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Research article

Li-Yorke chaotic property of cookie-cutter systems

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Abstract: In this paper, we investigate mean Li-Yorke chaos along some sequence and Li-Yorke chaos for cookie-cutter systems. By applying bounded distortion and a locally α-Hölder condition, we show that the cookie-cutter set contains a mean Li-Yorke scrambled set along some sequence in which the Hausdorff dimension equals the Hausdorff dimension of the cookie-cutter set. That is to say, a cookie-cutter system is mean Li-Yorke chaotic along some sequence. Meanwhile, we proved that every mean Li-Yorke scrambled set is also a scrambled set; hence a cookie-cutter system is also Li-Yorke chaotic.

Keywords: cookie-cutter sets; mean Li-Yorke chaotic set; Hausdorff dimension; bounded distortion; Li-Yorke chaos
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1. Introduction

The chaotic property is a characterization of the asymptotic behaviors between different orbits in the dynamical system. Until now, many versions of chaos have been studied, such as Devaney chaos [10], Li-Yorke chaos [22], distributional chaos [34] and mean Li-Yorke chaos [11]. These types of chaos are closely connected. Blanchard, Glasner, Kolyada and Maass [5] proved that positive entropy implied Li-Yorke chaos for a surjective continuous map. Huang and Ye [18] showed that Devaney chaos implies Li-Yorke chaos. Wang, Huang and Huan [36] introduced distributional chaos in a sequence, and showed that a continuous map of an interval is Li-Yorke chaotic if and only if it is distributively chaotic in a sequence. Downarowicz [11] observed that mean Li-Yorke chaos is equivalent to the DC2 chaos and proved that positive topological entropy implies mean Li-Yorke chaos. Liu, Wang and Chu [28] proved that Devaney chaos is stronger than distributional chaos in a sequence. Garcia-Ramos and Jin [16] found a new condition that implies mean Li-Yorke chaos. Cánovas [9] considered that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous interval maps that converges uniformly to a continuous map f, and showed that the chaoticity of f implied that the chaoticity of $\{f_n\}_{n=1}^{\infty}$. Mangang [30] studied the mean equicontinuity, sensitivity, expansiveness and distality of the product dynamical systems. The reader can refer to [6, 12, 13, 17, 19, 21, 33, 35, 37], which are related to Li-Yorke chaos.

Throughout this paper, we focus on Li-Yorke chaos and mean Li-Yorke chaos. Let us recall the definition of the scrambled set and mean Li-Yorke scrambled set.

Let (X,ρ) be a metric space and $f : X \to X$ be a self-map. The pair (X, f) is called a topological dynamical system.

Definition 1.1. [22] Let (X,ρ) be a metric space. For two points $x, y \in X$, (x, y) is a scrambled pair for the map $f : X \to X$, if

$$\limsup_{n\to\infty} \rho(f^n(x), f^n(y)) > 0 \text{ and } \liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0.$$

A subset $S \subseteq X$, containing at least two points, is a scrambled set of f, if for any $x, y \in S$ where $x \neq y$, (x, y) is a scrambled pair for f. If a scrambled set S for f is uncountable, we say that f is chaotic in the sense of Li-Yorke.

Definition 1.2. A subset $\mathcal{M}$ of X is called a mean Li-Yorke scrambled set if any two distinct points x, y in $\mathcal{M}$ satisfy

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \rho(f^j(x), f^j(y)) = 0 \text{ and } \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \rho(f^j(x), f^j(y)) > 0.$$

A system (X, f) is said to be mean Li-Yorke chaotic, if it contains an uncountable mean Li-Yorke scrambled set.

The sequence version of mean Li-Yorke chaos was introduced in [20].

Definition 1.3. Let $B = \{b_1 < b_2 < \cdots\}$ be a strictly increasing sequence of $\mathbb{N}$. According to [20], a subset $\mathcal{M}$ of X is called a mean Li-Yorke scrambled set along the sequence B if any two distinct points x, y in $\mathcal{M}$ satisfy

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \rho(f^{b_j}(x), f^{b_j}(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \rho(f^{b_j}(x), f^{b_j}(y)) > 0. \tag{1.1}$$

If the set $\mathcal{M}$ is uncountable, then we say that the system (X, f) is mean Li-Yorke chaotic along the sequence B.

The definitions of Li-Yorke chaos, and mean Li-Yorke chaos only require that the cardinality of the chaotic set is uncountable, so a natural further question is how 'large' the size of a scrambled set can be. There are three approaches to analyze the size: topological, measure-theoretic and dimension-theoretic. In [7], Blanchard, Huang and Snoha studied the topological size of scrambled sets extensively; see also the references therein. In [2], Balibrea and López surveyed the Lebesgue measure of scrambled sets for continuous maps on the interval. In [8], Bruin and López studied the Lebesgue measure of scrambled sets for C^2 and C^3 multimodal interval maps f with non-flat critical points. In [39], Xiong proved that there exists a scrambled set of the full Hausdorff dimension for the symbolic dynamics with finite symbols. Recently, the second author and his coauthors [25, 27] constructed a Li-Yorke chaotic set with full Hausdorff dimension for continued fractions and the β-transformation. Xiao [38] constructed a mean Li-Yorke chaotic set along a polynomial sequence with full Hausdorff dimension for the β-transformation. In 2021, the second author and his coauthor [26] studied the mean Li-Yorke chaotic property for continued fractions, and constructed a mean Li-Yorke chaotic set with full Hausdorff dimension for a sequence with some mild conditions.

