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Research article

Matching preclusion and conditional matching preclusion for hierarchical cubic networks

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Abstract: Matching preclusion originates from the measurement of interconnection network robustness in the event of edge failure. Conditional matching preclusion belongs to generalized matching preclusion. We obtain the matching preclusion number and conditional matching preclusion number for hierarchical cubic network(HCN_n). Additionally, all the optimal (conditional) matching preclusion sets of HCN_n are characterized, which generalize some related results of Birgham et al. and Cheng et al.

Keywords: matching; matching preclusion; conditional matching preclusion; hierarchical cubic network HCN_n
Mathematics Subject Classification: 05C05, 05C12, 05C76

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *perfect matching* in a graph is a set of edges such that each vertex is incident to exactly one of them, and an *almost-perfect matching* is a matching covering all but one vertex of G. A graph with an even number of vertices is an *even* graph, otherwise it is an odd graph. We use $G - F$ to denote the subgraph of G obtained by deleting all the vertices and/or the edges of $F \subseteq V(G) \cup E(G)$.

The *matching preclusion number* of graph G, denoted by $mp(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching or an almost-perfect matching. Any such optimal set is called an optimal matching preclusion set. Birgham et al. [1] first introduced the definition of matching preclusion as a measure of robustness of interconnection networks in the event of edge failure. But beyond that, it also has connections to a few concepts related to theoretical topics, such as the conditional connectivity and extremal graph theory. For more on these topics see the surveys [2–9].

Let G be a graph with an even number of vertices. If it also has the property that the unique optimal matching preclusion sets are those whose elements are incident to a single vertex, then the following Proposition 1.1 holds:

Proposition 1.1. *[4] Let G be a graph with an even number of vertices. Then $mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

We call an optimal solution of the form given in Proposition 1.1 a trivial optimal matching preclusion set. If $mp(G) = \delta(G)$, then G is maximally matched, and it is super matched if $mp(G) = \delta(G)$ and every optimal matching preclusion set is trivial. Obviously, the super matching network has better robustness.

The definition of conditional matching preclusion number was given in [4]. Let G be a graph in which $mp(G) > 0$. The *conditional matching preclusion number* of a graph G, denoted by $mp_1(G)$, is the minimum number of edges whose deletion leaves the resulting graph with no isolated vertices and without a perfect matching or an almost-perfect matching. Any such optimal set is called an *optimal conditional matching preclusion set*. If a conditional matching preclusion set does not exist in G, that is, we cannot delete edges to satisfy both conditions, we leave $mp_1(G)$ undefined.

In [4], Cheng et al. investigated a basic obstruction to a perfect matching in the resulting graph with no isolated vertices. They found that for a basic obstacle set of perfect matching there would be exist a path $u - w - v$, where the degree of u and the degree of v are both one. They considered any such path $u - w - v$ in the original graph and defined that $v_e(G) = min\{d_G(u) + d_G(v) - 2 - y_G(u, v)\colon$ there exists a 2-path between u and $v\}$, where $d_G(.)$ is the degree function, and $y_G(u, v) = 1$ if u and v are adjacent and 0 otherwise. Similar to Proposition 1.1, they obtained the following Proposition 1.2:

Proposition 1.2. *[4] Let G be a graph with an even number of vertices. Suppose that every vertex in G has degree at least* 3. *Then $mp_1(G) \leq v_e(G)$.*

We call an optimal solution of the form induced by a 2-path with value $v_e(G)$ a trivial optimal conditional matching preclusion set. If G is a super matched graph and $mp_1(G) = v_e(G)$, then it is conditionally maximally matched. If, in additional, all optimal solutions are trivial, then it is conditionally super matched.

At present, matching preclusion set(or conditional matching preclusion set) for various basic interconnection networks has attracted much attention and research interest. For instance, the matching preclusion number for balanced hypercubes was investigated in [10]. In [11], Cheng et al. considered conditional matching preclusion for the arrangement graphs. Wang et al. [12] showed that the conditional matching preclusion number for k-ary n-cube is $4n - 2$. For more details one can refer to [2–6, 10–13] and reference therein.

The hierarchical cubic network HCN_n was introduced by Ghose in [14]. For papers on hierarchical cubic network, we refer to the readers to [15] for a sample of results and additional references for HCN_n. The HCN_n has many desirable properties such as topological structure. The hierarchical cubic network $HCN_n(n \geq 2)$ can be decomposed into 2^n clusters, where each cluster is isomorphic to an n-dimensional hypercube Q_n. The matching preclusion and conditional matching preclusion of hypercube have been studied by [1] and [4] respectively. In this paper, we investigate the matching preclusion number and conditional matching preclusion number for hierarchical cubic networks. In Section 2, we introduce some notations, definitions and properties used throughout the paper. In Section 3, we discuss the matching preclusion number of HCN_n and characterize all the optimal matching preclusion sets of HCN_n. Finally, we consider the conditional matching preclusion number of HCN_n and all the optimal conditional matching preclusion sets of HCN_n are categorized in Section 4.

2. Preliminaries

For a path, the length of it is the number of edges contained in it. Denote the path of length of $k(k \geq 1)$ by k-path. For any vertex $u \in V(G)$, use $N_G(u)$ to denote the neighborhood of u, that is, the set of vertices adjacent to v. The distance $d_G(u, v)$, between a pair of vertices u and v in G, is the length of a shortest path joining u and v.

The Hamming distance, denoted by $d_H(u, v)$, between any two vertices u and v, is the number of different positions between the binary strings of u and v.

Let V_n be the set of binary sequence of length n, i.e., $V_n = \{x_1x_2 \cdots x_n \colon x_i \in \{0, 1\}, 1 \leq i \leq n\}$. For $x = x_1x_2 \cdots x_n \in V_n$, the element $\overline{x} = \overline{x}_1\overline{x}_2 \cdots \overline{x}_n \in V_n$ is called the bitwise complement of x, where $\overline{x}_i = \{0, 1\} \backslash \{x_i\}$ for each $i \in \{1, 2, \cdots, n\}$.

