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Research article

Lie analysis, conserved vectors, nonlinear self-adjoint classification and exact solutions of generalized $(N + 1)$-dimensional nonlinear Boussinesq equation

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Abstract: In this article, the generalized $(N + 1)$-dimensional nonlinear Boussinesq equation is analyzed via Lie symmetry method. Lie point symmetries of the considered equation and accompanying invariant groups are computed. After transforming the equation into a nonlinear ordinary differential equation (ODE), analytical solutions of various types are obtained using the $(G'/G, 1/G)$ expansion method. The concept of nonlinear self-adjointness is used in order to determine nonlocal conservation laws of the equation in lower dimensions. By selecting the appropriate parameter values, the study provides a graph of the solutions to the equation under study.

Keywords: generalized Boussinesq equation; Lie symmetry analysis; nonlinear self-adjointness; $(G'/G, 1/G)$ expansion method; conservation laws
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1. Introduction

The Boussinesq equation received a lot of attention in recent years. Originally, it was used to describe long waves in shallow water [1] in 1871. The peculiar behavior of the Boussinesq equation's soliton solutions explains our newfound interest in the equation. Solitons in integrable systems are commonly thought to be stable objects that interact trivially with altering only phase as a result of the interaction. Some recent studies on solitons solution to different partial differential equations (PDEs) can be found in [2–6]. The behavior of solitons in the Boussinesq equation, on the other hand, defies this stereotype. The Boussinesq equation aroused the interest of academics in both mathematics and physics [7–14] due to its profound importance and enticing mathematical properties, despite the fact that its solitons may decay under perturbations and cause a singularity in finite time [15].

Nonlinear evolution equations [16] play an important role in nonlinear science, particularly in plasma physics, ferromagnetic chains, water wave tanks, meta-materials, nonlinear wave propagation [17–20], Bose-Einstein condensates, and nonlinear optical fibers. In this article, we look at the extension of generalized Boussinesq equation [21] to $(N + 1)$ dimensions. Thus, we can write the $(N + 1)$-dimensional Boussinesq equation as:

$$u_{tt} = au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + \sum_{j=1}^{N-1} d_j u_{y_j y_j}, \quad b \neq 0, \tag{1.1}$$

where the scalars d_j denote the coefficients of dissipation terms in y_j directions and $u\left(t, x, y_1, \cdots, y_{N-1}\right)$ is a field function of spatial and temporal components. Equation (1.1) is similar to the $(N + 1)$-dimensional Boussinesq equation [22] for $n = 2$ and $d_j = 1 \ \forall\ j = 1, \cdots, N - 1$.

For $N = 1$, Eq (1.1) reduces to generalized $(1 + 1)$-dimensional Boussinesq equation [23]

$$u_{tt} = au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx}, \tag{1.2}$$

while for $N = 2$, Eq (1.1) exhibits generalized $(2 + 1)$-dimensional Boussinesq equation [24]

$$u_{tt} = au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + du_{yy}. \tag{1.3}$$

In addition, the $(3 + 1)$-dimensional Boussinesq equation considered in [21, 25], is

$$u_{tt} = au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + d_1 u_{yy} + d_2 u_{zz}. \tag{1.4}$$

For arbitrary parameter values, many special instances of Eq (1.1) for different dimensions are investigated [26–32].

Differential equations are unavoidable when dealing with numerous abnormalities in applied math, physical research, and design. A detailed grasp of the dynamic cycles portrayed in these scenarios is required in many branches of study. In financial, monetary, and sociological concerns, these conditions are utilized to mimic a wide range of nonlinear cycles. We can look at a variety of actual properties connected with these nonlinear circumstances thanks to their arrangements. An overall hypothesis clarifying the viability of numerous circumstances linked with them is utterly impracticable. As a result, a lot of focus has recently been paid to establishing various organizational techniques for these nonlinear situations. Researchers have devised many strategies for solving nonlinear partial differential equations.

In any nonlinear media, detailing with nonlinear PDEs and their exact solutions is an essential job. These solutions might help with a superior comprehension of intricate cycles and give knowledge into the actual attributes of the nonlinear models viable. This is mainly because they can supply a lot of physical information and help us understand how these physical models work. Many researchers focused on nonlinear elements, creating and applying an assortment of mathematical techniques for producing new exact analytical solutions for nonlinear differential equations. These methods include the Simplest equation method [33], the extended Trial equation method [34], the extended $(1/G')$ method [35], the direct extended algebraic method [36–38], the Hirota bilinear method [39], the extended sinh-Gordon equation expansion method [40], the meshless radial basis function method [41], the $\exp(-\phi(\xi))$ expansion method [42], the Painlevé-expansion method [43], modified Khater method [44], Jacobi elliptic function expansion method [45], etc. These analytical techniques proved significantly more trustworthy and efficient for obtaining solutions to many PDEs.

Numerous speculations meaning to distinguish procedures for getting explicit exact solutions went to the Lie group of transformation method of differential equations [46, 47]. The Lie group strategy is a standard methodology for deciding a nonlinear complex system's Lie symmetries. The method, specifically, permits us to reduce the dimension of the equation by one after only one application. Hence, the Lie symmetry technique is a typical, compelling, and exceptionally strong strategy with a variety of applications [48–54].

Any physical system's conservation laws play a critical role in characterizing its dynamics. These laws provide insight into the system's physical meaning. Conserved quantities such as energy, mass, and momentum are special cases of these laws. These are used to investigate integrability and linearization of mappings [55]. Many scholars contributed to the advancement of various method to construct conservation law for PDEs [56], making them a prominent research topic for mathematicians and physicists [57–60].

The rest of the manuscript is organized as follows. In Section 2, there is a brief review of some basic concepts and definitions used in this study. Lie point symmetries and group invariant solutions of considered Eq (1.1) are presented in Section 3. Section 5 contains the symmetry reduction and various exact solutions of Eq (1.1) along with their graphical portrays. In Section 6, nonlinear self-adjoint classification of Eq (1.1) is accomplished. Section 4 is devoted to the computation of nonlocal conserved vectors for lower dimensional Boussinesq equation. The study comes to an end with concluding remarks in Section 7.

2. Preliminaries

2.1. The Ibragimov's method

In this portion, we'll look at N. H. Ibragimov's method for constructing a conserved vector for a system of partial differential equations.

2.1.1. Nonlocal conservation law

Consider a system containing $\tilde{m}$ partial differential equations:

$$\mathfrak{U}_{\tilde{\rho}}\left(\tilde{x}, \tilde{u}, \tilde{u}_{(1)}, \tilde{u}_{(2)}, \cdots, \tilde{u}_{(r)}\right) = 0, \quad (\tilde{\rho} = 1, \cdots, \tilde{m}), \tag{2.1}$$

where $\mathfrak{U}_{\tilde{\rho}} \in \mathscr{V}$, $\mathscr{V}$ being the vector space of all differential functions of finite order.

In Eq (2.1), $\tilde{x} = \left(x^1, x^2, \cdots, x^{\tilde{n}}\right)$ denotes the $\tilde{n}$ independent variables and $\tilde{u} = \left(u^1, u^2, \cdots, u^{\tilde{m}}\right)$ are the $\tilde{m}$ dependent variables, while $\tilde{u}_i$ represents the partial derivative of $\tilde{u}$ w.r.t. the i^{th} component of $\tilde{x}$.

Definition 1. *The conservation law for the system of PDEs (2.1) is defined as:*

$$\left[\mathscr{D}_j\left(\mathscr{T}^j\right)\right]_{\mathfrak{U}_{\tilde{\rho}}=0} = 0, \tag{2.2}$$

where $\mathscr{D}_j$ represents the total derivative operators w.r.t. x^j $(1 \leq j \leq \tilde{n})$, given by:

$$\mathscr{D}_j = \frac{\partial}{\partial x^j} + u_j^{\tilde{\rho}} \frac{\partial}{\partial u^{\tilde{\rho}}} + u_{jk}^{\tilde{\rho}} \frac{\partial}{\partial u_k^{\tilde{\rho}}} + u_{jk\ell}^{\tilde{\rho}} \frac{\partial}{\partial u_{k\ell}^{\tilde{\rho}}} + \cdots, \quad \tilde{\rho} = 1, \cdots, \tilde{m}. \tag{2.3}$$

The repeated indices in Eq (2.2) denotes the standard summation convention.

Theorem 1. If the system (2.1) is invariant due to an infinitesimal generator

$$X = \xi^j(\tilde{x}, \tilde{u}) \frac{\partial}{\partial x^j} + \eta^{\tilde{\rho}}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial u^{\tilde{\rho}}}, \tag{2.4}$$

then the components of the conserved vector $\mathscr{T} = \left(\mathscr{T}^1, \mathscr{T}^2, \cdots, \mathscr{T}^{\tilde{n}}\right)$ for the system (2.1) satisfying the conservation law (2.2) are determined as follows:

$$\begin{aligned} \mathscr{T}^j = \xi^j \mathscr{L} + \mathcal{W}^{\tilde{\rho}} \left[\frac{\partial \mathscr{L}}{\partial u_j} - \mathscr{D}_k\left(\frac{\partial \mathscr{L}}{\partial u_{jk}}\right) + \mathscr{D}_k \mathscr{D}_\ell\left(\frac{\partial \mathscr{L}}{\partial u_{jk\ell}}\right) - \cdots\right] \\ + \mathscr{D}_k\left(\mathcal{W}^{\tilde{\rho}}\right)\left[\frac{\partial \mathscr{L}}{\partial u_{jk}} - \mathscr{D}_\ell\left(\frac{\partial \mathscr{L}}{\partial\, u_{jk\ell}}\right) + \cdots\right] + \mathscr{D}_k \mathscr{D}_\ell\left(\mathcal{W}^{\tilde{\rho}}\right)\left[\frac{\partial \mathscr{L}}{\partial u_{jk\ell}} - \cdots\right] + \cdots, \end{aligned} \tag{2.5}$$

where $\mathcal{W}^{\tilde{\rho}} = \eta^{\tilde{\rho}} - \xi^i u_i^{\tilde{\rho}}$ $(\tilde{\rho} = 1, 2, \cdots, \tilde{m})$, and $\mathscr{L} = \mathscr{P}^{\tilde{\rho}} \mathfrak{U}_{\tilde{\rho}}$ represents the formal Lagrangian.

Remark: Since the components of a conserved vector of system (2.1) depend on the nonlocal variables $\mathscr{P}^{\tilde{\rho}}$, $\tilde{\rho} = 1, \cdots, \tilde{m}$, therefore (2.2) is also refered as the *nonlocal conservation law* for the system (2.1).

The choice of nonlocal variables $\mathscr{P}^{\tilde{\rho}}$ for the construction of formal lagrangian leads to a concept of nonlinear selfadjointness.

2.1.2. Nonlinear self-adjointness

In this portion, we present some basic definitions related to the concept of nonlinear self-adjointness [61].