Both continued fractions and the β-transformations are interval mappings, which are defined on the unit $[0, 1)$. An interesting question is as follows: what about the dynamical system defined on a fractal set? For example, is a 3-adic transformation on the classical middle-third Cantor set Li-Yorke chaotic? that is to say, does the classic middle-third Cantor set contain an uncountable scrambled set?

Motivated by the ideas and results above, we focus on the size of the scrambled set for cookie-cutter sets from the dimensional sense. We first recall the definition of the cookie-cutter mappings, which was defined and studied by Bedford [4].

Definition 1.4. A mapping f is called a cookie-cutter map if there exists a finite collection of disjoint closed intervals $I_1, I_2, \cdots, I_q \subset I = [0, 1]$, such that

(1). f is defined in $\bigcup_{j=1}^{q} I_j$, the restriction of f on each I_j is 1 to 1 and surjective, the corresponding branch inverse is denoted by $\phi_j = (f|_{I_j})^{-1} : I \to I_j$;

(2). f is differentiable becoming the *Hölder* continuous derivative f', i.e. there exist constants $c > 0$ and $r \in (0, 1]$ such that $|f'(x) - f'(y)| \leq c|x - y|^r$ for any $x, y \in I_j$, $1 \leq j \leq q$;

(3). f is bounded expanding in the sense that

$$1 < b := \inf_x |f'(x)| \leq \sup_x |f'(x)| := B < +\infty.$$

The cookie-cutter set associated with f is

$$C = \{x \in [0, 1] : f^{n-1}(x) \in \bigcup_{j=1}^{q} I_j, \forall n \geq 1\} = \bigcap_{k=0}^{+\infty} f^{-k}(I),$$

where f^0 denotes identity mapping.

In [32], Nakata gave a method of the approximation of Hausdorff dimensions of generalized cookie-cutter Cantor sets using the thermodynamic formalism. In [23], Liang, Yu and Ren proved the existence of self-similar measures, conformal measures and Gibbs measures on cookie-cutter sets and analysed the dimension spectrum of each of these measures. Barral and Seuret [3] computed the singularity spectrum of the inverse measure of Gibbs measures on cookie-cutter sets. Baker [1] proposed a new multifractal zeta function and showed that under certain conditions the abscissa of convergence yields the Hausdorff multifractal spectrum for a class of measures supported on cookie-cutter sets. In [29], Ma, Rao and Wen defined the cookie-cutter-like sets by the limit sets of a sequence of classical cookie-cutter mappings, and they calculated the dimensions, Hausdorff and packing measures of the cookie-cutter-like sets. Liu [24] proved that for the Cookie-cutter-like dynamic system with unbounded expansion, the properties such as bounded variation and bounded distortion, the existence of a Gibbs-like measure still holds. Recently, Fan, Liao and Wu [15] studied the multifractal spectrum of some multiple ergodic averages in linear Cookie-Cutter dynamical systems.

Let "$\dim_H$" denote the Hausdorff dimension and $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure. Let $C(X)$ denote the continuous functions defined on X, and $P(\cdot)$ denote the pressure function on $C(X)$. The Hausdorff dimensions of a cookie-cutter set is given by the pressure function $P(\cdot)$.

Theorem 1.5. *[4] Suppose that C is the cookie-cutter set associated with f on* $[0,1]$. *Then* $0 < \mathcal{H}^{\beta}(C) < +\infty$ *and* $\dim_H C = \beta$*, where β is a unique real number with* $P(-\beta \log |f'(x)|) = 0$.

The main aim of this paper is to demonstrate the Hausdorff dimensions of scrambled sets and the mean Li-Yorke scrambled set along B in cookie-cutter sets. Let'$[\cdot]$' denote the integer part.

Theorem 1.6. *Let $G(x) = e_m x^m + e_{m-1}x^{m-1} + \cdots + e_1 x + e_0$ a polynomial with a degree $m \geq 3$ and $e_m > 0$. Let $B = \{b_1 < b_2 < \cdots\} \subset \{[G(n)] : n \geq 1\}$ be a sequence of positive integers. Let C be the cookie-cutter set associated with f on* $[0,1]$. *Then there exists a mean Li-Yorke scrambled set along B in C for which the Hausdorff dimension equals to* $\dim_H C$.

Corollary 1.7. Let C be the cookie-cutter set associated with f on $[0,1]$. Then there exists a scrambled set in C for which the Hausdorff dimension equals to $\dim_H C$. In particular, every cookie-cutter system is chaotic in the sense of Li-Yorke.

The rest of the paper is organized as follows. In Section 2, we collect and establish some elementary properties of the Hausdorff dimension, pressure function and cookie-cutter sets that will be used later. Section 3 is devoted to proving Theorem 1.6 and Corollary 1.7. In Section 4, Theorem 1.6 is applied to a linear example and a non-linear one. Finally, the authors' conclusions are given in Section 5.

2. Preliminaries

2.1. Basic concepts

Let $\mathcal{A} = \{1, 2, \cdots, q\}$ with $q \geq 2$. Denote the symbolic space of one-sided infinite sequences over $\mathcal{A}$ by

$$\mathcal{A}^{\mathbb{N}} = \{u = (u_1, u_2, \cdots) : \quad u_i \in \mathcal{A},\ \forall i \in \mathbb{N}\}.$$

The symbol u_i is called the i-th coordinate of u.