The n-dimensional hypercube network Q_n is an cube, shortly n-cube, with its vertex-set V_n, and two vertices are adjacent if and only if they differ exactly in one coordinate. Fig. 1 shows a Q_3. For the sake of simplicity, we use xQ_n to denote the Cartesian product $\{x\} \times Q_n$ of a vertex x and a hypercube network Q_n.

Definition 2.1. *[14] An n-dimensional hierarchical cubic network HCN_n with vertex-set $V_n \times V_n$ is obtained from 2^n n-cubes $\{xQ_n \colon x \in V_n\}$ by adding edges between two n-cubes, called crossing edges, according to the following rule. A vertex (x, y) in xQ_n is linked to*

(1) (y, x) in yQ_n if $x \neq y$ or

(2) $(\overline{x}, \overline{y})$ in $\overline{x}Q_n$ if $x = y$.

The vertex (y, x) in yQ_n or $(\overline{x}, \overline{y})$ in $\overline{x}Q_n$ is called on external neighbor of (x, y) in xQ_n.

The edges between any two different Q_n of HCN_2 are denoted by the crossing edges. And a 2-dimensional hierarchical cubic network HCN_2 is shown in Fig. 2.

From Definition 2.1, it is easy to obtain the following property about crossing edges in HCN_n.

(1) There are two crossing edges between two n-cubes xQ_n and yQ_n if and only if x and y are complementary, otherwise there is only.

(2) The set of crossing edges consists of a perfect matching of HCN_n.

For Q_n, the results on the matching preclusion and the conditional matching preclusion of it are as follows.

Lemma 2.2. *[1] Let $n \geq 2$. Then $mp(Q_n) = n$ and Q_n is super matched.*

Lemma 2.3. *[1] Let $n \geq 3$. Then $mp_1(Q_n) = 2n - 2$ and Q_n is conditionally super matched.*

Let $V_n = \{x^1, x^2, \cdots, x^{2^n}\}$, $x^1Q_n, x^2Q_n, \cdots, x^{2^n}Q_n$ be 2^n n-cubes in HCN_n. For $(z, z') \in E(HCN_n)$, and $z \in V(x^iQ_n)$, $z' \in V(x^jQ_n)$ where $1 \leq i \neq j \leq 2^n$, then z' is called the external neighbor of z. For notational simplicity, suppose that $x = x^1$. For a graph G, $v, w \in V(G)$, if there are $t(t \geq 2)$ paths

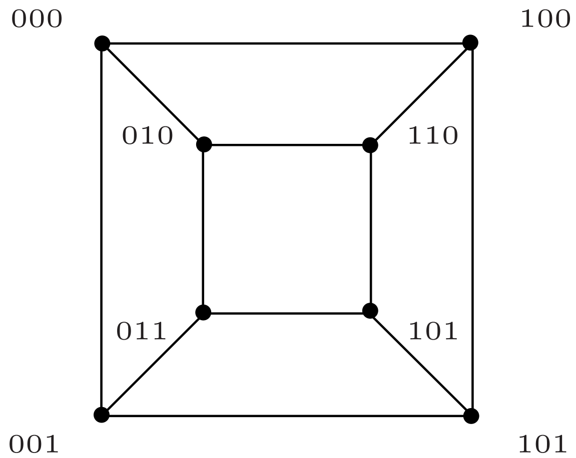

Figure 1. Q_3.

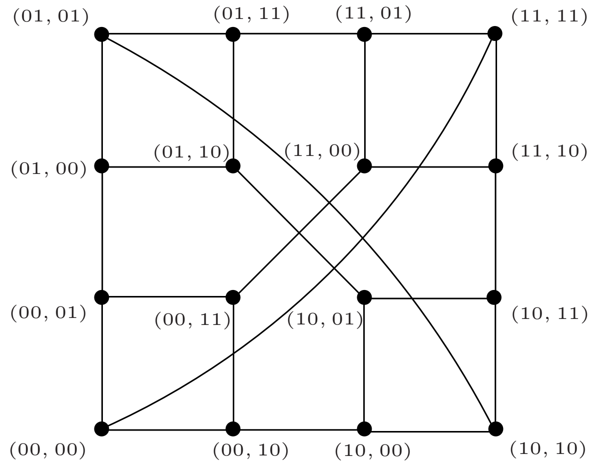

Figure 2. A 2-dimensional hierarchical cubic network HCN_2.

joining v and w, say $P_1, \ldots, P_t$ such that $V(P_i) \cap V(P_j) = \{v, w\}$ for $1 \leq i, j \leq t, i \neq j$, then these paths are denoted by internally vertex-disjoint paths.

Lemma 2.4. *Let u be a vertex of HCN_n, and u' be its external neighbor. Then there exist n internally vertex-disjoint 7-paths joining endpoints u and u'.*

Proof. Without loss of generality, suppose $u \in V(xQ_n), u = (x, y)$. Let $x_i, y_i, \overline{x}_i (1 \leq i \leq n)$ be the binary sequence of length n which differ from $x, y, \overline{x}$ in the ith bit only respectively, and $N_{xQ_n}(u) = \{u_i = (x, y_i), 1 \leq i \leq n\}$.

Suppose $d = d_H(x, y)$, then $0 \leq d \leq n$. We consider the following two cases.

Case 1. $d = 0$, that is $x = y$, $u = (x, x)$, $u' = (\overline{x}, \overline{x})$.

