Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 13104–13138.
DOI: 10.3934/math.2022724
Received: 12 March 2022
Revised: 17 April 2022
Accepted: 21 April 2022
Published: 10 May 2022

Research article

Generalization of rough sets using maximal right neighborhood systems and ideals with medical applications

Mona Hosny[1,2,*]

[1] Department of Mathematics, Faculty of Science for Girls, King Khalid University, Abha 61471, Saudi Arabia

[2] Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt

* **Correspondence:** Email: maly@kku.edu.sa, monahosny@edu.asu.edu.eg; Tel: +966540603780.

Abstract: Rough set theory is a mathematical technique to address the issues of uncertainty and vagueness in knowledge. An ideal is considered to be a crucial extension of this theory. It is an efficacious tool to dispose of vagueness and uncertainties by helping us to approximate the rough set in a more general manner. Minimizing the boundary region is one of the pivotal and substantial themes for studying the rough sets which consequently aim to maximize the accuracy measure. An ideal is one of the effective and successful followed methods to achieve this goal perfectly. So, the objective of this work is to present new methods for rough sets by using ideals. Some important characteristics of these methods are scrutinized and demonstrated to show that they yield accuracy measures greater and higher than the former ones in the other approaches. Finally, two medical applications are introduced to show the significance of utilizing the ideals in the proposed methods.

Keywords: rough sets; ideals; maximal right neighborhood; approximations spaces
Mathematics Subject Classification: 03E99, 54A05, 54A10, 54E99

1. Introduction

The problems of imprecision, ambiguity and incompleteness of information systems applied for data analysis has occupied the human mind for a long time. These problems exist in different fields such as economics, engineering, medical science and the social and environmental sciences. For years, mathematicians, engineers and scientists, particularly those who focus on artificial intelligence, have been seeking powerful tools to solve these problems. They suggested many techniques to achieve this aim such as the rough set theory [29, 30]. It relies mainly on two approximations namely, lower and upper approximations that are used to study the boundary region and accuracy measure. If the lower and upper approximations of the set are equal to each other, then it is called a crisp set; otherwise, it is known as a rough set. Therefore, the boundary region is defined as the difference between the upper and lower approximations. This boundary is usually associated with vagueness (i.e., the existence of objects that cannot be uniquely classified to the set or its complement). Hence, the accuracy of the set or ambiguity is dependent on whether the boundary region is empty or not respectively. A non-empty boundary region of a set means that our knowledge about the set is not sufficient to define the set precisely. So, one of the essential aims of this theory is to reduce the boundary region and increase the accuracy of the set. The classical rough approximations are based on an equivalence relation in a finite universe, but this relation is sometimes difficult to be obtained for real-world problems. Extensions of this theory were therefore studied by many approaches in order to deal with complex practical problems. One approach substituted the equivalence relations with tolerance or similarity relations [12, 13, 21, 25, 28, 32–35] or binary relations [3, 4, 8, 23, 31, 39–43].

Neighborhood systems are used to generalize the rough set theory by replacing an equivalence class with a neighborhood when defining approximations. Several types of neighborhoods were utilized to define the lower and upper approximations such as right and left neighborhoods [14, 37, 38], minimal right neighborhoods [2, 5] and minimal left neighborhoods [6]. Meanwhile, Abo-Tabl [1] defined the approximations by using minimal right neighborhoods, which were determined by reflexive relations that form the base of topological space. More recently, Dai et al. [11] presented three new types of approximations based on maximal right neighborhoods that were determined by similarity relations. The thing that distinguished Dai et al.'s approximations [11] from Abo-Tabl's approximations [1] was that the corresponding upper and lower approximations, boundary regions, accuracy measures and roughness measures for two types of Dai et al.'s approximations [11] had monotonicity. Therefore, Dai et al.'s approximations [11] were considered as an improvement of Abo-Tabl's approximations [1].

An ideal is a non-empty collection of sets that is closed under hereditary property and finite additivity [26, 36]. The interest in the idealized version of many rough set models has grown drastically in the past 10 years. The advantage and benefit of using an ideal in this theory is that it reduces the vagueness (uncertainty) of a concept to uncertainty areas at their borders by increasing the lower approximations and decreasing the upper approximations. Consequently, it minimizes the boundary region and improves the accuracy measure. So, the use of an ideal is a powerful method to demystify the concept and define it precisely. Accordingly, the study of this theory with ideals is an enjoyable topic that has received the attention of many researchers (see [9, 16–19, 22, 24, 27]). Therefore, ideals have been extensively applied to this theory.