We assign the discrete topology to $\mathcal{A}$ and the product topology to $\mathcal{A}^{\mathbb{N}}$. For $u, v \in \mathcal{A}^{\mathbb{N}}$, the distance d is defined by

$$d(u, v) = q^{-i}, \text{ where } i = \inf\{j \geq 0 : u_{j+1} \neq v_{j+1}\}.$$

Denote by $\mathcal{A}^n$ the set of all n-letter words and $\mathcal{A}^*$ for the set of all finite words over $\mathcal{A}$. For any u, $v \in \mathcal{A}^*$, uv denotes the concatenation of u and v. For two finite words u and v, u is called the prefix of v if there exists a finite word w such that $uw = v$. The finite word u is called the prefix of $v \in \mathcal{A}^{\mathbb{N}}$ if there exists an infinite sequence $w \in \mathcal{A}^{\mathbb{N}}$ such that $uw = v$. Both of them denote $u \sqsubseteq v$. The symbol "$|\cdot|$" means the diameter, the length and the absolute value with respect to a set, a word, and a real number respectively. For $i, j \in \mathbb{N}$ with $i < j$, we write $(u|_i^j) = (u_i, u_{i+1}, \cdots, u_j)$.

The shift map $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is defined by

$$(\sigma(u))_i = u_{i+1},\ \forall\, i \in \mathbb{N}.$$

2.2. *Hausdorff dimension and pressure function*

Let E be a subset of $\mathbb{R}$; a finite or countable collection of subsets $\{U_i\}_{i\geq 1}$ of $\mathbb{R}$ is called a δ-cover of a set $E \subset \mathbb{R}$ if $E \subset \cup_{i\geq 1} U_i$ and $|U_i| < \delta$ for all $i \geq 1$. For $s \geq 0$, $\delta \geq 0$, define

$$\mathcal{H}^s_\delta(E) = \inf\{\sum_{i\geq 1} |U_i|^s : \{U_i\}_{i\geq 1} \text{ is a } \delta\text{-cover of } E\}.$$

The s-dimensional Hausdorff measure of E is defined as

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E).$$

There exists a number $\dim_H E$, called the Hausdorff dimension of E, such that

$$\mathcal{H}^s(E) = \begin{cases} \infty & \text{if } s < \dim_H E, \\ 0 & \text{if } s > \dim_H E. \end{cases}$$

Thus

$$\dim_H E = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

The basics on the Hausdorff dimension can be found in [14], to which we refer the reader.

Let α be a positive real number. We say that a map $F : X \to \mathbb{R}$ $(X \subset \mathbb{R})$ satisfies the local α-Hölder condition if there exists a real number $r > 0$ and a constant $c > 0$ such that, for any $x, y \in X$ with $|x - y| < r$,

$$|F(x) - F(y)| \leq c|x - y|^\alpha.$$

The following well-known lemma can be easily deduced from the definitions of the Hausdorff dimension and the local α-Hölder condition. One can refer to [31] for more details.

Lemma 2.1. *Let X be a metric space and s, $\alpha > 0$ be real numbers. If a map $F : X \to \mathbb{R}$ satisfies the local α-Hölder condition, then $\mathcal{H}^s(F(X)) \leq c^s \mathcal{H}^{s\alpha}(X)$, where c is the constant in the definition of the locally α-Hölder condition. Moreover, $\alpha \dim_H(F(X)) \leq \dim_H(X)$.*

Let $f : X \to X$ and $\varphi : X \to \mathbb{R}$ be a continuous function. Denote

$$S_n\varphi(x) = \sum_{i=0}^{n-1} \varphi(f^i(x))$$

for $x \in X$. We define the pressure function $P : C(X) \to \mathbb{R}$ by

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in F(f^n)} e^{S_n\varphi(x)}$$

where $F(f^n)$ denotes the set of fixed points of f^n.

2.3. Cookie-cutter sets

In this subsection, we collect and establish some elementary properties on cookie-cutter sets. the reader can refer to [4] for more details.

Since $I_1, I_2, \cdots, I_q$ are disjoint closed intervals, as according to Definition 1.4, we can define a mapping π from $\{1, 2, \cdots, q\}^{\mathbb{N}}$ to C by

$$\pi(a_1, a_2, \cdots, a_n, \cdots) = \bigcap_{n>0} \phi_{u_1} \circ \phi_{u_2} \circ \cdots \circ \phi_{u_n}(I)$$

for any $(a_1, a_2, \cdots, a_n, \cdots) \in \{1, 2, \cdots, q\}^{\mathbb{N}}$. It can be verified that the mapping π is a continuous bijection.

For $w = u_1 u_2 \cdots u_n \in \{1, 2, \cdots, q\}^n$, define the basic interval of the order n corresponding to w by

$$I(w) = I(u_1 u_2 \cdots u_n) = \phi_{u_1} \circ \phi_{u_2} \circ \cdots \circ \phi_{u_n}(I).$$

For example, when $n = 1$, $I(i) = \phi_i(I) = I_i$ for $1 \le i \le q$.

The following diagram is commutative, that is, $f \circ \pi = \pi \circ \sigma$. Since π is bijective, the equation $\pi^{-1} \circ f = \sigma \circ \pi^{-1}$ also holds.

$$\begin{array}{ccc}
\{1, 2, \cdots, q\}^{\mathbb{N}} & \xrightarrow{\ \sigma\ } & \{1, 2, \cdots, q\}^{\mathbb{N}} \\
\pi \downarrow & & \downarrow \pi \\
C & \xrightarrow{\ f\ } & C
\end{array}$$

Remark 2.2. (1). Since $I_1, I_2, \cdots, I_q$ are disjoint closed intervals, as according to Definition 1.4, the distance of between two intervals $\rho(I_i, I_j) > 0$ for any $1 \le i \ne j \le q$; then, denote $\rho_{min} = \min_{1 \le i \ne j \le q} \rho(I_i, I_j)$.