Definition 2. *The adjoint system of equations corresponding to the Eq (2.1) is determined by taking the variational derivatives as:*

$$\mathfrak{U}_{\tilde{\rho}}^*\left(\tilde{x}, \tilde{u}, \tilde{\mathscr{P}}, \tilde{u}_{(1)}, \tilde{\mathscr{P}}_{(1)}, \tilde{u}_{(2)}, \tilde{\mathscr{P}}_{(2)}, \cdots, \tilde{u}_{(r)}, \tilde{\mathscr{P}}_{(r)}\right) = \frac{\delta \mathscr{L}}{\delta u^{\tilde{\rho}}} = 0, \quad (\tilde{\rho} = 1, \cdots, \tilde{m}), \tag{2.6}$$

where $\tilde{\mathscr{P}} = \left(\mathscr{P}^1, \mathscr{P}^2, \cdots, \mathscr{P}^{\tilde{m}}\right)$ are the new dependent variables termed as nonlocal variables, and the subscripts of $\tilde{\mathscr{P}}$ denotes the order of partial derivatives w.r.t. independent variables in $\tilde{x}$. Moreover, $\mathscr{L} = \mathscr{P}^{\varrho} \mathfrak{U}_{\varrho}$ is the formal Lagrangian while $\delta/\delta u^{\tilde{\rho}}$ is the Euler-Lagrange operator given by:

$$\frac{\delta}{\delta u^{\tilde{\rho}}} = \frac{\partial}{\partial u^{\tilde{\rho}}} + \sum_{i=1}^{\infty} (-1)^i\ \mathscr{D}_{j1} \cdots \mathscr{D}_{ji} \frac{\partial}{\partial u_{j1 \cdots ji}^{\tilde{\rho}}} \tag{2.7}$$

The total differentiation (2.3) is now extended to new variables $\left(\mathscr{P}^1, \cdots, \mathscr{P}^{\tilde{m}}\right)$ as:

$$\mathscr{D}_j = \frac{\partial}{\partial x^j} + u_j^{\tilde{\rho}} \frac{\partial}{\partial u^{\tilde{\rho}}} + \mathscr{P}_j^{\tilde{\rho}} \frac{\partial}{\partial \mathscr{P}^{\tilde{\rho}}} + u_{jk}^{\tilde{\rho}} \frac{\partial}{\partial u_k^{\tilde{\rho}}} + \mathscr{P}_{jk}^{\tilde{\rho}} \frac{\partial}{\partial \mathscr{P}_k^{\tilde{\rho}}} + u_{jk\ell}^{\tilde{\rho}} \frac{\partial}{\partial u_{k\ell}^{\tilde{\rho}}} + \mathscr{P}_{jk\ell}^{\tilde{\rho}} \frac{\partial}{\partial \mathscr{P}_{k\ell}^{\tilde{\rho}}} + \cdots, \ \tilde{\rho} = 1, \cdots, \tilde{m}. \quad (2.8)$$

Definition 3. *The system of PDEs (2.1) is said to be nonlinearly self-adjoint, if for some* $\tilde{\rho} = 1, \cdots, \tilde{m}$*, the substitution* $\mathscr{P}^{\tilde{\rho}} = \Psi^{\tilde{\rho}}(\tilde{x}, \tilde{u}) \neq 0$*, to the adjoint system yields:*

$$\mathfrak{U}_\rho^*\left(\tilde{x}, \tilde{u}, \tilde{\Psi}(\tilde{x}, \tilde{u}), \cdots, u_{(r)}, \tilde{\Psi}_{(r)}\right) = \lambda_\rho^\varrho \mathfrak{U}_\varrho\left(\tilde{x}, \tilde{u}, \cdots, \tilde{u}_{(r)}\right), \ \rho = 1, \cdots, m, \quad (2.9)$$

where $\lambda_\rho^\varrho \in \mathscr{V}$ are the undetermined coefficients and, $\mathscr{P}$ and Ψ are $\tilde{m}$ dimensional vectors.
Note that, if $\tilde{m} > m$, then system (2.1) is over-determined, while the corresponding adjoint system is *sub-definite*. Similarly, if $m > \tilde{m}$, then the system (2.1) is *sub-definite*, while its corresponding adjoint system is over-determined.

Definition 4. *The system of PDEs (2.1) is said to be quasi self-adjoint, if the substitution to the adjoint system defined by a continuously differentiable mapping* $\mathscr{P}^{\tilde{\rho}} = \Psi^{\tilde{\rho}}(\tilde{u})$ *from* $\tilde{m}$*-dimensional space of variables* $\tilde{u}$ *into* $\tilde{m}$*-dimensional space of variables* $\tilde{\mathscr{P}}$*, satisfies*

$$\mathfrak{U}_{\tilde{\rho}}^*\left(\tilde{x}, \tilde{u}, \tilde{\Psi}, \cdots, u_{(r)}, \tilde{\Psi}_{(r)}\right) = \lambda_{\tilde{\rho}}^\varrho \mathfrak{U}_\varrho\left(\tilde{x}, \tilde{u}, \cdots, \tilde{u}_{(r)}\right), \ \tilde{\rho} = 1, \cdots, \tilde{m}, \quad (2.10)$$

where the coefficients $\lambda_{\tilde{\rho}}^\varrho \in \mathscr{V}$ are undetermined while the components $\Psi^{\tilde{\rho}}(\tilde{u})$ of $\tilde{\Psi}$ are not all equals to zero simultaneously.

Definition 5. *The system (2.1) of partial differential equations is called strictly self-adjoint, if the substitution* $\tilde{\mathscr{P}} = \tilde{u}$ *to its adjoint system fulfills:*

$$\mathfrak{U}_{\tilde{\rho}}^*\left(\tilde{x}, \tilde{u}, \tilde{\Psi}, \cdots, u_{(r)}, \tilde{\Psi}_{(r)}\right) = \lambda_{\tilde{\rho}}^\varrho \mathfrak{U}_\varrho\left(\tilde{x}, \tilde{u}, \cdots, \tilde{u}_{(r)}\right), \ \tilde{\rho} = 1, \cdots, \tilde{m}, \ \lambda_{\tilde{\rho}}^\varrho \in \mathscr{V}. \quad (2.11)$$

However, the substitution $\tilde{\mathscr{P}} = \tilde{u}$ is not uniquely determined in the case of more than one dependent variables. For example in case of $\tilde{m}$ dependent variables it is possible that $\left(\mathscr{P}^1, \mathscr{P}^2, \cdots, \mathscr{P}^{\tilde{m}}\right) = \left(u^{\tilde{m}}, u^{\tilde{m}-1}, \cdots, u^1\right)$ yields Eq (2.11). Thus, in this case there is one out of $\tilde{m}!$ possible arrangements of the components of $\tilde{u}$, which satisfies Eq (2.11) upon substitution. In the situation of a single dependent variable, this definition is ideal.

Similarly, M. L. introduced the concept of weak selfadjointness [62]. However, it is important to note that Definitions 3 is the most generic, as it encompasses all of the other definitions related to the concept of self-adjointness. It's worth noting that all of the preceding Definitions 1–5 are applicable to the case of a single partial differential equation having only one dependent variable.
Remark: After determining the non-local variables by utilizing the knowledge of nonlinear self-adjointness, one will able to construct a formal Lagrangian and hence the components of the conserved vector for the system (2.1) by using Eq (2.5).

2.2. $(G'/G, 1/G)$-expansion method

Consider a nonlinear partial differential equation for $u(t, x, y_1, y_2, \cdots, y_{N-1})$ having general form:

$$U\left(u, u_t, u_x, u_{y_1}, u_{y_2}, \cdots\right) = 0. \quad (2.12)$$

Assuming that upon using the similarity transformation $u(t, x, y_1, y_2, \cdots, y_{N-1}) = g(\omega)$, the nonlinear PDE (2.12) is reduced into a nonlinear ODE:

$$U(\Theta, \Theta', \Theta'', \cdots) = 0, \tag{2.13}$$

where prime (′) denotes the ordinary derivative w.r.t. ω. Integrating Eq (2.13) can reduce its complexity.

According to the scheme of $(G'/G, 1/G)$-expansion method, Eq (2.13) possesses the solution of the form:

$$\Theta(\omega) = \sum_{i=0}^{m} p_i \Delta^i(\omega) + \sum_{i=1}^{m} q_i \Delta^{i-1}(\omega)\varphi(\omega), \tag{2.14}$$

where m is the positive integer obtained by taking the homogeneous balance between highest order nonlinear term and highest order linear term of ODE (2.13), p_0, p_i, q_i $(1 \leq i \leq m)$ are coefficients while $\Delta(\omega)$ and $\varphi(\omega)$ defined by:

$$\Delta = \frac{G'}{G}, \ \ \varphi = \frac{1}{G}, \tag{2.15}$$

in which $G(\omega)$ is a solution of second order ODE:

$$G''(\omega) + sG(\omega) = \upsilon, \tag{2.16}$$

where s and υ are arbitrary constants.

From Eqs (2.15) and (2.16), we have

$$\Delta' = -\Delta^2 + \upsilon\varphi - s, \ \ \varphi' = -\Delta\varphi. \tag{2.17}$$

Depending on the sign of the parameter s, the solution of Eq (2.16) is categorized into three cases:
Case 1. When s<0:
The general solution of Eq (2.16) is:

$$G(\omega) = \gamma_1 \sinh\left(\sqrt{-s}\omega\right) + \gamma_2 \cosh\left(\sqrt{-s}\omega\right) + \frac{\upsilon}{s}, \tag{2.18}$$

where γ_1 and γ_2 are the integration constants. Substituting Eq (2.18) into Eq (2.15) and using Eq (2.17), we get:

$$\varphi^2 = \frac{-s\left(\Delta^2 - 2\upsilon\varphi + s\right)}{s^2\sigma + \upsilon^2}, \ \ \sigma = \gamma_1^2 - \gamma_2^2. \tag{2.19}$$

Case 2. When s>0:
The general solution of Eq (2.16) is:

$$G(\omega) = \gamma_1 \sin\left(\sqrt{s}\omega\right) + \gamma_2 \cos\left(\sqrt{s}\omega\right) + \frac{\upsilon}{s}, \tag{2.20}$$

where γ_1 and γ_2 are the integration constants. Substituting Eq (2.20) into Eq (2.15) and using Eq (2.17), we get:

$$\varphi^2 = \frac{s\left(\Delta^2 - 2\upsilon\varphi + s\right)}{s^2\sigma - \upsilon^2}, \ \ \sigma = \gamma_1^2 + \gamma_2^2. \tag{2.21}$$

Case 3. When s=0:
In this case Eq (2.16) gives the solution of the form:

$$G(\omega) = \frac{\upsilon}{2}\omega^2 + \gamma_1\omega + \gamma_2, \tag{2.22}$$

where γ_1 and γ_2 are the integration constants. Thus Eq (2.15) by the use of Eqs (2.22) and (2.17), gives us:

$$\varphi^2 = \frac{\Delta^2 - 2\upsilon\varphi}{\gamma_1^2 - 2\upsilon\gamma_2}. \tag{2.23}$$

The following steps are used to obtain the solution of nonlinear PDE (2.12) corresponding to each case (say Case 1):
Step 1. After determining the value of positive integer m, Eq (2.14) is substituted into Eq (2.13) and, by the aid of Eqs (2.17) and (2.19), Eq (2.13) is changed into a polynomial of Δ^i, φ^j where $0 \leq i \leq n$, $j \leq 1$ (n is any integer).
Step 2. Equating the coefficients of $\Delta^i(\omega)$ and $\Delta^{i-1}\varphi(\omega)$ to zero, yields the system of algebraic equations involving p_i, q_i, s, σ, υ (and possibly the parameters of ODE (2.13)). After solving the system of algebraic equations and using Eq (2.15), Eq (2.14) gives the solution of ODE (2.13) (corresponding to Case 1).
Step 3. The above process is repeated for the evaluation of parameters for the remaining two cases and in a similar way solutions for ODE (2.13) are obtained.
Step 4. The solutions of nonlinear PDE (2.12) corresponding to the three cases are obtained by substituting the similarity variable (ω).

3. Lie analysis of Eq (1.1)

In this section, we shall find the Lie algebra admitted by Eq (1.1) using Lie theory. After the computation of the Lie point symmetries, we shall present the group invariant solutions, which are new types of solutions of Eq (1.1).

3.1. Lie point symmetries

Consider a one parameter (ε) Lie group of point transformation on the space of "$N+2$" variables, which preserves the invariance of Eq (1.1).

$$\begin{aligned}
t^\dagger &= t + \varepsilon\xi^0(t, x, y_1, \cdots, y_{N-1}, u) + O\left(\varepsilon^2\right),\\
x^\dagger &= x + \varepsilon\xi^1(t, x, y_1, \cdots, y_{N-1}, u) + O\left(\varepsilon^2\right),\\
y_j^\dagger &= y_j + \varepsilon\eta^j(t, x, y_1, \cdots, y_{N-1}, u) + O\left(\varepsilon^2\right), \quad j = 1, \cdots, N-1\\
u^\dagger &= u + \varepsilon\zeta(t, x, y_1, \cdots, y_{N-1}, u) + O\left(\varepsilon^2\right),
\end{aligned} \tag{3.1}$$

where ξ^0, ξ^1, η^j and ζ are known as infinitesimals of point transformation (3.1) while ε is the small parameter.