We are aware of the fact that ideals play an important role in the study of rough sets, particularly for removing the vagueness. So, one of the primary motivations of this work is to introduce new methods for rough sets by using ideals. Moreover, the present work is focused on expressing the main concepts of rough sets by using maximal right neighborhoods deduced by binary relations not similarity relations as in the previous studies [11]. Binary relations extend the application field of rough sets, but the similarity relations do not always hold in many real-life applications. Therefore, this restriction prevents the wide application of this set as it is shown at the end of this paper. Consequently, the present approach is an extension of Dai et al.'s approaches [11]. This paper comprises eight sections and its sequence is as follows. After the Introduction. Section 2 outlines the necessary concepts and preliminaries required for the sequel to this work. The purpose of Sections 3–6 is to construct four methods to approximate the set by using the notion of ideals and maximal right neighborhoods induced by binary relations. The properties of the current approximations are interjected and analyzed. It is proved that the boundary of a subset decreases and the accuracy increases as the ideal increases (see Theorems 3.1, 4.1, 5.1 and 6.1). Afterwards, it is elucidated that the corresponding upper and lower approximations, boundary regions, accuracy measures and roughness measures of three types of these approximations are monotonic (see Theorems 3.2, 4.2 and 6.2). Furthermore, the relationships among these approximations are interpreted and illustrated. Additionally, Theorems 3.3, 4.4, 5.3 and 6.4 explain that the present methods reduce the boundary region by increasing the lower approximations and decreasing the upper approximations. These methods reduce them more effectively than Dai et al.'s methods [11]. They also show that the current accuracy is greater than the previous one in [11]. Hence, this work is a generalization and an improvement of Dai et al.'s work [11]. It is worthwhile to note that, with the current approximations if the ideal equals the empty set and the binary relation is a similarity relation, then these approximations coincide with Dai et al.'s approximations [11]. So, Dai et al.'s approximations [11] are a special case of the current approximations. In the end, medical applications are presented to illustrate the importance of using ideals in the present approaches. It plays an intrinsic and a substantial role in the decision making problems of two real life applications. It is pointed out that the current techniques allow the medical staff to classify patients successfully in terms of infection of the new coronavirus COVID-19 (first application) and heart attacks (second application). In the first application, a similarity relation is used while in the second application a binary relation is applied. Therefore, Dai et al.'s methods [11] cannot be applied for the second application. This is because they are based on the similarity relations and this restricts the applications of the rough set theory. It emphasizes that the present methods open the way for more applications. Eventually, the conclusion of this paper and remarks for future research work are discussed in Section 8.

2. Preliminaries

Definition 2.1. [20] A non-empty collection $\mathcal{I}$ of subsets of a set U is called an ideal on U if it satisfies the following conditions:

(1) $A \in \mathcal{I}$ and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
(2) $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$.

i.e., $\mathcal{I}$ is closed under finite unions and subsets.

Definition 2.2. [22] Let $\mathcal{I}_1, \mathcal{I}_2$ be two ideals on a non-empty set U. The smallest collection generated by $\mathcal{I}_1, \mathcal{I}_2$ is denoted by $\mathcal{I}_1 \vee \mathcal{I}_2$ and defined as

$$\mathcal{I}_1 \vee \mathcal{I}_2 = \{G \cup F : G \in \mathcal{I}_1, F \in \mathcal{I}_2\}. \tag{2.1}$$

Proposition 2.1. *[22] If $\mathcal{I}_1, \mathcal{I}_2$ are two ideals on a non-empty set U and A, B are two subsets of U, then the collection $\mathcal{I}_1 \vee \mathcal{I}_2$ is satisfied by the following conditions:*

(1) $\mathcal{I}_1 \vee \mathcal{I}_2 \neq \phi$,
(2) $A \in \mathcal{I}_1 \vee \mathcal{I}_2,\ B \subseteq A \Rightarrow B \in \mathcal{I}_1 \vee \mathcal{I}_2$,
(3) $A, B \in \mathcal{I}_1 \vee \mathcal{I}_2 \Rightarrow A \cup B \in \mathcal{I}_1 \vee \mathcal{I}_2$.

It means that the collection $\mathcal{I}_1 \vee \mathcal{I}_2$ is an ideal on U.

Definition 2.3. [29] Let R be an equivalence relation on a universe U and $[x]_R$ be the equivalence class containing x. For any subset A of U, the lower approximation $\underline{apr}(A)$ and upper approximation $\overline{apr}(A)$ are defined by

$$\underline{apr}(A) = \{x \in U : [x]_R \subseteq A\}. \tag{2.2}$$
$$\overline{apr}(A) = \{x \in U : [x]_R \cap A \neq \phi\}. \tag{2.3}$$

These approximations satisfy the following properties:

($\mathcal{L}_1$) $\underline{apr}(A^c) = [\overline{apr}(A)]^c$, where A^c is the complement of A.

($\mathcal{L}_2$) $\underline{apr}(U) = U$.