So $d(x, x_i) = 1$, the external neighbor of $u_i = (x, x_i)(1 \leq i \leq n)$ is (x_i, x) in x_iQ_n respectively. For each vertex $(x_i, x)(1 \leq i \leq n)$, $N_{x_iQ_n}((x_i, x)) = \{(x_i, x_1), (x_i, x_2), \ldots, (x_i, x_n)\}$. For $(\overline{x}, \overline{x}) \in V(\overline{x}Q_n)$, $N_{\overline{x}Q_n}((\overline{x}, \overline{x})) = \{(\overline{x}, \overline{x}_1), (\overline{x}, \overline{x}_2), \ldots, (\overline{x}, \overline{x}_n)\}$. For each vertex $(\overline{x}, \overline{x}_i)(1 \leq i \leq n)$, its external neighbors is $(\overline{x}_i, \overline{x})$ in $\overline{x}_iQ_n$, $N_{\overline{x}_iQ_n}((\overline{x}_i, \overline{x})) = \{(\overline{x}_i, \overline{x}_1), (\overline{x}_i, \overline{x}_2), \cdots, (\overline{x}_i, \overline{x}_n)\}$. The paths $\{(x, x) - (x, x_i) - (x_i, x) - (x_i, x_i) - (\overline{x}_i, \overline{x}_i) - (\overline{x}_i, \overline{x}) - (\overline{x}, \overline{x}_i) - (\overline{x}, \overline{x}), 1 \leq i \leq n\}$ are the desired paths.

Case 2. $d \geq 1$, then $u' = (y, x)$.

Then $N_{xQ_n}(u) = \{(x, y_i), i = 1, 2, \cdots, n\}$, and $N_{yQ_n}(u') = \{(y, x_i), i = 1, 2, \cdots, n\}$.

If $d \geq 2$, the external neighbors of (x, y_i) and (y, x_i) are (y_i, x) and (x_i, y) respectively where $1 \leq i \leq n$. And $N_{y_iQ_n}((y_i, x)) = \{(y_i, x_1), (y_i, x_2), \cdots, (y_i, x_n)\}$, $N_{x_iQ_n}((x_i, y)) = \{(x_i, y_1), (x_i, y_2), \cdots, (x_i, y_n)\}$. The paths $\{(x, y) - (x, y_i) - (y_i, x) - (y_i, x_i) - (x_i, y_i) - (x_i, y) - (y, x_i) - (y, x), 1 \leq i \leq n\}$ are the desired paths.

If $d = 1$, there is some i such that $x = y_i$ and $y = x_i$. The ith path joining endpoints u and u' is $(x, y) - (x, x) - (\overline{x}, \overline{x}) - (\overline{x}, \overline{y}) - (\overline{y}, \overline{x}) - (\overline{y}, \overline{y}) - (y, y) - (y, x)$. And we can find the remaining $n - 1$ paths by the case $d \geq 2$, completing the proof. □

Let $Z \subseteq V(G)$ and $Y \subseteq V(G) \setminus Z$, the (Y, Z)-paths is a family of internally vertex-disjoint paths starting at a vertex $y \in Y$ ending at a vertex $z \in Z$ and whose internally vertices belong neither to Y nor Z. By Lemma 2.4, we can obtain the following corollary easily.

Corollary 2.5. *Let u be a vertex of HCN_n, and u' is its external neighbor, $U = N_{HCN_n}(u) - u'$. Then there are n internally vertex-disjoint paths joining u' and U.*

3. The matching preclusion number of HCN_n

Let C denote the set of all crossing edges of HCN_n. Then $E(HCN_n) = \cup_{i=1}^{2^n} E(x^i Q_n) \cup C$. Suppose $F \subseteq E(HCN_n)$ be a fault edge set of HCN_n, $F_i = F \cap E(x^i Q_n)(i = 1, 2, \cdots, 2^n)$, $F_c = F \cap C$, then $F = \cup_{i=1}^{2^n} F_i \cup F_c$.

Theorem 3.1. *Let $n \geq 2$. Then $mp(HCN_n) = n + 1$.*

Proof. As $\delta(HCN_n) = n + 1$, $mp(HCN_n) \leq n + 1$. And HCN_n is made up of 2^n n-cubes and a perfect matching, by Lemma 2.2, $mp(Q_n) = n$. Then $mp(HCN_n) \geq n + 1$. Thus $mp(HCN_n) = n + 1$. □

Theorem 3.2. *Let $n \geq 2$. Then $mp(HCN_n) = n + 1$. Moreover, HCN_n is super matched.*

Proof. By Theorem 3.1, $mp(HCN_n) = n + 1$, we now classify the optimal solutions. Let F be any optimal matching preclusion set of HCN_n, $|F| = n + 1$, we need to prove $HCN_n - F$ has no perfect matchings. As C is a perfect matching of HCN_n, $|F_c| \geq 1$ and $x^i Q_n - F_i$ has no perfect matching for some $i(i = 1, 2 \ldots, n)$. For notational convenience, we may assume that $xQ_n - F_1 (x = x^1)$.

xQ_n is isomorphic to Q_n, by Lemma 2.2, $|F_1| \geq n$. And $| \cup_{i=1}^{2^n} F_i| \leq n$, thus $|F_1| = n$, $|F_i| = 0(i = 2, \cdots, 2^n)$ and $xQ_n - F_1$ has an isolated vertex, say $u = (x, y)$. If the crossing edge $(u, u') \in F$, then u is an isolated vertex in $HCN_n - F$, thus $(u, u') \notin F$. If $d_H(x, y) \geq 2$. By Lemma 2.4, choose one path joining u and u', say $P : u = (x, y) - (x, y_i) - (y_i, x) - (y_i, x_i) - (x_i, y_i) - (x_i, y) - (y, x_i) - (y, x) = u'$ for some $i \in \{1, 2, \ldots, n\}$. And Q_n is made up of n edge-disjoint perfect matchings, so $xQ_n - \{(x, y), (x, y_i)\}$, $y_i Q_n - \{(y_i, x), (y_i, x_i)\}$, $x_i Q_n - \{(x_i, y_i), (x, y)\}$, $yQ_n - \{(y, x_i), (y, x)\}$ have perfect matchings, say M_x, M_{y_i}, M_{x_i}, M_y. And $x^i Q_n - F_i$ has a perfect matching for $x^i \in V_n \backslash \{x, y_i, x_i, y\}$, say M_{x^i}. Thus $M = M_x \cup M_{y_i} \cup M_{x_i} \cup M_y \bigcup_{x^i \in V_n \backslash \{x, y_i, x_i, y\}} M_{x^i} \cup ((x, y_i), (y_i, x)) \cup ((y_i, x_i), (x_i, y_i)) \cup ((x_i, y), (y, x_i)) \cup (u, u')$ be a perfect matching of $HCN_n - F$. If $d_H(x, y) \neq 2$, the perfect matching of $HCN_n - F$ can be obtained similarly, completing the proof. □

