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Research article

Numerical solution of system of fuzzy fractional order Volterra integro-differential equation using optimal homotopy asymptotic method

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Abstract: In this paper, an efficient technique called Optimal Homotopy Asymptotic Method has been extended for the first time to the solution of the system of fuzzy integro-differential equations of fractional order. This approach however, does not depend upon any small/large parameters in comparison to other perturbation method. This method provides a convenient way to control the convergence of approximation series and allows adjustment of convergence regions where necessary. The series solution has been developed and the recurrence relations are given explicitly. The fuzzy fractional derivatives are defined in Caputo sense. It is followed by suggesting a new result from Optimal Homotopy Asymptotic Method for Caputo fuzzy fractional derivative. We then construct a detailed procedure on finding the solutions of system of fuzzy integro-differential equations of fractional order and finally, we demonstrate a numerical example. The validity and efficiency of the proposed technique are demonstrated via these numerical examples which depend upon the parametric form of the fuzzy number. The optimum values of convergence control parameters are calculated using the well-known method of least squares, obtained results are compared with fractional residual power series method. It is observed from the results that the suggested method is accurate, straightforward and convenient for solving system of fuzzy Volterra integrodifferential equations of fractional order.

Keywords: OHAM; fuzzy Caputo operator; Riemann-Liouville fractional integral operator; approximate solution; system of fuzzy fractional order; Volterra integro-differential equations

Mathematics Subject Classification: 34K28, 47Gxx, 45Dxx

1. Introduction

Fractional calculus has been concerned with integration and differentiation of fractional (non-integer) order of the function. Riemann and Liouville defined the concept of fractional order intgro-differential equations [1]. Fractional calculus has developed an extensive attraction in current years in applied mathematics such as physics, medical, biology and engineering [2–8]. Whenever dealing with the fractional integro-differential equation many authors consider the terms Caputo fractional derivative, Riemann-Liouville and Grunwald-Letnikvo [9–13]. The subject fractional calculus has many applications in widespread and diverse field of science and engineering such as fractional dynamics in the trajectory control of redundant manipulators, viscoelasticity, electrochemistry, fluid mechanics, optics and signals processing etc.

Fractional integro-differential equations having some uncertainties in the form of boundary conditions, initial conditions and so on [14–16]. To resolve these type of uncertainties mathematicians introduced some concepts fuzzy set theory is one of them.

Zadeh introduced the concept of fuzzy set theory [17–20]. Later on Prade and Dubois [21,22], Nahmias [23], Tanaka and Mizumoto [24]. All of them experienced that the fuzzy number as a location of r-cut $0 \le r \le 1$.

Many authors investigated some numerical techniques related to these problem which include the existence of the solution for discontinuous [25], reproducing kernel algorithm [26], integro-differential under generalized Caputo differentiability [27], A domain decomposition method [28], fractional differential transform method [29], Jacobi polynomial operational matrix [30], global solutions for nonlinear fuzzy equations [31], radioactivity decay model [32], Caputo-Katugampola fractional derivative approach [33], two-dimensional legendre wavelet method [34], fuzzy Laplace transform [35], fuzzy sumudu transform [36]. Further we can see [37–40]

Optimal Homotopy Asymptotic Method (OHAM) is one of the powerful techniques introduced by Marinca at al. [41–43] for approximate solution of differential equations. OHAM attracted an enormous importance in solving various problems in different field of science. Iqbal et al. applied this technique to Klein-Gordon equations and singular Lane-Emden type equation [44]. Sheikholeslami et al. used the proposed method for investigation of the laminar viscous flow and magneto hydrodynamic flow in a permeable channel [45]. Hashmi et al. obtained the solution of nonlinear Fredholm integral equations using OHAM [46]. Nawaz at al. applied the proposed method for solution of fractional order integro-differential equations [47], fractional order partial differential equations [48] and three-dimensional integral equations [49].

Aim of our study is to extend OHAM for solution of system of fuzzy Volterra integro differential equation of fractional order of the following form

$$D_x^{\alpha} u(x) = h(x) + \int_a^x k(x,t)u(t)dt, \quad 0 \le \alpha \le 1,\ x \in [0,1], \tag{1.1}$$

with the given initial condition

$$u^k(0),\ \ u_0{}^k(x),\ \ k=1,2,3,....,\eta-1,\ \ \eta-1<\alpha<\eta,\ \ \eta\in N,$$

Where D_x^α represents the fuzzy fractional derivative in Caputo sense for fractional order of α with respect to x, $h:[a,b]\to\mathbb{R}_{\mathcal{F}}$ is fuzzy valued function, $k(x,t)$ is arbitrary kernel $u_0(x)\in\mathbb{R}_{\mathcal{F}}$ is an unknown solution. $\mathbb{R}_{\mathcal{F}}$ represent set of all fuzzy valued function on real line.

The remaining paper is structured as follows: A brief overview on some elementary concept, notations and definitions of fuzzy calculus and fuzzy fractional calculus are discussed in section 2. Analysis of the technique is presented in section 3. Proposed method is applied to solve fuzzy fractional order Volterra integro-differential equations in section 4. Result and discussion of the paper is given in section 5 and section 6 is the conclusion of the paper.

2. Preliminaries

In literature there exist various definitions of fuzzy calculus and fuzzy fractional calculus [50]. Some elementary concept, notations and definitions of fuzzy calculus and fuzzy fractional calculus related to this study are provided in this section.

Definition 2.1. The Riemann-Liouville fractional integral operator I_x^α of order α is [50]:

$$I_x^\alpha u(x)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^x (x-t)^{\alpha-1}u(t)dt=0, & \alpha>0,\\ u(x), & \alpha=0.\end{cases} \tag{2.1}$$

Definition 2.2. Caputo partial fractional Derivative operator D_x^α of order α with respect to x is defined as follow [50]:

$$D_x^\alpha u(x)=\begin{cases}\dfrac{1}{\Gamma(\eta-\alpha)}\displaystyle\int_0^x (x-t)^{\eta-\alpha-1}u^{(n)}(t)dt=0, & \eta-1<\alpha\le\eta,\\ \dfrac{d^\eta u(x)}{dx^\eta}, & \alpha=\eta\in N.\end{cases} \tag{2.2}$$

which clearly shows that

$$D_x^\alpha I_x^\alpha u(x)=u(x) \tag{2.3}$$

Definition 2.3. A fuzzy number σ is a mapping $\sigma:\mathbb{R}\to[0,1]$, satisfy the following property:

a. σ is normal that is, $\exists\ x_0\in\mathbb{R}$ with $u(x_0)=1$ [51,52].

b. σ is a convex fuzzy set that is, $u(\lambda x+(1-\lambda)y)\ge\min\{u(x),u(y)\}$ for all $x,y\in\mathbb{R},\ \lambda\in[0,1]$.

c. σ is upper semi-continuous in $\mathbb{R}$.

d. $\overline{\{x\in\mathbb{R}:u(x)>0\}}$ is compact.

Definition 2.4. Parametric form of fuzzy number σ represented by an order pair $(\underline{\sigma},\overline{\sigma})$ of the function $(\underline{\sigma}(r),\overline{\sigma}(r))$, satisfies the following conditions [52,53]:

a. $\underline{\sigma}(r)$ is bounded monotonic increasing left continuous $\forall\ r \in [0,1]$.
b. $\bar{\sigma}(r)$ is bounded monotonic decreasing left continuous $\forall\ r \in [0,1]$.
c. $\underline{\sigma}(r) \le \bar{\sigma}(r)\ \forall\ r \in [0,1]$.

Definition 2.5. Addition and scalar multiplication of fuzzy number is given as:

a. $\left(\sigma_1 \oplus \sigma_2\right) = \left(\underline{\sigma}_1(r) + \underline{\sigma}_2(r), \bar{\sigma}_1(r) + \bar{\sigma}_2(r)\right)$

b. $\left(k \otimes \sigma\right) = \begin{cases} \left(\underline{\sigma}(r), \bar{\sigma}(r)\right), & k \ge 0, \\ \left(\underline{\sigma}(r), \bar{\sigma}(r)\right), & k < 0. \end{cases}$

Definition *2.6.* A fuzzy real valued function $\sigma_1, \sigma_2 : [a,b] \to \mathbb{R}$, then in [54]:

$$D_U(\sigma_1, \sigma_2) = \sup\left\{D(\sigma_1(x), \sigma_2(x)) \mid x \in [a,b]\right\}.$$

Definition 2.7. Assume $u : [a,b] \to \mathbb{R}_{\mathcal{F}}$. For every partition $\mathrm{P} = \left\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, ..., \sigma_n\right\}$ and arbitrary $\ell_i : \sigma_{i-1} \le \ell_i \le \sigma_i,\ \ 2 \le i \le n$ consider

$\mathbb{R}_p = \sum_{i=2}^{n} u(\ell_j)(\sigma_i - \sigma_{i-1})$. The definite integral of $u(x)$ over $[\alpha, \beta]$ is

$$\int_\alpha^\beta u(x)dx = \lim \mathbb{R}_\rho,$$

which show existence of limit in metric [55].
Definite integral exist if $u(x)$ is continuous in metric D [51]:

$$\underline{\left(\int_\alpha^\beta u(x)dx\right)} = \int_\alpha^\beta \underline{u}(x)dx,$$
$$\overline{\left(\int_\alpha^\beta u(x)dx\right)} = \int_\alpha^\beta \bar{u}(x)dxt.$$

