Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 13208–13224.
DOI: 10.3934/math.2022728
Received: 08 February 2022
Revised: 11 March 2022
Accepted: 15 March 2022
Published: 11 May 2022

Research article

Laplace transform ordering of bivariate inactivity times

Mansour Shrahili* **and Mohamed Kayid**

Department of Statistics and Operations Research, College of Science, King Saud University, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: msharahili@ksu.edu.sa; Tel: +96614676258;
Fax: +966-1-4676274.

Abstract: In this paper we consider the Laplace transform of the bivariate inactivity time. We show that a weak bivariate reversed hazard rate order is characterized by the Laplace transform of the bivariate inactivity times in two different frames. The results are used to characterize the weak bivariate reversed hazard rate order using the weak bivariate mean inactivity time order. The results are also used to characterize the decreasing bivariate reversed hazard rate property using the Laplace transform of the bivariate inactivity time.

Keywords: bivariate reversed hazard rate order; bivariate Laplace transforms order; exponential distribution; characterization; BDRHR

Mathematics Subject Classification: 60E05, 62N05, 60E15

1. Introduction

In the context of reliability engineering and lifetime knowledge, there are several types of failures. One is a failure that is not immediately apparent to operations or maintenance personnel. This form of failure can result in equipment not performing a required function, such as protective functions in power plants and substations, standby equipment, backup power, or lack of capacity or power (see Wang and Pham [36], Tang et al. [34], Jiang et al. [16], and Ahmadi [2]). Predicting the exact timing of such a failure in complex systems is a significant problem in reliability engineering, as evidenced by the growing importance of the inactivity time of a lifespan (see, for instance, Zhang et al. [37] and Jia and Jeong [15]). Inactivity time has been useful in describing various maintenance strategies in reliability engineering (see Finkelstein [12]). It has also proven useful in biomedicine for estimating the incubation period of diseases and studying the behavior of lifetime distributions in retrospective survival studies (see Keiding and Gill [21], Keiding [22], and Andersen et al. [6]). Inactivity time has been used in forensic science and life insurance to predict the time of occurrence of events, such as the time of death of a person (see Gupta and Nanda [13]). This concept of reliability has also been used in risk analysis theory and econometrics (see Eeckhoudt and Gollier [10] and Kijima and Ohnishi [23]).

The concept of inactivity time provides a mathematical tool to measure the interval time between the exact time of failure and the time when the failure is realized. Since determining the exact failure time of a unit is necessary, the concept of inactivity time is useful. Let X be the random lifetime of a new unit that has a cumulative distribution function (CDF) F, and assume that the unit failed before the time t at which an inspection was performed. Then the time-dependent random variable (r.v.) $X_{(t)} := (t - X \mid X \leq t)$ which holds for all t for which $F(t) > 0$ is known in the context of reliability theory as the inactivity time of X or reversed residual lifetime of X (see Ruiz and Navarro [30], Nanda et al. [27], Kayid and Ahmad [18], and Kayid and Izadkhah [19]). The inactivity time can therefore be used to predict the timing of early failures of a component or product, which in turn can be of interest, for example, in finding the optimal time to perform burn-in processes (see Block and Savits [7]).

For any non-negative r.v. X with CDF F, the Laplace-Stieltjes transform is defined as

$$L_X(s) := \int_0^{+\infty} e^{-sx} dF(x), \quad s > 0. \tag{1.1}$$

It is obvious that $L_X(s)$ is non-increasing as a function of s. Let $\bar{F}$ with $\bar{F}(x) := 1 - F(x)$ denote the survival function (SF) of X which is a decreasing function on $[0, 1]$ calculating probabilities of X being strictly greater than x. The Laplace transform of $\bar{F}$ is given by

$$L_X^*(s) := \int_0^{+\infty} e^{-sx} \bar{F}(x) dx, \quad s > 0. \tag{1.2}$$

The values of r.v. X are non-negative, so the Laplace-Stieltjes transform (1.1) and the Laplace transform (1.2) always exist. Various stochastic orderings and reliability properties as well as some of their applications have been studied in the literature using the Laplace-Stieltjes transform and the Laplace transform (see Alzaid et al. [5], Nanda [26], Shaked and Wang [32], Ahmed and Kayid [3], Tepedelenlioglu et al. [35], Al-Gashgari et al. [4], Kayid et al. [20], and El-Arishy et al. [11]). If X has an absolutely continuous CDF, then

$$L_X^*(s) = \frac{1}{s}(1 - L_X(s)), \quad s > 0. \tag{1.3}$$

From the Eqs (1.1) and (1.2), one has $L_X(s) = E[e^{-sX}]$ and $L_X^*(s) = E[\min(X, E_s)]$ where E_s denotes an r.v. with exponential distribution having a mean equal with $1/s$.

Suppose that X and Y have Laplace transforms L_X and L_Y, respectively. It is then said that X is less or equal than Y in the Laplace transform order (denoted as $X \leq_{Lt} Y$) provided that $L_X(s) \geq L_Y(s)$ for all $s \geq 0$. From (1.3),

$$X \leq_{Lt} Y \text{ if, and only if, } L_X^*(s) \leq L_Y^*(s), \text{ for all } s > 0.$$

The inactivity times of X and Y with CDFs F and G are defined as $X_{(t)} = (t - X|X \leq t)$, $F(t) > 0$ and $Y_{(t)} = (t - Y|Y \leq t)$, $G(t) > 0$, respectively. The SFs of $X_{(t)}$ and $Y_{(t)}$ are given by $\bar{F}_{(t)}(x) = \frac{F(t-x)}{F(t)}$ and $\bar{G}_{(t)}(x) = \frac{G(t-x)}{G(t)}$ for $x \geq 0$. It is to be mentioned here that, when X and Y are non-negative r.v.s then $X_{(t)}$ and $Y_{(t)}$ have finite supports. To figure out this principle, let us assume that X and Y have supports $S_X = (l_X, u_X)$ and $S_Y = (l_Y, u_Y)$, respectively, where $l_X := \inf\{x|F(x) > 0\}$ and $u_X := \sup\{x|F(x) < 1\}$;

similarly l_Y and u_Y are defined. If $X_{(t)}$ and $Y_{(t)}$ have CDFs $F_{(t)}$ and $G_{(t)}$ which are valid, respectively, for values $t > l_X$ and values $t > l_Y$, then for the lower bound of the support of $X_{(t)}$ we get

$$\begin{aligned} l_{X_{(t)}} &:= \inf\{x|F_{(t)}(x) > 0\} \\ &= \inf\{x|1 - \frac{F(t-x)}{F(t)} > 0\} \\ &= \inf\{x|F(t-x) < F(t)\} \\ &= \inf\{x|x > t - F^{-1}oF(t) - t\} = 0, \end{aligned}$$

where $F^{-1}(u) := \inf\{x|F(x) > u\}$ for any $u \in [0, 1]$. Further, for the upper bound of the support of $X_{(t)}$ one obtains

$$\begin{aligned} u_{X_{(t)}} &:= \sup\{x|F_{(t)}(x) < 1\} \\ &= \sup\{x|1 - \frac{F(t-x)}{F(t)} < 1\} \\ &= \sup\{x|F(t-x) > 0\} \\ &= \sup\{x|x < t - l_X\} = t - l_X, \end{aligned}$$

and thus $X_{(t)}$ has support $S_{X_{(t)}} = (0, t - l_X)$ where $l_X \in [0, t)$ and analogously, $Y_{(t)}$ has support $S_{Y_{(t)}} = (0, t - l_Y)$ in which $l_Y \in [0, t)$. It is also concluded that the supports of $X_{(t)}$ and $Y_{(t)}$ do not depend on the upper bounds of the supports of X and Y, accordingly.

In what follows two well-known stochastic orderings are given.

Definition 1. *It is said that X with probability density function (PDF) f and CDF F is less than or equal with Y which has PDF g and CDF G in the reversed hazard rate order (denoted by $X \leq_{rhr} Y$) if $r_X(t) \leq r_Y(t)$ for all $t > \max\{l_X, l_Y\}$, where $r_X(t) = \frac{f(t)}{F(t)}$, $t > l_X$ and $r_Y(t) = \frac{g(t)}{G(t)}$, $t > l_Y$ are the reversed hazard rate functions of X and Y, respectively (see Finkelstein (2002)).*

The conditional r.v.s $X_{(t)}$ and $Y_{(t)}$, known as inactivity times of X and Y, can be applied to compare lifetime distributions from the perspective of the Laplace transform. Ahmad and Kayid [1] characterized the reversed hazard rate order as follows:

$$X \leq_{rhr} Y \text{ if, and only if, } X_{(t)} \geq_{Lt} Y_{(t)} \text{ for all } t > \max\{l_X, l_Y\}. \tag{1.4}$$

Definition 2. *It is said that X with mean inactivity time (MIT) function $\widetilde{m}_X$ given by $\widetilde{m}_X(t) = E(X_{(t)})$ is less than or equal with Y with MIT function $\widetilde{m}_Y$ given by $\widetilde{m}_Y(t) = E(Y_{(t)})$ (denoted by $X \leq_{mit} Y$) provided that $\widetilde{m}_X(t) \geq \widetilde{m}_Y(t)$ for all $t > \max\{l_X, l_Y\}$ (see Kayid and Izadkhah (2014)).*

Stochastic comparison of random lifetimes of units based on their inactivity times has attracted the attention of many researchers recently (see, e.g., Salehi and Tavangar [31], Patra and Kundu [29], Di Crescenzo et al. [9], Li and Li [24] and Guo et al. [14]).