($\mathcal{L}_3$) $\underline{apr}(\phi) = \phi$.

($\mathcal{L}_4$) $\underline{apr}(A) \subseteq A$.

($\mathcal{L}_5$) $\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$

($\mathcal{L}_6$) $\underline{apr}(A \cup B) \supseteq \underline{apr}(A) \cup \underline{apr}(B)$

($\mathcal{L}_7$) $A \subseteq B \Rightarrow \underline{apr}(A) \subseteq \underline{apr}(B)$.

($\mathcal{L}_8$) $\underline{apr}(\underline{apr}(A)) = \underline{apr}(A)$.

($\mathcal{L}_9$) $\overline{apr}(A) \subseteq \underline{apr}(\overline{apr}(A))$.

($\mathcal{U}_1$) $\overline{apr}(A^c) = [\underline{apr}(A)]^c$.

($\mathcal{U}_2$) $\overline{apr}(U) = U$.

($\mathcal{U}_3$) $\overline{apr}(\phi) = \phi$.

($\mathcal{U}_4$) $A \subseteq \overline{apr}(A)$.

($\mathcal{U}_5$) $\overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B)$.

($\mathcal{U}_6$) $\overline{apr}(A \cap B) \subseteq \overline{apr}(A) \cap \overline{apr}(B)$.

($\mathcal{U}_7$) $A \subseteq B \Rightarrow \overline{apr}(A) \subseteq \overline{apr}(B)$.

($\mathcal{U}_8$) $\overline{apr}(\overline{apr}(A)) = \overline{apr}(A)$.

($\mathcal{U}_9$) $\overline{apr}(\underline{apr}(A)) \subseteq \underline{apr}(A)$.

Definition 2.4. [11] Let R be an arbitrary binary relation on a non-empty finite set U and $x \in U$, then $< x > \breve{R} = \cup\{pR : x \in pR\}$. $< x > \breve{R}$ is the union of all right neighborhoods containing x.

Theorem 2.1. *[7] Let U be a universal set and R_1, R_2 be two binary relations on U. If $R_1 \subseteq R_2$, then $< x > \breve{R}_1 \subseteq < x > \breve{R}_2, \forall x \in U$.*

Definition 2.5. [11] Let R be a similarity relation on a non-empty set U. For any subset $A \subseteq U$, the first kind of lower and upper approximations, boundary regions, accuracy and roughness of A according to R are respectively defined by

$$\underline{apr}_R(A) = \{x \in U :< x > \breve{R} \subseteq A\}. \tag{2.4}$$

$$\overline{apr}_R(A) = \{x \in U :< x > \breve{R} \cap A \neq \phi\}. \tag{2.5}$$

$$Boundary_R(A) = \overline{apr}_R(A) - \underline{apr}_R(A). \tag{2.6}$$

$$Accuracy_R(A) = |\frac{\underline{apr}_R(A)}{\overline{apr}_R(A)}|, \overline{apr}_R(A) \neq \phi. \tag{2.7}$$

$$Roughness_R(A) = 1 - Accuracy_R(A). \tag{2.8}$$

Definition 2.6. [11] Let R be a similarity relation on a non-empty set U. For any subset $A \subseteq U$, the second kind of lower and upper approximations, boundary regions, accuracy and roughness of A according to R are respectively defined by

$$\underline{apr'}_R(A) = \cup\{< x > \breve{R} :< x > \breve{R} \subseteq A\}. \tag{2.9}$$

$$\overline{apr'}_R(A) = (\underline{apr'}_R(A^c))^c. \tag{2.10}$$

$$Boundary'_R(A) = \overline{apr'}_R(A) - \underline{apr'}_R(A). \tag{2.11}$$

$$Accuracy'_R(A) = |\frac{\underline{apr'}_R(A)}{\overline{apr'}_R(A)}|, \overline{apr'}_R(A) \neq \phi. \tag{2.12}$$

$$Roughness'_R(A) = 1 - Accuracy'_R(A). \tag{2.13}$$

Definition 2.7. [11] Let R be a similarity relation on a non-empty set U. For any subset $A \subseteq U$, the third kind of upper and lower approximations, boundary regions, accuracy and roughness of A according to R are respectively defined by

$$\overline{apr''}_R(A) = \cup\{< x > \breve{R} :< x > \breve{R} \cap A \neq \phi\}. \tag{2.14}$$

$$\underline{apr''}_R(A) = (\overline{apr''}_R(A^c))^c. \tag{2.15}$$

$$Boundary''_R(A) = \overline{apr''}_R(A) - \underline{apr''}_R(A). \tag{2.16}$$

$$Accuracy''_R(A) = |\frac{\underline{apr''}_R(A)}{\overline{apr''}_R(A)}|, \overline{apr''}_R(A) \neq \phi. \tag{2.17}$$

$$Roughness''_R(A) = 1 - Accuracy''_R(A). \tag{2.18}$$

3. First method to obtain generalized rough sets using ideals

In this section, the first type of generalized rough approximations is presented. The principle properties of these approximations are studied and compared to the previous ones in [11] and shown to be more general.