The infinitesimal generator corresponding to the above Lie group (3.1) is given by:

$$\begin{aligned} X = & \xi^0 (t, x, y_1, \cdots, y_{N-1}, u) \frac{\partial}{\partial t} + \xi^1 (t, x, y_1, \cdots, y_{N-1}, u) \frac{\partial}{\partial x} \\ & + \sum_{j=1}^{N-1} \eta^j (t, x, y_1, \cdots, y_{N-1}, u) \frac{\partial}{\partial y_j} + \zeta (t, x, y_1, \cdots, y_{N-1}, u) \frac{\partial}{\partial u}. \end{aligned} \tag{3.2}$$

The Lie invariance condition [47] for Eq (1.1) reads:

$$pr^{(4)} X (\Omega) |_{\Omega=0} = 0, \ \Omega = u_{tt} - a u_{xx} - b \left(u^2\right)_{xx} - c u_{xxxx} - \sum_{j=1}^{N-1} d_j u_{y_j y_j}, \tag{3.3}$$

where $pr^{(4)} X$ is the fourth prolongation of the infinitesimal generator (3.2) given by:

$$pr^{(3)} X = X + \zeta^{tt} \frac{\partial}{\partial u_{tt}} + \zeta^x \frac{\partial}{\partial u_x} + \zeta^{xx} \frac{\partial}{\partial u_{xx}} + \zeta^{xxxx} \frac{\partial}{\partial u_{xxxx}} + \sum_{j=1}^{N-1} \zeta^{y_j y_j} \frac{\partial}{\partial u_{y_j y_j}} , \tag{3.4}$$

in which ζ^{tt}, ζ^x, ζ^{xx}, ζ^{xxxx} and $\zeta^{y_j y_j}$ are the extended infinitesimals [46].

Using Eqs (3.3) and (3.4), we obtain an overdetermined system of PDEs about the infinitesimals of point transformation (3.1), which upon solving yields a finite dimensional Lie algebra of $(N+1)$ dimensional Boussinesq equation (1.1).

Theorem 2. *The* $(N + 1)$ *dimensional Boussinesq equation (1.1) admits a finite dimensional Lie algebra spanned by exactly* $\frac{N^2+N+4}{2}$ *Lie point symmetries given by:*

$$\begin{aligned} X_1 = & \frac{\partial}{\partial t}, \ X_2 = \frac{\partial}{\partial x}, \ X_{j+2} = \frac{\partial}{\partial y_j}, \ j = 1, \cdots, N-1. \\ X_{N+j+1} = & \frac{y_j}{d_j} \frac{\partial}{\partial t} + t \frac{\partial}{\partial y_j}, \ j = 1, \cdots, N-1 \\ X_{2N+\alpha+\beta-1} = & y_{\alpha+1} \frac{\partial}{\partial y_\beta} - \frac{d_{\alpha+1}}{d_\beta} y_\beta \frac{\partial}{\partial y_{\alpha+1}}, \ \alpha = 1, \cdots, N-2, \ \beta = 1, \cdots, \alpha. \\ X_{\frac{N^2+N+4}{2}} = & -2bt \frac{\partial}{\partial t} - bx \frac{\partial}{\partial x} - 2b \sum_{j=1}^{N-1} y_j \frac{\partial}{\partial y_j} + (a + 2bu) \frac{\partial}{\partial u}. \end{aligned} \tag{3.5}$$

It is evident from Eq (3.5) that Eq (1.1) admits infinitesimal generators X_{N+j+1} and $X_{2N+\alpha+\beta-1}$ only when $N > 1$ and $N > 2$, respectively.

- **Lie point symmetries of Eq (1.2)**

Equation (1.2) constitutes three dimensional Lie algebra $\left(\because (1^2 + 1 + 4)/2 = 3\right)$, given by (3.5) as follows

$$X_1 = \frac{\partial}{\partial t}, \ X_2 = \frac{\partial}{\partial x}, \ X_3 = -2bt \frac{\partial}{\partial t} - bx \frac{\partial}{\partial x} + (a + 2bu) \frac{\partial}{\partial u}. \tag{3.6}$$

- **Lie point symmetries of Eq (1.3)**

Equation (1.3) possesses five dimensional Lie algebra $\left(\because (2^2+2+4)/2=5\right)$, given by (3.5) as follows:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t},\;\; X_2 = \frac{\partial}{\partial x},\;\; X_3 = \frac{\partial}{\partial x},\;\; X_4 = \frac{y}{d}\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \\ X_5 &= -2bt\frac{\partial}{\partial t} - bx\frac{\partial}{\partial x} - 2by\frac{\partial}{\partial y} + (a+2bu)\frac{\partial}{\partial u}. \end{aligned} \tag{3.7}$$

- **Lie point symmetries of Eq (1.4)**

Equation (1.4) possesses eight dimensional Lie algebra $\left(\because (3^2+3+4)/2=8\right)$, given by (3.5) as follows:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t},\;\; X_2 = \frac{\partial}{\partial x},\;\; X_3 = \frac{\partial}{\partial y},\;\; X_4 = \frac{\partial}{\partial z}, \\ X_5 &= \frac{y}{d_1}\frac{\partial}{\partial t} + t\frac{\partial}{\partial y},\;\; X_6 = \frac{z}{d_2}\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \\ X_7 &= z\frac{\partial}{\partial y} - \frac{d_2}{d_1}y\frac{\partial}{\partial z}, \\ X_8 &= -2bt\frac{\partial}{\partial t} - bx\frac{\partial}{\partial x} - 2by\frac{\partial}{\partial y} - 2bz\frac{\partial}{\partial z} + (a+2bu)\frac{\partial}{\partial u}. \end{aligned} \tag{3.8}$$

3.2. Group invariant solutions of Eq (1.1)

Suppose the one parameter Lie group of transformations corresponding to the infinitesimal generators in (3.5) has the following form:

$$\begin{aligned} t^\dagger &= T(t, x, y_1, \cdots, y_{N-1}, u; \varepsilon)\;, \\ x^\dagger &= \chi(t, x, y_1, \cdots, y_{N-1}, u; \varepsilon), \\ y_j^\dagger &= Y_j(t, x, y_1, \cdots, y_{N-1}, u; \varepsilon), \\ u^\dagger &= U(t, x, y_1, \cdots, y_{N-1}, u; \varepsilon). \end{aligned} \tag{3.9}$$

To determine the group of point transformations (3.9) from the infinitesimal generators (3.5), we consider the following initial value problem:

$$\begin{aligned} \frac{dt^\dagger}{d\varepsilon} &= \xi^0\left(t^\dagger, x^\dagger, y_1^\dagger, \cdots, y_{N-1}^\dagger, u^\dagger\right)\;, \\ \frac{dx^\dagger}{d\varepsilon} &= \xi^1\left(t^\dagger, x^\dagger, y_1^\dagger, \cdots, y_{N-1}^\dagger, u^\dagger\right)\;, \\ \frac{dy_j^\dagger}{d\varepsilon} &= \eta^j\left(t^\dagger, x^\dagger, y_1^\dagger, \cdots, y_{N-1}^\dagger, u^\dagger\right),\;\; j = 1, \cdots, N-1, \\ \frac{du^\dagger}{d\varepsilon} &= \zeta\left(t^\dagger, x^\dagger, y_1^\dagger, \cdots, y_{N-1}^\dagger, u^\dagger\right), \end{aligned} \tag{3.10}$$

with the initial condition $(t^\dagger, x^\dagger, y_1^\dagger, \cdots, y_{N-1}^\dagger, u^\dagger)|_{\varepsilon=0} = (t, x, y_1, \cdots, y_{N-1}, u)$.

After providing the infinitesimals of each Lie point symmetry in (3.5), the solutions of the initial value problem (3.10) yield the following Lie groups.

$$G_1:\begin{cases}t^\dagger = t+\varepsilon,\\ x^\dagger = x,\\ y_j^\dagger = y_j,\ j=1,\cdots,N-1.\\ u^\dagger = u.\end{cases}\qquad G_2:\begin{cases}t^\dagger = t,\\ x^\dagger = x+\varepsilon,\\ y_j^\dagger = y_j,\ j=1,\cdots,N-1.\\ u^\dagger = u.\end{cases}$$

$$G_{j+2}:\begin{cases}t^\dagger = t,\\ x^\dagger = x,\\ y_j^\dagger = y_j+\varepsilon,\ j=1,\cdots,N-1.\\ u^\dagger = u.\end{cases}$$

$$G_{N+j+1}:\begin{cases}t^\dagger = t\cosh\left(\frac{\varepsilon}{\sqrt{d_j}}\right)+\frac{y_j}{\sqrt{d_j}}\sinh\left(\frac{\varepsilon}{\sqrt{d_j}}\right),\\ x^\dagger = x,\\ y_j^\dagger = t\sqrt{d_j}\sinh\left(\frac{\varepsilon}{\sqrt{d_j}}\right)+y_j\cosh\left(\frac{\varepsilon}{\sqrt{d_j}}\right),\\ u^\dagger = u.\end{cases}\qquad j=1,\cdots,N-1.$$

$$G_{2N+\alpha+\beta-1}:\begin{cases}t^\dagger = t,\\ x^\dagger = x,\\ \begin{cases}y_\beta^\dagger = y_\beta\cos\left(\sqrt{\frac{d_{\alpha+1}}{d_\beta}}\varepsilon\right)+y_{\alpha+1}\sqrt{\frac{d_\beta}{d_{\alpha+1}}}\sin\left(\sqrt{\frac{d_{\alpha+1}}{d_\beta}}\varepsilon\right),\\ y_{\alpha+1}^\dagger = y_{\alpha+1}\cos\left(\sqrt{\frac{d_{\alpha+1}}{d_\beta}}\varepsilon\right)-y_\beta\sqrt{\frac{d_{\alpha+1}}{d_\beta}}\sin\left(\sqrt{\frac{d_{\alpha+1}}{d_\beta}}\varepsilon\right),\\ y_k^\dagger = y_k,\ (k\neq\alpha+1,\beta),\ \alpha=1,\cdots,N-2,\ \beta=1,\cdots,\alpha.\end{cases}\\ u^\dagger = u.\end{cases}$$

$$G_{\frac{N^2+N+4}{2}}:\begin{cases}t^\dagger = te^{-2b\varepsilon},\\ x^\dagger = xe^{-b\varepsilon},\\ y_j^\dagger = y_je^{-2b\varepsilon},\ j=1,\cdots,N-1.\\ u^\dagger = ue^{2b\varepsilon}+\frac{a}{2b}\left(e^{2b\varepsilon}-1\right).\end{cases}$$

Theorem: *If $u(t,x,y_1,\cdots,y_{N-1})$ is a solution of $(N+1)$-dimensional generalized Boussinesq equation (1.1), then new solutions $u^\ell = u^\ell(t,x,y_1,\cdots,y_{N-1})$, $\left(\ell=1,\cdots,\frac{N^2+N+4}{2}\right)$ of Eq (1.1) corresponding to each group are given by:*

For G_1 :

$$u^1 = u(t-\varepsilon,x,y_1,\cdots,y_{N-1}).$$

For G_2 :

$$u^2 = u(t,x-\varepsilon,y_1,\cdots,y_{N-1}).$$

For G_{j+2} $(j = 1, \cdots, N-1)$:

$$u^3 = u\left(t, x, y_1 - \varepsilon, \cdots, y_{N-1}\right),$$

$$\cdots \qquad \cdots$$

$$u^{N+1} = u\left(t, x, y_1, \cdots, y_{N-1} - \varepsilon\right).$$

For G_{N+j+1} $(j = 1, \cdots, N-1)$:

$$u^{N+2} = u\left(t\cosh\left(\frac{\varepsilon}{\sqrt{d_1}}\right) - \frac{y_1}{\sqrt{d_1}}\sinh\left(\frac{\varepsilon}{\sqrt{d_1}}\right), x, -t\sqrt{d_1}\sinh\left(\frac{\varepsilon}{\sqrt{d_1}}\right) + y_1\cosh\left(\frac{\varepsilon}{\sqrt{d_1}}\right), y_2, \cdots, y_{N-1}\right),$$

$$u^{N+3} = u\left(t\cosh\left(\frac{\varepsilon}{\sqrt{d_2}}\right) - \frac{y_2}{\sqrt{d_2}}\sinh\left(\frac{\varepsilon}{\sqrt{d_2}}\right), x, y_1, -t\sqrt{d_2}\sinh\left(\frac{\varepsilon}{\sqrt{d_2}}\right) + y_2\cosh\left(\frac{\varepsilon}{\sqrt{d_2}}\right), \cdots, y_{N-1}\right),$$

$$\cdots \qquad \cdots$$

$$u^{2N} = u\left(t\cosh\left(\frac{\varepsilon}{\sqrt{d_{N-1}}}\right) - \frac{y_{N-1}}{\sqrt{d_{N-1}}}\sinh\left(\frac{\varepsilon}{\sqrt{d_{N-1}}}\right), x, y_1, \cdots, -t\sqrt{d_{N-1}}\sinh\left(\frac{\varepsilon}{\sqrt{d_{N-1}}}\right) + y_{N-1}\cosh\left(\frac{\varepsilon}{\sqrt{d_{N-1}}}\right)\right).$$