It is well-known that $X \leq_{rhr} Y$ implies $\phi(X) \leq_{rhr} \phi(Y)$ for every increasing function ϕ, and, therefore, $\phi(X) \leq_{mit} \phi(Y)$ since the mean inactivity time order is weaker than the reversed hazard rate order. By taking $\phi(x) = e^{sx}$, $s > 0$ as an increasing function, it is concluded that $X \leq_{rhr} Y$ implies $e^{sX} \leq_{mit} e^{sY}$, for all $s > 0$. To develop this implication in the revered direction it was proved by Ortega (2008) that

$$X \leq_{rhr} Y \text{ if, and only if, } e^{sX} \leq_{mit} e^{sY} \text{ for all } s > 0. \tag{1.5}$$

In this paper, we further develop the characterizations given in (1.4) and (1.5) to obtain several results for the ordering of bivariate distributions, and the main results serve as an essential complement to those obtained in univariate cases. We also examine some characterizations of a relevant reliability class of bivariate lifetime distributions. To our knowledge, the Laplace transform for bivariate inactivity times has not been considered anywhere in the literature. Therefore, the aim of this study is to present the use of this measure to derive various stochastic order properties and to characterize bivariate lifetime patterns of distributions.

The paper is organized as follows. Section 2 introduces the bivariate inactivity time and its distributional properties, as well as some preliminary concepts used in the rest of the paper. A weak bivariate order of the reversed hazard rate and a weak bivariate order of the mean inactivity time are defined and the relationship between these orders is presented. Section 3 presents the main results of the paper, including three characterizations of the weak bivariate order of the reversed hazard rate using the Laplace transform of the bivariate inactivity times and the Laplace transform of the marginal inactivity times, as well as a characterization of a bivariate property of the decreasing failure rate. Finally, Section 4 concludes the paper with further illustrative explanations and descriptions.

2. Reliability measures of bivariate inactivity time

Suppose that $\mathbf{X} = (X_1, X_2)$ is a pair of random lifetimes of two devices that has a common CDF F. Consider a situation where accurate information about the times at which early failures occurred in the past is not available. Assume that the failure of the first (resp. second) device occurs at time t_1 (resp. t_2). The time-dependent random pair

$$\mathbf{X}_{(t_1,t_2)} = (t_1 - X_1, t_2 - X_2 | X_1 \leq t_1, X_2 \leq t_2),\ (t_1, t_2) : F(t_1, t_2) > 0, \tag{2.1}$$

in which (t_1, t_2) in which $t_i > 0, i = 1, 2$ is the pair of observation times, is called the bivariate inactivity time associated with $\mathbf{X} = (X_1, X_2)$ (cf. Mulero and Pellerey [25]). The random pair $\mathbf{X}_{(t_1,t_2)}$ has SF

$$\bar{F}_{(t_1,t_2)}(x_1, x_2) = \frac{F(t_1 - x_1, t_2 - x_2)}{F(t_1, t_2)},\ x_i > 0, i = 1, 2. \tag{2.2}$$

Since F is, in general, supported on $\mathbb{R}^2_+$, thus when $x_i > t_i$ at least for one $i = 1, 2$ then $\bar{F}_{(t_1,t_2)}(x_1, x_2) = 0$. This can be also acknowledged by the fact that $\mathbf{X}_{(t_1,t_2)}$ in (2.1) has a support in $[0, t_1] \times [0, t_2]$. If F is assumed to be absolutely continues with the corresponding joint PDF f, then the joint PDF of $\mathbf{X}_{(t_1,t_2)}$ is obtained as

$$f_{(t_1,t_2)}(x_1, x_2) = \frac{f(t_1 - x_1, t_2 - x_2)}{F(t_1, t_2)},\ x_i > 0, i = 1, 2. \tag{2.3}$$

The marginal distributions are the ones associated with the conditional r.v.

$$\mathbf{X}_{i,t_i} = (t_i - X_i | X_1 \leq t_1, X_2 \leq t_2), i = 1, 2;\ (t_1, t_2) : F(t_1, t_2) > 0. \tag{2.4}$$

From (2.4), one can rewrite the bivariate inactivity time in (2.1) as $\mathbf{X}_{(t_1,t_2)} = (\mathbf{X}_{1,t_1}, \mathbf{X}_{2,t_2})$. It can be seen that the marginal distributions of the conditional random vector $(\mathbf{X}|\mathbf{X} \in C)$ where $\mathbf{X} = (X_1, X_2, \ldots, X_p)$ for any arbitrary set $C \in \mathbb{R}^p$ depend on joint probability of $P(\mathbf{X} \in C)$ which in turn is affected by the dependence between elements of $\mathbf{X}$. Indeed, (2.4) is a particular case where $p = 2$ and $P(\mathbf{X} \in C) = F(t_1, t_2)$.

Suppose that X_i has marginal CDF F_i and SF $\bar{F}_i \equiv 1 - F_i$ for $i = 1, 2$. The PDF of X_i (whenever it exists) is given by $f_i, i = 1, 2$. The conditional SF of $\mathbf{X}_{i,t_i}$ is acquired as

$$\bar{F}_{1,t_1}(x_1) = \frac{F(t_1 - x_1, t_2)}{F(t_1, t_2)} \text{ and } \bar{F}_{2,t_2}(x_2) = \frac{F(t_1, t_2 - x_2)}{F(t_1, t_2)} \tag{2.5}$$

and the corresponding conditional PDF is, therefore, obtained by

$$f_{1,t_1}(x_1) = \frac{-\frac{\partial}{\partial x_1} F(t_1 - x_1, t_2)}{F(t_1, t_2)} \text{ and } f_{2,t_2}(x_2) = \frac{-\frac{\partial}{\partial x_2} F(t_1, t_2 - x_2)}{F(t_1, t_2)}. \tag{2.6}$$

If we suppose that X_1 and X_2 are independent then the r.v. $\mathbf{X}_{i,t_i} = (t_i - X_i | X_1 \leq t_1, X_2 \leq t_2)$ is dietetically distributed with the univariate well-known inactivity time $X_{i(t_i)} = (t_i - X_i | X_i \leq t_i), i = 1, 2$. The conditional r.v.s X_{1,t_1} and X_{2,t_2} can be used for further stochastic comparisons of X_1 and X_2 on the left tail of their distributions in the case they are possibly dependent.

Domma [8] extended some important results of reliability theory based on the bivariate hazard rate to the case of the bivariate reversed hazard rate. The gradient of the reversed hazard rate of $\mathbf{X} = (X_1, X_2)$ is given by

$$\begin{aligned} \mathbf{r_X}(t_1, t_2) &= \triangle \ln(F(t_1, t_2)) \\ &= (r_{1,\mathbf{X}}(t_1, t_2), r_{2,\mathbf{X}}(t_1, t_2)), \quad t_i > 0, i = 1, 2, \end{aligned}$$

where $\triangle = \left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}\right)$ and

$$r_{1,\mathbf{X}}(t_1, t_2) = \lim_{\delta \to 0^+} \frac{P(\mathbf{X}_{1,t_1} \in (0, \delta]}{\delta} = \frac{\partial}{\partial t_1} \ln(F(t_1, t_2))$$

and

$$r_{2,\mathbf{X}}(t_1, t_2) = \lim_{\delta \to 0^+} \frac{P(\mathbf{X}_{2,t_2} \in (0, \delta])}{\delta} = \frac{\partial}{\partial t_2} \ln(F(t_1, t_2)).$$

Let us denote by $\widetilde{m}_{\mathbf{X}}$ the bivariate MIT function of $\mathbf{X} = (X_1, X_2)$ which is given by $\widetilde{m}_{\mathbf{X}}(t_1, t_2) = (\widetilde{m}_{1,\mathbf{X}}(t_1, t_2), \widetilde{m}_{2,\mathbf{X}}(t_1, t_2))$ so that, the conditional MITs are obtained as (cf. Domma [8])

$$\widetilde{m}_{1,\mathbf{X}}(t_1, t_2) = E[\mathbf{X}_{1,t_1}] = \frac{\int_0^{t_1} F(x_1, t_2)\, dx_1}{F(t_1, t_2)}$$

and

$$\widetilde{m}_{2,\mathbf{X}}(t_1, t_2) = E[\mathbf{X}_{1,t_2}] = \frac{\int_0^{t_2} F(t_1, x_2)\, dx_2}{F(t_1, t_2)}.$$

On the basis of the bivariate inactivity time and also the foregoing reliability measures a couple of bivariate ordering properties are defined as follows.

Definition 3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two non-negative random pairs with reversed hazard rates gradients* $\mathbf{r_X}$ *and* $\mathbf{r_Y}$*, respectively. It is then said that* $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in weak bivariate reversed hazard rate order (denoted by* $\mathbf{X} \leq_{wrhr} \mathbf{Y}$*) whenever* $r_{i,\mathbf{X}}(t_1, t_2) \leq r_{i,\mathbf{Y}}(t_1, t_2), i = 1, 2$ *for all* $(t_1, t_2) \in \mathbb{R}_+^2$*, or equivalently if*

$$\frac{G(t_1, t_2)}{F(t_1, t_2)} \text{ is non-decreasing in } (t_1, t_2) \in \{(t_1, t_2) \in \mathbb{R}_+^2 : G(t_1, t_2) > 0\}.$$

Definition 4. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two non-negative random pairs with bivariate MIT functions* $\widetilde{\mathbf{m}}_{\mathbf{X}}$ *and* $\widetilde{\mathbf{m}}_{\mathbf{Y}}$*, respectively. Then, it is said that* $\mathbf{X}$ *is less than* $\mathbf{Y}$ *in weak bivariate mean inactivity time order (denoted by* $\mathbf{X} \leq_{wmit} \mathbf{Y}$*) if* $\widetilde{m}_{i,\mathbf{X}}(t_1, t_2) \geq \widetilde{m}_{i,\mathbf{Y}}(t_1, t_2), i = 1, 2$ *for all* $(t_1, t_2) \in \mathbb{R}_+^2$*, or equivalently if*

$$\frac{\int_0^{t_1} G(x_1, t_2)\, dx_1}{\int_0^{t_1} F(x_1, t_2)\, dx_1} \text{ is non-decreasing in } t_1 > 0, \text{ for every } t_2 \geq 0$$

and

$$\frac{\int_0^{t_2} G(t_1, x_2)\, dx_2}{\int_0^{t_2} F(t_1, x_2)\, dx_2} \text{ is non-decreasing in } t_2 > 0, \text{for every } t_1 \geq 0.$$

The next example illustrates a situation where a family of bivariate distributions fulfills the weak bivariate reversed hazard rate order based on ordering conditions on the set of parameters of the distribution.