Definition 3.1. Let R be a binary relation on a non-empty set U and $\mathcal{I}$ be an ideal on U. For any subset $A \subseteq U$, the first kind of generalized lower and upper approximations, boundary regions, accuracy and roughness of A using ideal and according to R are respectively defined by

$$\underline{apr}_R^{\mathcal{I}}(A) = \{x \in U :< x > \breve{R} \cap A^c \in \mathcal{I}\}. \tag{3.1}$$

$$\overline{apr}_R^{\mathcal{I}}(A) = \{x \in U :< x > \breve{R} \cap A \notin \mathcal{I}\}. \tag{3.2}$$

$$Boundary_R^{\mathcal{I}}(A) = \overline{apr}_R^{\mathcal{I}}(A) - \underline{apr}_R^{\mathcal{I}}(A). \tag{3.3}$$

$$Accuracy_R^{\mathcal{I}}(A) = \frac{|\underline{apr}_R^{\mathcal{I}}(A)|}{|\overline{apr}_R^{\mathcal{I}}(A)}|, \overline{apr}_R^{\mathcal{I}}(A) \neq \phi. \tag{3.4}$$

$$Roughness_R^{\mathcal{I}}(A) = 1 - Accuracy_R^{\mathcal{I}}(A). \tag{3.5}$$

Proposition 3.1. *Let $A, B \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. Then, the following properties hold:*

(1) $\overline{apr}_R^{\mathcal{I}}(\phi) = \phi$.

(2) $A \subseteq B \Rightarrow \overline{apr}_R^{\mathcal{I}}(A) \subseteq \overline{apr}_R^{\mathcal{I}}(B)$.

(3) $\overline{apr}_R^{\mathcal{I}}(A \cap B) \subseteq \overline{apr}_R^{\mathcal{I}}(A) \cap \overline{apr}_R^{\mathcal{I}}(B)$.

(4) $\overline{apr}_R^{\mathcal{I}}(A \cup B) = \overline{apr}_R^{\mathcal{I}}(A) \cup \overline{apr}_R^{\mathcal{I}}(B)$.

(5) $\overline{apr}_R^{\mathcal{I}}(A) = (\underline{apr}_R^{\mathcal{I}}(A^c))^c$.

(6) If $A \in \mathcal{I}$, *then* $\overline{apr}_R^{\mathcal{I}}(A) = \phi$.

(7) If $\mathcal{I} \subseteq \mathcal{J}$, *then* $\overline{apr}_R^{\mathcal{J}}(A) \subseteq \overline{apr}_R^{\mathcal{I}}(A)$.

(8) If $\mathcal{I} = P(U)$, *then* $\overline{apr}_R^{\mathcal{I}}(A) = \phi$.

(9) $\overline{apr}_R^{\mathcal{I} \cap \mathcal{J}}(A) = \overline{apr}_R^{\mathcal{I}}(A) \cup \overline{apr}_R^{\mathcal{J}}(A)$.

(10) $\overline{apr}_R^{\mathcal{I} \vee \mathcal{J}}(A) = \overline{apr}_R^{\mathcal{I}}(A) \cap \overline{apr}_R^{\mathcal{J}}(A)$.

Proof.

(1) $\overline{apr}_R^{\mathcal{I}}(\phi) = \{x \in U :< x > \breve{R} \cap \phi \notin \mathcal{I}\} = \phi$.

(2) Let $x \in \overline{apr}_R^{\mathcal{I}}(A)$. Then, $< x > \breve{R} \cap A \notin \mathcal{I}$. Since $A \subseteq B$ and $\mathcal{I}$ is an ideal. It follows that $< x > \breve{R} \cap B \notin \mathcal{I}$. Therefore, $x \in \overline{apr}_R^{\mathcal{I}}(B)$. Hence, $\overline{apr}_R^{\mathcal{I}}(A) \subseteq \overline{apr}_R^{\mathcal{I}}(B)$.

(3) The proof is immediately by (2).