4. The conditional matching preclusion of HCN_n

Theorem 4.1. *Let $n \geq 3$. Then $mp_1(HCN_n) = 2n$.*

Proof. Since $\nu_e(HCN_n) = 2n$, $mp_1(HCN_n) \leq 2n$. Let F be any conditional matching preclusion set of HCN_n and $|F| \leq 2n - 1$, it is enough to show that $HCN_n - F$ has a perfect matching if $HCN_n - F$ has no isolated vertices. By the structure of HCN_n, we know that $|F_c| \geq 1$. We can claim that there is only one of $\{x^i Q_n - F_i, 1 \leq i \leq 2^n\}$ with no prefect matching. Otherwise, without loss of generality, suppose both $xQ_n - F_1$ and $x^2 Q_n - F_2$ have no prefect matching, then $|F_1| \geq n$ and $|F_2| \geq n$ by Lemma 2.2. And $|F| \geq |F_1| + |F_2| + |F_c| \geq 2n + 1$, a contradiction with $|F| \leq 2n - 1$. For notational convenience, we may assume that $xQ_n - F_1$. Thus $|F_1| \geq n$. And as $|F| = |F_1| + \cdots + |F_{2^n}| + |F_c| = 2n - 1$, then $|F_i| \leq n - 2$ for $i \in \{2, 3, \ldots, 2^n\}$. We consider the following two cases.

Case 1. $|F_c| \geq 2$.

Then $|F_1| + \cdots + |F_{2^n}| \leq 2n - 3$, $|F_1| \leq 2n - 3$. Let $|F_c| = i(i \geq 2)$, $|F_1| \leq 2n - (i + 1)$, and $|F_c| \leq n - 1$ for $|F_1| \geq n$. From Lemma 2.3, $xQ_n - F_1$ has an isolated vertex, otherwise $xQ_n - F_1$ has a perfect matching. For xQ_n consists of n edge-disjoint perfect matchings, then $xQ_n - u$ has n edge-disjoint almost perfect matchings. And $|F_1 \cap E(xQ_n - u)| \leq n - (i + 1)$, there exists at least $(i + 1)$ edge-disjoint almost perfect matchings of $xQ_n - (F_1 \cup \{u\})$. Without loss of generality, suppose the $i + 1$ unmatched vertices are

$u_1, u_2, \cdots, u_{i+1}$. As there is no isolated vertex in $HCN_n - F$, $(u, u') \notin F$. By Corollary 2.5, there exist $(i+1)$ internally vertex-disjoint paths joining u' and $\{u_1, \ldots, u_{i+1}\}$, say $P'_1, \ldots, P'_{i+1}$. And $|F_c| = i$, thus there is at least one path of $\{P'_1, \ldots, P'_{i+1}\}$ and none of its crossing edges is in F_c, say $i = 1$. Thus P_1 is a 7-path joining endpoints u and u', we can obtain the perfect matching of $HCN_n - F$ by the proof of Theorem 3.2.

Case 2. $|F_c| = 1$.

Then $|F_1| + \cdots + |F_{2^n}| = 2n - 2$. If $xQ_n - F_1$ has an isolated vertex. Similar to the proof above, we can obtain the perfect matching of $HCN_n - F$ easily.

Now suppose $xQ_n - F_1$ has no isolated vertex. As $xQ_n - F_1$ has no perfect matching, then F_1 is a trivial conditional matching preclusion set of HCN_n, namely $|F_1| = 2n - 2$, $|F_i| = 0$ for $2 \le i \le 2^n$. For notational simplicity, we may assume that it is induced by $u_1 - u - u_2$. Let $(x, y_i^j)(j = 2, \cdots, n)$ be adjacent to $(x, y_i)(i = 1, 2)$ such that y_i^j differs from y_i in the jth bit only, and u'_1, u'_2 be the external neighbors of u_1 and u_2. By $|F_c| = 1$, either $(u_1, u'_1) \notin F_c$ or $(u_2, u'_2) \notin F_c$, say (u_1, u'_1). Let $N_{xQ_n}(u_1) = \{u, v_1, \ldots, v_{n-1}\}$. By Corollary 2.5, there are $n - 1$ internally vertex-disjoint 7-paths joining u'_1 and $\{v_1, \ldots, v_{n-1}\}$, say R_i for $1 \le i \le n-1$. As $n \ge 3$ and $|F_c| = 1$, there is at least one path of $\{R_1, \ldots, R_{n-1}\}$ such that none of its crossing edges is in F_c, say R_1. The perfect matching induced by R_1 is denoted as M_1. And $xQ_n - (F_1 \cup (u_1, v_1))$ has a perfect matching, say M_{x^1}. $\cup_{i=1}^{2^n} x^i Q_n - (R_1 - \{u_1, v_1\})$ also has a perfect matching, say M', then $M = M_{x^1} \cup M_1 \cup M'$ is a perfect matching of $HCN_n - F$.