Example 5. *Suppose that* $\mathbf{X}$ *follows the bivariate CDF*

$$F(t_1, t_2) = \exp\{-\frac{a_1}{t_1} - \frac{a_2}{t_2} \exp\{\frac{a_1}{t_1}\}\}, \quad t_i \geq 0, i = 1, 2$$

where $a_i > 0$ *for* $i = 1, 2$*. Further, assume that* $\mathbf{Y}$ *follows the bivariate CDF*

$$G(t_1, t_2) = \exp\{-\frac{b_1}{t_1} - \frac{b_2}{t_2} \exp\{\frac{b_1}{t_1}\}\}, \quad t_i \geq 0, i = 1, 2$$

where $b_i > 0$ *for* $i = 1, 2$*. If* $a_i \leq b_i$*, for every* $i = 1, 2$ *then by routine algebraic calculations it can be shown that* $\frac{G(t_1, t_2)}{F(t_1, t_2)}$ *is non-decreasing in* t_1 *for every* $t_2 \geq 0$ *and also it is proved that* $\frac{G(t_1, t_2)}{F(t_1, t_2)}$ *is non-decreasing in* t_2 *for every* $t_1 \geq 0$*, i.e.,* $\mathbf{X} \leq_{wrhr} \mathbf{Y}$.

Proposition 6. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two non-negative random pairs with absolutely continuous distributions* F *and* G*. If* $\mathbf{X} \leq_{wrhr} \mathbf{Y}$ *then* $\mathbf{X} \leq_{wmit} \mathbf{Y}$.

Proof. Denote $H_1 = F$ and $H_2 = G$, thus H_i is a joint CDF for each $i = 1, 2$. In one hand, since by the definition, $\mathbf{X} \leq_{wrhr} \mathbf{Y}$ implies that

$$\frac{G(t_1, t_2)}{F(t_1, t_2)} \text{ is non-decreasing in } t_1, \text{ for every } t_2 \geq 0,$$

thus the function

$$H_{t_2} : (i, x) \mapsto H_i(x, t_2) \text{ is } TP_2 \text{ in } (i, x) \in \{1, 2\} \times \mathbb{R}_+, \text{ for all } t_2 \geq 0. \tag{2.7}$$

On the other hand, $\mathbf{X} \leq_{wrhr} \mathbf{Y}$ also implies by the definition that

$$\frac{G(t_1, t_2)}{F(t_1, t_2)} \text{ is non-decreasing in } t_2, \text{ for every } t_1 \geq 0,$$

i.e., the function

$$H^*_{t_1} : (i, y) \mapsto H_i(t_1, y) \text{ is } TP_2 \text{ in } (i, y) \in \{1, 2\} \times \mathbb{R}_+, \text{ for all } t_1 \geq 0. \tag{2.8}$$

Using (2.7) and the fact that the heavy-side function $I:(x,t_1)\mapsto I[x\le t_1]$ is TP_2 in $(x,t_1)\in\mathbb{R}_+\times\mathbb{R}_+$, it follows by the general composition theorem of Karlin [17] that, for all $t_2\ge 0$, $\int_0^{+\infty} H_{t_2}(i,x)I(x,t_1)\,dx$ is TP_2 in $(i,t_1)\in\{1,2\}\times\mathbb{R}_+$ which means that

$$\frac{\int_0^{t_1} G(x_1,t_2)\,dx_1}{\int_0^{t_1} F(x_1,t_2)\,dx_1} \text{ is non-decreasing in } t_1>0, \text{ for every } t_2\ge 0.$$

Further, from (2.8), since $I^*:(y,t_2)\mapsto I[y\le t_2]$ is TP_2 in $(y,t_2)\in\mathbb{R}_+\times\mathbb{R}_+$, thus repeated application of the general composition theorem of Karlin [17] provides that, for all $t_1\ge 0$, $\int_0^{+\infty} H^*_{t_1}(i,y)I^*(y,t_2)\,dy$ is TP_2 in $(i,t_2)\in\{1,2\}\times\mathbb{R}_+$ which is equivalent to saying that

$$\frac{\int_0^{t_1} G(x_1,t_2)\,dx_1}{\int_0^{t_1} F(x_1,t_2)\,dx_1} \text{ is non-decreasing in } t_2>0, \text{ for every } t_1\ge 0.$$

Thus, it is concluded that $\mathbf{X}\le_{wmit}\mathbf{Y}$. □ □

The following example is used to indicate that the reversed implication in Proposition 6 is not satisfied.

Example 7. *Suppose that* $\mathbf{X}$ *has the following bivariate CDF*

$$F(x_1,x_2)=\begin{cases}\dfrac{x_1x_2}{4} & \textit{if } \ 0\le x_i\le 2, i=1,2\\[2ex] \dfrac{x_i}{2} & \textit{if } \ 0\le x_i\le 2, x_{3-i}\ge 2, i=1,2\\[2ex] 1 & \textit{if } \ x_i\ge 2, i=1,2\end{cases}$$

Let $\mathbf{Y}$ *follow the bivariate CDF*

$$G(x_1,x_2)=\begin{cases}\dfrac{(x_1x_2)^2}{4} & \textit{if } \ 0\le x_i\le 1, i=1,2\\[2ex] \dfrac{x_i^2}{2} & \textit{if } \ 0\le x_i\le 1, x_{3-i}\ge 2, i=1,2\\[2ex] \dfrac{x_i^2(x_{3-i}^2+2)}{12} & \textit{if } \ 0\le x_i\le 1, 1\le x_{3-i}\le 2, i=1,2\\[2ex] \dfrac{(x_1^2+2)(x_2^2+2)}{36} & \textit{if } \ 1\le x_i\le 2, i=1,2\\[2ex] \dfrac{(x_i^2+2)}{6} & \textit{if } \ 1\le x_i\le 2, x_{3-i}\ge 2, i=1,2\\[2ex] 1 & \textit{if } \ x_i\ge 2, i=1,2\end{cases}$$

It can be verified that $\dfrac{G(x_1,x_2)}{F(x_1,x_2)}$ *is not non-decreasing in* $x_1 \geq 0$ *for every* $x_2 \geq 0$ *and also it is not non-decreasing in* $x_2 \geq 0$ *for every* $x_1 \geq 0$. *Therefore,* $\mathbf{X} \nleq_{wrhr} \mathbf{Y}$. *On the other hand,*

$$\widetilde{m}_{i,\mathbf{X}}(t_1,t_2) = \begin{cases} \dfrac{t_i}{2} & \text{if } \; 0 \leq t_i \leq 2 \\ \\ t_i - 1 & \text{if } \; t_i \geq 2 \end{cases}, i = 1,2.$$

and, moreover,

$$\widetilde{m}_{i,\mathbf{Y}}(t_1,t_2) = \begin{cases} \dfrac{t_i}{3} & \text{if } \; 0 \leq t_i \leq 1 \\ \\ \dfrac{t_i^3 + 6t_i - 4}{3(t_i^2+2)} & \text{if } \; 1 \leq t_i \leq 2 \\ \\ t_i - 1 & \text{if } \; t_i \geq 2 \end{cases}, i = 1,2.$$

It can be plainly observed that $\widetilde{m}_{i,\mathbf{X}}(t_1,t_2) \geq \widetilde{m}_{i,\mathbf{Y}}(t_1,t_2)$, *for every* $i = 1,2$ *and, as a result,* $\mathbf{X} \leq_{wmit} \mathbf{Y}$.

The bivariate Laplace-Stieltjes transform associated with the non-negative random pair $\mathbf{X} = (X_1, X_2)$ with joint density $f(x_1,x_2)$ is given by

$$L_{\mathbf{X}}(\mathbf{s}) = \int_0^{+\infty}\int_0^{+\infty} e^{-s_1X_1 - s_2X_2} f(x_1,x_2)dx_1dx_2, \;\; \mathbf{s} = (s_1,s_2) \in \mathbb{R}_+^2. \tag{2.9}$$

Definition 8. *For two non-negative random pairs* $\mathbf{X} = (X_1,X_2)$ *and* $\mathbf{Y} = (Y_1,Y_2)$ *with respective Laplace transforms* $L_{\mathbf{X}}$ *and* $L_{\mathbf{Y}}$ *it is said that* $\mathbf{X}$ *is less than or equal with* $\mathbf{Y}$ *in the bivariate Laplace transform (denoted by* $\mathbf{X} \leq_{BLt} \mathbf{Y}$*) whenever* $L_{\mathbf{X}}(\mathbf{s}) \geq L_{\mathbf{Y}}(\mathbf{s})$, *for all* $s \in \mathbb{R}_+^2$.