(4) $\overline{apr}_R^{\mathcal{I}}(A) \cup \overline{apr}_R^{\mathcal{I}}(B) \subseteq \overline{apr}_R^{\mathcal{I}}(A \cup B)$ according to (2). Let $x \in \overline{apr}_R^{\mathcal{I}}(A \cup B)$. Then, $< x > \breve{R} \cap (A \cup B) \notin \mathcal{I}$. It follows that $((< x > \breve{R} \cap A) \cup (< x > \breve{R} \cap B)) \notin \mathcal{I}$. Therefore, $< x > \breve{R} \cap A \notin I$ or $< x > \breve{R} \cap B \notin \mathcal{I}$, which gives $x \in \overline{apr}_R^{\mathcal{I}}(A)$ or $x \in \overline{apr}_R^{\mathcal{I}}(B)$. Then, $x \in \overline{apr}_R^{\mathcal{I}}(A) \cup \overline{apr}_R^{\mathcal{I}}(B)$. Thus, $\overline{apr}_R^{\mathcal{I}}(A \cup B) \subseteq \overline{apr}_R^{\mathcal{I}}(A) \cup \overline{apr}_R^{\mathcal{I}}(B)$. Hence, $\overline{apr}_R^{\mathcal{I}}(A \cup B) = \overline{apr}_R^{\mathcal{I}}(A) \cup \overline{apr}_R^{\mathcal{I}}(B)$.

(5) $(\underline{apr}_R^{\mathcal{I}}(A^c))^c = (\{x \in U :< x > \breve{R} \cap A \in \mathcal{I}\})^c = \{x \in U :< x > \breve{R} \cap A \notin \mathcal{I}\} = \overline{apr}_R^{\mathcal{I}}(A)$.

(6) The proof is straightforward by Definition 3.1.

(7) Let $x \in \overline{apr}_R^{\mathcal{J}}(A)$. Then, $< x > \breve{R} \cap A \notin \mathcal{J}$. Since $\mathcal{I} \subseteq \mathcal{J}$. So, $< x > \breve{R} \cap A \notin \mathcal{I}$. Therefore, $x \in \overline{apr}_R^{\mathcal{I}}(A)$. Hence, $\overline{apr}_R^{\mathcal{J}}(A) \subseteq \overline{apr}_R^{\mathcal{I}}(A)$.

(8) The proof is straightforward by Definition 3.1.

(9)

$$\begin{aligned}\overline{apr}_R^{\mathcal{I}\cap\mathcal{J}}(A) &= \{x \in U :< x > \breve{R} \cap A \notin \mathcal{I} \cap \mathcal{J}\}\\ &= \{x \in U :< x > \breve{R} \cap A \notin \mathcal{I}\} \text{ or } \{x \in U :< x > \breve{R} \cap A \notin \mathcal{J}\}\\ &= \{x \in U :< x > \breve{R} \cap A \notin \mathcal{I}\} \cup \{x \in U :< x > \breve{R} \cap A \notin \mathcal{J}\}\\ &= \overline{apr}_R^{\mathcal{I}}(A) \cup \overline{apr}_R^{\mathcal{J}}(A).\end{aligned}$$

(10)

$$\begin{aligned}\overline{apr}_R^{\mathcal{I}\vee\mathcal{J}}(A) &= \{x \in U :< x > \breve{R} \cap A \notin \mathcal{I} \vee \mathcal{J}\}\\ &= \{x \in U :< x > \breve{R} \cap A \notin \mathcal{I} \cup \mathcal{J}\}\\ &= \{x \in U :< x > \breve{R} \cap A \notin \mathcal{I}\} \text{ and } \{x \in U :< x > \breve{R} \cap A \notin \mathcal{J}\}\\ &= \{x \in U :< x > \breve{R} \cap A \notin \mathcal{I}\} \cap \{x \in U :< x > \breve{R} \cap A \notin \mathcal{J}\}\\ &= \overline{apr}_R^{\mathcal{I}}(A) \cap \overline{apr}_R^{\mathcal{J}}(A).\end{aligned}$$

Proposition 3.2. *Let $A, B \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. Then, the following properties hold:*

(1) $\underline{apr}_R^{\mathcal{I}}(U) = U$.

(2) $A \subseteq B \Rightarrow \underline{apr}_R^{\mathcal{I}}(A) \subseteq \underline{apr}_R^{\mathcal{I}}(B)$.

(3) $\underline{apr}_R^{\mathcal{I}}(A) \cup \underline{apr}_R^{\mathcal{I}}(B) \subseteq \underline{apr}_R^{\mathcal{I}}(A \cup B)$.

(4) $\underline{apr}_R^{\mathcal{I}}(A \cap B) = \underline{apr}_R^{\mathcal{I}}(A) \cap \underline{apr}_R^{\mathcal{I}}(B)$.

(5) $\underline{apr}_R^{\mathcal{I}}(A) = (\overline{apr}_R^{\mathcal{I}}(A^c))^c$.

(6) If $A^c \in \mathcal{I}$, *then* $\underline{apr}_R^{\mathcal{I}}(A) = U$.

(7) If $\mathcal{I} \subseteq \mathcal{J}$, *then* $\underline{apr}_R^{\mathcal{I}}(A) \subseteq \underline{apr}_R^{\mathcal{J}}(A)$.