□

Lemma 4.2. *[1] Let $n \ge 3$. Let F be a conditional matching preclusion set in Q_n with $|F| = 2n - 1$. Then there exists $f, f' \in F$ such that f and f' are independent and both $Q_n - (F - \{f\})$ and $Q_n - (F - \{f'\})$ contain a perfect matching.*

From the structure of Q_n, we know that there is a fact: for $n \ge 3$, suppose P be a 3-path of Q_n, then $Q_n - P$ has a perfect matching. Thus we can obtain the following lemmas.

Lemma 4.3. *Let $v, w \in V(Q_n)$ and $d(v, w) = 3$. Then $Q_n - \{v, w\}$ has a perfect matching.*

Lemma 4.4. *Let F' be a fault edge set of Q_3 with $|F'| \le 4$, $v, w \in V(Q_3)$, $d(v, w) = 3$. If v is an isolated vertex of $Q_3 - F'$, then $Q_3 - F'$ has an almost perfect matching such that w is unmatched.*

Proof. Without loss of generality, let $v = 000$, then $w = 111$. As $d(v) = 3$, $|E(Q_3 - v) \cap F'| = 1$. For any edge of $E(Q_3 - v)$, we can find an almost perfect matching and w is unmatched. □

Lemma 4.5. *For $n \ge 3$, let F' be a fault edge set of Q_n with $|F'| \le 2n - 2$, $v, w_1, \ldots, w_{n-1} \in V(Q_n)$, $d(v, w_i) = 3(i = 1, \ldots, n-2)$. If v is an isolated vertex of $Q_n - F'$, then $Q_n - F'$ has an almost perfect matching and $w_i(i = 1, \ldots, n-2)$ is unmatched.*

Proof. We complete the lemma by induction on n. For $n = 3$, the conclusion holds by Lemma 4.4, now suppose $n \ge 4$ and the Lemma is true for $n - 1$.

Q_n contains two $(n-1)$-dimensional Q_{n-1}, say Q^0_{n-1} and Q^1_{n-1}, with the first bit is 0 and 1 respectively, the perfect matching between Q^0_{n-1} and Q^1_{n-1} is defined by M. Let $F'_i = F' \cap Q^i_{n-1}(i = 0, 1)$ and $F_M = F' \cap M$. Hence $F' = F'_0 \cup F'_1 \cup F_M$. Without loss of generality, we can assume that $v \in V(Q^0_{n-1})$. As v is an isolated vertex of $Q_n - F$, $|F_M| \ge 1$. If $|F_M| \ge 2$, hence $|F'_0| + |F'_1| \le 2n - 4$ and $|F'_i| \le 2n - 4$ for $i = 0, 1$. By inductive assumption, the conclusion holds. Now suppose $|F_M| = 1$, $|F'_0| = 2n - 3$ and $|F'_1| = 0$.

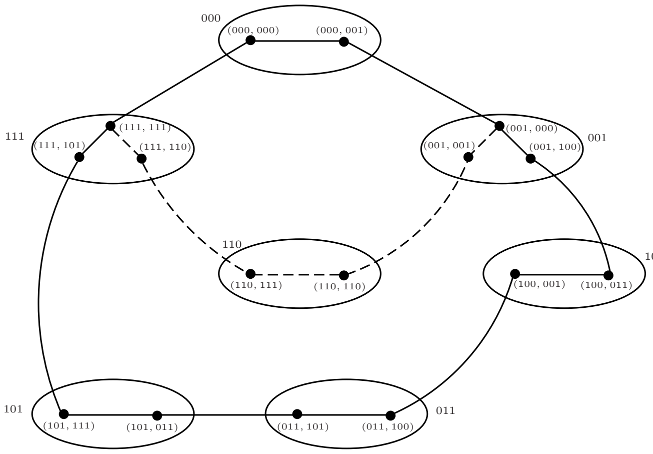

Figure 3. The illustration of case 1 of Lemma 4.7.

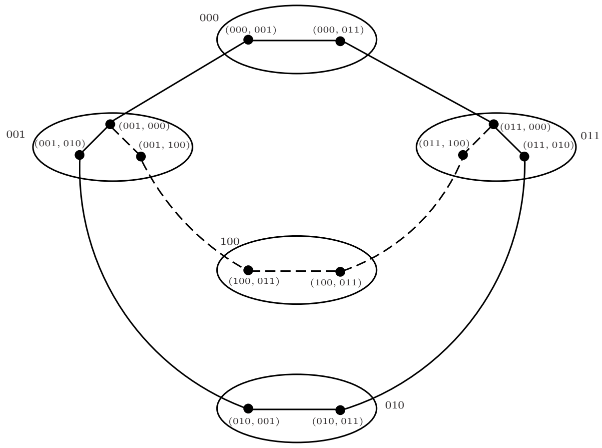

Figure 4. The illustration of case 2 of Lemma 4.7.

Denote the neighbors of v in Q^0_{n-1} by $v_i(1 \leq i \leq n-1)$. Let the external neighbors of v and $v_i(1 \leq i \leq n-1)$ be v' and v'_i. For Q^0_{n-1} consists of $(n-1)$-edge disjoint perfect matchings, $Q^0_{n-1} - v$ must contain $(n-1)$-edge disjoint almost perfect matchings, and $|E(Q^0_{n-1} - v) \cap F| \leq n-2$, there exists an almost perfect matching such that v_i is unmatched for some i, say v_1. And $Q^0_{n-1} - \{v, v_1\}$ has a perfect matching, say M_0. As $|F_M| = 1$, then $(v_1, v'_1) \notin F_M$. Let $N_{Q^1_{n-1}}(v'_1) = \{v', w_2, \ldots, w_{n-1}\}$ and $w_1 = v'$. As $|F_1| = 0$, $Q^1_{n-1} - \{v'_1\}$ have $(n-1)$-edge disjoint almost perfect matching such that w_i is unmatched, and $d(v, w_i) = 3$, completing the proof. □

Lemma 4.6. *For $n \geq 3$ and any two vertices of $V(x^i Q_n)(1 \leq i \leq 2^n)$ such that the distance between them is 3. Then there is a cycle L of even order and $V(L)$ contains the two vertices.*

Proof. For notational convenience, let $i = 1$, $u = (x, y)$, $v = (x, y')$. As $d(u, v) = 3$, then $d_H(y, y') = 3$. We complete the proof according to the value of $d_H(x, y)$.

