \int_\Omega g(x)\varphi_k(x)dx = \lim_{n\to\infty}\langle S(u_n), \varphi_k\rangle = \lim_{n\to\infty}\int_\Omega f(u_n)\varphi_k(x)dx \\ &= \lim_{n\to\infty}\int_\Omega f(u_n)(\varphi_k^+(x) - \varphi_k^-(x))dx = \int_\Omega [f(+\infty)\varphi_k^+(x) - f(-\infty)\varphi_k^-(x)]dx.\end{aligned}$$

Case 3.2. *$\omega = -\varphi_k$, i.e., $\omega_n \to -\varphi_k$. Hence, $u_n \to -\infty$ in $\Omega \cap \{x : \varphi_k(x) \geq 0\}$, $u_n \to +\infty$ in $\Omega \cap \{x : \varphi_k(x) \leq 0\}$. Let $n \to \infty$ and $v = \varphi_k$, and from* (3.19) *we have*

$$\begin{aligned}\langle g^*, \varphi_k\rangle &= \int_\Omega g(x)\varphi_k(x)dx = \lim_{n\to\infty}\langle S(u_n), \varphi_k\rangle = \lim_{n\to\infty}\int_\Omega f(u_n)\varphi_k(x)dx \\ &= \lim_{n\to\infty}\int_\Omega f(u_n)(\varphi_k^+(x) - \varphi_k^-(x))dx = \int_\Omega [f(-\infty)\varphi_k^+(x) - f(+\infty)\varphi_k^-(x)]dx.\end{aligned}$$

The above two cases contradict (3.17) or (3.18). This proves that the homotopy $T_\tau(u)$ is admissible within the ball $Q(0, R)$. Hence, from Lemma 2.3(C3) we obtain

$$\deg(A - \lambda_k B - S + g^*, Q(0, R), 0) = \deg(A - (\lambda_k + \delta_k)B, Q(0, R), 0). \tag{3.21}$$

Note that the right-hand side of (3.21) is an odd number by Lemma 2.3(C2) and $\lambda_k + \delta_k \in (\lambda_k, \lambda_{k+1})$. Hence $\deg(A - \lambda_k B - S + g^*, Q(0, R), 0) \neq 0$, and Lemma 2.3(C1) implies that (3.16) has a weak solution. This completes the proof. □

Example 3.1. *Let $f(x,t) = b|t|^{p-2}t$, $t \in \mathbb{R}, x \in \Omega$. When $b \neq \lambda_k, k = 1, 2, \cdots$, (H1) holds. When $b > \mu_1^{-1}$, (H1) and (H2) hold.*

Example 3.2. *Let $f(t) = \arctan t, t \in \mathbb{R}$ and $g(x) \equiv 1, x \in \Omega$. Then $f(-\infty) = \lim_{t\to-\infty} f(t) = -\frac{\pi}{2}$, $f(+\infty) = \lim_{t\to+\infty} f(t) = \frac{\pi}{2}$, and*

$$\begin{aligned}&\int_\Omega -\frac{\pi}{2}\varphi_k^+(x)dx - \int_\Omega \frac{\pi}{2}\varphi_k^-(x)dx = -\frac{\pi}{2}\int_\Omega |\varphi_k(x)|dx \\ &< \int_\Omega \varphi_k(x)dx < \int_\Omega \frac{\pi}{2}\varphi_k^+(x)dx - \int_\Omega -\frac{\pi}{2}\varphi_k^-(x)dx = \frac{\pi}{2}\int_\Omega |\varphi_k(x)|dx.\end{aligned}$$

Note that (H3) and (3.18) *hold.*

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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