3. Application of OHAM

By considering definition 2.4. as discussed in section 2, Eq (1.1) becomes:

$$\begin{cases} D_x^\alpha \underline{u}(x,r) - \underline{h}(x,r) - \int_a^x k(x,t)\underline{u}(t,r)dt = 0, \\ D_x^\alpha \bar{u}(x,r) - \underline{h}(x,r) - \int_a^x k(x,t)\bar{u}(t,r)dt = 0, \end{cases} \quad 0 \le \alpha \le 1, 0 \le r \le 1,\ \ x \in [0,1], \tag{3.1}$$

with the given initial condition

$$\left[u^k(0)\right]^r,\ \left(\underline{u}_0{}^k(x,r), \bar{u}_0{}^k(x,r)\right),\ k = 1,2,3,....,\eta - 1,\ \ \eta - 1 < \alpha < \eta,\ \ \eta \in N, \tag{3.2}$$

The homotopy of OHAM [41–43], constructed as follow:

$$\begin{cases} (1-\rho)\left(\dfrac{\partial^\alpha \underline{\upsilon}(x,r;\rho)}{\partial t^\alpha}-\underline{h}(x,r)\right)=\mathcal{H}(\rho)\left(\dfrac{\partial^\alpha \underline{\upsilon}(x,r;\rho)}{\partial t^\alpha}-\underline{h}(x,r)-\underline{\delta}(\underline{\upsilon},r)\right), \\ (1-\rho)\left(\dfrac{\partial^\alpha \overline{\upsilon}(x,r;\rho)}{\partial t^\alpha}-\overline{h}(x,r)\right)=\mathcal{H}(\rho)\left(\dfrac{\partial^\alpha \overline{\upsilon}(x,r;\rho)}{\partial t^\alpha}-\overline{h}(x,r)-\overline{\delta}(\overline{\upsilon},r)\right). \end{cases} \tag{3.3}$$

where $\rho\in[0,1]$, $\mathcal{H}(\rho)=\sum_{m\geq 1} c_m\rho^m$ for all $\rho\neq 0$ is an auxiliary function, if $\rho=0$ then $\mathcal{H}(0)=0$ where

$$\begin{cases} \underline{\upsilon}(x,r,0)=\underline{u}_0(x,r) & \underline{\upsilon}(x,r;1)=\underline{u}(x,r), \\ \overline{\upsilon}(x,r,0)=\overline{u}_0(x,r) & \overline{\upsilon}(x,r;1)=\overline{u}(x,r). \end{cases}$$

and c_m represent auxiliary constants. Using Taylor's series to expand $\upsilon(x,r;\rho)$ about ρ we get

$$\begin{cases} \underline{\upsilon}(x,r;\rho)=\underline{u}_0(x,r)+\sum_{m\geq 1}\underline{u}_m(x,r)\rho^m, \\ \overline{\upsilon}(x,r;\rho)=\overline{u}_0(x,r)+\sum_{m\geq 1}\overline{u}_m(x,r)\rho^m. \end{cases} \tag{3.4}$$

Inserting Eq (3.4) into Eq (3.3) we get series of the problems by comparing the like power of ρ given as follow:

$$\rho^0:\begin{cases} \underline{u}_0(x,r)-\underline{h}(x,r)=0, \\ \overline{u}_0(x,r)-\overline{h}(x,r)=0. \end{cases} \tag{3.5}$$

$$\rho^1:\begin{cases} \underline{u}_1(x,r)+c_1\delta(\underline{u}_0)+(1+c_1)+\underline{u}_0(x,r)=0, \\ \overline{u}_1(x,r)+c_1\delta(\overline{u}_0)+(1+c_1)+\overline{u}_0(x,r)=0. \end{cases} \tag{3.6}$$

$$\rho^2:\begin{cases} \underline{u}_2(x,r)+c_1\delta(\underline{u}_1)+c_2\delta(\underline{u}_0)+c_2(\underline{h}(x,r)-\underline{u}_0(x,r)) \\ -(1+c_1)\underline{u}_1(x,r)=0, \\ \overline{u}_2(x,r)+c_1\delta(\overline{u}_1)+c_2\delta(\overline{u}_0)+c_2(\overline{h}(x,r)-\overline{u}_0(x,r)) \\ -(1+c_1)\overline{u}_1(x,r)=0. \end{cases} \tag{3.7}$$

$$\rho^n:\begin{cases} \underline{u}_n(x,r)+c_1\delta(\underline{u}_n)+c_2\delta(\underline{u}_{n-1})+c_3(h+\delta(\underline{u}_0))\ldots-c_2\underline{u}_{n-1}(x,r)-(1+c_1)\underline{u}_n(x,r)=0, \\ \overline{u}_n(x,r)+c_1\delta(\overline{u}_n)+c_2\delta(\overline{u}_{n-1})+c_3(h+\delta(\overline{u}_0))\ldots-c_2\overline{u}_{n-1}(x,r)-(1+c_1)\overline{u}_n(x,r)=0. \end{cases} \tag{3.8}$$

For calculating the constants $c_1, c_2, c_3\ldots$, m^{th} order optimum solution becomes

$$\begin{cases} \underline{u}^m(x,r,c_l)=\underline{u}_0(x,r)+\sum_{k=1}^{m}\underline{u}_k(x,r,c_l), \quad l=1,2,3,\ldots m, \\ \overline{u}^m(x,r,c_l)=\overline{u}_0(x,r)+\sum_{k=1}^{m}\overline{u}_k(x,r,c_l), \quad l=1,2,3,\ldots m. \end{cases} \tag{3.9}$$

Putting Eq (3.9) into Eq (3.1), we can found our residual given as follow:

$$\begin{cases} \underline{R}\left(x,r;c_l\right) = \underline{u}^m(x,r;c_l) - \underline{h}(x,r) - \delta(\underline{u}),\ l = 1,2,... \\ \overline{R}\left(x,r;c_l\right) = \overline{u}^m(x,r;c_l) - \overline{h}(x,r) - \delta(\overline{u}),\ l = 1,2,... \end{cases} \tag{3.10}$$

If $R\left(x,r;c_l\right) = 0$, then $\underline{u}^m(x,r;c_l)$ & $\overline{u}^m(x,r;c_l)$ will be the exact solutions.

Optimum solution contains some auxiliary constants; the optimal values of these constants are obtained through various techniques. In the present work, we have used the least square method [56,57]. The method of least squares is a powerful technique for obtaining the values of auxiliary constants. By putting the optimal values of these constants in Eq (8), we obtain the OHAM solution.

4. Application and accuracy

Problem 4.1. Consider system of fuzzy fractional order Volterra integro-differential equation as [58]:

$$\begin{cases} D_x^\alpha \underline{u}(x,r) = \left(r-1\right) + \int\limits_0^x \underline{u}(t,r)dt \\ D_x^\alpha \overline{u}(x,r) = \left(1-r\right) + \int\limits_0^x \overline{u}(t,r)dt \end{cases},\quad 0 < \alpha \le 1,\ x \in [0,1], \tag{4.1}$$

subject to the fuzzy initial condition $\left[u(0)\right]^r = \left[r-1, 1-r\right]$, and for $\alpha = 1$ fuzzy fractional order Volterra integro-differential equations the exact solution is $\left[u(x)\right]^r = \left[r-1, 1-r\right] Sinh(x)$ and $0 \le r \le 1$.

By follow the technique as discussed in section 3, we get series of problems and their solutions as:

$$\begin{cases} D_x^{\alpha} \underline{u}_0(x,r) + (1-r) = 0, \\ D_x^{\alpha} \overline{u}_0(x,r) + (r-1) = 0. \end{cases} \tag{4.2}$$

$$\begin{cases} D_x^{\alpha} \underline{u}_1(x,r) - 1 + r - c_1 + rc_1 + \left(\int\limits_0^x \underline{u}_0(t,r)dt \right) c_1 - D_x^{\alpha} \underline{u}_0(x,r) - c_1 D_x^{\alpha} \underline{u}_0(x,r) = 0, \\ D_x^{\alpha} \overline{u}_1(x,r) + 1 - r + c_1 - rc_1 + \left(\int\limits_0^x \overline{u}_0(t,r)dt \right) c_1 - D_x^{\alpha} \overline{u}_0(x,r) - c_1 D_x^{\alpha} \overline{u}(x,r) = 0. \end{cases} \tag{4.3}$$

$$\begin{cases} D_x^{\alpha} \underline{u}_2(x,r) + \left(\int\limits_0^x \underline{u}_1(t,r)dt \right) c_1 - c_2 + rc_2 + \left(\int\limits_0^x \underline{u}_0(t,r)dt \right) c_2 - c_2 D_x^{\alpha} \underline{u}_0(x,r) - \\ D_x^{\alpha} \underline{u}_1(x,r) - c_1 D_x^{\alpha} \underline{u}_0(x,r) = 0, \\ D_x^{\alpha} \overline{u}_2(x,r) + \left(\int\limits_0^x \overline{u}_1(t,r)dt \right) c_1 + c_2 - rc_2 + \left(\int\limits_0^x \overline{u}_0(t,r)dt \right) c_2 - c_2 D_x^{\alpha} \overline{u}_0(x,r) - \\ D_x^{\alpha} \overline{u}_1(x,r) - c_1 D_x^{\alpha} \overline{u}_0(x,r) = 0. \end{cases} \tag{4.4}$$