Case 1. $d_H(x, y) = 0$ or 3.

Then $d_H(x, y') = 3$ or 0, we only need to consider that $d_H(x, y) = 0$.

For $n = 3$, we can assume that $u = (000, 000)$, $v = (000, 111)$. Hence $L : (000, 000) - (111, 111) - (111, 110) - (111, , 100) - (111, 000) - (000, 111) - (000, 011) - (000, 001) - (000, 000)$.

For $n \geq 4$, without loss of generality, suppose that $u = (0^n, 0^n), v = (0^n, 0^{n-3}111)$. Then L : $(0^n, 0^n) - (1^n, 1^n) - (1^n, 1^{n-3}110) - (1^n, 1^{n-3}100) - (1^n, 1^{n-3}000) - (1^{n-3}000, 1^n) - (1^{n-3}000, 1^{n-3}110) - (1^{n-3}000, 1^{n-3}100) - (1^{n-3}000, 1^{n-3}000) - (0^{n-3}111, 0^{n-3}111) - (0^{n-3}111, 0^{n-3}110) - (0^{n-3}111, 0^{n-3}100) - (0^{n-3}111, 0^{n-3}000) - (0^n, 0^{n-3}111) - (0^n, 0^{n-3}110) - (0^n, 0^{n-3}100) - (0^n, 0^n)$.

Case 2. $d_H(x, y) \neq 0, 3$.

Let $P : u = (x, y) - (x, z) - (x, w) - (x, y')$ denote a 3-path of xQ_n. Then we can obtain the cycle $L : (x, y) - (x, z) - (x, w) - (x, y') - (y', x) - (y', x') - (x', y') - (x', w) - (x', z) - (x', y) - (y, x') - (y, x) - (x, y)$ with $d_H(x, x') = 1$ and $x' \neq y$, completing the proof. □

By the above lemma, we obtain that there is a path of length odd joining endpoints u and v. The following Lemmas show that there are 2 internally vertex-disjoint paths between the external neighbors of two vertices of HCN_n.

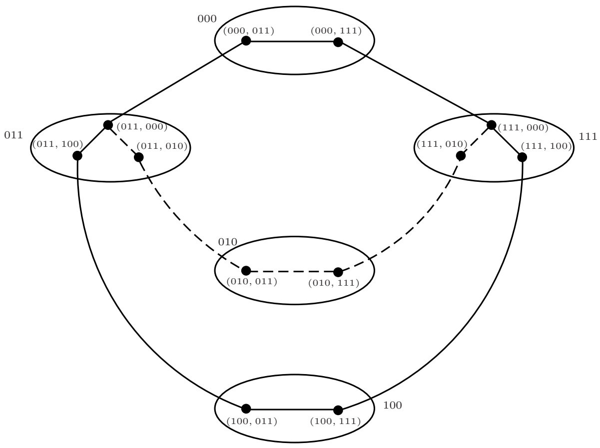

Figure 5. The illustration of case 3 of Lemma 4.7.

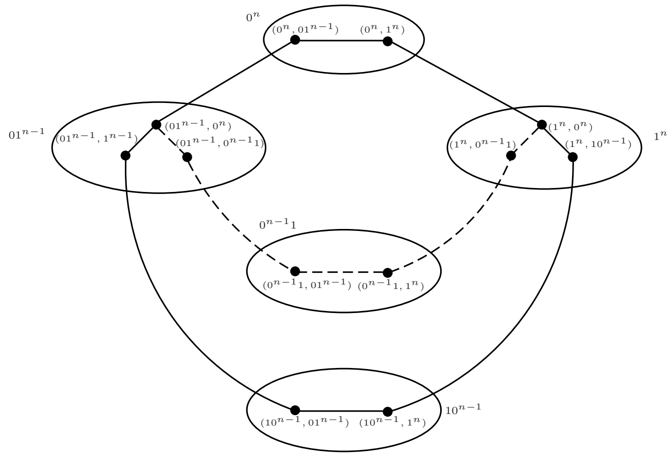

Figure 6. The illustration of Lemma 4.8.

Lemma 4.7. *For $n = 3$, $w, q \in V(xQ_3)$ and w is adjacent to q, w', q' are the external neighbors of w and q respectively. There exist 2 internally vertex-disjoint paths of length odd, joining w' and q'.*

Proof. Without loss of generality, suppose $x = 000$, $w = (x, z), q = (x, z_1)$, and z_1 differs from z in the first bit. We consider the following cases by the value of $d_H(x, z)$.

Case 1. $d_H(x, z) = 0$.

That is $z = 000$, then $d_H(x, z_1) = 1$. Let $z_1 = 001$. $w = (000, 000), q = (000, 001)$, $w' = (111, 111), q' = (001, 000)$ are the external neighbor of w and q respectively. There are two internally vertex-disjont paths of length 5 joining w' and q', denoted by P_1, P_2, $P_1 : (111, 111) - (111, 110) - (110, 111) - (110, 110) - (001, 001) - (001, 000)$, $P_2 : (111, 111) - (111, 101) - (101, 111) - (101, 011) - (011, 101) - (011, 100) - (100, 011) - (100, 001) - (001, 100) - (001, 000)$, see Fig. 3.

Case 2. $d_H(x, z) = 1$.