(8) If $\mathcal{I} = P(U)$, *then* $\underline{apr}_R^{\mathcal{I}}(A) = U$.

(9) $\underline{apr}_R^{\mathcal{I}\cap\mathcal{J}}(A) = \underline{apr}_R^{\mathcal{I}}(A) \cap \underline{apr}_R^{\mathcal{J}}(A)$.

Proof.

(1) $\underline{apr}_R^{\mathcal{I}}(U) = \{x \in U :< x > \breve{R} \cap \phi \in \mathcal{I}\} = U$.

(2) Let $x \in \underline{apr}_R^{\mathcal{I}}(A)$. Then, $< x > \breve{R} \cap A^c \in \mathcal{I}$. Since $B^c \subseteq A^c$ and $\mathcal{I}$ is an ideal. So, $< x > \breve{R} \cap B^c \in \mathcal{I}$. Therefore, $x \in \underline{apr}_R^{\mathcal{I}}(B)$. Hence, $\underline{apr}_R^{\mathcal{I}}(A) \subseteq \underline{apr}_R^{\mathcal{I}}(B)$.

(3) The proof is immediately by (2).

(4) $\underline{apr}_R^{\mathcal{I}}(A) \cap \underline{apr}_R^{\mathcal{I}}(B) \supseteq \underline{apr}_R^{\mathcal{I}}(A \cap B)$ according to (2). Let $x \in \underline{apr}_R^{\mathcal{I}}(A) \cap \underline{apr}_R^{\mathcal{I}}(B)$. Then, $< x > \breve{R} \cap A^c \in \mathcal{I}$ and $< x > \breve{R} \cap B^c \in \mathcal{I}$. It follows that $(< x > \breve{R} \cap (A^c \cup B^c)) \in \mathcal{I}$. So, $(< x > \breve{R} \cap (A \cap B)^c) \in \mathcal{I}$. Therefore, $x \in \underline{apr}_R^{\mathcal{I}}(A \cap B)$. Thus, $\underline{apr}_R^{\mathcal{I}}(A) \cap \underline{apr}_R^{\mathcal{I}}(B) \subseteq \underline{apr}_R^{\mathcal{I}}(A \cap B)$. Hence, $\underline{apr}_R^{\mathcal{I}}(A) \cap \underline{apr}_R^{\mathcal{I}}(B) = \underline{apr}_R^{\mathcal{I}}(A \cap B)$.

(5) $(\overline{apr}_R^{\mathcal{I}}(A^c))^c = (\{x \in U :< x > \breve{R} \cap A^c \notin \mathcal{I}\})^c = \{x \in U :< x > \breve{R} \cap A^c \in \mathcal{I}\} = \underline{apr}_R^{\mathcal{I}}(A)$.

(6) The proof is straightforward by Definition 3.1.

(7) Let $x \in \underline{apr}_R^{\mathcal{I}}(A)$. Then, $< x > \breve{R} \cap A^c \in \mathcal{I}$. Since $\mathcal{I} \subseteq \mathcal{J}$. It follows that $< x > \breve{R} \cap A^c \in \mathcal{J}$. Therefore, $x \in \underline{apr}_R^{\mathcal{J}}(A)$. Hence, $\underline{apr}_R^{\mathcal{I}}(A) \subseteq \underline{apr}_R^{\mathcal{J}}(A)$.

(8) The proof is straightforward by Definition 3.1.

(9)

$$\begin{aligned}\underline{apr}_R^{\mathcal{I}\cap\mathcal{J}}(A) &= \{x \in U :< x > \breve{R} \cap A^c \in \mathcal{I} \cap \mathcal{J}\}\\ &= \{x \in U :< x > \breve{R} \cap A^c \in \mathcal{I}\} \ and\ \{x \in U :< x > \breve{R} \cap A^c \in \mathcal{J}\}\\ &= \{x \in U :< x > \breve{R} \cap A^c \in \mathcal{I}\} \cap \{x \in U :< x > \breve{R} \cap A^c \in \mathcal{J}\}\\ &= \underline{apr}_R^{\mathcal{I}}(A) \cap \underline{apr}_R^{\mathcal{J}}(A).\end{aligned}$$

Remark 3.1. The following examples show that

(1) The converse of (2), (6), (7) and (8) in Propositions 3.1 and 3.2 is not necessarily true in general.

(2) The inclusion of (3) in Propositions 3.1 and 3.2 cannot be replaced by an equality relation in general.