$$\begin{cases} D_x^{\alpha}\underline{u}_3(x,r)+\left(\int_0^x \underline{u}_2(t,r)dt\right)c_1+\left(\int_0^x \underline{u}_1(t,r)dt\right)c_2-c_3+rc_3+\left(\int_0^x \underline{u}_0(t,r)dt\right)c_3-c_3D_x^{\alpha}\underline{u}_0(x,r) \\ -c_2D_x^{\alpha}\underline{u}_1(x,r)-D_x^{\alpha}\underline{u}_2(x,r)-c_1D_x^{\alpha}\underline{u}_2(x,r)=0, \\ D_x^{\alpha}\overline{u}_3(x,r)+\left(\int_0^x \overline{u}_2(t,r)dt\right)c_1+\left(\int_0^x \overline{u}_1(t,r)dt\right)c_2+c_3-rc_3+\left(\int_0^x \overline{u}_0(t,r)dt\right)c_3-c_3D_x^{\alpha}\overline{u}_0(x,r) \\ -c_2D_x^{\alpha}\overline{u}_1(x,r)-D_x^{\alpha}\overline{u}_2(x,r)-c_1D_x^{\alpha}\overline{u}_2(x,r)=0. \end{cases} \tag{4.5}$$

Their solutions are

$$\begin{cases} \underline{u}_0(x,r)=\dfrac{(-1+r)x^{\alpha}}{\alpha\Gamma(\alpha)} \\ \overline{u}_0(x,r)=-\dfrac{(-1+r)x^{\alpha}}{\alpha\Gamma(\alpha)} \end{cases}, \tag{4.6}$$

$$\begin{cases} \underline{u}_1(x,r)=-\dfrac{(-1+r)x^{1+2\alpha}c_1}{\Gamma(2+2\alpha)}, \\ \overline{u}_1(x,r)=\dfrac{(-1+r)x^{1+2\alpha}c_1}{\Gamma(2+2\alpha)}. \end{cases} \tag{4.7}$$

$$\begin{cases} \underline{u}_2(x,r)=(-1+r)x^{1+2\alpha}\left(\dfrac{x^{1+\alpha}c_1^2}{\Gamma(3+3\alpha)}-\dfrac{c_1+c_1^2+c_2}{\Gamma(2+2\alpha)}\right), \\ \overline{u}_2(x,r)=(-1+r)x^{1+\alpha}\left(-\dfrac{x^{1+\alpha}c_1^2}{\Gamma(3+3\alpha)}+\dfrac{c_1+c_1^2+c_2}{\Gamma(2+2\alpha)}\right). \end{cases} \tag{4.8}$$

$$\begin{cases} \underline{u}_3(x,r)=(-1+r)x^{1+2\alpha}\left(-\dfrac{x^{2+2\alpha}c_1^3}{\Gamma(4+4\alpha)}+\dfrac{2x^{1+\alpha}c_1(c_1+c_1^2+c_2)}{\Gamma(3+3\alpha)}-\dfrac{c_1+2c_1^2+c_1^3+c_2+2c_1c_2+c_3}{\Gamma(2+2\alpha)}\right), \\ \overline{u}_3(x,r)=(-1+r)x^{1+2\alpha}\left(\dfrac{x^{2+2\alpha}c_1^3}{\Gamma(4+4\alpha)}-\dfrac{2x^{1+\alpha}c_1(c_1+c_1^2+c_2)}{\Gamma(3+3\alpha)}+\dfrac{c_1+2c_1^2+c_1^3+c_2+2c_1c_2+c_3}{\Gamma(2+2\alpha)}\right). \end{cases} \tag{4.9}$$

Adding (4.6), (4.7), (4.8) and (4.9), one can construct $\underline{u}(x,r)\,\&\,\overline{u}(x,r)$:

$$\begin{cases} \underline{u}(x,r)=(-1+r)x^{\alpha}\left(\begin{array}{l}\dfrac{1}{\Gamma(1+\alpha)}-\dfrac{x^{3+3\alpha}c_1^3}{\Gamma(4+4\alpha)}+\dfrac{x^{2+2\alpha}c_1(c_1(3+2c_1)+2c_2)}{\Gamma(3+3\alpha)}- \\ \dfrac{x^{1+\alpha}(2c_2+c_1(3+c_1(3+c_1)+2c_2)+c_3)}{\Gamma(2+2\alpha)}\end{array}\right), \\ \overline{u}(x,r)=(-1+r)x^{\alpha}\left(\begin{array}{l}-\dfrac{1}{\Gamma(1+\alpha)}+\dfrac{x^{3+3\alpha}c_1^3}{\Gamma(4+4\alpha)}-\dfrac{x^{2+2\alpha}c_1(c_1(3+2c_1)+2c_2)}{\Gamma(3+3\alpha)}+ \\ \dfrac{x^{1+\alpha}(2c_2+c_1(3+c_1(3+c_1)+2c_2)+c_3)}{\Gamma(2+2\alpha)}\end{array}\right). \end{cases} \tag{4.10}$$

Values of c_1, c_2 and c_3 contain is in Eq (4.10)

Substituting the values from Table 1 into Eq (4.10), the approximate solutions for $\underline{u}(x,r)$ & $\overline{u}(x,r)$ at different values of α taking $r=0.75$ respectively is as follow

Table 1. at r=0.75.

α	$\underline{c}_1$ & $\overline{c}_1$	$\underline{c}_2$ & $\overline{c}_2$	$\underline{c}_3$ & $\overline{c}_3$
0.7	-1.0257850714449026	$5.298291106236844\times10^{-4}$	$-3.040859671410477\times10^{-5}$
0.8	-1.0193406988378892	$3.249294721058776\times10^{-4}$	$-1.5364294422415488\times10^{-5}$
0.9	-1.014446487354385	$1.9694362983845834\times10^{-4}$	$-7.63055570435551\times10^{-6}$
1	-1.0107450504316333	$1.1791102776455743\times10^{-4}$	$-3.779171763451589\times10^{-6}$

$\alpha=0.7$

$$\begin{cases}\underline{u}(x,r)\approx-0.2751369x^{0.7}-0.08602097x^{2.4}-0.25x^{2.4}(-0.00904995+0.0376716x^{1.7})\\ -0.25x^{2.4}(0.000425859-0.00198165x^{1.7}+0.0021734872x^{3.4}),\\ \overline{u}(x,r)\approx 0.2751369x^{0.7}+0.08602097x^{2.4}-0.25x^{2.4}(0.00904995-0.0376716x^{1.7})-\\ 0.25x^{2.4}(-0.000425859+0.00198165x^{1.7}-0.0021734872x^{3.4}).\end{cases}\tag{4.11}$$

$\alpha=0.8$

$$\begin{cases}\underline{u}(x,r)\approx-0.2684178x^{0.8}-0.0685589x^{2.6}-0.25x^{2.6}(-0.005391327+0.023297809x^{1.8})\\ -0.25x^{2.6}(0.000197513-0.0009160448x^{1.8}+0.001008410504x^{3.6}),\\ \overline{u}(x,r)\approx 0.2684178x^{0.8}+0.0685589x^{2.6}-0.25x^{2.6}(0.0053913-0.023297809x^{1.8})\\ -0.25x^{2.6}(-0.000197513+0.0009160448x^{1.8}-0.001008410504x^{3.6}).\end{cases}\tag{4.12}$$

$\alpha=0.9$

$$\begin{cases}\underline{u}(x,r)\approx-0.2599385x^{0.9}-0.0540269x^{2.8}-0.25x^{2.8}(-0.0031639+0.01418901x^{1.9})\\ -0.25x^{2.8}(0.00008989-0.0004154745x^{1.9}+0.000458881x^{3.8}),\\ \overline{u}(x,r)\approx 0.2599385x^{0.9}+0.0540269x^{2.8}-0.25x^{2.8}(0.0031639-0.014189097x^{1.9})\\ -0.25x^{2.8}(-0.00008989+0.0004154745x^{1.9}-0.000458881x^{3.8}).\end{cases}\tag{4.13}$$

$\alpha=1$

$$\begin{cases}\underline{u}(x,r)\approx-0.25x-0.042114377x^{3}-0.25x^{3}(-0.001829736+0.0085133796x^{2})\\ -0.25x^{3}(0.0000401535-0.0001849397x^{2}+0.00020487753x^{4}),\\ \overline{u}(x,r)\approx 0.25x+0.042114377x^{3}-0.25x^{3}(0.001829736-0.0085133796x^{2})\\ -0.25x^{3}(-0.0000401535+0.0001849397x^{2}-0.0002048775x^{4}).\end{cases}\tag{4.14}$$

Substituting the values from Table 2 into Eq (4.10), the approximate solutions for $\underline{u}(x,r)$ & $\overline{u}(x,r)$ at different values of α taking $r=0.5$ respectively is as follow

Table 2. at r=0.5.