By Theorem 7.D.1. of Shaked and Shanthikumar [33], $\mathbf{X} \leq_{BLt} \mathbf{Y}$ gives $X_i \leq_{Lt} Y_i, i = 1,2$. The reversed implication does not hold in the general case.

3. Characterization results using bivariate Laplace transform of inactivity times

In this section, the bivariate Laplace transform (2.9) is used to compare the inactivity times of non-negative random pairs $\mathbf{X} = (X_1,X_2)$ and $\mathbf{Y} = (Y_1,Y_2)$. Appealing to the notations introduced in (1.1), (1.2) and (2.9), it can be seen that

$$L_{\mathbf{X}}(\mathbf{s}) = 1 - s_1L^*_{X_1}(s_1) - s_2L^*_{X_2}(s_2) + s_1s_2L^*_{\mathbf{X}}(\mathbf{s}), \;\; s_i > 0, i = 1,2, \tag{3.1}$$

in which

$$L^*_{\mathbf{X}}(\mathbf{s}) = \int_0^{+\infty}\int_0^{+\infty} e^{-s_1x_1 - s_2x_2}\bar{F}(x_1,x_2)\, dx_1dx_2. \tag{3.2}$$

The bivariate Laplace transform of $\mathbf{X}_{(\mathbf{t})} = (\mathbf{X}_{1,t_1}, \mathbf{X}_{2,t_2})$ is obtained here. Firstly, because of (3.1) the Laplace transforms of $\mathbf{X}_{1,t_1}$ and $\mathbf{X}_{2,t_2}$ are acquired. The Eqs (2.3) and (2.5) lead to expressions for (3.1) and (3.2) as follows:

$$L^*_{\mathbf{X}_{1,t_1}}(s_1) := \frac{\int_0^{t_1} e^{+s_1x_1}F(x_1,t_2)dx_1}{e^{+s_1t_1}F(t_1,t_2)} \;\; \text{and} \;\; L^*_{\mathbf{X}_{2,t_2}}(s_2) := \frac{\int_0^{t_2} e^{+s_2x_2}F(t_1,x_2)dx_2}{e^{+s_2t_2}F(t_1,t_2)}. \tag{3.3}$$

The Laplace transform of the bivariate inactivity time is then obtained as

$$L_{\mathbf{X}_{(\mathbf{t})}}(s_1, s_2) := 1 - s_1 \frac{\int_0^{t_1} e^{+s_1 x_1} F(x_1, t_2) dx_1}{e^{+s_1 t_1} F(t_1, t_2)} - s_2 \frac{\int_0^{t_2} e^{+s_2 x_2} F(t_1, x_2) dx_2}{e^{+s_2 t_2} F(t_1, t_2)} + s_1 s_2 \frac{\int_0^{t_1} \int_0^{t_2} e^{+s_1 x_1 + s_2 x_2} F(x_1, x_2) dx_2 dx_1}{F(t_1, t_2)}. \tag{3.4}$$

Necessary and sufficient conditions for the Laplace transform ordering between $\mathbf{X}_{i,t_i}$ and $\mathbf{Y}_{i,t_i}, i = 1, 2$ are secured as follows:

Proposition 9. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two non-negative random pairs having joint CDFs* F *and* G *for which the bivariate inactivity times are* $\mathbf{X}_{(\mathbf{t})} = (\mathbf{X}_{1,t_1}, \mathbf{X}_{2,t_2})$ *and* $\mathbf{Y}_{(\mathbf{t})} = (\mathbf{Y}_{1,t_1}, \mathbf{Y}_{2,t_2})$*, respectively. Then*

(i) $\mathbf{X}_{1,t_1} \geq_{Lt} \mathbf{Y}_{1,t_1}$*, for all* $\mathbf{t} \in \mathbb{R}_+^2$ *if, and only if,* $\frac{\int_0^{t_1} e^{+s_1 x_1} G(x_1,t_2) dx_1}{\int_0^{t_1} e^{+s_1 x_1} F(x_1,t_2) dx_1}$ *is non-decreasing in* $t_1 \geq 0$ *for every* $t_2 \geq 0$.

(ii) $\mathbf{X}_{2,t_2} \geq_{Lt} \mathbf{Y}_{2,t_2}$ *for all* $\mathbf{t} \in \mathbb{R}_+^2$ *if, and only if,* $\frac{\int_0^{t_2} e^{+s_2 x_2} G(t_1,x_2) dx_2}{\int_0^{t_2} e^{+s_2 x_2} F(t_1,x_2) dx_2}$ *is non-decreasing in* $t_2 \geq 0$ *for every* $t_1 \geq 0$.

Proof. The proof of assertion (i) is only given, since the assertion (ii) can be proved similarly. For all $t_i \geq 0, i = 1, 2$ and for every $s_1 \geq 0$,

$$\frac{\partial}{\partial t_1} \frac{\int_0^{t_1} e^{+s_1 x_1} G(x_1, t_2) dx_1}{\int_0^{t_1} e^{+s_1 x_1} F(x_1, t_2) dx_1} \overset{sgn}{=} + e^{+s_1 t_1} G(t_1, t_2) \int_0^{t_1} e^{+s_1 x_1} F(x_1, t_2) dx_1 - e^{+s_1 t_1} F(t_1, t_2) \int_0^{t_1} e^{+s_1 x_1} G(x_1, t_2) dx_1,$$

where $\overset{sgn}{=}$ indicates the equality in sign. Thus, one concludes for all $t_i \geq 0, i = 1, 2$ that

$$\frac{\partial}{\partial t_1} \frac{\int_0^{t_1} e^{+s_1 x_1} G(x_1, t_2) dx_1}{\int_0^{t_1} e^{+s_1 x_1} F(x_1, t_2) dx_1} \geq 0, \ \forall \ s_1 \geq 0,$$

if, and only if, for all $\mathbf{t} \in \mathbb{R}_+^2$ (see, Eq (3.3))

$$L^*_{\mathbf{X}_{1,t_1}}(s_1) = \frac{\int_0^{t_1} e^{+s_1 x_1} F(x_1, t_2) dx_1}{e^{+s_1 t_1} F(t_1, t_2)} \geq \frac{\int_0^{t_1} e^{+s_1 x_1} G(x_1, t_2) dx_1}{e^{+s_1 t_1} G(t_1, t_2)} = L^*_{\mathbf{Y}_{1,t_1}}(s_1), \ \forall \ s_1 \geq 0.$$

This holds if, and only if, $\mathbf{X}_{1,t_1} \geq_{Lt} \mathbf{Y}_{1,t_1}$, for all $\mathbf{t} \in \mathbb{R}_+^2$. □ □

The main result of the paper is the following theorem.

Theorem 10. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two non-negative random pairs having joint CDFs* F *and* G*, respectively. Then*

$$\mathbf{X} \leq_{wrhr} \mathbf{Y} \quad \text{if, and only if,} \quad \mathbf{X}_{i,t_i} \geq_{Lt} \mathbf{Y}_{i,t_i}, i = 1, 2, \quad \text{for all } \mathbf{t} \in \mathbb{R}_+^2. \tag{3.5}$$

Proof. Firstly, it is shown that $\mathbf{X}_{i,t_i} \geq_{Lt} \mathbf{Y}_{i,t_i}, i = 1, 2,$ for all $\mathbf{t} \in \mathbb{R}_+^2$ is a sufficient condition to conclude $\mathbf{X} \leq_{wrhr} \mathbf{Y}$. For every $i = 1, 2$, we have

$$L^*_{\mathbf{X}_{i,t_i}}(s_i) = \frac{1}{s_i}(1 - L_{\mathbf{X}_{i,t_i}}(s_i)) \text{ and } L^*_{\mathbf{Y}_{i,t_i}}(s_i) = \frac{1}{s_i}(1 - L_{\mathbf{Y}_{i,t_i}}(s_i)). \tag{3.6}$$

By applying the Eqs (3.6) and (3.3) and also using (2.6) we obtain

$$\begin{aligned}
\frac{\int_0^{t_1} e^{+s_1x_1}G(x_1,t_2)dx_1}{\int_0^{t_1} e^{+s_1x_1}F(x_1,t_2)dx_1} &= \frac{G(t_1,t_2)L^*_{\mathbf{Y}_{1,t_1}}(s_1)}{F(t_1,t_2)L^*_{\mathbf{Y}_{1,t_1}}(s_1)} \\
&= \frac{G(t_1,t_2) - G(t_1,t_2)\int_0^{+\infty} e^{-s_1x_1}f_{\mathbf{X}_{1,t_1}}(x_1)dx_1}{F(t_1,t_2) - F(t_1,t_2)\int_0^{+\infty} e^{-s_1x_1}f_{\mathbf{Y}_{1,t_1}}(x_1)dx_1} \\
&= \frac{G(t_1,t_2) - \int_0^{t_1} e^{-s_1(t_1-x_1)}\left(\frac{\partial}{\partial x_1}G(x_1,t_2)\right)dx_1}{F(t_1,t_2) - \int_0^{t_1} e^{-s_1(t_1-x_1)}\left(\frac{\partial}{\partial x_1}F(x_1,t_2)\right)dx_1},
\end{aligned} \tag{3.7}$$

and analogously,

$$\begin{aligned}
\frac{\int_0^{t_2} e^{+s_2x_2}G(t_1,x_2)dx_2}{\int_0^{t_2} e^{+s_2x_2}F(t_1,x_2)dx_2} &= \frac{G(t_1,t_2)L^*_{\mathbf{Y}_{2,t_2}}(s_2)}{F(t_1,t_2)L^*_{\mathbf{X}_{2,t_2}}(s_2)} \\
&= \frac{G(t_1,t_2) - G(t_1,t_2)\int_0^{+\infty} e^{-s_2x_2}f_{\mathbf{X}_{2,t_2}}(x_2)dx_2}{F(t_1,t_2) - F(t_1,t_2)\int_0^{+\infty} e^{-s_2x_2}f_{\mathbf{Y}_{2,t_2}}(x_2)dx_2} \\
&= \frac{G(t_1,t_2) - \int_0^{t_2} e^{-s_2(t_2-x_2)}\left(\frac{\partial}{\partial x_2}G(t_1,x_2)\right)dx_2}{F(t_1,t_2) - \int_0^{t_2} e^{-s_2(t_2-x_2)}\left(\frac{\partial}{\partial x_2}F(t_1,x_2)\right)dx_2}.
\end{aligned} \tag{3.8}$$