Without loss of generality, let $z = 001$, then $d_H(x, z_1) = 0$ or $d_H(x, z_1) = 2$. If $d_H(x, z_1) = 0$, the result can be obtained by Case 1. Thus $d_H(x, z_1) = 2$, let $z_1 = 011$. $w = (000, 001), q = (000, 011), w' = (001, 000), q' = (011, 000)$. There are two internally vertex-disjoint paths of length 5 joining $(001, 000)$ and $(011, 000)$, denoted by P_1, P_2, $P_1 : (001, 000) - (001, 010) - (010, 001) - (010, 011) - (011, 010) - (011, 000)$, $P_2 : (001, 000) - (001, 100) - (100, 001) - (100, 011) - (011, 100) - (011, 000)$, see Fig.4.

Case 3. $d_H(x, z) = 2$.

Without loss of generality, suppose $z = 011$, then $d_H(x, z_1) = 1$ or $d_H(x, z_1) = 3$. If $d_H(x, z_1) = 1$, we can obtain the result by case 2. Thus $d_H(x, z_1) = 3$, $(x, z) = (000, 011)$, and let $(x, z_1) = (000, 111)$. $w = (000, 011), q = (000, 111)$, $w' = (011, 000)$ and $q' = (111, 000)$. There exist two internally vertex-disjoint 5-paths joining w' and q', denoted by $P_1, P_2, P_1 : (011, 000) - (011, 010) - (010, 011) - (010, 111) - (111, 010) - (111, 000)$, $P_2 : (011, 000) - (011, 100) - (100, 011) - (100, 111) - (111, 100) - (111, 000)$, see Fig.5.

Case 4. $d_H(x, z) = 3$.

Then $d_H(x, z') = 2$, the proof is similar to case 3. □

Lemma 4.8. *For $n \geq 3$, let $w, q \in V(xQ_n)$ and w is adjacent to q, w', q' are the external neighbors of w and q respectively. There exist 2 internally vertex-disjoint paths of length odd, joining w' and q'.*

Proof. Suppose $w = (x, z), q = (x, z_1)$, z_1 differs from z in the first bit. The conclusion holds clearly for $n = 3$ by Lemma 4.7. Assume $n \geq 4$ below. We complete the proof by induction on n. Now we suppose the result holds for $n - 1$. Then $0 \leq d_H(x, z) \leq n$. If $0 \leq d_H(x, z) \leq n - 2$, the result can be obtained by induction hypothesis easily. Thus $d_H(x, z) = n - 1$ or n. If $d_H(x, z) = n$, then $d_H(x, z_1)$ must be $n - 1$, , the case is the same as $d_H(x, z) = n - 1$ and $d_H(x, z_1) = n$. Now we suppose $d_H(x, z) = n - 1$, then $d_H(x, z_1) = n - 2$ or n, the case $d_H(x, z_1) = n - 2$ holds by induction, thus $d_H(x, z_1) = n$. Without loss of generality, suppose $x = 0^n, z = 01^{n-1}, z_1 = 1^n$. $w = (x, z) = (0^n, 01^{n-1}), q = (x, z_1) = (0^n, 1^n)$, their external neighbors are $w' = (01^{n-1}, 0^n)$ and $q' = (1^n, 0^0)$ respectively. We can find two vertex-disjoint paths joining w' and q', denoted by P_1 and P_2, $P_1 : (01^{n-1}, 0^n) - (01^{n-1}, 0^{n-1}1) - (0^{n-1}1, 01^{n-1}) - (0^{n-1}1, 1^n) - (1^n, 0^{n-1}1) - (1^n, 0^n)$, $P_2 : (01^{n-1}, 0^n) - (01^{n-1}, 10^{n-1}) - (10^{n-1}, 01^{n-1}) - (10^{n-1}, 1^n) - (1^n, 10^{n-1}) - (1^n, 0^n)$, see Fig.6. □

Theorem 4.9. *For $n \geq 4$, $mp_1(HCN_n) = 2n$, and it is conditionally super matched.*

Proof. By Theorem 4.1, we know that $mp_1(HCN_n) = 2n$. Now we only need to prove that HCN_n is conditionally super matched. Let $|F| = 2n$, it is enough to show that one of the following holds: (1) $HCN_n - F$ has an isolated vertex; (2) F is a conditional matching preclusion set; (3) $HCN_n - F$ has a perfect matching. We use the same notation as in the proof of Theorem 4.1. As $|F_1| \geq n$, $|F_c| \geq 1$, then $|F_j| \leq n - 1$ for $j \in \{2, \ldots, 2^n\}$. We claim that $|F_c| \leq n$. If $|F_c| \geq n + 1$, then $|F_i| \leq n - 1$ for each $i(1 \leq i \leq 2^n)$, thus $x^iQ_n - F_i$ has a perfect matching, say M_i, the $M = \cup_{i=1}^{2n} M_i$ is a perfect matching of $HCN_n - F$. We consider the following cases.

Case 1. $2 \leq |F_c| \leq n$, then $n \leq |F_1| \leq 2n - 2$.

If $xQ_n - F_1$ has an isolated vertex, say u. Let u_i' be the external neighbors of $u_i, (i = 1, 2, \cdots, n)$. And $(u, u') \notin F$, otherwise u is an isolated vertex in $xQ_n - F$. Let $|F_c| = \ell(\ell \geq 2)$, then $|F_1| \leq 2n - \ell$, and $|F_1 \cap E(xQ_n - u)| \leq n - \ell$. As $xQ_n - u$ consists of n edge-disjoint almost perfect matchings, $xQ_n - u$ has at least ℓ edge-disjoint almost perfect matchings. Suppose the ℓ vertices unmatched are $u_1, u_2, \cdots, u_\ell$. By corollary 2.5, there are ℓ paths $P_1', P_2', \ldots, P_\ell'$. If there is $P_i'(1 \leq i \leq \ell)$ such that none of crossing edge of $E(P_i')$ is in F_c, say P_1', we can also find a perfect matching of $HCN_n - F$. If for each $P_i'(1 \leq i \leq \ell)$, one crossing edge of $E(P_i')$ is in F_c. From Lemma 4.5, there is an almost perfect matching such that a vertex v with $d(u, v) = 3$ is unmatched. From Lemma 4.6, there is a cycle L of even order and $V(L)$ contains u, v. The perfect matching induced by L is denoted by M_L, and $xQ_n - (F_1 \cup (V(L) \cap V(xQ_n)))$ has a perfect matching, say M_{x^1}, $\cup_{i=2}^{2^n} xQ_n - (V(L) - (V(L) \cap V(xQ_n)))$ also has a perfect matching, say M', then $M = M_L \cup M_{x^1} \cup M'$ is a perfect matching of $HCN_n - F$.