α	$\underline{c}_1$ & $\bar{c}_1$	$\underline{c}_2$ & $\bar{c}_2$	$\underline{c}_3$ & $\bar{c}_3$
0.7	-1.0257850714449026	$5.298291106236844\times10^{-4}$	$-3.040859671410477\times10^{-5}$
0.8	-1.0193406988378892	$3.249294721058776\times10^{-4}$	$-1.5364294422415488\times10^{-5}$
0.9	-1.014446487354385	$1.9694362983845834\times10^{-4}$	$-7.630555570435551\times10^{-6}$
1	-1.0107453381292266	$1.1800167363027721\times10^{-4}$	$-3.726389827252244\times10^{-6}$

$\alpha = 0.7$

$$\begin{cases} \underline{u}(x,r) \approx -0.5502737x^{0.7} - 0.172041940x^{2.4} - 0.5x^{2.4}(-0.00904995 + 0.0376716x^{1.7}) \\ -0.5x^{2.4}(0.0004258594 - 0.0019816476x^{1.7} + 0.0021734872x^{3.4}), \\ \bar{u}(x,r) \approx 0.5502737x^{0.7} + 0.172041940x^{2.4} - 0.5x^{2.4}(0.00904995 - 0.03767164x^{1.7}) \\ -0.5x^{2.4}(-0.0004258594 + 0.0019816476x^{1.7} - 0.0021734872x^{3.4}). \end{cases} \tag{4.15}$$

$\alpha = 0.8$

$$\begin{cases} \underline{u}(x,r) \approx -0.53683564x^{0.8} - 0.137117858x^{2.6} - 0.5x^{2.6}(-0.00539133 + 0.02329781x^{1.8}) \\ -0.5x^{2.6}(0.00019751 - 0.0009160448x^{1.8} + 0.0010084105x^{3.6}), \\ \bar{u}(x,r) \approx 0.53683564x^{0.8} + 0.137117858x^{2.6} - 0.5x^{2.6}(0.00539133 - 0.02329781x^{1.8}) \\ -0.5x^{2.6}(-0.00019751 + 0.0009160448x^{1.8} - 0.0010084105x^{3.6}). \end{cases} \tag{4.16}$$

$\alpha = 0.9$

$$\begin{cases} \underline{u}(x,r) \approx -0.5198771x^{0.9} - 0.10805378x^{2.8} - 0.5x^{2.8}(-0.0031640 + 0.0141891x^{1.9}) \\ -0.5x^{2.8}(0.0000898945 - 0.0004154745x^{1.9} + 0.000458881x^{3.8}), \\ \bar{u}(x,r) \approx 0.5198771x^{0.9} + 0.10805378x^{2.8} - 0.5x^{2.8}(0.0031640 - 0.0141891x^{1.9}) \\ -0.5x^{2.8}(-0.0000898945 + 0.0004154745x^{1.9} - 0.000458881x^{3.8}). \end{cases} \tag{4.17}$$

$\alpha = 1$

$$\begin{cases} \underline{u}(x,r) \approx -0.5x - 0.0842288x^{3} - 0.5x^{3}(-0.0018298 + 0.008513x^{2}) - 0.5x^{3} \\ (0.0000401612 - 0.00018494622x^{2} + 0.0002048777x^{4}), \\ \bar{u}(x,r) \approx 0.5x + 0.0842288x^{3} - 0.5x^{3}(0.0018298 - 0.008513x^{2}) - 0.5x^{3} \\ (-0.000040162 + 0.00018494622x^{2} - 0.0002048777x^{4}). \end{cases} \tag{4.18}$$

Problem 4.2. Consider system of fuzzy fractional order Volterra integro-differential equation as [59]:

$$D_x^{\alpha} u(x,r) + \int_0^t u(t,r)dt = 0, \quad 0 < \alpha \le 1, \ x \in [0,1], \tag{4.19}$$

subject to the fuzzy initial condition $[u(0)]^r = [r-1, 1-r]$, and the exact solution is $\underline{u}(x,r) = (r-1)E_{\alpha+1}\left(-t^{\alpha+1}\right), \bar{u}(x,r) = (1-r)E_{\alpha+1}\left(-t^{\alpha+1}\right)$,

where $E_{\alpha+1}$ is a Mittag-Leffler function and $0 \le r \le 1$.
By follow the technique as discussed in section 3, we get series of problems and their solutions as:

$$\begin{cases} D_x^{\alpha} \underline{u}_0(x,r) = 0, \\ D_x^{\alpha} \overline{u}_0(x,r) = 0. \end{cases} \tag{4.20}$$

$$\begin{cases} D_x^{\alpha} \underline{u}_1(x,r) + \left(-\int_0^x \underline{u}_0(t,r)dt \right) c_1 - D_x^{\alpha} \underline{u}_0(x,r) - c_1 D_x^{\alpha} \underline{u}_0(x,r) = 0, \\ D_x^{\alpha} \overline{u}_1(x,r) + \left(-\int_0^x \overline{u}_0(t,r)dt \right) c_1 - D_x^{\alpha} \overline{u}_0(x,r) - c_1 D_x^{\alpha} \overline{u}(x,r) = 0. \end{cases} \tag{4.21}$$

$$\begin{cases} D_x^{\alpha} \underline{u}_2(x,r) + \left(\int_0^x \underline{u}_1(t,r)dt \right) c_1 - \left(\int_0^x \underline{u}_0(t,r)dt \right) c_2 - c_2 D_x^{\alpha} \underline{u}_0(x,r) - D_x^{\alpha} \underline{u}_1(x,r) - c_1 D_x^{\alpha} \underline{u}_0(x,r) = 0, \\ D_x^{\alpha} \overline{u}_2(x,r) + \left(-\int_0^x \overline{u}_1(t,r)dt \right) c_1 - \left(\int_0^x \overline{u}_0(t,r)dt \right) c_2 - c_2 D_x^{\alpha} \overline{u}_0(x,r) - D_x^{\alpha} \overline{u}_1(x,r) - c_1 D_x^{\alpha} \overline{u}_0(x,r) = 0. \end{cases} \tag{4.22}$$

$$\begin{cases} D_x^{\alpha} \underline{u}_3(x,r) - \left(\int_0^x \underline{u}_2(t,r)dt \right) c_1 - \left(\int_0^x \underline{u}_1(t,r)dt \right) c_2 - \left(\int_0^x \underline{u}_0(t,r)dt \right) c_3 - c_3 D_x^{\alpha} \underline{u}_0(x,r) \\ -c_2 D_x^{\alpha} \underline{u}_1(x,r) - D_x^{\alpha} \underline{u}_2(x,r) - c_1 D_x^{\alpha} \underline{u}_2(x,r) = 0, \\ D_x^{\alpha} \overline{u}_3(x,r) - \left(\int_0^x \overline{u}_2(t,r)dt \right) c_1 - \left(\int_0^x \overline{u}_1(t,r)dt \right) c_2 - \left(\int_0^x \overline{u}_0(t,r)dt \right) c_3 - c_3 D_x^{\alpha} \overline{u}_0(x,r) \\ -c_2 D_x^{\alpha} \overline{u}_1(x,r) - D_x^{\alpha} \overline{u}_2(x,r) - c_1 D_x^{\alpha} \overline{u}_2(x,r) = 0. \end{cases} \tag{4.23}$$

And their solutions are

$$\begin{cases} \underline{u}_0(x,r) = r - 1, \\ \overline{u}_0(x,r) = 1 - r. \end{cases} \tag{4.24}$$

$$\begin{cases} \underline{u}_1(x,\underline{r}) = \dfrac{(-1+r)x^{1+\alpha} c_1}{(\alpha+\alpha^2)\Gamma(\alpha)}, \\ \bar{u}_1(x,\bar{r}) = -\dfrac{(-1+r)x^{1+\alpha} c_1}{\alpha(1+\alpha)\Gamma(\alpha)}, \end{cases} \tag{4.25}$$

$$\begin{cases} \underline{u}_2(x,r) = (-1+r)x^{1+\alpha} \left(\dfrac{x^{1+\alpha} c_1^2}{\Gamma(3+2\alpha)} + \dfrac{c_1 + c_1^2 + c_2}{\Gamma(2+\alpha)} \right), \\ \overline{u}_2(x,r) = (-1+r)x^{1+\alpha} \left(-\dfrac{x^{1+\alpha} c_1^2}{\Gamma(3+2\alpha)} - \dfrac{c_1 + c_1^2 + c_2}{\Gamma(2+\alpha)} \right). \end{cases} \tag{4.26}$$

$$\begin{cases} \underline{u}_3(x,r) = \dfrac{(-1+r)x^{1+\alpha}\left(c_2 + c_1\left(1 + \dfrac{x^{2+2\alpha}c_1^2}{\Gamma(4+2\alpha)} + c_1(2+c_1) + 2c_2 + \dfrac{2x^{1+\alpha}(c_1+c_1^2+c_2)}{\Gamma(3+3\alpha)}\right) + c_3\right)}{\Gamma(1+\alpha)}, \\ \overline{u}_3(x,r) = (-1+r)x^{1+\alpha}\left(-\dfrac{x^{2+2\alpha}c_1^3}{\Gamma(4+3\alpha)} - \dfrac{2x^{1+\alpha}c_1(c_1+c_1^2+c_2)}{\Gamma(3+2\alpha)} - \dfrac{c_2 + c_1((1+c_1)^2 + 2c_2) + c_3}{\Gamma(2+\alpha)}\right), \end{cases} \tag{4.27}$$