By Proposition 9, we deduce that $\mathbf{X}_{1,t_1} \geq_{Lt} \mathbf{Y}_{1,t_1}$ for all $\mathbf{t} \in \mathbb{R}_+^2$ if, and only if, (3.7) is non-decreasing in t_1 for every $t_2 \geq 0$ and for all $s_1 \geq 0$ and specially when $s_1 \to +\infty$, $\mathbf{X}_{1,t_1} \geq_{Lt} \mathbf{Y}_{1,t_1}$ for all $\mathbf{t} \in \mathbb{R}_+^2$ concludes that (3.7) is non-decreasing in t_1 for every $t_2 \geq 0$. In this framework, $\mathbf{X}_{2,t_2} \geq_{Lt} \mathbf{Y}_{2,t_2}$ for all $\mathbf{t} \in \mathbb{R}_+^2$ holds if, and only if, (3.8) is non-decreasing in t_2 for every $t_1 \geq 0$ and for all $s_2 \geq 0$ and in particular when $s_2 \to +\infty$, $\mathbf{X}_{2,t_2} \geq_{Lt} \mathbf{Y}_{2,t_2}$ for all $\mathbf{t} \in \mathbb{R}_+^2$ further implies (3.8) is non-decreasing in t_2 for every $t_1 \geq 0$. For all $s_1 \geq 0$ and, also, for all $\mathbf{t} \in \mathbb{R}_+^2$, we observe that

$$\left| e^{-s_1(t_1-x_1)}I[x_1 \leq t_1]\left(\frac{\partial}{\partial x_1}F(x_1,t_2)\right)\right| \leq \frac{\partial}{\partial x_1}F(x_1,t_2),$$

where $\frac{\partial}{\partial x_1}F(x_1,t_2)$ is an integrable function for all $\mathbf{t} \in \mathbb{R}_+^2$ because

$$\int_0^{+\infty}\left(\frac{\partial}{\partial x_1}F(x_1,t_2)\right)dx_1 = F_2(t_2) \leq 1 < +\infty.$$

The Lebesgue's dominated convergence theorem concludes that

$$\lim_{s_1\to+\infty}\int_0^{t_1}\xi(x_1,t_2)e^{-s_1(t_1-x_1)}dx_1 = \lim_{s_1\to+\infty}\int_0^{+\infty}\xi(x_1,t_2)e^{-s_1(t_1-x_1)}I[x_1 \leq t_1]dx_1$$

$$= \int_0^{+\infty} \lim_{s_1 \to +\infty} \xi(x_1, t_2) e^{-s_1(t_1 - x_1)} I[x_1 \le t_1] dx_1 = 0,$$

where $\xi(x_1, t_2) = \frac{\partial}{\partial x_1} F(x_1, t_2)$. Similarly,

$$\lim_{s_1 \to +\infty} \int_0^{t_1} \eta(x_1, t_2) e^{-s_1(t_1 - x_1)} dx_1 = 0,$$

where $\eta(x_1, t_2) = \frac{\partial}{\partial x_1} G(x_1, t_2)$. Therefore,

$$\mathbf{X}_{1,t_1} \ge_{Lt} \mathbf{Y}_{1,t_1}, \ \forall\, \mathbf{t} \in \mathbb{R}_+^2 \ \text{ implies that } \ \frac{G(t_1, t_2)}{F(t_1, t_2)} \ \text{ is non-decreasing in } t_1, \forall\, t_2 \ge 0. \tag{3.9}$$

In addition,

$$\mathbf{X}_{2,t_2} \ge_{Lt} \mathbf{Y}_{2,t_2}, \ \forall\, \mathbf{t} \in \mathbb{R}_+^2 \ \text{ implies that } \ \frac{G(t_1, t_2)}{F(t_1, t_2)} \ \text{ is non-decreasing in } t_2, \forall\, t_1 \ge 0. \tag{3.10}$$

The Eqs (3.9) and (3.10) on the whole imply that $\mathbf{X} \le_{wrhr} \mathbf{Y}$. To prove the reversed implication we proceed as in the proof of Proposition 6. Suppose that $\mathbf{X} \le_{wrhr} \mathbf{Y}$ holds true. Fix $t_2 \ge 0$ and also fix $s_1 \ge 0$ as two arbitrary points. Define K and L as

$$(t_1, x_1) \mapsto K(t_1, x_1) = e^{+s_1 x_1} I[x_1 \le t_1], t_1, x_1 \ge 0 \ \text{ and } \ (x_1, i) \mapsto L(x_1, i), i = 1, 2, x_1 \ge 0,$$

respectively, in the way $L(x_1, 1) = F(x_1, t_2)$ and $L(x_1, 2) = G(x_1, t_2)$. By assumption, $\frac{G(x_1,t_2)}{F(x_1,t_2)}$ is non-decreasing in $x_1 \ge 0$. Hence, L is TP_2 in $(x_1, i) \in \mathbb{R}_+ \times \{1, 2\}$ and also it is readily seen that K is also TP_2 in $(t_1, x_1) \in \mathbb{R}_+ \times \mathbb{R}_+$. By the general composition theorem of Karlin [17], $\int_0^{+\infty} K(t_1, x_1) L(x_1, i) dx_1$ is TP_2 in $(t_1, i) \in \mathbb{R}_+ \times \{1, 2\}$. The conclusion not being affected by the choice of $t_2 \ge 0$ and also by the choice of $s_1 \ge 0$, thus $\frac{\int_0^{t_1} e^{+s_1 x_1} G(x_1,t_2) dx_1}{\int_0^{t_1} e^{+s_1 x_1} F(x_1,t_2) dx_1}$ is non-decreasing in $t_1 \ge 0$, for all $s_1 \ge 0$ and for every $t_2 \ge 0$. By Proportion 9 (i), it follows that $\mathbf{X}_{1,t_1} \ge_{Lt} \mathbf{Y}_{1,t_1}$ for all $\mathbf{t} \in \mathbb{R}_+^2$. It is also acknowledged that $\mathbf{X} \le_{wfr} \mathbf{Y}$ implies that $\mathbf{X}_{2,t_2} \ge_{Lt} \mathbf{Y}_{2,t_2}$. Hence the theorem's proof is closed. □ □

In Theorem 7.D.5 of Shaked and Shanthikumar [33] it was established that $\mathbf{X}_{(\mathbf{t})} \ge_{BLt} \mathbf{Y}_{(\mathbf{t})}$ yields $\mathbf{X}_{i,t_i} \ge_{Lt} \mathbf{Y}_{i,t_i}, i = 1, 2$. Therefore, Theorem 10 further concludes that if $\mathbf{X}_{(\mathbf{t})} \ge_{BLt} \mathbf{Y}_{(\mathbf{t})}$ for all $\mathbf{t} \in \mathbb{R}_+^2$ then $\mathbf{X} \le_{wrhr} \mathbf{Y}$. The converse implication is not, however, concluded and it may be questioned whether $\mathbf{X} \le_{wrhr} \mathbf{Y}$ leads to $\mathbf{X}_{(\mathbf{t})} \ge_{BLt} \mathbf{Y}_{(\mathbf{t})}$ for all $\mathbf{t} \in \mathbb{R}_+^2$. We want to show that this result is also valid without imposing any further assumption which will be an interesting observation strengthening the results of Theorem 10.

Proposition 11. *Suppose that* $\mathbf{X}$ *and* $\mathbf{Y}$ *are non-negative pairs having joint CDFs* F *and* G *with the bivariate inactivity times* $\mathbf{X}_{(\mathbf{t})} = (\mathbf{X}_{1,t_1}, \mathbf{X}_{2,t_2})$ *and* $\mathbf{Y}_{(\mathbf{t})} = (\mathbf{Y}_{1,t_1}, \mathbf{Y}_{2,t_2})$, *respectively. Then* $\mathbf{X} \le_{wrhr} \mathbf{Y}$ *implies* $\mathbf{X}_{(\mathbf{t})} \ge_{BLt} \mathbf{Y}_{(\mathbf{t})}$ *for all* $\mathbf{t} \in \mathbb{R}_+^2$.