If $xQ_n - F_1$ has no isolated vertex, then F_1 is a conditional super matching preclusion set of $xQ_n - F_1$ with $|F_1| = 2n - 2$ and $|F_c| \leq 2$. We may assume that it is induced by $u_1 - u - u_2$. Let $u, v_1, \cdots, v_{n-1}$ be the neighbors of u_1 in xQ_n, and $u, w_1, \cdots, w_{n-1}$ be the neighbors of u_2 in xQ_n, $v_i{}'$, $w_i{}'$ be the external neighbor of v_i and $w_i(i = 1, 2, \ldots, n - 1)$ respectively. There is at least one of $\{(u_1, u_1'), (u_2, u_2')\}$ not in F_c, otherwise, F is a conditional super matching preclusion, say $(u_1, u_1') \notin F$. By Corollary 2.5, there are $n - 1$ internally vertex-disjoint 7-paths joining u_1' and $\{v_1, \ldots, v_{n-1}\}$, say $Q_1, \ldots, Q_{n-1}$. As $|F_c| \geq 2$, $|F_c| = 2$. And $n \geq 4$, there is at least one path $Q_i(i = 1, 2, \ldots, n - 1)$ such that none of crossing edge of $E(Q_i)$ is in F_c, say Q_1. By the proof above, we know that $HCN_n - (V(Q_1) \cup \{u, u_2\} \cup F)$ has a perfect matching, say M_0, and Q_1 induces a perfect matching, say M'. Then $M_0 \cup M' \cup (u, u_2)$ is a perfect matching of $HCN_n - F$.

Case 2. $|F_c| = 1$.

If $|F_1| \leq 2n-2$, the perfect matching of $HCN_n - F$ can be obtained as the Case 1. Thus we only consider $|F_1| = 2n-1$, and $xQ_n - F_1$ has at most two isolated vertices.

If $xQ_n - F_1$ has two isolated vertices. We can claim that the two isolated vertices are adjacent. Otherwise $|F_1| \geq 2n$ for $\delta(Q_n) = n$, a contradiction. Without loss of generality, suppose such two isolated vertices are u and u_1. As both u and u_1 are not isolated vertices in $HCN_n - F$, then $(u, u') \notin F_c$ and $(u_1, u_1') \notin F_c$. From Lemma 4.8, there are 2 internally vertex-disjoint paths of even order joining endpoints u' and u_1', say R_1, R_2. As $|F_c| = 1$, there is at least one path $R_i (i = 1, 2)$ such that none of crossing edges of $E(R_i)$ is in F_c, say R_1. And $HCN_n - (V(R_1) \cup \{u', u_1'\} \cup F)$ has a perfect matching, say M_0, $V(R_1) \cup \{u', u_1'\}$ induces a perfect matching, say M_1, then $M = M_0 \cup M_1$ is a perfect matching of $HCN_n - F$.

If $xQ_n - F_1$ has an isolated vertex. As there are n edge-disjoint almost perfect matchings of $xQ_n - u$, then there exists at least one almost perfect matching of $xQ_n - F_1$, say M', and u_1 is unmatched. Moreover, $(u, u') \notin F_c$, otherwise u is an isolated vertex of $HCN_n - F$. If $(u_1, u_1{}') \in F_c$, as u_1 is not an isolated vertex of $xQ_n - F_1$, there exists a vertex adjacent to u_1, say v_1. Let $v \in V(xQ_n)$, and $(v_1, v) \in M'$, then $d(u, v) = 3$. From Lemma 4.6, there is a cycle L of even order and $V(L)$ contains u and v. The perfect matching of $HCN_n - (V(L) \cup F)$ is denoted by M_0, and $V(L)$ induces a perfect matching, say M_1, then $M = M_0 \cup M_1$ is a perfect matching of $HCN_n - F$. If $(u_1, u_1{}') \notin F_c$, by the proof case 1, we can find a perfect matching of $HCN_n - F$ similarly.

If $xQ_n - F_1$ has no isolated vertex. Suppose $f = (u_1, v_1), f' = (u_2, w_1)$, and f, f' are independent, by Lemma 4.2, $xQ_n - (F_1 - f)$ and $xQ_n - (F_1 - f')$ contain a perfect matching. For $|F_c| = 1$, there is at most one of $\{(u_1, u_1'), (v_1, v_1'), (w, w'), (w_1, w_1{}')\}$ in F_c. Without loss of generality, $(u_1, u_1') \notin F_c$, $(v_1, v_1{}') \notin F_c$. By the proof of case 1, we can find a perfect matching of $HCN_n - F$. □

5. Conclusions

Hierarchical cube network is a very important network constructed based on 2^n hypercube networks and a perfect matching between them. The matching preclusion number of a graph is the minimum number of edges whose deletion results in a graph that has neither perfect matchings nor almost-perfect matchings, and the conditional matching preclusion number of a graph is the minimum number of edges whose deletion leaves a resulting graph with no isolated vertices that has neither perfect matchings nor almost perfect matchings. In this paper, we find these two numbers for the hierarchical cubic network(HCN_n), characterize all its optimal matching preclusion sets and conditional matching exclusion sets, and prove that the hierarchical cube network has the property of super matching. These results generalize some related results of Birgham et al. [1] and E. Cheng et al. [4] from hypercube network to hierarchical cubic network.

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Conflict of interest

The authors declare no conflict of interest.

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