Example 3.1. (i) *Let*

$$U = \{a, b, c, d\},$$

$$\mathcal{I} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

and

$$R = \{(a, b), (a, c), (b, c), (b, d), (c, a), (c, d)\}$$

be a binary relation defined on U thus $< a > \breve{R} = \{a, d\}, < b > \breve{R} = \{b, c\}, < c > \breve{R} = \{b, c, d\}$ *and* $< d > \breve{R} = \{a, c, d\}$. *For (2), take*

(a) $A = \{a\}$ *and* $B = \{d\}$; *then,* $\overline{apr}_R^{\mathcal{I}}(A) = \phi$ *and* $\overline{apr}_R^{\mathcal{I}}(B) = \{a, c, d\}$. *Therefore,* $\overline{apr}_R^{\mathcal{I}}(A) \subseteq \overline{apr}_R^{\mathcal{I}}(B)$, *but* $A \nsubseteq B$.

(b) $A = \{b\}$ *and* $B = \{a, c, d\}$; *then,* $\underline{apr}_R^{\mathcal{I}}(A) = \{b\}$ *and* $\underline{apr}_R^{\mathcal{I}}(B) = U$. *Therefore,* $\underline{apr}_R^{\mathcal{I}}(A) \subseteq \underline{apr}_R^{\mathcal{I}}(B)$, *but* $A \nsubseteq B$.

(ii) *Let* $U = \{a, b, c, d\}, \mathcal{J} = \{\phi, \{a\}\}, \mathcal{I} = \{\phi, \{d\}\}$ *and* $R = \{(a, a), (b, b), (c, c)\}$ *be a binary relation defined on* U; *thus,* $< a > \breve{R} = \{a\}, < b > \breve{R} = \{b\}, < c > \breve{R} = \{c\}$ *and* $< d > \breve{R} = \phi$.

(1) For (6), take

(a) $A = \{a, d\}$; *then,* $\overline{apr}_R^{\mathcal{J}}(A) = \phi$. *Therefore,* $\overline{apr}_R^{\mathcal{J}}(A) = \phi$, *but* $A \notin \mathcal{J}$.

(b) $A = \{b, c\}$; *then,* $\underline{apr}_R^{\mathcal{J}}(A) = U$. *Therefore,* $\underline{apr}_R^{\mathcal{J}}(A) = U$, *but* $A^c \notin \mathcal{J}$.

(2) For (7), take

(a) $A = \{a, d\}$; *then,* $\overline{apr}_R^{\mathcal{I}}(A) = \{a\}$ *and* $\overline{apr}_R^{\mathcal{J}}(A) = \phi$. *Therefore,* $\overline{apr}_R^{\mathcal{J}}(A) \subseteq \overline{apr}_R^{\mathcal{I}}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(b) $A = \{b, c\}$; *then,* $\underline{apr}_R^{\mathcal{I}}(A) = \{b, c, d\}$ *and* $\underline{apr}_R^{\mathcal{J}}(A) = U$. *Therefore,* $\underline{apr}_R^{\mathcal{I}}(A) \subseteq \underline{apr}_R^{\mathcal{J}}(A)$, *but* $\mathcal{I} \nsubseteq \mathcal{J}$.

(3) For (8), take

(a) *$A = \{a, d\}$; then, $\overline{apr}^{\mathcal{J}}_R(A) = \phi$, but $\mathcal{J} \neq P(U)$.*

(b) *$A = \{b, c\}$; then, $\underline{apr}^{\mathcal{J}}_R(A) = U$, but $\mathcal{J} \neq P(U)$.*

(iii) *Let $U = \{a, b, c, d\}$, $\mathcal{I} = \{\phi, \{d\}\}$ and $R = \Delta \cup \{(b, a), (c, a), (d, a)\}$ be a binary relation defined on U, (where Δ is the identity relation and equal to $\{(a, a), (b, b), (c, c), (d, d)\}$); thus, $< a > \breve{R} = U$, $< b > \breve{R} = \{a, b\}$, $< c > \breve{R} = \{a, c\}$ and $< d > \breve{R} = \{a, d\}$. For (3), take*

(a) *$A = \{a, d\}$, $B = \{b, c\}$ and $A \cap B = \phi$; then, $\overline{apr}^{\mathcal{I}}_R(A) = U$, $\overline{apr}^{\mathcal{I}}_R(B) = \{a, b, c\}$ and $\overline{apr}^{\mathcal{I}}_R(A \cap B) = \phi$. Therefore, $\overline{apr}^{\mathcal{I}}_R(A) \cap \overline{apr}^{\mathcal{I}}_R(B) = \{a, b, c\} \neq \phi = \overline{apr}^{\mathcal{I}}_R(A \cap B)$.*

(b) *$A = \{a, d\}$, $B = \{b, c\}$ and $A \cup B = U$; then, $\underline{apr}^{\mathcal{I}}_R(A) = \{d\}$, $\underline{apr}^{\mathcal{I}}_R(B) = \phi$ and $\overline{apr}^{\mathcal{I}}_R(A \cup B) = U$. Therefore, $\underline{apr}^{\mathcal{I}}_R(A) \cup \underline{apr}^{\mathcal{I}}_R(B) = \{d\} \neq U = \underline{apr}^{\mathcal{I}}_R(A \cup B)$.*

Remark 3.2. There are some properties that are not held or satisfied for the first type.