(2). According to the condition (3) of Definition 1.4, the map f is monotonous on each I_j $(1 \le j \le q)$.

(3). For any n and any $w \in \{1, 2, \cdots, q\}^n$, $|I(w)| \le \dfrac{1}{b^n}$.

The following lemmas are useful for proving Theorem 1.6.

Lemma 2.3. *[14] Let C be a cookie-cutter set and $\rho_{min} = \min_{1 \le i \ne j \le q} \rho(I_i, I_j)$. For any $w \in \{1, 2, \cdots, q\}^*$, there exists a constant d_1 such that*

$$\rho_{min} d_1 |I(w)| \le \rho(I(wi), I(wj)) \le |I(w)|,$$

for any $1 \le i \ne j \le q$.

Lemma 2.4. *[14](bounded distortion) Let C be a cookie-cutter set associated with f on $[0, 1]$. There exists a constant $K_1 > 0$ such that for any $N \ge 1$ and any $x, y \in I(w)$, $w \in \{1, 2, \cdots, q\}^*$,*

$$K_1^{-1} \le \frac{|(f^N)'(x)|}{|(f^N)'(y)|} \le K_1.$$

3. Proof of Theorem 1.6 and Corollary 1.7

The idea of proving Theorem 1.6 is to construct a mean Li-Yorke scrambled set in the symbolic space $\{1, \cdots, q\}^{\mathbb{N}}$, then project such set onto the unit interval $[0, 1]$ where the projection is a mean Li-Yorke scrambled set along B, and finally calculate the Hausdorff dimension of the projection set.

3.1. Construction of a mean Li-Yorke scrambled set

Actually, the idea for constructing a mean Li-Yorke scrambled set was inspired by [28]. In the following, we write $\Sigma_q = \{1, 2, \cdots, q\}^{\mathbb{N}}$.

We define a positive integer sequence $\{c_k\}_{k\geq 1}$ such that $c_1 = 1$ and $c_2 = 2^{c_1}$ for any $k \geq 2$ and

$$c_{2k-1} = c_1 + c_2 + \cdots + c_{2k-2}$$

and

$$c_{2k} = 2^{c_1+c_2+\cdots+c_{2k-2}+c_{2k-1}}.$$

Let

$$\mathcal{J}_1 = \{j \in \mathbb{N} : \sum_{i=1}^{2k-1} c_i < j \leq \sum_{i=1}^{2k} c_i \text{ for some } k \geq 1\}$$

and

$$\mathcal{J}_2 = \{j \in \mathbb{N} : \sum_{i=1}^{2k} c_i < j \leq \sum_{i=1}^{2k+1} c_i \text{ for some } k \geq 1\}.$$

For $u = (u_1, u_2, \cdots) \in \Sigma_q$, and denote $(1^j) = (\underbrace{11\cdots 1}_{j})$, we set a sequence of finite words $\{V_j(u)\}_{j\geq 1}$ as follows.

$$V_j(u) = \begin{cases} u_1 & \text{if } j = 1, \\ 1^j & \text{if } j \in \mathcal{J}_1, \\ u_1 u_2 \cdots u_j & \text{if } j \in \mathcal{J}_2. \end{cases}$$

Recall that $B = \{b_j\}_{j\geq 1}$, $(v|_i^j) = (v_i, v_{i+1}, \cdots, v_j)$ and $\left| v|_i^j \right| = j - i + 1$. For $u = (u_1, u_2, \cdots) \in \Sigma_q$, we define a mapping $g : \Sigma_q \to \Sigma_q$ by

$$g(u) = (u\Big|_1^{b_1}, V_1(u), u\Big|_{b_1+1}^{b_2-|V_1(u)|}, V_2(u), \cdots, V_{n-1}(u), u\Big|_{b_{n-1}-\sum_{i=1}^{n-2}|V_i(u)|+1}^{b_n-\sum_{i=1}^{n-1}|V_i(u)|}, V_n(u), \cdots)$$

Denote $W_1(u) = u\Big|_1^{b_1}$, and $W_n(u) = u\Big|_{b_{n-1}-\sum_{i=1}^{n-2}|V_i(u)|+1}^{b_n-\sum_{i=1}^{n-1}|V_i(u)|}$ for any $n \geq 2$. The mapping $g : \Sigma_q \to \Sigma_q$ can be written as:

$$g(u) = (W_1(u), V_1(u), W_2(u), V_2(u), \cdots, V_{n-1}(u), W_n(u), V_n(u), \cdots).$$

Remark 3.1. (1). The mapping g is continuous and injective.

(2). For $j \in \mathcal{J}_1$, $\sigma^{b_j}(g(u)) = (1^j, W_{j+1}(u), V_{j+1}(u), \cdots)$.

(3). For $j \in \mathcal{J}_2$, $\sigma^{b_j}(g(u)) = (u_1 u_2 \cdots u_j, W_{j+1}(u), V_{j+1}(u), \cdots)$.

Remark 3.2. (1). When $j \to +\infty$, the length of the finite word 1^j tends to infinity which can guarantee the equality of (1.1).