For $G_{2N+\alpha+\beta-1}$ $(\alpha = 1, \cdots, N-2, \ \beta = 1, \cdots, \alpha)$:

$$u^{2N+1} = u\left(t, x, y_1\cos\left(\sqrt{\frac{d_2}{d_1}}\varepsilon\right) - y_2\sqrt{\frac{d_1}{d_2}}\sin\left(\sqrt{\frac{d_2}{d_1}}\varepsilon\right), y_2\cos\left(\sqrt{\frac{d_2}{d_1}}\varepsilon\right) + y_1\sqrt{\frac{d_2}{d_1}}\sin\left(\sqrt{\frac{d_2}{d_1}}\varepsilon\right), y_3, \cdots, y_{N-1}\right),$$

$$u^{2N+2} = u\left(t, x, y_1\cos\left(\sqrt{\frac{d_3}{d_1}}\varepsilon\right) - y_3\sqrt{\frac{d_1}{d_3}}\sin\left(\sqrt{\frac{d_3}{d_1}}\varepsilon\right), y_2, y_3\cos\left(\sqrt{\frac{d_3}{d_1}}\varepsilon\right) + y_1\sqrt{\frac{d_3}{d_1}}\sin\left(\sqrt{\frac{d_3}{d_1}}\varepsilon\right), \cdots, y_{N-1}\right),$$

$$\cdots \qquad \cdots$$

$$u^{(N^2+N+2)/2} = u\left(t, x, y_1, \cdots, y_{N-2}\cos\left(\sqrt{\frac{d_{N-1}}{d_{N-2}}}\varepsilon\right) - y_{N-1}\sqrt{\frac{d_{N-2}}{d_{N-1}}}\sin\left(\sqrt{\frac{d_{N-1}}{d_{N-2}}}\varepsilon\right), y_{N-1}\cos\left(\sqrt{\frac{d_{N-1}}{d_{N-2}}}\varepsilon\right) + y_{N-2}\sqrt{\frac{d_{N-1}}{d_{N-2}}}\sin\left(\sqrt{\frac{d_{N-1}}{d_{N-2}}}\varepsilon\right)\right).$$

For $G_{(N^2+N+4)/2}$:

$$u^{(N^2+N+4)/2} = \frac{a}{2b}\left(e^{-2b\varepsilon} - 1\right) + e^{-2b\varepsilon}u\left(te^{2b\varepsilon}, xe^{b\varepsilon}, y_1e^{2b\varepsilon}, \cdots, y_{N-1}e^{2b\varepsilon}\right).$$

4. Symmetry reduction and solutions of Eq (1.1)

In this section, we shall find the closed form solutions of Eq (1.1). To accomplish this, we'll first conduct symmetry reduction in such a way that Eq (1.1) becomes a nonlinear ODE.

4.1. Reduction of Eq (1.1) using abelian sub-algebra

After computing the commutation relations [46] for Eq (1.1), it is found that the Lie algebra (3.5) is closed w.r.t. the Lie bracket for every N. As the Lie bracket is commutative for all the translational Lie point symmetries in (3.5). i.e., $\left[X_i, X_j\right] = 0, \ \forall \ i, j = 1, \cdots, N+1$. Therefore, these translational symmetries create an abelian subalgebra, and the associated one dimensional optimal system of Eq (1.1) comprises conjugacy classes [46] that are formed by these symmetries. One of them is the conjugacy class, that makes advantage of all translational symmetries is $< X_1 + \mathscr{H} X_2 + \sum_{j=1}^{N-1} \theta_j X_{j+2} >$, corresponding to which the similarity solution is obtained by solving the characteristic equation:

$$\frac{dt}{1} = \frac{dx}{\mathscr{H}} = \frac{dy_1}{\theta_1} = \cdots = \frac{dy_{N-1}}{\theta_{N-1}},$$

which yields the similarity variables

$$\omega = t - \mathscr{H} x - \sum_{j=1}^{N-1} \theta_j y_j, \quad u = \Theta(\omega). \tag{4.1}$$

Using Eq (4.1) in Eq (1.1) we obtain the nonlinear ODE:

$$\mathscr{H}^4 c \Theta^{(4)} + \mathscr{H}^2 b \left(2\Theta\Theta'' + 2\Theta'^2\right) + \left(\mathscr{H}^2 a - 1 + \sum_{j=1}^{N-1} d_j \theta_j^2\right) \Theta'' = 0. \tag{4.2}$$

For convenience, we take

$$\mu = \mathscr{H}^2 a - 1 + \sum_{j=1}^{N-1} d_j \theta_j^2. \tag{4.3}$$

Thus, Eq (4.2) takes the form

$$\mathscr{H}^4 c \Theta^{(4)} + \mathscr{H}^2 b \left(2\Theta\Theta'' + 2\Theta'^2\right) + \mu \Theta'' = 0, \quad \left(\prime = \frac{d}{d\omega}\right). \tag{4.4}$$

Integrating twice and neglecting the constants of integration each time, we obtain a second order nonlinear ODE:

$$\mathscr{H}^4 c \Theta'' + \mathscr{H}^2 b \Theta^2 + \mu \Theta = 0. \tag{4.5}$$

4.2. Application of $(G'/G, 1/G)$-method to Eq (1.1)

By taking homogeneous balance between highest order linear term (Θ'') and the highest order nonlinear term $\left(\Theta^2\right)$ of Eq (4.5), we find the value of the positive integer $m = 2$, thus from Eq (2.14), we assume the solution of the form:

$$\Theta(\omega) = p_0 + p_1 \Delta(\omega) + p_2 \Delta^2(\omega) + q_1 \varphi(\omega) + q_2 \Delta(\omega) \varphi(\omega). \tag{4.6}$$

We get a system of algebraic equations by substituting Eq (4.6) into Eq (4.5) and comparing the coefficients of like powers of $\Delta(\omega)$ and $\varphi(\omega)$ using the approach outlined in Section 2.2. Solving that algebraic system using *Maple software*, yields the values of parameters μ, υ, p_0, p_1, p_2, q_1, and q_2.

4.2.1. Hyperbolic solutions ($s < 0$)

For $s < 0$, the following two sets of parameters yields the *hyperbolic solutions* for Eq (1.1).

- **Set 1:**

$$p_0 = -\frac{6\mathscr{H}^2 cs}{b},\ p_1 = 0,\ p_2 = -\frac{6\mathscr{H}^2 c}{b},\ q_1 = q_2 = 0,\ \mu = 4s\mathscr{H}^4 c,\ \upsilon = 0. \tag{4.7}$$

- **Set 2:**

$$p_0 = -\frac{2\mathscr{H}^2 cs}{b},\ p_1 = 0,\ p_2 = -\frac{6\mathscr{H}^2 c}{b},\ q_1 = q_2 = 0,\ \mu = -4s\mathscr{H}^4 c,\ \upsilon = 0. \tag{4.8}$$

By substituting these values of the parameters in Eq (4.6), we get solutions for Eq (4.5), which, when combined with the similarity variables (4.1), produces the solutions for Eq (1.1).

For Set 1**:**

We get the following solution for Eq (1.1) by plugging Eq (4.7) into Eq (4.6) and considering Eq (4.1).

$$u_1\left(t, x, y_1, y_2, \cdots, y_{N-1}\right) = \frac{6\mathscr{H}^2 \sigma sc}{b\left(\gamma_1 \sinh\left(\sqrt{-s}\omega\right) + \gamma_2 \cosh\left(\sqrt{-s}\omega\right)\right)^2}, \tag{4.9}$$

where $\sigma = \gamma_1^2 - \gamma_2^2$ and $\omega = t - \mathscr{H} x - \theta_1 y_1 - \cdots - \theta_1 y_{N-1}$.

For Set 2**:**

We get the following solution for Eq (1.1) by plugging Eq (4.8) into Eq (4.6) and considering Eq (4.1).

$$u_2\left(t, x, y_1, y_2, \cdots, y_{N-1}\right) = \frac{6\mathscr{H}^2 sc}{b}\left[\left(\frac{\gamma_1 \cosh\left(\sqrt{-s}\omega\right) + \gamma_2 \sinh\left(\sqrt{-s}\omega\right)}{\gamma_1 \sinh\left(\sqrt{-s}\omega\right) + \gamma_2 \cosh\left(\sqrt{-s}\omega\right)}\right)^2 - \frac{1}{3}\right], \tag{4.10}$$

where $\omega = t - \mathscr{H} x - \theta_1 y_1 - \cdots - \theta_{N-1} y_{N-1}$.

The hyperbolic exact solutions u_1 and u_2 represented by Eqs (4.9) and (4.10) are portrayed in Figures 1 and 2, respectively.

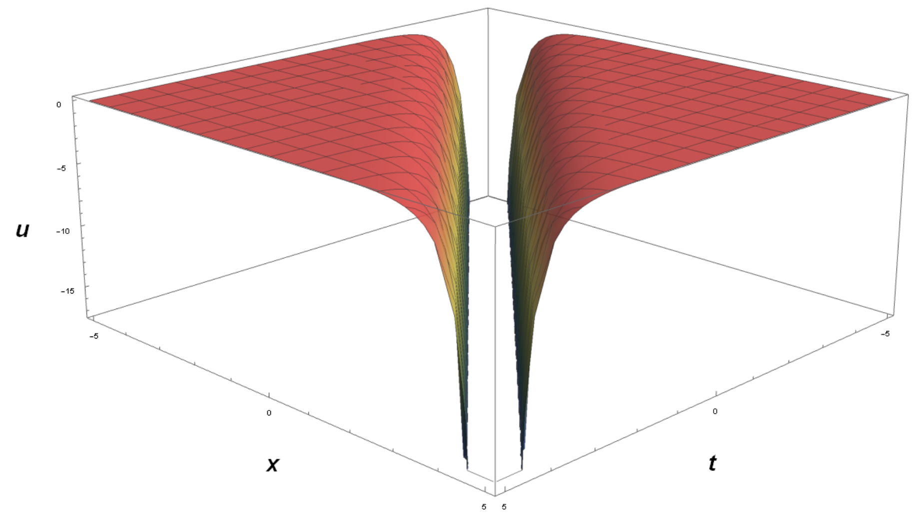

Figure 1. 3D plot.

Graph of u_1 **by choosing** $a = -1,\ b = -1,\ c = -1,\ s = -1,\ \mathscr{H} = 1,\ \gamma_1 = 1,\ \gamma_2 = 0, y_1 = y_2 = \cdots = y_{N-1} = 0$.

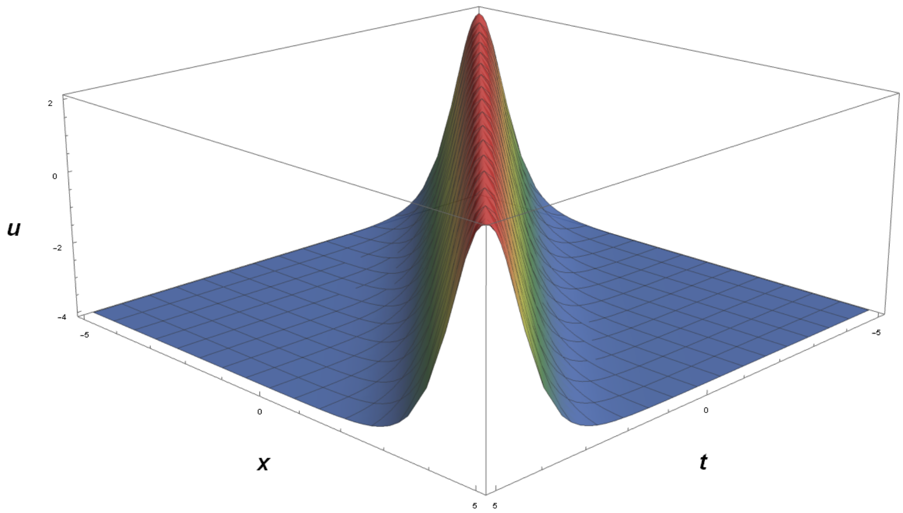

Figure 2. 3D plot.

Graph of u_2 **by choosing** $a = -1,\ b = -1,\ c = -1,\ s = -1,\ \mathscr{H} = 1,\ \gamma_1 = 1,\ \gamma_2 = 0, y_1 = y_2 = \cdots = y_{N-1} = 0.$

4.2.2. Trigonometric solutions ($s > 0$)

For $s > 0$, the following two sets of parameters yields the hyperbolic solutions for Eq (1.1).