Adding (4.24), (4.25), (4.26) and (4.27), one can construct $\underline{u}(x,r)$ & $\overline{u}(x,r)$:

$$\begin{cases} \underline{u}(x,r) = -1 + r + (-1+r)x^{1+\alpha}\left(\begin{array}{l} \dfrac{x^{1+\alpha}c_1^2}{\Gamma(3+2\alpha)} + \dfrac{x^{2+2\alpha}c_1^3}{\Gamma(1+\alpha)\Gamma(4+2\alpha)} + \dfrac{2x^{1+\alpha}c_1(c_1+c_1^2+c_2)}{\Gamma(1+\alpha)\Gamma(3+\alpha)} \\ + \dfrac{c_1(2+c_1)+c_2}{\Gamma(2+\alpha)} + \dfrac{c_2 + c_1((1+c_1)^2 + 2c_2) + c_3}{\Gamma(1+\alpha)} \end{array}\right), \\ \overline{u}(x,r) = 1 - r + (-1+r)x^{1+\alpha}\left(\begin{array}{l} -\dfrac{x^{2+2\alpha}c_1^3}{\Gamma(4+3\alpha)} - \dfrac{x^{1+\alpha}c_1(c_1(3+2c_1)+2c_2)}{\Gamma(3+2\alpha)} - \\ \dfrac{2c_2 + c_1(3 + c_1(3+c_1) + 2c_2) + c_3}{\Gamma(2+\alpha)} \end{array}\right). \end{cases} \tag{4.28}$$

Values of c_1, c_2 and c_3 contain in Eq (4.28)

Substituting the values from Tables 3 and 4 into Eq (4.28), the approximate solutions for $\underline{u}(x,r)$ & $\overline{u}(x,r)$ at different values of α taking $r = 0.5$ is as follow

Table 3. at r=0.5.

α	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}_3$
0.2	−0.8038238618267683	$7.6178003377104005\times10^{-3}$	$-1.334733828882206\times10^{-3}$
0.4	−0.739676946061329	0.02530379950927192	$-3.78307542584476\times10^{-3}$
0.6	−0.6725325561865596	0.04771922184372877	$-8.140310215705777\times10^{-3}$
0.8	−0.6062340661192892	0.06889015391501904	−0.015746150088743013
1.0	−0.5432795308983783	0.08615024033359142	−0.026582381449582644

Table 4. at r=0.5.

α	$\overline{c}_1$	$\overline{c}_2$	$\overline{c}_3$
0.2	−0.9072542694138958	$3.5727393445527333\times10^{-3}$	$3.637119200533134\times10^{-4}$
0.4	−0.9409187563211361	$1.9803045412863643\times10^{-3}$	$1.7807647588483674\times10^{-4}$
0.6	−0.9636043521097131	$9.808463578935658\times10^{-4}$	$7.369503633679291\times10^{-5}$
0.8	−0.9781782948007163	$4.4492367269725435\times10^{-4}$	$2.6729929232143615\times10^{-5}$
1.0	−0.9872029432879605	$1.896941835601795\times10^{-4}$	$1.021455544474314\times10^{-5}$

$\alpha = 0.2$

$$\begin{cases} \underline{u}(x,r) \approx -0.5 - 0.5x^{1.2}(-0.905948 + 0.325139x^{1.2} - 0.055807x^{2.4}), \\ \overline{u}(x,r) \approx 0.5 - 0.5x^{1.2}(0.905948 - 0.325139x^{1.2} + 0.055807x^{2.4}). \end{cases} \tag{4.29}$$

$\alpha = 0.4$

$$\begin{cases} \underline{u}(x,r) \approx -0.5 - 0.5x^{1.4}(-0.804545 + 0.210093x^{1.4} - 0.025570x^{2.8}), \\ \overline{u}(x,r) \approx 0.5 - 0.5x^{1.4}(0.80454545 - 0.210093x^{1.4} + 0.025570x^{2.8}). \end{cases} \tag{4.30}$$

$\alpha = 0.6$

$$\begin{cases} \underline{u}(x,r) \approx -0.5 - 0.5x^{1.6}(-0.699349 + 0.128177x^{1.6} - 0.010450x^{3.2}), \\ \overline{u}(x,r) \approx 0.5 - 0.5x^{1.6}(0.699349 - 0.128177x^{1.6} + 0.010450x^{3.2}). \end{cases} \tag{4.31}$$

$\alpha = 0.8$

$$\begin{cases} \underline{u}(x,r) \approx -0.5 - 0.5x^{1.8}(-0.596450 + 0.074561x^{1.8} - 0.0038863x^{3.6}), \\ \overline{u}(x,r) \approx 0.5 - 0.5x^{1.8}(0.5964503 - 0.074561x^{1.8} + 0.0038863x^{3.6}). \end{cases} \tag{4.32}$$

$\alpha = 1$

$$\begin{cases} \underline{u}(x,r) \approx -0.5 - 0.5x^{2}(-0.499992 + 0.04163089x^{2} - 0.001336253x^{4}), \\ \overline{u}(x,r) \approx 0.5 - 0.5x^{2}(0.4999914 - 0.04163076x^{2} + 0.001336247x^{4}). \end{cases} \tag{4.33}$$

Substituting the values from Tables 5 and 6 into Eq (4.28), the approximate solutions for $\underline{u}(x,r)$ & $\overline{u}(x,r)$ at different values of r taking $\alpha = 0.5$ is as follow

Table 5. at α =0.5.

r	$\underline{c}_1$	$\underline{c}_2$	$\underline{c}_3$
0	–0.7062087686601037	0.036389918752432 72	$-5.6065011933001474\times10^{-3}$
0.2	–0.7062087687083373	0.03638991875755982	$-5.606501189519195\times10^{-3}$
0.4	–0.7062087686911187	0.03638991875566347	$-5.606501190876118\times10^{-3}$
0.6	–0.7062087687083373	0.03638991875755982	$-5.606501189519195\times10^{-3}$
0.8	–0.7062087686312865	0.036389918749298394	$-5.606501195566148\times10^{-3}$

Table 6. at α =0.5.

r	$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3$
0	–0.9534544876637709	$1.4110239362094313\times10^{-3}$	$1.1669890178958486\times10^{-4}$
0.2	–0.9534544876175205	$1.4110239407705756\times10^{-3}$	$1.1669890245071123\times10^{-4}$
0.4	–0.9534544874104965	$1.4110239614371703\times10^{-3}$	$1.1669890547775563\times10^{-4}$
0.6	–0.9534544876175205	$1.4110239407705756\times10^{-3}$	$1.1669890245071123\times10^{-4}$
0.8	–0.9534544876289686	$1.4110239398903034\times10^{-3}$	$1.1669890235330204\times10^{-4}$

$r=0$

$$\begin{cases} \underline{u}(x,r)\approx -1-x^{1.5}(-0.75199032254+0.165168588631x^{1.5}-0.016559344247x^{3.}), \\ \overline{u}(x,r)\approx 1-x^{1.5}(0.75199032256-0.165168588653x^{1.5}+0.016559344262x^{3.}). \end{cases} \tag{4.34}$$

$r=0.2$

$$\begin{cases} \underline{u}(x,r)\approx -0.8-0.8x^{1.5}(-0.751990322550+0.165168588x^{1.5}-0.01655934425x^{3.}), \\ \overline{u}(x,r)\approx 0.8-0.8x^{1.5}(0.751990322554-0.165168588x^{1.5}+0.01655934426x^{3.}). \end{cases} \tag{4.35}$$

$r=0.4$

$$\begin{cases} \underline{u}(x,r)\approx -0.6-0.6x^{1.5}(-0.7519903225+0.1651685886x^{1.5}-0.016559344250x^{3.}), \\ \overline{u}(x,r)\approx 0.6-0.6x^{1.5}(0.7519903225-0.1651685886x^{1.5}+0.0165593442496x^{3.}). \end{cases} \tag{4.36}$$

$r=0.6$

$$\begin{cases} \underline{u}(x,r)\approx -0.4-0.4x^{1.5}(-0.751990322550+0.16516858863x^{1.5}-0.01655934425x^{3.}), \\ \overline{u}(x,r)\approx 0.4-0.4x^{1.5}(0.751990322554-0.16516858865x^{1.5}+0.01655934426x^{3.}). \end{cases} \tag{4.37}$$

$r=0.8$

$$\begin{cases} \underline{u}(x,r)\approx -0.1999910-0.1999910x^{1.5}(-0.7519903+0.16516859x^{1.5}-0.0165593x^{3.}), \\ \overline{u}(x,r)\approx 0.19999910-0.19999910x^{1.5}(0.7519903-0.16516859x^{1.5}+0.0165593x^{3.}). \end{cases} \tag{4.38}$$