Proof. The proof obtains if we prove that $L_{\mathbf{X}_{(\mathbf{t})}}(\mathbf{s}) \le L_{\mathbf{Y}_{(\mathbf{t})}}(\mathbf{s})$, for all $\mathbf{s} \in \mathbb{R}_+^2$ and for all $\mathbf{t} \in \mathbb{R}_+^2$. Since $\mathbf{X} \le_{wrhr} \mathbf{Y}$ thus by (3.5), $\mathbf{X}_{2,t_2} \ge_{Lt} \mathbf{Y}_{2,t_2}$ and, equivalently, $1 - s_2 L^*_{\mathbf{X}_{2,t_2}}(s_2) \le 1 - s_2 L^*_{\mathbf{Y}_{2,t_2}}(s_2)$, for all $\mathbf{t} \in \mathbb{R}_+^2$ and for all $s_2 \ge 0$. This ordering condition is equivalent to

$$s_2 L^*_{\mathbf{X}_{(\mathbf{t})}}(\mathbf{s}) - L^*_{\mathbf{X}_{1,t_1}}(s_1) \le s_2 L^*_{\mathbf{Y}_{(\mathbf{t})}}(\mathbf{s}) - L^*_{\mathbf{Y}_{1,t_1}}(s_1), \ \ \forall\, \mathbf{t} \in \mathbb{R}_+^2, \ \forall\, s_i \ge 0, i = 1, 2. \tag{3.11}$$

The Eq (3.4) then fulfills the desired result. Let us develop that

$$\begin{aligned}
L^*_{\mathbf{X}_{(\mathbf{t})}}(\mathbf{s}) &= \frac{\int_0^{t_1}\int_0^{t_2} e^{+s_1x_1+s_2x_2}F(x_1,x_2)dx_2dx_1}{e^{+s_1t_1+s_2t_2}F(t_1,t_2)}\\
&= e^{-s_1t_1}\int_0^{t_1} e^{+s_1x_1}\left(\frac{\int_0^{t_2} e^{+s_2x_2}F(x_1,x_2)dx_2}{e^{+s_2t_2}F(x_1,t_2)}\right)\frac{F(x_1,t_2)}{F(t_1,t_2)}dx_1\\
&= e^{-s_1t_1}\int_0^{t_1} e^{+s_1x_1}L^*_{\mathbf{X}_{2,t_2^*}}(s_2)\frac{F(x_1,t_2)}{F(t_1,t_2)}dx_1,
\end{aligned}$$

where $\mathbf{X}_{2,t_2^*} = (t_2 - X_2 | X_1 \leq x_1, X_2 \leq t_2^*)$ which is the first component of the inactivity time $\mathbf{X}_{(\mathbf{t}^*)}$ in which $\mathbf{t}^* = (x_1, t_2)$. Moreover, $L^*_{X_{2,t_2^*}}(s_2)$ is the Laplace transform of X_{2,t_2^*}, can be grasped by substituting (2.5) by utilizing $\mathbf{X}_{2,t_2^*}$ into (3.3). Let us write

$$\begin{aligned}
\Delta_{\mathbf{X}}(\mathbf{t},\mathbf{s}) &= L^*_{\mathbf{X}_{1,t_1}}(s_1) - s_2L^*_{\mathbf{X}_{(\mathbf{t})}}(\mathbf{s})\\
&= \frac{\int_0^{t_1} e^{+s_1x_1}F(x_1,t_2)dx_1}{e^{+s_1t_1}F(t_1,t_2)} - s_2\frac{\int_0^{t_1}\int_0^{t_2} e^{+s_1x_1+s_2x_2}F(x_1,x_2)dx_2dx_1}{F(x_1,t_2)}\\
&= e^{-s_1t_1}\int_0^{t_1} e^{+s_1x_1}(1 - s_2L^*_{\mathbf{X}_{2,t_2^*}}(s_2))\frac{F(x_1,t_2)}{F(t_1,t_2)}dx_1\\
&= e^{-s_1t_1}\int_0^{t_1} e^{+s_1x_1}L_{\mathbf{X}_{2,t_2^*}}(s_2)\frac{F(x_1,t_2)}{F(t_1,t_2)}dx_1,
\end{aligned}$$

where $L_{\mathbf{X}_{2,t_2^*}}(s_2) = E(e^{-s_2\mathbf{X}_{2,t_2^*}})$ for every $s_2 \geq 0$, and for every $t_2 \geq 0$ and also for all $x_1 \geq 0$. By doing similar steps, we get

$$\Delta_{\mathbf{Y}}(\mathbf{t},\mathbf{s}) = L^*_{\mathbf{Y}_{1,t_1}}(s_1) - s_2L^*_{\mathbf{Y}_{(\mathbf{t})}}(\mathbf{s}) = e^{-s_1t_1}\int_0^{t_1} e^{+s_1x_1}L_{\mathbf{Y}_{2,t_2^*}}(s_2)\frac{G(x_1,t_2)}{G(t_1,t_2)}dx_1,$$

where $\mathbf{Y}_{2,t_2^*} = (t_2 - Y_2 | Y_1 \leq x_1, Y_2 \leq t_2^*)$ which is the first component of $\mathbf{Y}_{(\mathbf{t}^*)}$ in which $\mathbf{t}^* = (x_1, t_2)$. Now in this modified setting, if we show that $\Delta_{\mathbf{Y}}(\mathbf{t},\mathbf{s}) \leq \Delta_{\mathbf{X}}(\mathbf{t},\mathbf{s})$, for all $\mathbf{t},\mathbf{s} \in \mathbb{R}_+^2$ then (3.11) is secured. Note that $\mathbf{X} \leq_{wrhr} \mathbf{Y}$ gives $\mathbf{X}_{2,t_2^*} \geq_{Lt} \mathbf{Y}_{2,t_2^*}$ for all $x_1 \geq 0$ and for every $t_2 \geq 0$. It thus follows that

$$\mathbf{X} \leq_{wrhr} \mathbf{Y} \text{ implies } L_{\mathbf{Y}_{2,t_2^*}}(s_2) \leq L_{\mathbf{X}_{2,t_2^*}}(s_2),\ \forall\ x_1 \geq 0,\ \forall\ t_2,\ \forall\ s_2 \geq 0.$$

Furthermore,

$$\mathbf{X} \leq_{wrhr} \mathbf{Y} \text{ implies } \frac{G(x_1,t_2)}{G(t_1,t_2)} - \frac{F(x_1,t_2)}{F(t_1,t_2)} \leq 0,\ \forall\ x_1 \leq t_1 \in \mathbb{R}_+,\ \forall\ t_2 \geq 0.$$

Hence, it can be obtained that

$$\begin{aligned}
\Delta_{\mathbf{Y}}(\mathbf{t},\mathbf{s}) - \Delta_{\mathbf{X}}(\mathbf{t},\mathbf{s}) &= \int_0^{t_1} e^{-s_1(t_1-x_1)}\left(L_{\mathbf{Y}_{2,t_2^*}}(s_2)\frac{G(x_1,t_2)}{G(t_1,t_2)} - L_{\mathbf{X}_{2,t_2^*}}(s_2)\frac{F(x_1,t_2)}{F(t_1,t_2)}\right)dx_1\\
&\leq \int_0^{t_1} e^{-s_1(t_1-x_1)}L_{X_{2,t_2^*}}(s_2)\left(\frac{G(x_1,t_2)}{G(t_1,t_2)} - \frac{F(x_1,t_2)}{F(t_1,t_2)}\right)dx_1\\
&\leq 0,\ \text{for all } \mathbf{t} \in \mathbb{R}_+^2,\ \text{for all } s_2 \geq 0. \qquad \square
\end{aligned}$$

□

Theorem 10 and Proposition 11 on the whole concludes the following result as a characterization property of the weak bivariate reversed hazard rate order using bivariate Laplace transform ordering of inactivity time. The proof is simply obtained and hence we omit it.

Theorem 12. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two non-negative random pairs. Then*

$$\mathbf{X} \leq_{wrhr} \mathbf{Y} \quad \text{if, and only if,} \quad \mathbf{X}_{(\mathbf{t})} \geq_{BLt} \mathbf{Y}_{(\mathbf{t})}, i = 1, 2, \quad \text{for all } \mathbf{t} \in \mathbb{R}_+^2. \tag{3.12}$$

The result we obtained in Theorem 10 is useful to characterize the weak bivariate reversed hazard rate order of $\mathbf{X}$ and $\mathbf{Y}$ via the weak bivariate mean inactivity time order applied to $\eta_{\mathbf{s}}(\mathbf{X}) = (e^{s_1 X_1}, e^{s_2 X_2})$ and $\eta_{\mathbf{s}}(\mathbf{Y}) = (e^{s_1 Y_1}, e^{s_2 Y_2})$ as two $\mathbf{s}$-dependent random pairs on $[1, +\infty)$.