- **Set 1:**

$$p_0 = -\frac{6\mathscr{H}^2 cs}{b},\ p_1 = 0,\ p_2 = -\frac{6\mathscr{H}^2 c}{b},\ q_1 = q_2 = 0,\ \mu = 4s\mathscr{H}^4 c,\ \upsilon = 0. \tag{4.11}$$

- **Set 2:**

$$p_0 = -\frac{2\mathscr{H}^2 cs}{b},\ p_1 = 0,\ p_2 = -\frac{6\mathscr{H}^2 c}{b},\ q_1 = q_2 = 0,\ \mu = -4s\mathscr{H}^4 c,\ \upsilon = 0. \tag{4.12}$$

By substituting these values of the parameters in Eq (4.6), we get solutions for Eq (4.5), which, when combined with the similarity variables (4.1), produces the solutions for Eq (1.1).

For Set 1**:**

We get the following solution for Eq (1.1) by plugging Eq (4.11) into Eq (4.6) and considering Eq (4.1).

$$u_3\left(t, x, y_1, y_2, \cdots, y_{N-1}\right) = -\frac{6\mathscr{H}^2 \sigma sc}{b\left(\gamma_1 \sin\left(\sqrt{s}\omega\right) + \gamma_2 \cos\left(\sqrt{s}\omega\right)\right)^2}, \tag{4.13}$$

where $\sigma = \gamma_1^2 + \gamma_2^2$ and $\omega = t - \mathscr{H}x - \theta_1 y_1 - \cdots - \theta_1 y_{N-1}$.

For Set 2**:**

We get the following solution for Eq (1.1) by plugging Eq (4.12) into Eq (4.6) and considering Eq (4.1).

$$u_4\left(t, x, y_1, y_2, \cdots, y_{N-1}\right) = -\frac{6\mathscr{H}^2 sc}{b}\left[\left(\frac{\gamma_1 \cos\left(\sqrt{s}\omega\right) - \gamma_2 \sin\left(\sqrt{s}\omega\right)}{\gamma_1 \sin\left(\sqrt{s}\omega\right) + \gamma_2 \cosh\left(\sqrt{s}\omega\right)}\right)^2 + \frac{1}{3}\right], \tag{4.14}$$

where $\omega = t - \mathscr{H}x - \theta_1 y_1 - \cdots - \theta_{N-1} y_{N-1}$.

The trigonometric exact solutions u_3 and u_4 represented by Eqs (4.13) and (4.14) are portrayed in Figures 3 and 4 respectively.

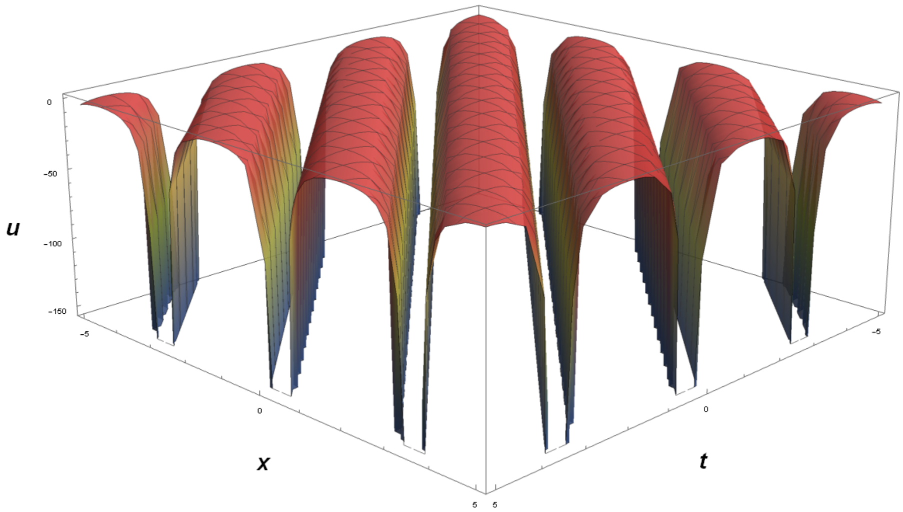

Figure 3. 3D plot.

Graph of u_3 **by choosing** $a = -1,\ b = -1,\ c = -1,\ s = 1,\ \mathscr{H} = 1,\ \gamma_1 = 1,\ \gamma_2 = 0, y_1 = y_2 = \cdots = y_{N-1} = 0$.

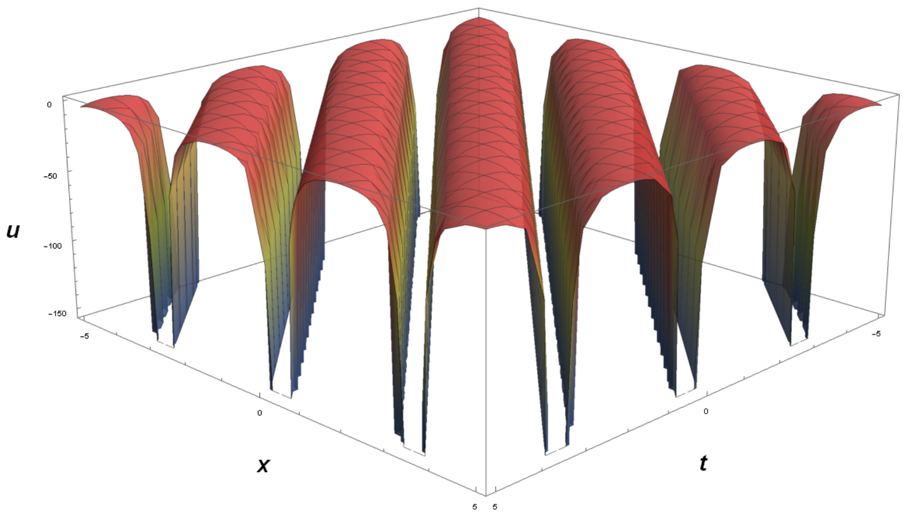

Figure 4. 3D plot.

Graph of u_4 **by choosing** $a = -1,\ b = -1,\ c = -1,\ s = 1,\ \mathscr{H} = 1,\ \gamma_1 = 1,\ \gamma_2 = 0, y_1 = y_2 = \cdots = y_{N-1} = 0$.

4.2.3. Rational solutions ($s = 0$)

When $s = 0$, We get a trivial solution

$$u_5\left(t, x, y_1, y_2, \cdots, y_{N-1}\right) = p_0, \tag{4.15}$$

for the set of parameters:

$$p_0 = p_0,\ p_1 = 0,\ p_2 = 0,\ q_1 = q_2 = 0,\ \mu = -\mathscr{H}^2 b p_0,\ \upsilon = 0. \tag{4.16}$$

Thus, Eq (1.1) does not have any nontrivial rational function solutions.

5. Nonlinear self-adjointness classification of Eq (1.1)

Suppose,

$$\mathfrak{U}\left(t, x, u, u_t, u_x, u_{xx}, u_{y_1y_1}, u_{y_2y_2}, \cdots, u_{y_{N-1}y_{N-1}}\right) = 0 \tag{5.1}$$

represents Eq (1.1) where

$$\mathfrak{U} = u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx} - \sum_{j=1}^{N-1} d_j u_{y_jy_j}\,. \tag{5.2}$$

Considering the nonlocal variable $\mathscr{P}\left(t, x, y_1, y_2, \cdots, y_{N-1}\right)$,the formal Lagrangian for Eq (1.1) is of the form:

$$\mathscr{L} = \mathscr{P}\left(t, x, y_1, y_2, \cdots, y_{N-1}\right)\left(u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx} - \sum_{j=1}^{N-1} d_j u_{y_jy_j}\right). \tag{5.3}$$

The corresponding adjoint equation of Eq (1.1) is:

$$\mathfrak{U}^* = \frac{\delta}{\delta u}\left[\mathscr{P}\left(u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx} - \sum_{j=1}^{N-1} d_j u_{y_jy_j}\right)\right] = 0, \tag{5.4}$$

where the Euler Lagrange operator $\delta/\delta u$ has the following form for Eq (1.1):

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - \mathscr{D}_x\frac{\partial}{\partial u_x} - \mathscr{D}_t^2\frac{\partial}{\partial u_{tt}} - \mathscr{D}_x^2\frac{\partial}{\partial u_{xx}} - \sum_{j=1}^{N-1}\mathscr{D}_{y_j}^2\frac{\partial}{\partial u_{y_jy_j}} - \mathscr{D}_x^4\frac{\partial}{\partial u_{xxxx}}, \tag{5.5}$$

in which the total derivatives are as follows:

$$\begin{aligned}
\mathscr{D}_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xxx}\frac{\partial}{\partial u_{xx}} + u_{xxxx}\frac{\partial}{\partial u_{xxx}} + u_{xxxxx}\frac{\partial}{\partial u_{xxxx}},\\
\mathscr{D}_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{ttt}\frac{\partial}{\partial u_{tt}},\\
\mathscr{D}_{y_j} &= \frac{\partial}{\partial y_j} + u_{y_j}\frac{\partial}{\partial u} + u_{y_jy_j}\frac{\partial}{\partial u_{y_j}} + u_{y_jy_jy_j}\frac{\partial}{\partial u_{y_jy_j}}\ \ (j = 1, \cdots, N-1)\,.
\end{aligned} \tag{5.6}$$

Using Eqs (5.5) and (5.6), Eq (5.4) becomes:

$$\mathfrak{U}^* = \mathscr{P}_{tt} - (a + 2bu)\,\mathscr{P}_{xx} - c\,\mathscr{P}_{xxxx} - \sum_{j=1}^{N-1} d_j \mathscr{P}_{y_j y_j} = 0. \tag{5.7}$$

Invoking the Definitions 3–5 for the pair of Eqs (5.1) and (5.7), we state the following theorem after reckoning.

Theorem: *Equation (1.1) is not strictly, quasi or weak self-adjoint, rather it is nonlinear self-adjoint (i.e.,* $\mathfrak{U} = \lambda \mathfrak{U}$, $\lambda = \Psi_u$*), with the nonlocal variable* $\mathscr{P} = \Psi$ *determined from the following system of PDEs:*

$$\Psi_{xx} = 0, \quad \Psi_{tt} = \sum_{j=1}^{N-1} d_j \Psi_{y_j y_j}, \quad \Psi_u = 0. \tag{5.8}$$

Remark: From above system of PDEs (5.8), it is clear that the nonlocal variable $\mathscr{P} = \Psi$ is independent of variable u, i.e., $\mathscr{P} = \Psi\,(t, x, y_1, \cdots, y_{N-1})$.

Consider the scenario when Eq (1.1) reflects the $(1 + 1)$ and $(2 + 1)$ dimensional Boussinesq equations (1.2) and (1.3).

- ***Nonlocal variable for*** $N = 1$***:***

For $N = 1$, the system of PDEs (5.8) takes the form:

$$\Psi_{xx} = 0, \quad \Psi_{tt} = 0, \quad \Psi_u = 0, \tag{5.9}$$

which upon solving yields

$$\mathscr{P} = \Psi\,(t, x, u) = C_1 xt + C_2 x + C_3 t + C_4. \tag{5.10}$$

- ***Nonlocal variable for*** $N = 2$***:***

For $N = 2$, the system of PDEs (5.8) takes the form:

$$\Psi_{xx} = 0, \quad \Psi_{tt} = d_1 \Psi_{y_1 y_1}, \quad \Psi_u = 0, \tag{5.11}$$

For sake of convenience, we remove the indices in Eq (5.11) (*i.e.,* taking $d_1 = d$ and $y_1 = y$). Thus solving

$$\Psi_{xx} = 0, \quad \Psi_{tt} = d\Psi_{yy}, \quad \Psi_u = 0,$$

yields

$$\mathscr{P} = \Psi\,(t, x, y, u) = (x + 1)\left[f\left(t + \frac{y}{\sqrt{d}}\right) + g\left(t - \frac{y}{\sqrt{d}}\right)\right]. \tag{5.12}$$

where f and g are arbitrary functions of its arguments.

6. Conservation laws

Components of conserved vector for Eq (1.1) corresponding to Lie symmetries (3.5) can be determined using Ibragimov's Theorem (1) using Eq (2.5).