5. Result and discussion

Tables 1–6 show the values of auxiliary constant at different values of $r\,\&\,\alpha$ for both lower and upper solution of OHAM for the solved problems. Tables 7 and 8 show the comparison of absolute error of 3rd order OHAM with Fractional Residual Power Series (FRPS) Method for 5-approximated solution and $k=5$ for both lower and upper solutions of OHAM at different value of α for problem 1. Comparison of absolute error of 3rd orders OHAM for both lower and upper solution of OHAM are shown in Tables 9 and 10. Numerical result show that OHAM provide more accuracy as compared to the other method and as $\alpha \to 1$ the approximate solution become very close to the exact solution. Graphical representation confirmed the convergence of fractional order solution towards the integer order solution. In Figure 1 graphical representation of OHAM at $\alpha = 0.7, 0.8, 0.9, 1$, $r = 0.75$ and $\alpha = 0.7, 0.8, 0.9, 1$, $r = 0.50$ are discussed for both $\underline{u}(x,r)$ & $\bar{u}(x,r)$ for problem 1. Figures 2 and 3 show the comparison of OHAM with the exact solution at different values of and taking *r=0.75* & *r=0.5* respectively for problem 1. Figure 4 represent the comparison of OHAM at $\alpha = 0.2, 0.4, 0.6, 0.8, 1$, $r = 0.5$ and $r = 0, 0.2, 0.4, 0.6, 0.8$, $\alpha = 0.5$ for both $\underline{u}(x,r)$ and $\bar{u}(x,r)$ for problem 2. Figure 5 shows the comparison of OHAM with the exact solution at different values of and *r=0.5* while Figure 6 shows the comparison of OHAM with the exact solution at different values of r and *=0.5* for problem 2.

Table 7. Comparison of Absolute Error (Abs Err.) of 3rd order OHAM for $\underline{u}(x,r)$ and Fractional Residual Power Series (FRPS) [54] Method for 5-approximated solution and $k=5$ for problem 1.

r	x	FRPS [58] $\alpha=0.7$	OHAM	FRPS [58] $\alpha=0.8$	OHAM	FRPS [58] $\alpha=0.9$	OHAM	FRPS [58] $\alpha=1$	OHAM
0.75	0.2	0.042797	0.040621	0.025514	0.024764	0.011512	0.011321	6.35273×10^{-10}	2.14676×10^{-9}
	0.4	0.059664	0.051698	0.035840	0.032584	0.016392	0.015405	8.14507×10^{-8}	1.00832×10^{-8}
	0.6	0.075769	0.058997	0.045171	0.037635	0.020545	0.018031	1.39554×10^{-6}	1.11336×10^{-8}
	0.8	0.094364	0.066136	0.055863	0.04232	0.025182	0.020362	1.04955×10^{-5}	6.21567×10^{-9}
0.50	0.2	0.085595	0.081241	0.051027	0.049528	0.011321	0.022643	1.27055×10^{-9}	4.25946×10^{-9}
	0.4	0.119328	0.103396	0.071680	0.065167	0.015405	0.030811	1.62901×10^{-7}	1.98877×10^{-8}
	0.6	0.151537	0.117994	0.090342	0.075269	0.018031	0.036062	2.79107×10^{-6}	2.12923×10^{-8}
	0.8	0.188728	0.132271	0.111723	0.084640	0.020362	0.040725	2.09911×10^{-5}	1.00120×10^{-8}

Table 8. Comparison of Absolute Error (Abs Err.) of 3rd order OHAM for $\bar{u}(x,r)$ and Fractional Residual Power Series (FRPS) [58] Method for 5-approximated solution and $k=5$ for problem 1.

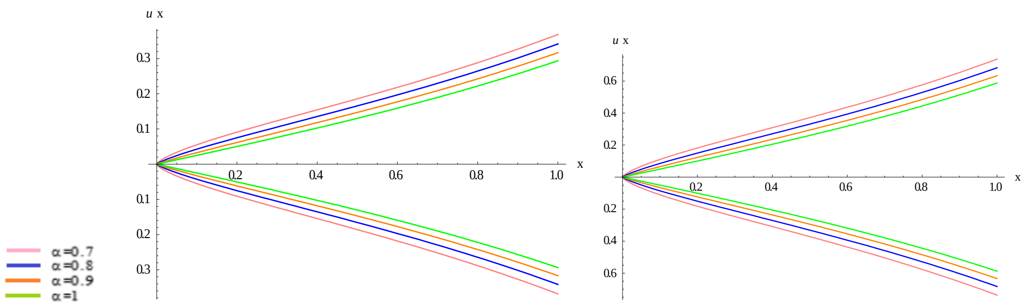
r	x	FRPS [58] $\alpha=0.7$	OHAM	FRPS [58] $\alpha=0.8$	OHAM	FRPS [58] $\alpha=0.9$	OHAM	FRPS [58] $\alpha=1$	OHAM
0.75	0.2	0.085595	0.081242	0.025514	0.024764	0.011512	0.011321	6.35273×10^{-10}	2.14676×10^{-9}
	0.4	0.119328	0.103396	0.035840	0.032584	0.016392	0.015405	8.14507×10^{-8}	1.00832×10^{-8}
	0.6	0.151537	0.117994	0.045171	0.037635	0.020545	0.018031	1.39554×10^{-6}	1.11336×10^{-8}
	0.8	0.188728	0.132271	0.055862	0.04232	0.025182	0.020362	1.04955×10^{-5}	6.21567×10^{-9}
0.50	0.2	0.042797	0.040621	0.051027	0.049528	0.023025	0.022643	1.27055×10^{-9}	4.25946×10^{-9}
	0.4	0.059664	0.051698	0.071680	0.065167	0.032784	0.030811	1.62901×10^{-7}	1.98877×10^{-8}
	0.6	0.075769	0.058997	0.090342	0.075269	0.041090	0.036062	2.79107×10^{-6}	2.12923×10^{-8}
	0.8	0.094364	0.066136	0.111723	0.084640	0.050364	0.040725	2.09911×10^{-5}	1.0012×10^{-8}

Table 9. Comparison of Absolute Error (Abs Err.) of 3rd order OHAM for $\underline{u}(x,\underline{r})$ & $\overline{u}(x,r)$ at different values of α taking $r = 0.5$ for problem 2.

x	$\underline{u}(x,r)$ $\alpha = 0.4$	$\overline{u}(x,r)$	$\underline{u}(x,r)$ $\alpha = 0.6$	$\overline{u}(x,r)$	$\underline{u}(x,r)$ $\alpha = 0.8$	$\overline{u}(x,r)$	$\underline{u}(x,r)$ $\alpha = 1$	$\overline{u}(x,r)$
0.2	1.2736×10^{-5}	1.2736×10^{-5}	3.2527×10^{-6}	3.2527×10^{-6}	6.93075×10^{-7}	6.93076×10^{-7}	1.29476×10^{-7}	1.44585×10^{-7}
0.4	2.3337×10^{-6}	2.3337×10^{-6}	2.4426×10^{-6}	2.4426×10^{-6}	9.50573×10^{-7}	9.50573×10^{-7}	2.67538×10^{-7}	3.26771×10^{-7}
0.6	9.9993×10^{-6}	9.9993×10^{-6}	1.8451×10^{-6}	1.8451×10^{-6}	3.8544×10^{-8}	3.85438×10^{-8}	1.10198×10^{-7}	2.39031×10^{-7}
0.8	4.1251×10^{-6}	4.1251×10^{-6}	1.3416×10^{-7}	1.3416×10^{-7}	6.55017×10^{-8}	6.55017×10^{-8}	9.64628×10^{-9}	2.27883×10^{-7}
1.0	1.312×10^{-5}	1.312×10^{-5}	1.8373×10^{-6}	1.8373×10^{-6}	5.49341×10^{-8}	5.4934×10^{-8}	7.7356×10^{-8}	3.9735×10^{-7}

Table 10. Comparison of Absolute Error (Abs Err.) of 3rd order OHAM for $\underline{u}(x,r)$ & $\overline{u}(x,r)$ at different values of r taking $\alpha = 0.5$ for problem 2.

x	$\underline{u}(x,r)$ $r = 0.4$	$\overline{u}(x,r)$	$\underline{u}(x,r)$ $r = 0.6$	$\overline{u}(x,r)$	$\underline{u}(x,r)$ $r = 0.8$	$\overline{u}(x,r)$	$\underline{u}(x,r)$ $r = 1$	$\overline{u}(x,r)$
0.	0.	0.	0.	0.	0.	0.	0.	0.
0.2	1.0578×10^{-5}	1.0578×10^{-5}	7.9338×10^{-6}	7.9338×10^{-6}	5.28921×10^{-6}	5.28921×10^{-6}	2.64461×10^{-6}	2.64461×10^{-6}
0.4	4.8940×10^{-6}	4.8940×10^{-6}	3.6705×10^{-6}	3.6705×10^{-6}	2.44698×10^{-6}	2.44698×10^{-6}	1.22349×10^{-6}	1.22349×10^{-6}
0.6	7.4825×10^{-6}	7.4825×10^{-6}	5.6119×10^{-6}	5.6119×10^{-6}	3.74125×10^{-6}	3.74125×10^{-6}	1.87062×10^{-6}	1.87062×10^{-6}
1.8	1.8038×10^{-6}	1.8038×10^{-6}	1.35291×10^{-6}	1.35291×10^{-6}	9.01939×10^{-7}	9.01939×10^{-7}	4.5097×10^{-7}	4.5097×10^{-7}



$\alpha = 0.7,\ 0.8,\ 0.9,\ 1,\ r = 0.50$ $\qquad$ $\alpha = 0.7,\ 0.8,\ 0.9,\ 1,\ r = 0.75$

Figure 1. Solution plot of OHAM for $\underline{u}(x,r)$ & $\overline{u}(x,r)$ at different values of r & α for problem 1.