Theorem 13. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two lifetime random pairs with CDFs* F *and* G*, respectively. Then,*

$$\mathbf{X} \leq_{wrhr} \mathbf{Y} \ \text{if, and only if,} \ \eta_{\mathbf{s}}(\mathbf{X}) \leq_{wmit} \eta_{\mathbf{s}}(\mathbf{Y}), \text{ for all } \mathbf{s} \in \mathbb{R}_+^2.$$

Proof. Firstly, we show that $\mathbf{X} \leq_{wrhr} \mathbf{Y}$ implies $\eta_{\mathbf{s}}(\mathbf{X}) \leq_{wmit} \eta_{\mathbf{s}}(\mathbf{Y})$ for every non-negative pair $\mathbf{s} = (s_1, s_2)$. If $\mathbf{X} \leq_{wrhr} \mathbf{Y}$ and if we denote $\eta_{s_1}(a) = e^{+s_1 a}, a > 0$ and $\eta_{s_2}(b) = e^{s_2 b}, b > 0$, then since $\eta_{s_1}(a)$ and $\eta_{s_2}(b)$ are increasing in a and b, respectively, thus it is not hard to conclude that $\eta_{\mathbf{s}}(\mathbf{X}) \leq_{wrfr} \eta_{\mathbf{s}}(\mathbf{Y})$ and, as a result of Proposition 6, $\eta_{\mathbf{s}}(\mathbf{X}) \leq_{wmit} \eta_{\mathbf{s}}(\mathbf{Y})$. To prove the converse part of the theorem, assume that $\eta_{\mathbf{s}}(\mathbf{X})$ and $\eta_{\mathbf{s}}(\mathbf{Y})$ have respective CDFs

$$F_{\eta_{\mathbf{s}}(\mathbf{X})}(\mathbf{u}) = F(\frac{1}{s_1} \ln(u_1), \frac{1}{s_2} \ln(u_2))$$

and

$$G_{\eta_{\mathbf{s}}(\mathbf{Y})}(\mathbf{u}) = G(\frac{1}{s_1} \ln(u_1), \frac{1}{s_2} \ln(u_2)),$$

where $\mathbf{u} = (u_1, u_2)$ and $u_i \in [1, +\infty), i = 1, 2$. If $\eta_{\mathbf{s}}(\mathbf{X}) \leq_{wmit} \eta_{\mathbf{s}}(\mathbf{Y})$ for every non-negative $\mathbf{s} = (s_1, s_2)$ then

$$\begin{aligned}
\widetilde{m}_{1,\eta_{\mathbf{s}}(\mathbf{X})}(\mathbf{u}) &= \int_1^{u_1} \frac{F(\frac{1}{s_1} \ln(w_1), \frac{1}{s_2} \ln(u_2))}{F(\frac{1}{s_1} \ln(u_1), \frac{1}{s_2} \ln(u_2))} dw_1 \\
&= \int_0^{\frac{1}{s_1} \ln(u_1)} \frac{s_1 e^{s_1 x_1} F(x_1, \frac{1}{s_2} \ln(u_2))}{F(\frac{1}{s_1} \ln(u_1), \frac{1}{s_2} \ln(u_2))} dx_1 \\
&\geq \int_0^{\frac{1}{s_1} \ln(u_1)} \frac{s_1 e^{+s_1 x_1} G(x_1, \frac{1}{s_2} \ln(u_2))}{G(\frac{1}{s_1} \ln(u_1), \frac{1}{s_2} \ln(u_2))} dx_1 \\
&= \widetilde{m}_{1,\eta_{\mathbf{s}}(\mathbf{Y})}(\mathbf{u}), \quad \text{for all } u_i \in [1, +\infty), \mathbf{s} \in \mathbb{R}_+^2
\end{aligned} \tag{4.3}$$

and, moreover,

$$\begin{aligned}
m_{2,\eta_{\mathbf{s}}(\mathbf{X})}(\mathbf{u}) &= \int_1^{u_2} \frac{F(\frac{1}{s_1} \ln(u_1), \frac{1}{s_2} \ln(w_2))}{F(\frac{1}{s_1} \ln(u_1), \frac{1}{s_2} \ln(u_2))} dw_2 \\
&= \int_0^{\frac{1}{s_2} \ln(u_2)} \frac{s_2 e^{+s_2 x_2} F(\frac{1}{s_1} \ln(u_1), x_2)}{F(\frac{1}{s_1} \ln(u_1), \frac{1}{s_2} \ln(u_2))} dx_2
\end{aligned}$$

$$\geq \int_0^{\frac{1}{s_2}\ln(u_2)} \frac{s_2 e^{+s_2 x_2} G(\frac{1}{s_1}\ln(u_1), x_2)}{G(\frac{1}{s_1}\ln(u_1), \frac{1}{s_2}\ln(u_2))} dx_2$$

$$= \widetilde{m}_{2,\eta_{\mathbf{s}}(\mathbf{Y})}(\mathbf{u}), \quad \text{for all } u_i \in [1, +\infty), \mathbf{s} \in \mathbb{R}_+^2. \tag{4.4}$$

Let us take $u_i = e^{+s_i t_i}$ where $t_i \geq 0, i = 1, 2$, then the Eq (4.3) implies that for all $\mathbf{s}, \mathbf{t} \in \mathbb{R}_+^2$

$$L^*_{\mathbf{X}_{1,t_1}}(s_1) = \frac{1}{s_1}\widetilde{m}_{1,\eta_{\mathbf{s}}(\mathbf{X})}(e^{s_1 t_1}, e^{s_2 t_2}) \geq L^*_{\mathbf{Y}_{1,t_1}}(s_1) = \frac{1}{s_1}\widetilde{m}_{1,\eta_{\mathbf{s}}(\mathbf{Y})}(e^{s_1 t_1}, e^{s_2 t_2}),$$

and, simultaneously, the Eq (4.4) for all $\mathbf{s}, \mathbf{t} \in \mathbb{R}_+^2$ gives

$$L^*_{\mathbf{X}_{2,t_2}}(s_2) = \frac{1}{s_2}\widetilde{m}_{2,\eta_{\mathbf{s}}(\mathbf{X})}(e^{s_1 t_1}, e^{s_2 t_2}) \geq L^*_{\mathbf{Y}_{2,t_2}}(s_2) = \frac{1}{s_2}\widetilde{m}_{2,\eta_{\mathbf{s}}(\mathbf{Y})}(e^{s_1 t_1}, e^{s_2 t_2})$$

Therefore, an application of Theorem 10 guarantees the desired result. □ □

Potential applications of stochastic orderings lie in the characterization of different reliability classes of lifetime distributions (cf. Shaked and Shanthikumar [33]). Below, we give further descriptions of some bivariate reliability notions in terms of the Laplace transform order of the inactivity time of $\mathbf{X}$ and a characterization result for a bivariate decreasing reversed hazard rate (BDRHR) property.

The following class of bivariate distributions can be considered.

Definition 14. *Let* $\mathbf{X}$ *be a random pair of lifetimes which has joint SF* F*, joint reversed hazard rate function* $\mathbf{r} \equiv (r_{1,\mathbf{X}}, r_{2,\mathbf{X}})$*. Then,* $\mathbf{X}$ *is said to have BDRHR property whenever* $\frac{F(t_1+a_1, t_2+a_2)}{F(t_1,t_2)}$ *is non-increasing in* $t_i \geq 0$ $(i = 1, 2)$ *for every* $\mathbf{a} = (a_1, a_2) \in \mathbb{R}_+^2$ *or equivalently if,* $r_{i,\mathbf{X}}(t_1, t_2) \geq r_{i,\mathbf{X}}(t_1+a_1, t_2+a_2)$ *for all* $t_i \geq 0$ *and for every* $a_i \geq 0$*,* $i = 1, 2$*.*

Note that $\mathbf{X}(\mathbf{a}) = (X_1 - a_1, X_2 - a_2)$ where $\mathbf{a} = (a_1, a_2) \in \mathbb{R}_+^2$ has CDF $F(t_1 + a_1, t_2 + a_2)$ at the point $\mathbf{t} = (t_1, t_2)$. Thus, from Definition 14, the distribution of $\mathbf{X}$ belongs to BDRHR class if, and only if, $\mathbf{X}(\mathbf{a}) \leq_{wrhr} \mathbf{X}$, for all $\mathbf{a} \in \mathbb{R}_+^2$. Let us denote by $(\mathbf{X}(\mathbf{a}))_{(\mathbf{t})}$ the bivariate inactivity time of $\mathbf{X}(\mathbf{a})$. It is straightforward to see $(\mathbf{X}(\mathbf{a}))_{(\mathbf{t})}$ and $\mathbf{X}_{(\mathbf{t}+\mathbf{a})}$ are identical in distribution. From Theorem 12, it thus follows that $\mathbf{X}$ has BDRHR property, if, and only if,

$$\begin{aligned}\mathbf{X}_{(\mathbf{t}+\mathbf{a})} &= (t_1 + a_1 - X_1, t_2 + a_2 - X_2 | X_1 \leq t_1 + a_1, X_2 \leq t_2 + a_2) \\ &\geq_{BLt} (t_1 - X_1, t_2 - X_2 | X_1 \leq t_1, X_2 \leq t_2) = \mathbf{X}_{(\mathbf{t})}, \quad \text{for all } \mathbf{t}, \mathbf{a} \in \mathbb{R}_+^2,\end{aligned}$$

which, equivalently, holds if

$$L_{\mathbf{X}_{(\mathbf{t}+\mathbf{a})}}(\mathbf{s}) \leq L_{\mathbf{X}_{(\mathbf{t})}}(\mathbf{s}), \quad \text{for all } \mathbf{t}, \mathbf{a}, \mathbf{s} \in \mathbb{R}_+^2,$$

i.e., $\mathbf{X}$ has BDRHR property if, and only if, $L_{\mathbf{X}_{(\mathbf{t})}}(\mathbf{s})$ is non-increasing in $\mathbf{t}$ for all $\mathbf{s} \in \mathbb{R}_+^2$. From Theorem 10, it is realized that $\mathbf{X}$ has BDRHR property if, and only if,

$$\begin{aligned}\mathbf{X}_{i,t_i+a_i} &= (t_i + a_i - X_i | X_1 \leq t_1 + a_1, X_2 \leq t_2 + a_2) \\ &\geq_{Lt} (t_i - X_i | X_1 \leq t_1, X_2 \leq t_2) = \mathbf{X}_{i,t_i}, \quad \text{for all } t_i, a_i \in \mathbb{R}_+, i = 1, 2,\end{aligned}$$

which means that

$$L_{\mathbf{X}_{i,t_i}}(s_i) \text{ (resp. } L^*_{\mathbf{X}_{i,t_i}}(s_i)) \text{ is non-increasing (resp. non-decreasing) in } t_i, i = 1, 2.$$

It is eventually concluded that the BDRHR property of a bivariate lifetime distribution is characterized by the decreasing property of conditional and bivariate Laplace transform associated with the bivariate inactivity time $\mathbf{X}_{(\mathbf{t})}$ in terms of $\mathbf{t} = (t_1, t_2)$.