(2). For $u, v \in \Sigma_q$ with $u \neq v$, there exists $k \geq 1$ such that $u_k \neq v_k$. For $j \in \mathcal{J}_2$ and $j \to +\infty$, u_k and v_k appear infinitely in $V_j(u)$ and $V_j(u)$ which can guarantee the inequality of (1.1).

Let $\mathcal{M} := g(\Sigma_q)$. Recall that the mapping π from $\{1, 2, \cdots, q\}^{\mathbb{N}}$ to C is

$$\pi(a_1, a_2, \cdots, a_n, \cdots) = \bigcap_{n>0} \phi_{u_1} \circ \phi_{u_2} \circ \cdots \circ \phi_{u_n}(I)$$

for any $(a_1, a_2, \cdots, a_n, \cdots) \in \Sigma_q$, and that the equation $f \circ \pi = \pi \circ \sigma$ holds.

The following lemma indicates $\pi(\mathcal{M})$ is a mean Li-Yorke scrambled set of f on C.

Lemma 3.3. *The set $\pi(\mathcal{M})$ is a mean Li-Yorke scrambled set along B of f on C.*

Proof. For any $u, v \in \mathcal{M}$ with $u \neq v$, we shall show that $(\pi(u), \pi(v))$ is a mean Li-Yorke scrambled pair for f.

Since the map π is continuous and satisfies $f \circ \pi = \pi \circ \sigma$, we obtain

$$\left|f^{b_j} \circ \pi(u) - f^{b_j} \circ \pi(v)\right| = \left|\pi(\sigma^{b_j}(u)) - \pi(\sigma^{b_j}(v))\right|.$$

(1). **Lower limits**. Taking $n = \sum_{i=1}^{2k} c_i$, since $\left|\pi(\sigma^{b_j}(u)) - \pi(\sigma^{b_j}(v))\right| \leq 1$ for $1 \leq j \leq n - c_{2k}$, we get

$$\sum_{j=1}^{n} \left|\pi(\sigma^{b_j}(u)) - \pi(\sigma^{b_j}(v))\right| \leq n - c_{2k} + \sum_{j=n-c_{2k}+1}^{n} \left|\pi(\sigma^{b_j}(u)) - \pi(\sigma^{b_j}(v))\right|.$$

By Remark 2.2(3) and Remark 3.1(2),

$$\sum_{j=n-c_{2k}+1}^{n} \left|\pi(\sigma^{b_j}(u)) - \pi(\sigma^{b_j}(v))\right| \leq \sum_{j=n-c_{2k}+1}^{n} |I(1^j)| \leq \sum_{j=n-c_{2k}+1}^{n} b^{-j} \leq \frac{1}{b-1}.$$

Therefore

$$\frac{1}{n} \sum_{j=1}^{n} \left|f^{b_j} \circ \pi(u) - f^{b_j} \circ \pi(v)\right| \leq \frac{n - c_{2k} + \frac{1}{b-1}}{n}.$$

Note that $c_{2k} = 2^{\sum_{i=1}^{2k-1} c_i}$ and $n = \sum_{i=1}^{2k} c_i$; taking the lim inf gives

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \left|f^{b_j} \circ \pi(u) - f^{b_j} \circ \pi(v)\right| = 0.$$

(2). **Upper limits**.

For any $u, v \in \mathcal{M}$ with $u \neq v$, there exists $\xi \neq \eta \in \Sigma_q$ satisying $g(\xi) = u$ and $g(\eta) = v$. Since the mapping g is injective, it follows that $\xi \neq \eta$. Let $t \geq 1$ be the least positive integer such that $\xi_t \neq \eta_t$ and $\xi_i = \eta_i$ for any $1 \leq i \leq t-1$.

For $k > t$ large enough, taking $n = \sum_{i=1}^{2k+1} c_i$, we obtain

$$\sum_{j=1}^{n} \left|\pi(\sigma^{b_j}(u)) - \pi(\sigma^{b_j}(v))\right| \geq \sum_{j=n-c_{2k+1}+1}^{n} \left|\pi(\sigma^{b_j}(u)) - \pi(\sigma^{b_j}(v))\right|$$

By Remark 3.1(3), for $j \in \mathcal{J}_2$,

$$\sigma^{b_j}(u) = \sigma^{b_j}(g(\xi)) = (\xi_1\xi_2 \cdots \xi_j, W_{j+1}(\xi), V_{j+1}(\xi), \cdots)$$

and

$$\sigma^{b_j}(v) = \sigma^{b_j}(g(\eta)) = (\eta_1\eta_2 \cdots \eta_j, W_{j+1}(\eta), V_{j+1}(\eta), \cdots).$$

When $n - c_{2k+1} + 1 \leq j \leq n$, it follows that $j \in \mathcal{J}_2$ and $j > t$. By the definition of g, the distinct symbols ξ_t and η_t appear infinitely often in the same location of $g(\xi)$ and $g(\eta)$ respectively. Hence

$$\sum_{j=n-c_{2k+1}+1}^{n} \left|\pi(\sigma^{b_j}(u)) - \pi(\sigma^{b_j}(v))\right| \geq \sum_{j=n-c_{2k+1}+1}^{n} h_t = c_{2k+1}h_t,$$

where $h_t = \rho(I(\xi_1\xi_2\cdots\xi_t), I(\eta_1\eta_2\cdots\eta_t)) \geq \rho_{min}d_1|I(\xi_1\xi_2\cdots\xi_{t-1})|$. Therefore

$$\frac{1}{n}\sum_{j=1}^{n}\left|f^{b_j}\circ\pi(u) - f^{b_j}\circ\pi(v)\right| \geq \frac{c_{2k+1}h_t}{n}.$$