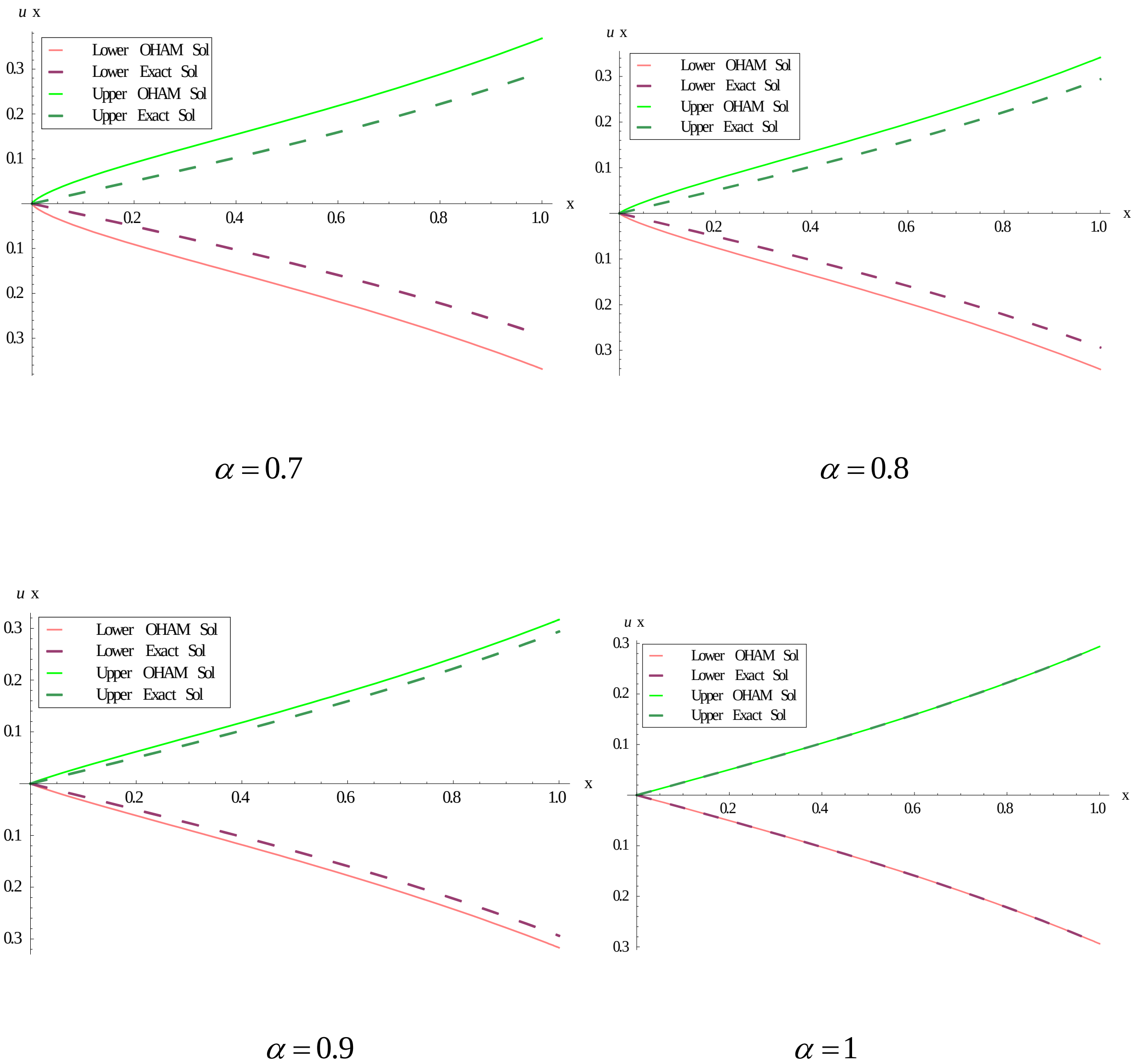


Figure 2. Solution plot of OHAM and Exact for $\underline{u}(x,r)$ & $\overline{u}(x,r)$ at different values of α taking $r = 0.75$ for problem 1.

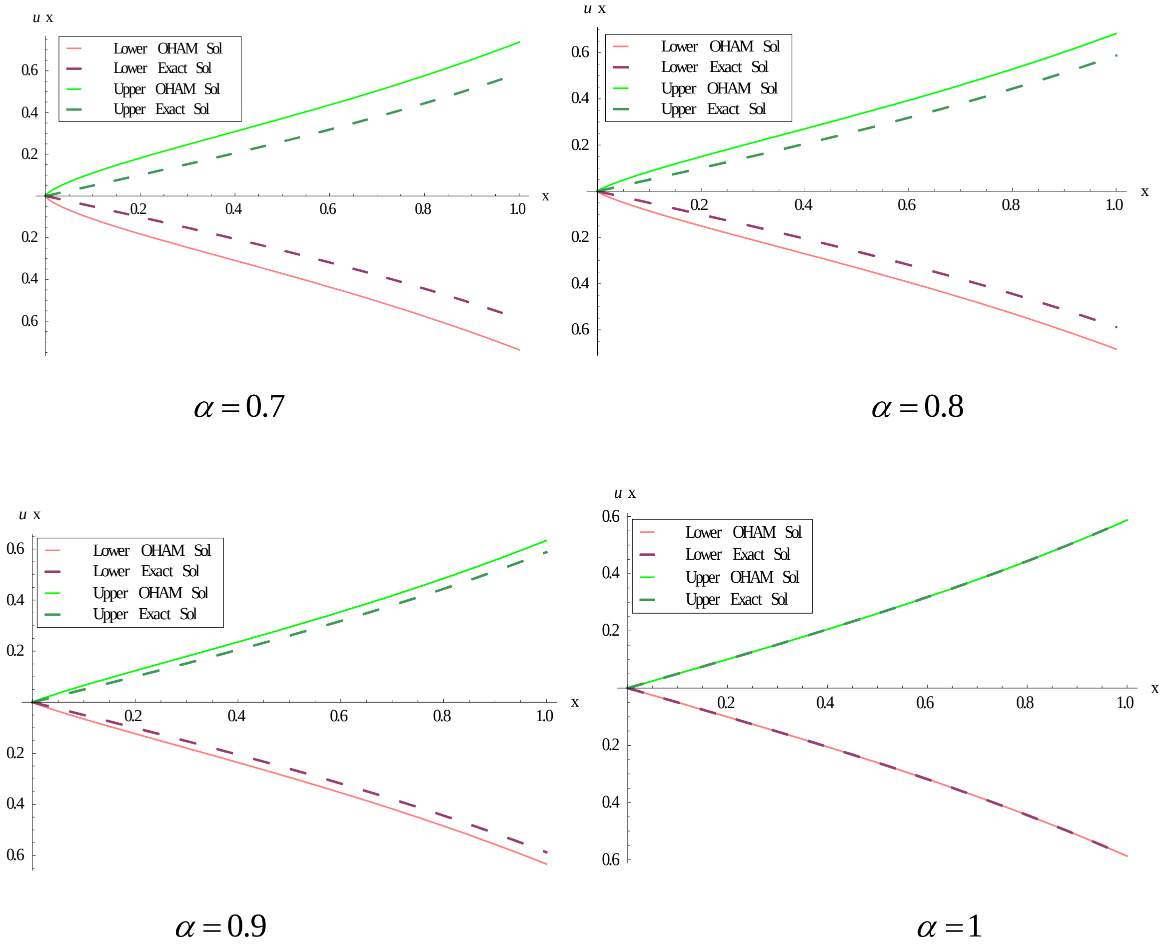


Figure 3. Solution plot of OHAM and Exact for $\underline{u}(x,r)$ & $\overline{u}(x,r)$ at different values of α taking $r = 0.50$ for problem 1.