4. Conclusions

In this paper, two existing characterizations of the reversed hazard rate order using the Laplace transform order and the mean inactivity time order in univariate cases in literature have been developed to bivariate cases. There is another aspect to bivariate distributions compared to univariate distributions, which is the dependence structure of a random pair with a bivariate distribution. The results obtained in this work confirm that the dependence structure in random pairs does not matter when the characterizations are made in bivariate cases. Considering bivariate inactivity times of a random pair of lifetimes instead of inactivity times of single random lifetime is significant in the context of lost lives, as the former provides an extended framework for the latter to be evaluated in a similar setting where lifetimes are influenced by each other and give rise to some dependencies.

Future studies may determine possible generalizations of the characterization results in multivariate cases. The concept of inactivity time could be defined for random lifetime vectors where the dimensions of the lifetimes exceed two. Then using the multivariate Laplace transform and applying it for comparison of inactivity time in higher dimensions we may be able to characterize multivariate reversed hazard rate order and/or multivariate notions of lifetime distributions.

Acknowledgments

This work was supported by Researchers Supporting Project number (RSP2022R464), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

There is no conflict of interest declared by the authors.

References

1. I. A. Ahmad, M. Kayid, Characterizations of the RHR and MIT orderings and the DRHR and IMIT classes of life distributions, *Prcbab. Eng. Inf. Sci.,* **19** (2005), 447–461. https://doi.org/10.1017/S026996480505028X
2. R. Ahmadi, Reliability and maintenance modeling for a load-sharing k-out-of-n system subject to hidden failures, *Comput. Ind. Eng.*, **150** (2020), 106894. https://doi.org/10.1016/j.cie.2020.106894
3. H. Ahmed, M. Kayid, Preservation properties for the Laplace transform ordering of residual lives, *Stat. Pap.*, **45** (2004), 583–590. https://doi.org/10.1007/BF02760570
4. F. H. Al-Gashgari, A. I. Shawky, M. A. W. Mahmoud, A nonparametric test for testing exponentiality against NBUCA class of life distributions based on Laplace transform, *Qual. Reliab. Eng. Int.*, **32** (2016), 29–36. https://doi.org/10.1002/qre.1723
5. A. Alzaid, J. S. Kim, F. Proschan, Laplace ordering and its applications, *J. Appl. Probab.*, **28** (1991), 116–130. https://doi.org/10.2307/3214745
6. P. Andersen, O. Borgan, R. Gill, N. Keiding, *Statistical models based on counting processes*, Springer Series in Statistics, 1991. https://doi.org/10.1007/978-1-4612-4348-9

7. H. W. Block, T. H. Savits, Burn-in, *Stat. Sci.*, **12** (1997), 1–19. https://doi.org/10.1214/ss/1029963258

8. F. Domma, Bivariate reversed hazard rate, notions, and measures of dependence and their relationships, *Commun. Stat.-Theor. M.*, **40** (2011), 989–999. https://doi.org/10.1080/03610920903511777

9. A. Di Crescenzo, P. Di Gironimo, S. Kayal, Analysis of the past lifetime in a replacement model through stochastic comparisons and differential entropy, *Mathematics*, **8** (2020), 1203. https://doi.org/10.3390/math8081203

10. L. Eeckhoudt, C. Gollier, Demand for risky assets and the monotone probability ratio order, *J. Risk Uncertain.*, **11** (1995), 113–122. https://doi.org/10.1007/BF01067680

11. S. M. El-Arishy, L. S. Diab, E. S. El-Atfy, Characterizations on decreasing Laplace transform of time to failure class and hypotheses testing, *Comput. Sci. Comput. Math.*, **10** (2020), 49–54. https://doi.org/10.20967/jcscm.2020.03.002

12. M. Finkelstein, On the reversed hazard rate, *Reliab. Eng. Syst. Saf.*, **78** (2002), 71–75. https://doi.org/10.1016/S0951-8320(02)00113-8

13. R. Gupta, A. K. Nanda, Some results on reversed hazard rate ordering, *Commun. Stat.-Theor. M.,* **30** (2001), 2447–2457. https://doi.org10.1081/STA-100107697

14. Z. Guo, J. Zhang, R. Yan, On inactivity times of failed components of coherent system under double monitoring, *Prcbab. Eng. Inf. Sci.,* 2021, 1–18. https://doi.org/10.1017/S0269964821000152

15. Y. Jia, J. H. Jeong, Cause-specific quantile regression on inactivity time, *Stat. Med.*, **40** (2021), 1811–1824. https://doi.org/10.1002/sim.8871

16. J. Jiang, Z. J. Zhou, X. X. Han, B. C. Zhang, X. D. Ling, A new BRB based method to establish hidden failure prognosis model by using life data and monitoring observation, *Knowl. Based Syst.*, **67** (2014), 270–277. https://doi.org/10.1016/j.knosys.2014.04.045

17. S. Karlin, *Total positivity* , Stanford University Press, 1968.

18. M. Kayid, I. A. Ahmad, On the mean inactivity time ordering with reliability applications, *Prcbab. Eng. Inf. Sci.,* **18** (2004), 395–409. https://doi.org/10.1017/S0269964804183071

19. M. Kayid, S. Izadkhah, Mean inactivity time function, associated orderings, and classes of life distributions, *IEEE Trans. Reliab.*, **63** (2014), 593–602. https://doi.org/10.1109/TR.2014.2315954

20. M. Kayid, S. Izadkhah, S. Alshami, Laplace transform ordering of time to failure in age replacement models, *J. Korean Stat. Soc.*, **45** (2016), 101–113. https://doi.org/10.1016/j.jkss.2015.08.001

21. N. Keiding, R. Gill, Random truncation models and Markov processes, *Ann. Stat.*, **18** (1990), 582–602. https://doi.org/10.1214/aos/1176347617

22. N. Keiding, Age-specific incidence and prevalence: A statistical perspective, *J. R. Stat. Soc. A Stat.*, **154** (1991), 371–412. https://doi.org/10.2307/2983150

23. M. Kijima, M. Ohnishi, Stochastic orders and their applications in financial optimization, *Math. Methods Oper. Res.*, **50** (1999), 351–372. https://doi.org/10.1007/s001860050102

24. C. Li, X. Li, On stochastic dependence in residual lifetime and inactivity time with some applications, *Stat. Probab. Lett.*, **177** (2021), 109120. https://doi.org/10.1016/j.spl.2021.109120

25. J. Mulero, F. Pellerey, Bivariate aging properties under Archimedean dependence structures, *Commun. Stat.-Theor. M.*, **39** (2010), 3108–3121. https://doi.org/10.1080/03610920903199987

26. A. K. Nanda, Stochastic orders in terms of Laplace transforms, *Bull. Calcutta Stat. Assoc.*, **45** (1995), 195–202. https://doi.org/10.1177/0008068319950306

27. A. K. Nanda, H. Singh, N. Misra, P. Paul, Reliability properties of reversed residual lifetime, *Commun. Stat.-Theor. M.*, **32** (2003), 2031–2042. https://doi.org/10.1081/STA-120023264

28. E. M. Ortega, A note on some functional relationships involving the mean inactivity time order, *IEEE Trans. Reliab.*, **58** (2008), 172–178. https://doi.org/10.1109/TR.2008.2006576

29. A. Patra, C. Kundu, Further results on residual life and inactivity time at random time, *Commun. Stat.-Theor. M.*, **49** (2020), 1261–1271. https://doi.org/10.1080/03610926.2018.1563170

30. J. M. Ruiz, J. Navarro, Characterizations based on conditional expectations of the double truncated distribution, *Ann. Inst. Stat. Math.*, **48** (1996), 563–572. https://doi.org/10.1007/BF00050855

31. E. Salehi, M. Tavangar, Stochastic comparisons on conditional residual lifetime and inactivity time of coherent systems with exchangeable components, *Stat. Probab. Lett.*, **145** (2019), 327–337. https://doi.org/10.1016/j.spl.2018.10.007

32. M. Shaked, T. Wong, Stochastic orders based on ratios of Laplace transforms, *J. Appl. Probab.*, **34** (1997), 404–419. https://doi.org/10.2307/3215380

33. M. Shaked, J. G. Shanthikumar, *Stochastic orders*, Springer, New York, 2007. https://doi.org/10.1007/978-0-387-34675-5

34. T. Tang, D. Lin, D. Banjevic, A. K. Jardine, Availability of a system subject to hidden failure inspected at constant intervals with non-negligible downtime due to inspection and downtime due to repair/replacement, *J. Stat. Plan. Infer.*, **143** (2013), 176–185. https://doi.org/10.1016/j.jspi.2012.05.011

35. C. Tepedelenlioglu, A. Rajan, Y. Zhang, Applications of stochastic ordering to wireless communications, *IEEE Trans. Wirel. Commun.*, **10** (2011), 4249–4257. https://doi.org/10.1109/TWC.2011.093011.110187

36. Y. Wang, H. Pham, A multi-objective optimization of imperfect preventive maintenance policy for dependent competing risk systems with hidden failure, *IEEE Trans. Reliab.*, **60** (2011), 770–781. https://doi.org/10.1109/TR.2011.2167779

37. Y. Zhang, Z. Sun, R. Qin, H. Xiong, Idle duration prediction for manufacturing system using a gaussian mixture model integrated neural network for energy efficiency improvement, *IEEE Trans. Autom. Sci. Eng.*, **18** (2019), 47–55. https://doi.org/10.1109/TASE.2019.2938662