Note that $c_{2k+1} = \sum_{i=1}^{2k} c_i$ and $n = \sum_{i=1}^{2k+1} c_i = 2c_{2k+1}$, taking the lim sup gives

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left|f^{b_j}\circ\pi(u) - f^{b_j}\circ\pi(v)\right| = \frac{h_t}{2} > 0.$$

□

3.2. Estimation of $\dim_H \pi(\mathcal{M})$

Note that $\pi(\mathcal{M}) \subseteq C$, so we have $\dim_H \pi(\mathcal{M}) \leq \dim_H C$.

Consider a mapping $h : \pi(\mathcal{M}) \to C$ defined by

$$h(a) = \pi\circ g^{-1}\circ\pi^{-1}(a),$$

for $a \in \pi(\mathcal{M})$. Since π is continuous and bijective from Σ_q to C and the mapping g is continuous and bijective from Σ_q to $\mathcal{M}$, the mapping h is a continuous bijection on $\pi(\mathcal{M})$.

Proposition 3.4. We have $\dim_H \pi(\mathcal{M}) \geq \dim_H C$.

By Lemma 2.1, Proposition 3.4 is the corollary of the following lemma.

Lemma 3.5. *For any $\epsilon > 0$, the mapping h satisfies the local $\frac{1}{1+\epsilon}$-Hölder condition.*

Before proving Lemma 3.5, we make use of a kind of symbolic space described as follows:

For any $n \geq 1$, we can check that

$$\pi(\mathcal{M}) = \bigcap_{n\geq 1}\bigcup_{(u_1,u_2,\cdots,u_n)\in A_n} I(u_1, u_2, \cdots, u_n),$$

where

$$A_n = \{(u_1, u_2, \cdots, u_n) \in \{1, \cdots, q\}^n : (u_1, \cdots, u_n) = (x_1, \cdots, x_n), \forall (x_1, x_2, \cdots) \in \mathcal{M}\}.$$

For any $n \geq 1$, let $t(n)$ be the sum of the length of all of the inserted pieces before the element in the position of n for any $g(u) \in \mathcal{M}$, that is,

$$t(n) = \begin{cases} 0 & \text{if } 1 \leq n \leq b_1, \\ n - b_1 & \text{if } b_1 + 1 \leq n \leq b_1 + |V_1(u)|, \\ \sum_{j=1}^{k} |V_j(u)| & \text{if } b_k + |V_k(u)| + 1 \leq n \leq b_{k+1} \text{ for some } k \geq 1, \\ \sum_{j=1}^{k-1} |V_j(u)| + n - b_k & \text{if } b_k + 1 \leq n \leq b_k + |V_k(u)| \text{ for some } k \geq 2. \end{cases}$$

Since

$$t(b_{k+1}) = \sum_{j=1}^{k} |V_j(u)| = \sum_{j=1}^{k} j = \frac{k(k+1)}{2},$$

according to the condition of Theorem 1.6, $G(x) = e_m x^m + e_{m-1}x^{m-1} + \cdots + e_0$ is a polynomial with degree $m \geq 3$ and $B = \{b_1 < b_2 < \cdots\} \subset \{[G(n)] : n \geq 1\}$; thus, we have $\lim_{k\to\infty} \frac{t(b_{k+1})}{b_k} = 0$.

For any $n > 1$, there exists $l \geq 1$ such that $b_l \leq n < b_{l+1}$; thus, $\frac{t(n)}{n} \leq \frac{t(b_{l+1})}{b_l}$, which implies $\lim_{n\to\infty} \frac{t(n)}{n} = 0$.

For any finite word $(u_1, u_2, \cdots, u_n)$, let $\overline{(u_1, u_2, \cdots, u_n)}$ be the finite word by eliminating the finite words $\{V_j(u)\}_{j\geq 1}$. Then

$$\overline{(u_1, u_2, \cdots, u_n)} \in \{1, 2, \cdots, q\}^{n-t(n)}.$$

The word $\overline{(u_1, u_2, \cdots, u_n)}$ has different cases for n and $t(n)$, that is,

$$\overline{(u_1, u_2, \cdots, u_n)} = \overline{(u|_1^n)} = \begin{cases} (u|_1^n) & \text{if } 1 \leq n \leq b_1, \\ (u|_1^{b_1}) & \text{if } b_1 + 1 \leq n \leq b_1 + |V_1(u)|, \\ (u|_1^{n-\sum_{j=1}^{k}|V_j(u)|}) & \text{if } b_k + |V_k(u)| + 1 \leq n \leq b_{k+1} \text{ for some } k \geq 1, \\ (u|_1^{b_k-\sum_{j=1}^{k-1}|V_j(u)|}) & \text{if } b_k + 1 \leq n \leq b_k + |V_k(u)| \text{ for some } k \geq 2. \end{cases}$$

For any $u = (u_1, u_2, \cdots), v = (v_1, v_2, \cdots) \in \mathcal{M}$ and $u \neq v$, there exists $n \geq 1$ such that $(u_1, \cdots, u_n) = (v_1, \cdots, v_n)$ and $u_{n+1} \neq v_{n+1}$.