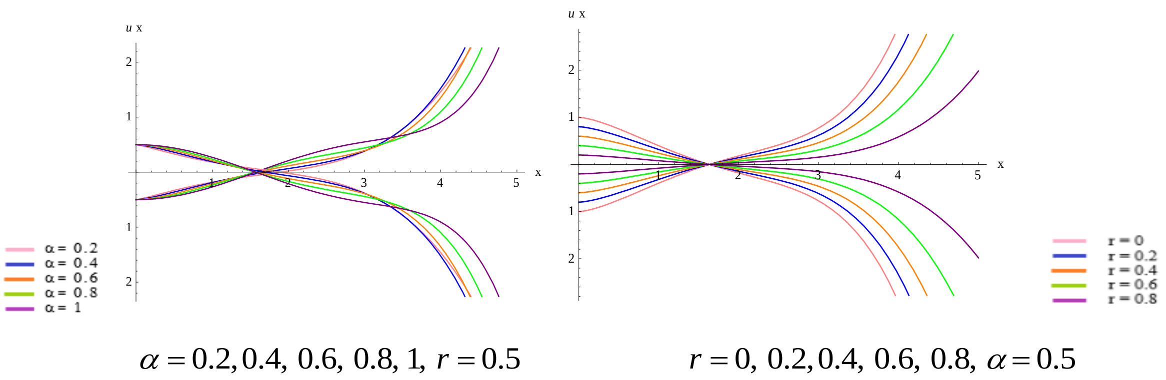
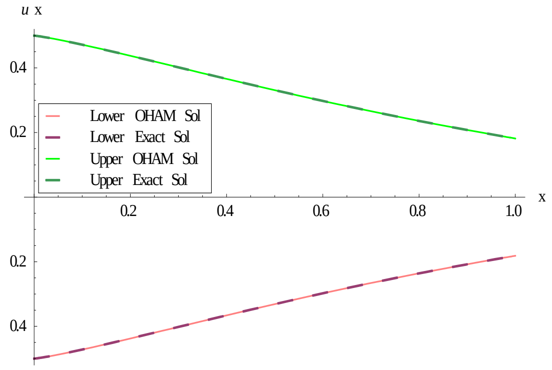
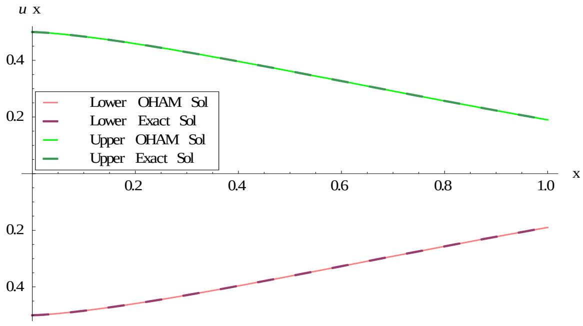


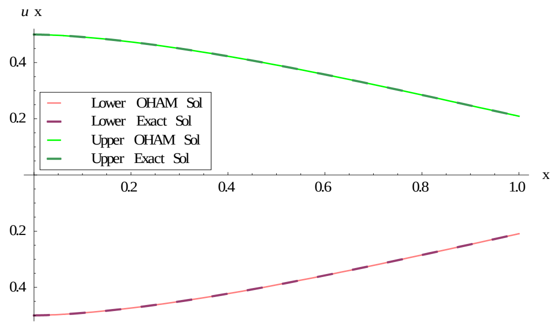
Figure 4. Solution plot of OHAM for $\underline{u}(x,r)$ & $\overline{u}(x,r)$ at different values of r & α for problem 2.



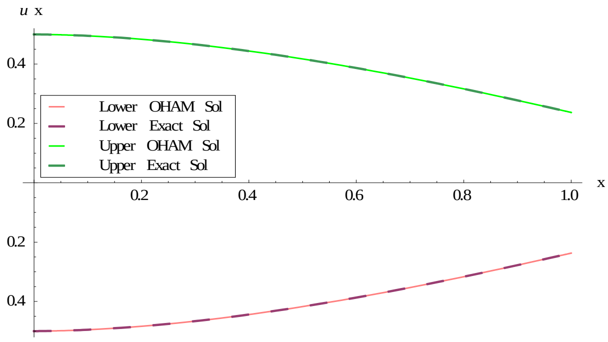
$\alpha = 0.2$



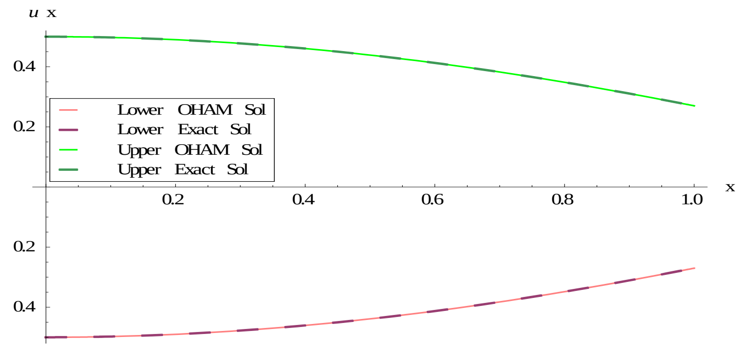
$\alpha = 0.4$



$\alpha = 0.6$



$\alpha = 0.8$



$\alpha = 1$

Figure 5. Solution plot of OHAM and Exact for $\underline{u}(x, r)$ & $\overline{u}(x, r)$ at different values of α taking $r = 0.5$ for problem 2.

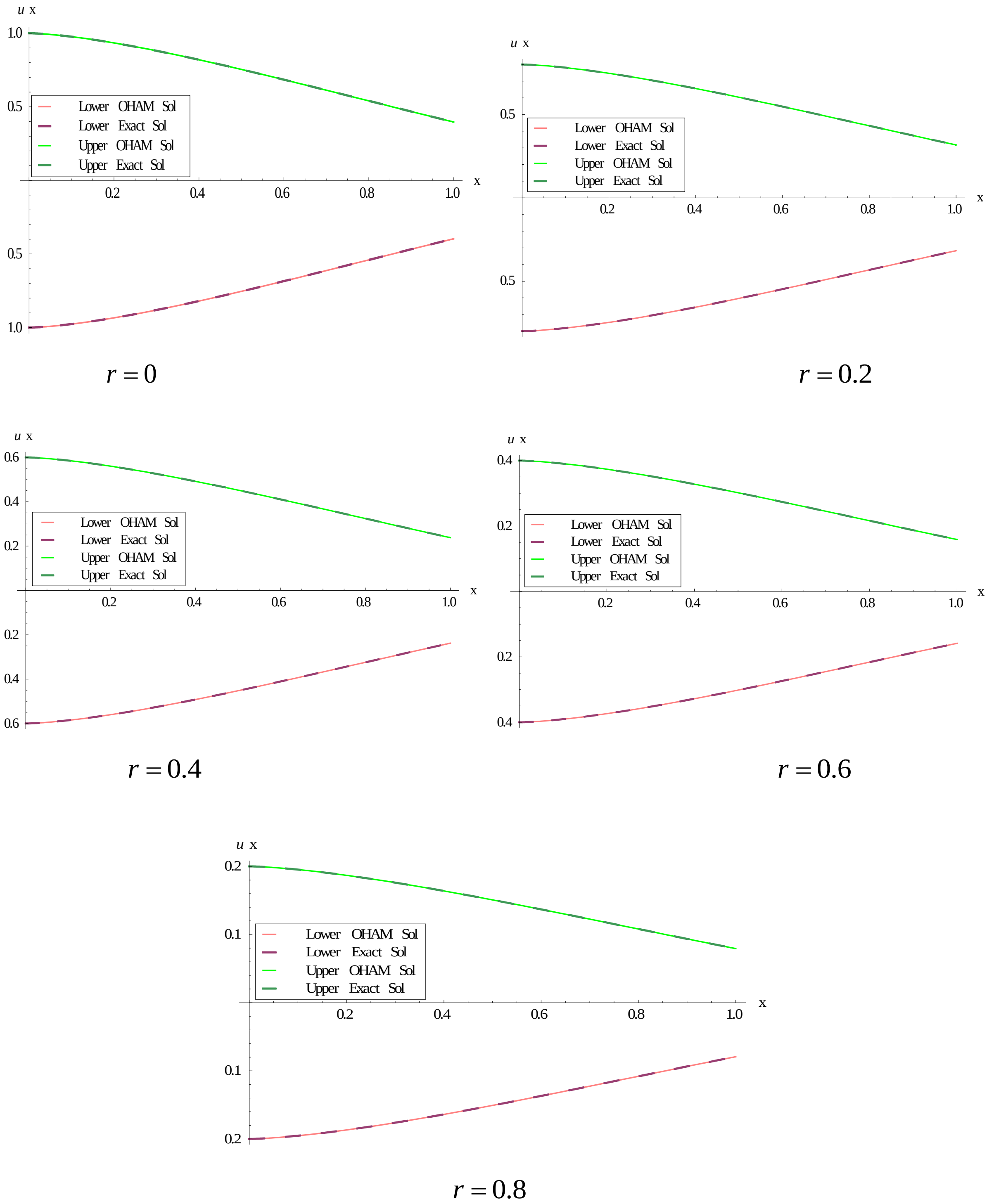


Figure 6. Solution plot of OHAM and Exact for $\underline{u}(x,r)$ & $\overline{u}(x,r)$ at different values of r taking $\alpha = 0.5$ for problem 2.

6. Conclusions

In the research paper, a powerful technique known as Optimal Homotopy Asymptotic Method (OHAM) has been extended to the solution of system of fuzzy integro differential equations of fractional order. The obtained results are quite interesting and are in good agreement with the exact solution. Two numerical equations are taken as test examples which show the behavior and reliability of the proposed method. The extension of OHAM to system of fuzzy integro differential equations of fractional order is more accurate and as a result this technique will more appealing for the researchers for finding out optimum solutions of system of fuzzy integro differential equations of fractional order.

Conflict of interest

The authors declare no conflict of interest.

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