If $\mathscr{T} = (\mathscr{T}^t, \mathscr{T}^x, \mathscr{T}^{y_1}, \cdots, \mathscr{T}^{y_{N-1}})$ is a conserved vector for Eq (1.1), then it must satisfy the conservation law:

$$\left[\mathscr{D}_t\left(\mathscr{T}^t\right) + \mathscr{D}_x\left(\mathscr{T}^x\right) + \mathscr{D}_{y_1}\left(\mathscr{T}^{y_1}\right) + \cdots + \mathscr{D}_{y_{N-1}}\left(\mathscr{T}^{y_{N-1}}\right)\right]_{(1.1)} = 0. \tag{6.1}$$

Equation (2.5) yields the following components of conserved vector $\mathscr{T}$ for Eq (1.1):

$$\begin{aligned}
\mathscr{T}^t &= \xi_\ell^0 \mathscr{L} - \mathcal{W}_\ell \mathscr{D}_t\left(\frac{\partial \mathscr{L}}{\partial u_{tt}}\right) + \mathscr{D}_t\left(\mathcal{W}_\ell\right)\left(\frac{\partial \mathscr{L}}{\partial u_{tt}}\right),\\
\mathscr{T}^x &= \xi_\ell^1 \mathscr{L} + \mathcal{W}_\ell\left[\frac{\partial \mathscr{L}}{\partial u_x} - \mathscr{D}_x\left(\frac{\partial \mathscr{L}}{\partial u_{xx}}\right) - \mathscr{D}_x^3\left(\frac{\partial \mathscr{L}}{\partial u_{xxxx}}\right)\right] + \mathscr{D}_x^3\left(\mathcal{W}_\ell\right)\left(\frac{\partial] \mathscr{L}}{\partial u_{xxxx}}\right)\\
&\quad + \mathscr{D}_x\left(\mathcal{W}_\ell\right)\left[\frac{\partial \mathscr{L}}{\partial u_{xx}} + \mathscr{D}_x^2\left(\frac{\partial \mathscr{L}}{\partial u_{xxxx}}\right)\right] - \mathscr{D}_x^2\left(\mathcal{W}_\ell\right)\mathscr{D}_x\left(\frac{\partial \mathscr{L}}{\partial u_{xxxx}}\right),\\
\mathscr{T}^{y_j} &= \eta_\ell^{y_j} \mathscr{L} - \mathcal{W}_\ell \mathscr{D}_{y_j}\left(\frac{\partial \mathscr{L}}{\partial u_{y_j y_j}}\right) + \mathscr{D}_{y_j}\left(\mathcal{W}_\ell\right)\left(\frac{\partial \mathscr{L}}{\partial u_{y_j y_j}}\right), \quad \begin{array}{l} j = 1, \cdots, N-1 \\ \ell = 1, \cdots, \frac{N^2+N+4}{2} \end{array},
\end{aligned} \tag{6.2}$$

where

$$\mathcal{W}_\ell = \zeta_\ell - \xi_\ell^0 \frac{\partial u}{\partial t} - \xi_\ell^1 \frac{\partial u}{\partial x} - \sum_{j=1}^{N-1} \eta_\ell^{y_j} \frac{\partial^2 u}{\partial y_j^2} \quad (\ell = 1, \cdots, \frac{N^2+N+4}{2}), \tag{6.3}$$

denotes the Lie characteristic function corresponding to Lie point symmetries X_ℓ (3.5) and $\mathscr{L} = \mathscr{P}\mathfrak{U}$ represents the formal Lagrangian in which the nonlocal variable $\mathscr{P}$ satisfies the PDE system (5.8).

Conservation laws of generalized fourth order Boussinesq equation [63] have been computed using multiplier approach. Moreover, infinite conservation laws for Eq (1.3) have also computed using Bell polynomials [64].

In the following subsections, we shall compute the components of conserved vectors for Eqs (1.2) and (1.3) corresponding to the classical Lie point symmetries (3.6) and (3.7), respectively by using the nonlocal conservation laws.

6.1. Conserved vectors for Eq (1.2)

The nonlocal conservation law for Eq (1.2) is given by:

$$\left[\mathscr{D}_t\left(\mathscr{T}^t\right) + \mathscr{D}_x\left(\mathscr{T}^x\right)\right]_{(1.2)} = 0, \tag{6.4}$$

where $\mathscr{T} = (\mathscr{T}^t, \mathscr{T}^x)$ represents the conserved vector for Eq (1.2).

The components of conserved vector $\mathscr{T}^t$ and $\mathscr{T}^x$ are followed by Eq (6.2) as:

$$\begin{aligned}
\mathscr{T}^t &= \xi_\ell^0 \mathscr{L} - \mathcal{W}_\ell \mathscr{D}_t\left(\frac{\partial \mathscr{L}}{\partial u_{tt}}\right) + \mathscr{D}_t\left(\mathcal{W}_\ell\right)\left(\frac{\partial \mathscr{L}}{\partial u_{tt}}\right),\\
\mathscr{T}^x &= \xi_\ell^1 \mathscr{L} + \mathcal{W}_\ell\left[\frac{\partial \mathscr{L}}{\partial u_x} - \mathscr{D}_x\left(\frac{\partial \mathscr{L}}{\partial u_{xx}}\right) - \mathscr{D}_x^3\left(\frac{\partial \mathscr{L}}{\partial u_{xxxx}}\right)\right] + \mathscr{D}_x^3\left(\mathcal{W}_\ell\right)\left(\frac{\partial] \mathscr{L}}{\partial u_{xxxx}}\right)\\
&\quad + \mathscr{D}_x\left(\mathcal{W}_\ell\right)\left[\frac{\partial \mathscr{L}}{\partial u_{xx}} + \mathscr{D}_x^2\left(\frac{\partial \mathscr{L}}{\partial u_{xxxx}}\right)\right] - \mathscr{D}_x^2\left(\mathcal{W}_\ell\right)\mathscr{D}_x\left(\frac{\partial \mathscr{L}}{\partial u_{xxxx}}\right),
\end{aligned} \tag{6.5}$$

where $\mathcal{W}_\ell = \zeta_\ell - \xi^0_\ell u_t - \xi^1_\ell u_x$, $(\ell = 1, 2, 3)$ are the Lie characteristic functions corresponding to Lie point symmetries X_ℓ (3.6), given by:

$$\mathcal{W}_1 = -u_t, \;\; \mathcal{W}_2 = -u_x, \;\; \mathcal{W}_3 = a + 2bu + 2btu_t + bxu_x, \tag{6.6}$$

and $\mathscr{L}$ is the the formal Lagrangian

$$\mathscr{L} = \mathscr{P}\mathfrak{U}, \;\; \mathfrak{U} = u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx}, \tag{6.7}$$

with nonlocal variable $\mathscr{P}$ given by Eq (5.11).
Case 1. For $X_1 = \frac{\partial}{\partial t}$

Substituting $\xi^0_1 = 1$, $\xi^1_1 = \zeta_1 = 0$, $\mathcal{W}_1 = -u_t$ and $\mathscr{L}$ from Eq (6.7), in the pair of Eq (6.5), yields

$$\begin{aligned} \mathscr{T}^t &= (C_1x + C_3)\,u_t - (C_1xt + C_2x + C_3t + C_4)\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx}\right), \\ \mathscr{T}^x &= \left(au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}\right)(C_1xt + C_2x + C_3t + C_4) - (cu_{txx} - (a + 2bu)\,u_t)\,(C_1t + C_2)\,. \end{aligned} \tag{6.8}$$

Surely, the conservation law (6.4) is satisfied, since

$$\mathscr{D}_t\left(\mathscr{T}^t\right) + \mathscr{D}_x\left(\mathscr{T}^x\right) = (C_1x + C_3)\,\mathfrak{U}.$$

Four pairs of components of conserved vectors of Eq (1.2) can be obtained from Eq (6.8), w.r.t. set of arbitrary constants $\{C_1, C_2, C_3, C_4\}$.

- ***For*** C_1***=1,*** $C_2 = C_3 = C_4 = 0$

$$\begin{cases} \mathscr{T}^t = xu_t - xt\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx}\right), \\ \mathscr{T}^x = xt\left(au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}\right) - t\,(cu_{txx} - (a + 2bu)\,u_t)\,. \end{cases}$$

- ***For*** C_2***=1,*** $C_1 = C_3 = C_4 = 0$

$$\begin{cases} \mathscr{T}^t = -x\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx}\right), \\ \mathscr{T}^x = x\left(au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}\right) + (a + 2bu)\,u_t - cu_{txx}. \end{cases}$$

- ***For*** C_3***=1,*** $C_1 = C_2 = C_4 = 0$

$$\begin{cases} \mathscr{T}^t = u_t - t\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx}\right), \\ \mathscr{T}^x = t\left(au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}\right). \end{cases}$$

- ***For*** C_4***=1,*** $C_1 = C_2 = C_3 = 0$

$$\begin{cases} \mathscr{T}^t = -au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx}, \\ \mathscr{T}^x = au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}. \end{cases}$$

Case 2. For $X_2 = \frac{\partial}{\partial x}$ Substituting $\xi^1_2 = 1$, $\xi^0_2 = \zeta_2 = 0$, $\mathcal{W}_2 = -u_x$ and $\mathscr{L}$ from Eq (6.7), in the pair of Eq (6.5), yields

$$\begin{aligned} \mathscr{T}^t &= (C_1x + C_3)\,u_x - (C_1xt + C_2x + C_3t + C_4)\,u_{tx}, \\ \mathscr{T}^x &= (C_1xt + C_2x + C_3t + C_4)\,u_{tt} - (cu_{xxx} + (a + 2bu)\,u_x)\,(C_1t + C_2)\,. \end{aligned} \tag{6.9}$$

Clearly, these components satisfy conservation law (6.4), since

$$\mathscr{D}_t(\mathscr{T}^t)+\mathscr{D}_x(\mathscr{T}^x)=(C_1t+C_2)\,\mathfrak{U}.$$

Four pairs of components of conserved vectors of Eq (1.2) can be obtained from Eq (6.9), w.r.t. set of arbitrary constants $\{C_1, C_2, C_3, C_4\}$.

- ***For** C_1**=1,** $C_2=C_3=C_4=0$

$$\begin{cases}\mathscr{T}^t=xu_x-xtu_{tx},\\ \mathscr{T}^x=xtu_{tt}-t\left(cu_{xxx}+(a+2bu)\,u_x\right).\end{cases}$$

- ***For** C_2**=1,** $C_1=C_3=C_4=0$

$$\begin{cases}\mathscr{T}^t=-xtu_{tx},\\ \mathscr{T}^x=xu_{tt}-cu_{xxx}-(a+2bu)\,u_x.\end{cases}$$

- ***For** C_3**=1,** $C_1=C_2=C_4=0$

$$\begin{cases}\mathscr{T}^t=u_x-tu_{tx},\\ \mathscr{T}^x=tu_{tt}.\end{cases}$$

- ***For** C_4**=1,** $C_1=C_2=C_3=0$

$$\begin{cases}\mathscr{T}^t=-u_{tx},\\ \mathscr{T}^x=u_{tt}.\end{cases}$$

Case 3. For $X_3=-2bt\frac{\partial}{\partial t}-bx\frac{\partial}{\partial x}+(a+2bu)\frac{\partial}{\partial u}$
Substituting $\xi_3^0=-2bt,\ \xi_3^1=-bx,\ \zeta_3=a+2bu,\ \mathcal{W}_3=a+2bu+2btu_t+bxu_x$ and $\mathscr{L}$ from Eq (6.7), in the pair of Eq (6.5), yields

$$\begin{aligned}\mathscr{T}^t=&b\left(xu_{tx}+2t\left(au_{xx}+b\left(u^2\right)_{xx}+cu_{xxxx}\right)+4u_t\right)(C_1xt+C_2x+C_3t+C_4)\\ &-(a+2bu+2btu_t+bxu_x)(C_1x+C_3),\\ \mathscr{T}^x=&\left[(a+2bu)(a+2bu+bxu_x+2btu_t)+bc\left(xu_{xxx}+2tu_{txx}+4u_{xx}\right)\right](C_1t+C_2)\\ &-b\left[xu_{tt}+(a+2bu)(5u_x+2tu_{tx})+4btu_tu_x+c\left(5u_{xxx}+2tu_{txxx}\right)\right](C_1xt+C_2x+C_3t+C_4).\end{aligned}\tag{6.10}$$

Evidently, these components satisfy conservation law (6.4), since

$$\mathscr{D}_t(\mathscr{T}^t)+\mathscr{D}_x(\mathscr{T}^x)=(2C_2bx+C_3bt+3C_4b)\,\mathfrak{U}.$$

Four pairs of components of conserved vectors of Eq (1.2) can be obtained from Eq (6.10), w.r.t. set of arbitrary constants $\{C_1, C_2, C_3, C_4\}$.