Lemma 3.6. *We have* $|\pi(u) - \pi(v)| \geq \rho_{min} d_1 |I(u_1, \cdots, u_n)|$ *where* d_1 *is defined in Lemma* 2.3.

Proof. Applying Lemma 2.3, we immediately obtain the conclusion.

□

Proof of Lemma 3.5:

According to Lagrange's mean value theorem, for any n and any finite word $(u_1, \cdots, u_n)$, there exist $\xi \in I(u_1, \cdots, u_n)$ and $\eta \in I\overline{(u_1, \cdots, u_n)}$ such that

$$(f^n)'(\xi)|I(u_1, \cdots, u_n)| = 1$$

and

$$(f^{n-t(n)})'(\eta)|I\overline{(u_1, \cdots, u_n)}| = 1.$$

Recall that $u \sqsubseteq v$ means that u is the prefix of v. By the form of $g(u)$,

$$g(u) = (W_1(u), V_1(u), W_2(u), V_2(u), \cdots, W_n(u), V_n(u), \cdots)$$

for any $n > 1$; thus, we can find the maximum integer n_0, where $n_0 \leq \sqrt{2n} + 1$, satisfying

$$(W_1(u), V_1(u), \cdots, W_{n_0}(u), V_{n_0}(u)) \sqsubseteq g(u)|_1^n$$

and

$$g(u)|_1^n \sqsubseteq (W_1(u), V_1(u), \cdots, W_{n_0+1}(u), V_{n_0+1}(u)).$$

We denote the location of the finite word $W_i(u)$ $(1 \le i \le n_0 + 1)$ by $[N_{2i-2}, N_{2i-1}]$ and

$$R = [1, n] \setminus \bigcup_{i=1}^{n_0+1} [N_{2i-2}, N_{2i-1}].$$

Let $\epsilon > 0$; then, by $\lim_{n\to\infty} \frac{t(n)}{n} = 0$, there exists $K = K(\epsilon)$ such that

$$B^{t(n)} K_1^{n_0+1} < b^{(n-t(n))\epsilon}$$

for any $n \ge K$, where b is the lower bound and B is the upper bound in Definition 1.4, and K_1 is defined as Lemma 2.4.

For any $(u_1, u_2, \cdots, u_n) \in A_n$ with $n \ge K$,

$$\begin{aligned}
|I(u_1, u_2, \cdots, u_n)| &= \frac{1}{(f^n)'(\xi)} = \frac{1}{\prod_{k=0}^{n-1} f'(f^k(\xi))} \\
&\ge \frac{1}{B^{t(n)}} \frac{1}{\prod_{0\le k\le n-1, k\notin R} f'(f^k(\xi))} \\
&= \frac{1}{B^{t(n)}} \frac{1}{\prod_{1\le k\le n_0+1} (f^{N_{2k-1}-N_{2k-2}+1})'(f^{N_{2k-2}}(\xi))} \\
&= \frac{1}{B^{t(n)}} \frac{(f^{n-t(n)})'(\eta)|\overline{I(u_1, \cdots, u_n)}|}{\prod_{1\le k\le n_0+1} (f^{N_{2k-1}-N_{2k-2}+1})'(f^{N_{2k-2}}(\xi))} \\
&\ge \frac{1}{B^{t(n)}} \frac{1}{K_1^{n_0+1}} |\overline{I(u_1, \cdots, u_n)}| \\
&\ge \frac{1}{b^{(n-t(n))\epsilon}} |\overline{I(u_1, \cdots, u_n)}| \\
&\ge |\overline{I(u_1, \cdots, u_n)}|^{1+\epsilon}
\end{aligned}$$

where the first inequality holds according to the condition (3) of Definition 1.4, the second inequality holds by Lemma 2.4, and the last inequality holds according to (3) of Remark 2.2.

Let $r < \rho_{min} d_1 \min_{(u_1,\cdots,u_K)\in A_K} |I(u_1, \cdots, u_K)|$. For any $\pi(v) \in (\pi(u) - r, \pi(u) + r)$, there exists n such that $(u_1, \cdots, u_n) = (v_1, \cdots, v_n)$ and $u_{n+1} \ne v_{n+1}$; then

$$\begin{aligned}
|h(\pi(u)) - h(\pi(v))| &= |\pi \circ g^{-1} \circ \pi^{-1}(\pi(u)) - \pi \circ g^{-1} \circ \pi^{-1}(\pi(v))| \\
&\le |\overline{I(u_1, \cdots, u_n)}| \\
&\le |I(u_1, \cdots, u_n)|^{\frac{1}{1+\epsilon}} \\
&\le (\rho_{min} d_1)^{-\frac{1}{1+\epsilon}} |\pi(u) - \pi(v)|^{\frac{1}{1+\epsilon}}
\end{aligned}$$

where the last inequality holds by Lemma 3.6.

Proof of Theorem 1.6: Proposition 3.4 implies that $\dim_H \pi(\mathcal{M}) = \dim_H C$. □

The following lemma indicates that every mean Li-Yorke scrambled set along B is a scrambled set. Hence Corollary 1.7 is proved by Theorem 1.6 and Lemma 3.7.

Lemma 3.7. *Let (X, ρ) be a metric space and $f : X \to X$ be a self-map. Suppose that $\mathcal{M}$ is a mean Li-Yorke scrambled set along some sequence $B \subset \mathbb{N}$; then, $\mathcal{M}$ is a scrambled set of f.*

Proof. For any $x, y \in \mathcal{M}$ with $x \neq y$, that is, x and y satisfy the condition (1.1), the proof is divided into two parts.

(1). Suppose that

$$\limsup_{n\to\infty} \rho(f^n(x), f^n(y)) = 0.$$

Thus

$$\lim_{n\to\infty} \rho(f^n(x), f^n(y)) = 0,$$

that is, we have

$$\lim_{n\to\infty} \rho(f^{b_n}(x), f^{b_n}(y)) = 0,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \rho(f^{b_j}(x), f^{b_j}(y)) = 0.$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \rho(f^{b_j}(x), f^{b_j}(y)) = 0.$$

This is a contradiction. So $\limsup\limits_{n\to\infty} \rho(f^n(x), f^n(y)) > 0$.

(2). Given

$$\liminf_{n\to\infty} \rho(f^n(x), f^n(y)) > 0,$$

there exist a constant $\delta > 0$ and an integer $N > 1$ such that, for any $n \geq N$, we have

$$\rho(f^n(x), f^n(y)) > \delta.$$

Therefore,

$$\frac{1}{n} \sum_{j=1}^{n} \rho(f^{b_j}(x), f^{b_j}(y)) \geq \frac{(n-N)\delta}{n},$$

which implies that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \rho(f^{b_j}(x), f^{b_j}(y)) > \delta > 0.$$

This is also a contradiction. Hence, $\liminf\limits_{n\to\infty} \rho(f^n(x), f^n(y)) = 0$.

Combining (1) and (2), we know that $\mathcal{M}$ is a scrambled set of f.

□

4. Applications

In this section, we give a linear example and a non-linear one by applying Theorem 1.6.

4.1. Similarity transformations

Firstly, we recall the classical middle-third Cantor set C as follows.

Let $I = [0, 1]$, $I_0 = [0, 1/3]$ and $I_2 = [2/3, 1]$. Define the mapping $f : I_0 \sqcup I_2 \to I$ by

$$f(x) = \begin{cases} 3x, & x \in I_0, \\ 3x - 2, & x \in I_2, \end{cases}$$

where '$\sqcup$' denotes the disjoint union. We can check that $C_{1/3} = \bigcap_{k=0}^{+\infty} f^{-k}(I)$ is a cookie-cutter set.

On the symbolic space $\{0, 2\}^{\mathbb{N}}$, we can define the projection mapping $\pi : \{0, 2\}^{\mathbb{N}} \to [0, 1]$ by

$$\pi(x_1, x_2, \cdots) = \sum_{i=1}^{+\infty} \frac{x_i}{3^i}$$

for $(x_1, x_2, \cdots) \in \{0, 2\}^{\mathbb{N}}$. The middle-third Cantor set can also be described by the projection mapping π, that is,

$$C_{1/3} = \pi\Big(\{0, 2\}^{\mathbb{N}}\Big) = \{x \in [0, 1) : x = \sum_{i=1}^{+\infty} \frac{x_i}{3^i}, x_i \in \{0, 2\}, \forall i \geq 1\}.$$

It is well known that $\dim_H C_{1/3} = \dfrac{\log 2}{\log 3}$; thus, applying Theorem 1.6, the middle-third Cantor set contains a scrambled set for which the Hausdorff dimension is $\dfrac{\log 2}{\log 3}$.

4.2. Non-linear example

The example in Section 4.1 is piecewise linear. Here, we give a non-linear one that is from [14].

Let $I = [0, 1]$. Define the mappings ϕ_1 and ϕ_2 from I to I as

$$\phi_1(x) = \frac{1}{3}x + \frac{1}{10}x^2, \qquad \phi_2(x) = \frac{1}{3}x + \frac{2}{3} - \frac{1}{10}x^2.$$

Then

$$f(x) = \begin{cases} \phi_1^{-1}(x), & x \in \phi_1(I), \\ \phi_2^{-1}(x), & x \in \phi_2(I). \end{cases}$$

We can check that $C = \bigcap_{k=0}^{+\infty} f^{-k}(I)$ is a cookie-cutter set. By Theorem 1.5, $\dim_H C$ is the unique solution of $P(-\beta \log |f'(x)|) = 0$.

Applying Theorem 1.6, we see that the above cookie-cutter set contains a scrambled set for which the Hausdorff dimension equals to $\dim_H C$.

5. Further problems

In this paper, we demonstrated the Li-Yorke chaos of the cookie-cutter sets. In fact, according to the definitions of scrambled sets and mean Li-Yorke scrambled sets along the sequence B, mean Li-Yorke chaos along the sequence B is stronger than Li-Yorke chaos. Downarowicz [11] observed that mean Li-Yorke chaos is equivalent to the DC2 chaos and proved that positive topological entropy implies mean Li-Yorke chaos for a continuous interval map. A natural question is below.

Question 5.1. Is every cookie-cutter system mean Li-Yorke chaotic?

6. Conclusions

In this paper, we demonstrated mean Li-Yorke chaos along some sequence and Li-Yorke chaos for cookie-cutter system. For a given polynomial sequence B, by constructing a mean Li-Yorke scrambled set along B, we prove that the cookie-cutter set contains a mean Li-Yorke scrambled set along B for which the Hausdorff dimension equals the Hausdorff dimension of the cookie-cutter set. That is to say, cookie-cutter system is mean Li-Yorke chaotic along B. Meanwhile, we proved that every mean Li-Yorke scrambled set is also a scrambled set, hence, cookie-cutter system is also Li-Yorke chaotic.

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Conflict of interest

The authors declare no conflict of interest.

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