- ***For** C_1**=1,** $C_2=C_3=C_4=0$

$$\begin{cases}\mathscr{T}^t=bxt\left(xu_{tx}+2t\left(au_{xx}+b\left(u^2\right)_{xx}+cu_{xxxx}\right)+2u_t\right)-x\left(a+2bu+bxu_x\right),\\ \mathscr{T}^x=\quad t\left[(a+2bu)\left(a+2bu-2b\left(2xu_x-tu_t\right)\right)+bc\left(2tu_{txx}+4u_{xx}\right)\right]\\ \qquad -bxt\left[xu_{tt}+2t(a+2bu)\,u_{tx}+4btu_tu_x+2c\left(2cu_{xxx}+tu_{txxx}\right)\right].\end{cases}$$

- ***For*** C_2***=1,*** $C_1 = C_3 = C_4 = 0$

$$\begin{cases} \mathscr{T}^t = bx\left(xu_{tx} + 2t\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx}\right) + 4u_t\right), \\ \mathscr{T}^x = \quad t\,(a+2bu)\,(a+2bu-2b\,(2xu_x - tu_t)) + bc\,(2tu_{txx} + 4u_{xx}) \\ \qquad\quad -bx\,[xu_{tt} + 2t\,(a+2bu)\,u_{tx} + 4btu_tu_x + 2c\,(2cu_{xxx} + tu_{txxx})]\,. \end{cases}$$

- ***For*** C_3***=1,*** $C_1 = C_2 = C_4 = 0$

$$\begin{cases} \mathscr{T}^t = bt\left(xu_{tx} + 2t\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx}\right) + 2u_t\right) - (a + 2bu + bxu_x)\,, \\ \mathscr{T}^x = -bt\,(xu_{tt} + (a+2bu)\,(5u_x + 2tu_{tx}) + c\,(5u_{xxx} + 2tu_{txxx}) + 4btu_tu_x)\,. \end{cases}$$

- ***For*** C_4***=1,*** $C_1 = C_2 = C_3 = 0$

$$\begin{cases} \mathscr{T}^t = bt\left(xu_{tx} + 2t\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx}\right) + 2u_t\right) - (a + 2bu + bxu_x)\,, \\ \mathscr{T}^x = -b\,(xu_{tt} + (a+2bu)\,(5u_x + 2tu_{tx}) + c\,(5u_{xxx} + 2tu_{txxx}) + 4btu_tu_x)\,. \end{cases}$$

6.1.1. Conserved vectors for Eq (1.3)

The nonlocal conservation law for Eq (1.3) is given by:

$$\left[\mathscr{D}_t\left(\mathscr{T}^t\right) + \mathscr{D}_x\left(\mathscr{T}^x\right) + \mathscr{D}_y\left(\mathscr{T}^y\right)\right]_{(1.3)} = 0, \tag{6.11}$$

where $\mathscr{T} = (\mathscr{T}^t, \mathscr{T}^x, \mathscr{T}^y)$ represents the conserved vector for Eq (1.3).

The components of conserved vector $\mathscr{T}^t$, $\mathscr{T}^x$ and $\mathscr{T}^y$ are followed by Eq (6.2) as:

$$\begin{aligned} \mathscr{T}^t &= \xi^0_\ell \mathscr{L} - \mathcal{W}_\ell \mathscr{D}_t\left(\frac{\partial \mathscr{L}}{\partial u_{tt}}\right) + \mathscr{D}_t\left(\mathcal{W}_\ell\right)\left(\frac{\partial \mathscr{L}}{\partial u_{tt}}\right), \\ \mathscr{T}^x &= \xi^1_\ell \mathscr{L} + \mathcal{W}_\ell\left[\frac{\partial \mathscr{L}}{\partial u_x} - \mathscr{D}_x\left(\frac{\partial \mathscr{L}}{\partial u_{xx}}\right) - \mathscr{D}^3_x\left(\frac{\partial \mathscr{L}}{\partial u_{xxxx}}\right)\right] + \mathscr{D}^3_x\left(\mathcal{W}_\ell\right)\left(\frac{\partial]\mathscr{L}}{\partial u_{xxxx}}\right) \\ &\quad + \mathscr{D}_x\left(\mathcal{W}_\ell\right)\left[\frac{\partial \mathscr{L}}{\partial u_{xx}} + \mathscr{D}^2_x\left(\frac{\partial \mathscr{L}}{\partial u_{xxxx}}\right)\right] - \mathscr{D}^2_x\left(\mathcal{W}_\ell\right)\mathscr{D}_x\left(\frac{\partial \mathscr{L}}{\partial u_{xxxx}}\right), \\ \mathscr{T}^y &= \eta_\ell \mathscr{L} - \mathcal{W}_\ell \mathscr{D}_y\left(\frac{\partial \mathscr{L}}{\partial u_{yy}}\right) + \mathscr{D}_y\left(\mathcal{W}_\ell\right)\left(\frac{\partial \mathscr{L}}{\partial u_{yy}}\right), \end{aligned} \tag{6.12}$$

where $\mathcal{W}_\ell = \zeta_\ell - \xi^0_\ell u_t - \xi^1_\ell u_x - \eta_\ell u_y$, $(\ell = 1, 2, 3, 4, 5)$ are the Lie characteristic functions corresponding to Lie point symmetries X_ℓ (3.7), given by:

$$\mathcal{W}_1 = -u_t,\ \mathcal{W}_2 = -u_x,\ \mathcal{W}_3 = -u_y,\ \mathcal{W}_4 = -\frac{y}{d}u_t - tu_y,\ \mathcal{W}_5 = a + 2bu + 2btu_t + bxu_x + 2byu_y, \tag{6.13}$$

and $\mathscr{L}$ is the the formal Lagrangian

$$\mathscr{L} = \mathscr{P}\mathfrak{U},\ \ \mathfrak{U} = u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx} - du_{yy}, \tag{6.14}$$

with nonlocal variable $\mathscr{P}$ given by Eq (5.12).

Case 1. For $X_1 = \frac{\partial}{\partial t}$

Substituting $\xi_1^0 = 1,\ \xi_1^1 = \eta_1 = \zeta_1 = 0, \mathcal{W}_1 = -u_t$ and $\mathscr{L}$ from Eq (6.14), in the pair of Eq (6.12), yields

$$\begin{aligned} \mathscr{T}^t &= (x+1)\left[(f'+g')\,u_t - (f+g)\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + du_{yy}\right)\right], \\ \mathscr{T}^x &= (f+g)\left[(x+1)\left(au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}\right) - \left(au_t + b\left(u^2\right)_t + cu_{txx}\right)\right], \\ \mathscr{T}^y &= (x+1)\left[d\,(f+g)\,u_{ty} - \sqrt{d}\,(f'-g')\,u_t\right], \end{aligned} \tag{6.15}$$

where $f = f\left(t + \frac{y}{\sqrt{d}}\right)$ and $g = g\left(t - \frac{y}{\sqrt{d}}\right)$ are the arbitrary functions and, f' and g' denote their first order explicit derivatives.

Surely, the conservation law (6.11) is satisfied, since

$$\mathscr{D}_t\left(\mathscr{T}^t\right) + \mathscr{D}_x\left(\mathscr{T}^x\right) + \mathscr{D}_y\left(\mathscr{T}^y\right) = (x+1)\,(f'+g')\,\mathfrak{U}.$$

The components of conserved vectors of Eq (1.3), w.r.t the arbitrary functions f and g are followed by Eq (6.15) as:

- ***For*** $f \neq 0$, ***and*** $g = 0$:

$$\begin{cases} \mathscr{T}^t = (x+1)\left(f'u_t - f\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + du_{yy}\right)\right), \\ \mathscr{T}^x = f\,(x+1)\left(au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}\right) - f\left(au_t + b\left(u^2\right)_t + cu_{txx}\right), \\ \mathscr{T}^y = (x+1)\left(f\,du_{ty} - f'\,\sqrt{d}u_t\right). \end{cases}$$

- ***For*** $f = 0$, ***and*** $g \neq 0$:

$$\begin{cases} \mathscr{T}^t = (x+1)\left(g'u_t - g\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + du_{yy}\right)\right), \\ \mathscr{T}^x = g\,(x+1)\left(au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}\right) - g\left(au_t + b\left(u^2\right)_t + cu_{txx}\right), \\ \mathscr{T}^y = (x+1)\left(g\,du_{ty} + g'\,\sqrt{d}u_t\right). \end{cases}$$

Case 2. For $X_2 = \frac{\partial}{\partial x}$

Substituting $\xi_2^1 = 1,\ \xi_2^0 = \eta_2 = \zeta_2 = 0, \mathcal{W}_2 = -u_x$ and $\mathscr{L}$ from Eq (6.14), in the pair of Eq (6.12), yields

$$\begin{aligned} \mathscr{T}^t &= (x+1)\left[(f'+g')\,u_x - (f+g)\,u_{tx}\right], \\ \mathscr{T}^x &= (f+g)\left[(x+1)\left(u_{tt} - du_{yy}\right) - \left(au_x + b\left(u^2\right)_x + cu_{xxx}\right)\right], \\ \mathscr{T}^y &= (x+1)\left[d\,(f+g)\,u_{xy} - \sqrt{d}\,(f'-g')\,u_x\right], \end{aligned} \tag{6.16}$$

where $f = f\left(t + \frac{y}{\sqrt{d}}\right)$ and $g = g\left(t - \frac{y}{\sqrt{d}}\right)$ are the arbitrary functions and, f' and g' represent their first order explicit derivatives.

Clearly, the conservation law (6.11) is satisfied, since

$$\mathscr{D}_t\left(\mathscr{T}^t\right) + \mathscr{D}_x\left(\mathscr{T}^x\right) + \mathscr{D}_y\left(\mathscr{T}^y\right) = (f+g)\,\mathfrak{U}.$$

The components of conserved vectors of Eq (1.3), w.r.t the arbitrary functions f and g are followed by Eq (6.16) as:

- ***For*** $f \neq 0$, ***and*** $g = 0$**:**

$$\begin{cases} \mathscr{T}^t = (x+1)\,(f' u_x - f u_{tx})\,, \\ \mathscr{T}^x = f\,(x+1)\left(u_{tt} - d u_{yy}\right) - f\left(a u_x + b\left(u^2\right)_x + c u_{xxx}\right), \\ \mathscr{T}^y = (x+1)\left(f\, d u_{xy} - f'\ \sqrt{d} u_x\right). \end{cases}$$

- ***For*** $f = 0$, ***and*** $g \neq 0$**:**

$$\begin{cases} \mathscr{T}^t = (x+1)\,(g' u_x - g u_{tx})\,, \\ \mathscr{T}^x = g\,(x+1)\left(u_{tt} - d u_{yy}\right) - g\left(a u_x + b\left(u^2\right)_x + c u_{xxx}\right), \\ \mathscr{T}^y = (x+1)\left(g\, d u_{xy} + g'\ \sqrt{d} u_x\right). \end{cases}$$

Case 3. For $X_3 = \frac{\partial}{\partial y}$
Substituting $\eta_3 = 1$, $\xi_3^0 = \xi_3^1 = \zeta_3 = 0$, $\mathcal{W}_3 = -u_y$ and $\mathscr{L}$ from Eq (6.14), in the pair of Eq (6.12), yields

$$\begin{aligned} \mathscr{T}^t &= (x+1)\left[(f'+g')\,u_y - (f+g)\,u_{ty}\right], \\ \mathscr{T}^x &= (f+g)\left[(x+1)\left(a u_{xy} + b\left(u^2\right)_{xy} + c u_{xxxy}\right) - \left(a u_y + b\left(u^2\right)_y + c u_{xxy}\right)\right], \\ \mathscr{T}^y &= (x+1)\left[(f+g)\left(u_{tt} - a u_{xx} - b\left(u^2\right)_{xx} - c u_{xxxx}\right) - \sqrt{d}\,(f'-g')\,u_y\right], \end{aligned} \tag{6.17}$$

where $f = f\left(t + \frac{y}{\sqrt{d}}\right)$ and $g = g\left(t - \frac{y}{\sqrt{d}}\right)$ are the arbitrary functions and, f' and g' denote their first order explicit derivatives.

Clearly, the conservation law (6.11) is satisfied, since

$$\mathscr{D}_t\,(\mathscr{T}^t) + \mathscr{D}_x\,(\mathscr{T}^x) + \mathscr{D}_y\,(\mathscr{T}^y) = -\,(x+1)\left(\frac{f'-g'}{\sqrt{d}}\right)\mathfrak{U}.$$

The components of conserved vectors of Eq (1.3), w.r.t the arbitrary functions f and g are followed by Eq (6.17) as:

- ***For*** $f \neq 0$, ***and*** $g = 0$**:**

$$\begin{cases} \mathscr{T}^t = (x+1)\left(f' u_y - f u_{ty}\right), \\ \mathscr{T}^x = f\,(x+1)\left(a u_{xy} + b\left(u^2\right)_{xy} + c u_{xxxy}\right) - f\left(a u_y + b\left(u^2\right)_y + c u_{xxy}\right), \\ \mathscr{T}^y = (x+1)\left[f\left(u_{tt} - a u_{xx} - b\left(u^2\right)_{xx} - c u_{xxxx}\right) - f'\ \sqrt{d} u_y\right]. \end{cases}$$

- ***For*** $f = 0$, ***and*** $g \neq 0$**:**

$$\begin{cases} \mathscr{T}^t = (x+1)\left(g' u_y - g u_{ty}\right), \\ \mathscr{T}^x = g\,(x+1)\left(a u_{xy} + b\left(u^2\right)_{xy} + c u_{xxxy}\right) - g\left(a u_y + b\left(u^2\right)_y + c u_{xxy}\right), \\ \mathscr{T}^y = (x+1)\left[g\left(u_{tt} - a u_{xx} - b\left(u^2\right)_{xx} - c u_{xxxx}\right) + g'\ \sqrt{d} u_y\right]. \end{cases}$$

Case 4. For $X_4 = \frac{\partial}{\partial y}$

Substituting $\xi_4^0 = \frac{y}{d}$, $\eta_4 = t$, $\xi_4^1 = \zeta_4 = 0$, $\mathcal{W}_4 = -\frac{y}{d}u_t - tu_y$ and $\mathscr{L}$ from Eq (6.14), in the pair of Eq (6.12), yields

$$\begin{aligned}\mathscr{T}^t &= (x+1)(f'+g')\left(\frac{y}{d}u_t + tu_y\right) - (x+1)(f+g)\left(\left(tu_y\right)_t + \frac{y}{d}\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + du_{yy}\right)\right),\\ \mathscr{T}^x &= (f+g)(x+1)\left[\frac{y}{d}\left(au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}\right) + t\left(au_{xy} + b\left(u^2\right)_{xy} + cu_{xxxy}\right)\right]\\ &\quad - (f+g)\left[\frac{y}{d}\left(au_t + b\left(u^2\right)_t + cu_{txx}\right) + t\left(au_y + b\left(u^2\right)_y + cu_{xxy}\right)\right],\\ \mathscr{T}^y &= (x+1)(f+g)\left((yu_t)_y + t\left(u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx}\right)\right) - (x+1)(f'-g')\left(t\sqrt{d}u_y + \frac{y}{\sqrt{d}}u_t\right),\end{aligned} \tag{6.18}$$

where $f = f\left(t + \frac{y}{\sqrt{d}}\right)$ and $g = g\left(t - \frac{y}{\sqrt{d}}\right)$ are the arbitrary functions and, f' and g' represent their first order explicit derivatives.

Evidently, the conservation law (6.11) is satisfied, since

$$\mathscr{D}_t\left(\mathscr{T}^t\right) + \mathscr{D}_x\left(\mathscr{T}^x\right) + \mathscr{D}_y\left(\mathscr{T}^y\right) = (x+1)\left(\frac{y}{d}(f'+g') + \frac{t}{\sqrt{d}}(f'-g')\right)\mathfrak{U}.$$

The components of conserved vectors of Eq (1.3), w.r.t the arbitrary functions f and g are followed by Eq (6.18) as:

- ***For*** $f \neq 0$, ***and*** $g = 0$:

$$\begin{cases}\mathscr{T}^t = (x+1)\left[f'\left(\frac{y}{d}u_t + tu_y\right) - f\left(\left(tu_y\right)_t + \frac{y}{d}\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + du_{yy}\right)\right)\right],\\ \mathscr{T}^x = \; f(x+1)\left[\frac{y}{d}\left(au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}\right) + t\left(au_{xy} + b\left(u^2\right)_{xy} + cu_{xxxy}\right)\right]\\ \qquad\quad -f\left[\frac{y}{d}\left(au_t + b\left(u^2\right)_t + cu_{txx}\right) + t\left(au_y + b\left(u^2\right)_y + cu_{xxy}\right)\right],\\ \mathscr{T}^y = (x+1)\left[f\left((yu_t)_y + t\left(u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx}\right)\right) - f'\left(t\sqrt{d}u_y + \frac{y}{\sqrt{d}}u_t\right)\right].\end{cases}$$

- ***For*** $f = 0$, ***and*** $g \neq 0$:

$$\begin{cases}\mathscr{T}^t = (x+1)\left[g'\left(\frac{y}{d}u_t + tu_y\right) - g\left(\left(tu_y\right)_t + \frac{y}{d}\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + du_{yy}\right)\right)\right],\\ \mathscr{T}^x = \; g(x+1)\left[\frac{y}{d}\left(au_{tx} + b\left(u^2\right)_{tx} + cu_{txxx}\right) + t\left(au_{xy} + b\left(u^2\right)_{xy} + cu_{xxxy}\right)\right]\\ \qquad\quad -g\left[\frac{y}{d}\left(au_t + b\left(u^2\right)_t + cu_{txx}\right) + t\left(au_y + b\left(u^2\right)_y + cu_{xxy}\right)\right],\\ \mathscr{T}^y = (x+1)\left[g\left((yu_t)_y + t\left(u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx}\right)\right) + g'\left(t\sqrt{d}u_y + \frac{y}{\sqrt{d}}u_t\right)\right].\end{cases}$$

Case 5. For $X_5 = -2bt\frac{\partial}{\partial t} - bx\frac{\partial}{\partial x} - 2by\frac{\partial}{\partial y} + (a+2bu)\frac{\partial}{\partial u}$
Substituting $\xi_5^0 = -2bt$, $\xi_5^1 = -bx$, $\eta_4 = -2by$, $\zeta_4 = a + 2bu$, $\mathcal{W}_5 = a + 2bu + 2btu_t + bxu_x + 2byu_y$

and $\mathscr{L}$ from Eq (6.14), in the pair of Eq (6.12), yields

$$\begin{aligned}
\mathscr{T}^t =& b(x+1)(f+g)\left(4u_t + xu_{tx} + 2yu_{ty}\right) - (x+1)(f'+g')\left(a + bxu_x + 2b\left(u + tu_t + yu_y\right)\right) \\
&+ 2bt(x+1)(f+g)\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + du_{yy}\right), \\
\mathscr{T}^x =& (a+2bu)(f+g)\left(a + 2bu + bxu_x + 2b\left(tu_t + yu_y\right)\right) + bc(f+g)\left(xu_{xxx} + 2\left(tu_{txx} + yu_{xxy}\right) + 4u_{xx}\right) \\
&- b(f+g)(x+1)\left[xu_{tt} - dxu_{yy} + (a+2bu)\left(5u_x + 2tu_{tx} + 2yu_{xy}\right)\right] \\
&- b(f+g)(x+1)\left[4b\left(tu_t + yu_y\right)u_x + c\left(5u_{xxx} + 2tu_{txxx} + 2yu_{xxxy}\right)\right], \\
\mathscr{T}^y =& -(f+g)(x+1)\left[2by\left(u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx}\right) - d\left(bxu_{xy} + 4bu_y + 2btu_{ty}\right)\right] \\
&+ \sqrt{d}\left(a + 2bu + 2b\left(tu_t + yu_y\right) + bxu_x\right)(f'-g')(x+1).
\end{aligned} \tag{6.19}$$

where $f = f\left(t + \frac{y}{\sqrt{d}}\right)$ and $g = g\left(t - \frac{y}{\sqrt{d}}\right)$ are the arbitrary functions and, f' and g' represent their first order explicit derivatives.

Surely, the conservation law (6.11) is satisfied, since

$$\mathscr{D}_t(\mathscr{T}^t) + \mathscr{D}_x(\mathscr{T}^x) + \mathscr{D}_y(\mathscr{T}^y) = b\left[f + g - 2(x+1)\left(t(f'+g') + \frac{y}{\sqrt{d}}(f'-g')\right)\right]\mathfrak{U}.$$

The components of conserved vectors of Eq (1.3), w.r.t the arbitrary functions f and g are followed by Eq (6.19) as:

- ***For*** $f \neq 0$, ***and*** $g = 0$**:**

$$\begin{cases}
\mathscr{T}^t = (x+1)\left[bf\left(4u_t + xu_{tx} + 2yu_{ty}\right) - f'\left(a + bxu_x + 2b\left(u + tu_t + yu_y\right)\right)\right] \\
\qquad + 2btf(x+1)\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + du_{yy}\right), \\
\mathscr{T}^x = f(a+2bu)\left(a + 2bu + bxu_x + 2b\left(tu_t + yu_y\right)\right) + bcf\left(xu_{xxx} + 2\left(tu_{txx} + yu_{xxy}\right) + 4u_{xx}\right) \\
\qquad - bf(x+1)\left[xu_{tt} - dxu_{yy} + (a+2bu)\left(5u_x + 2tu_{tx} + 2yu_{xy}\right)\right] \\
\qquad - bf(x+1)\left[4b\left(tu_t + yu_y\right)u_x + c\left(5u_{xxx} + 2tu_{txxx} + 2yu_{xxxy}\right)\right], \\
\mathscr{T}^y = f'\sqrt{d}(x+1)\left(a + 2bu + 2b\left(tu_t + yu_y\right) + bxu_x\right) \\
\qquad - f(x+1)\left[2by\left(u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx}\right) - d\left(bxu_{xy} + 4bu_y + 2btu_{ty}\right)\right].
\end{cases}$$

- ***For*** $f = 0$, ***and*** $g \neq 0$**:**

$$\begin{cases}
\mathscr{T}^t = (x+1)\left[bg\left(4u_t + xu_{tx} + 2yu_{ty}\right) - g'\left(a + bxu_x + 2b\left(u + tu_t + yu_y\right)\right)\right] \\
\qquad + 2btg(x+1)\left(au_{xx} + b\left(u^2\right)_{xx} + cu_{xxxx} + du_{yy}\right), \\
\mathscr{T}^x = g(a+2bu)\left(a + 2bu + bxu_x + 2b\left(tu_t + yu_y\right)\right) + bcg\left(xu_{xxx} + 2\left(tu_{txx} + yu_{xxy}\right) + 4u_{xx}\right) \\
\qquad - bfg(x+1)\left[xu_{tt} - dxu_{yy} + (a+2bu)\left(5u_x + 2tu_{tx} + 2yu_{xy}\right)\right] \\
\qquad - bg(x+1)\left[4b\left(tu_t + yu_y\right)u_x + c\left(5u_{xxx} + 2tu_{txxx} + 2yu_{xxxy}\right)\right], \\
\mathscr{T}^y = -g'\sqrt{d}(x+1)\left(a + 2bu + 2b\left(tu_t + yu_y\right) + bxu_x\right) \\
\qquad - g(x+1)\left[2by\left(u_{tt} - au_{xx} - b\left(u^2\right)_{xx} - cu_{xxxx}\right) - d\left(bxu_{xy} + 4bu_y + 2btu_{ty}\right)\right].
\end{cases}$$

7. Conclusions

In this article, the $(N + 1)$-dimensional generalized Boissinesq equation was studied using the classical Lie theory of differential equations. Lie point symmetries and the corresponding group invariant solutions of the equation were obtained with the help of which new form of solutions were detected. Using the appropriate similarity variables, the considered equation was transformed into nonlinear ordinary differential equation, whose solutions were constructed by the aid of $(G'/G, 1/G)$ expansion method. Some of the significant solutions were graphically depicted. The concept of nonlinear self-adjointness was employed with the help of which the nonlocal conservation laws were obtained. This article's research findings are novel, more generic, and physically applicable.

Conflict of interest

The authors declare no conflict of interest.

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