(i) Considering Example 3.1 (i), take

(1) $A = \{a\}$; then, $\overline{apr}^{\mathcal{I}}_R(A) = \phi$. Hence, $A \nsubseteq \overline{apr}^{\mathcal{I}}_R(A)$.

(2) $A = \{b, c, d\}$; then, $\underline{apr}^{\mathcal{I}}_R(A) = U$. Hence, $\underline{apr}^{\mathcal{I}}_R(A) \nsubseteq A$.

(3) $A = U$; then, $\overline{apr}^{\mathcal{I}}_R(U) = \{a, c, d\}$. Hence, $\overline{apr}^{\mathcal{I}}_R(U) \neq U$.

(4) $A = \phi$; then, $\underline{apr}^{\mathcal{I}}_R(\phi) = \{b\}$. Hence, $\underline{apr}^{\mathcal{I}}_R(\phi) \neq \phi$.

(ii) Considering Example 3.1 (iii), take

(1) $A = \{b, c\}$; then, $\overline{apr}^{\mathcal{I}}_R(A) = \{a, b, c\}$ and $\overline{apr}^{\mathcal{I}}_R(\overline{apr}^{\mathcal{I}}_R(A)) = U$. Hence, $\overline{apr}^{\mathcal{I}}_R(A) \neq \overline{apr}^{\mathcal{I}}_R(\overline{apr}^{\mathcal{I}}_R(A))$.

(2) $A = \{a, d\}$; then, $\underline{apr}^{\mathcal{I}}_R(A) = \{d\}$ and $\underline{apr}^{\mathcal{I}}_R(\underline{apr}^{\mathcal{I}}_R(A)) = \phi$. Hence, $\underline{apr}^{\mathcal{I}}_R(A) \neq \underline{apr}^{\mathcal{I}}_R(\underline{apr}^{\mathcal{I}}_R(A))$.

(iii)

Example 3.2. *Let $U = \{a, b, c, d\}$, $\mathcal{I} = \{\phi, \{a\}\}$ and $R = \Delta \cup \{(a, b), (b, a), (c, a), (c, b), (d, a), (d, b)\}$ be a binary relation defined on U thus $< a > \breve{R} =< b > \breve{R} = U$, $< c > \breve{R} = \{a, b, c\}$ and $< d > \breve{R} = \{a, b, d\}$. It is clear that, if*

(1) $A = \{c\}$; then, $\overline{apr}^{\mathcal{I}}_R(A) = \{a, b, c\}$ and $\underline{apr}^{\mathcal{I}}_R(\overline{apr}^{\mathcal{I}}_R(A)) = \{c\}$. Hence, $\overline{apr}^{\mathcal{I}}_R(A) \nsubseteq \underline{apr}^{\mathcal{I}}_R(\overline{apr}^{\mathcal{I}}_R(A))$.

(2) $A = \{a, b, d\}$; then, $\underline{apr}^{\mathcal{I}}_R(A) = \{d\}$ and $\overline{apr}^{\mathcal{I}}_R(\underline{apr}^{\mathcal{I}}_R(A)) = \{a, b, d\}$. Hence,

$$\overline{apr}^{\mathcal{I}}_R(\underline{apr}^{\mathcal{I}}_R(A)) \nsubseteq \underline{apr}^{\mathcal{I}}_R(A).$$

Theorem 3.1. *Let $A \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. If $\mathcal{I} \subseteq \mathcal{J}$, then*

(1) $Boundary^{\mathcal{J}}_R(A) \subseteq Boundary^{\mathcal{I}}_R(A)$.

(2) $Accuracy^{\mathcal{I}}_R(A) \leq Accuracy^{\mathcal{J}}_R(A)$.

Proof.

(1) Let $x \in Boundary_R^{\mathcal{J}}(A)$. Then, $x \in \overline{apr}_R^{\mathcal{J}}(A) - \underline{apr}_R^{\mathcal{J}}(A)$. So, $x \in \overline{apr}_R^{\mathcal{J}}(A)$ and $x \in (\underline{apr}_R^{\mathcal{J}}(A))^c$. Hence, $x \in \overline{apr}_R^{\mathcal{I}}(A)$ and $x \in (\underline{apr}_R^{\mathcal{I}}(A))^c$ according to (7) of Propositions 3.1 and 3.2. It follows that $x \in Boundary_R^{\mathcal{I}}(A)$. Therefore, $Boundary_R^{\mathcal{J}}(A) \subseteq Boundary_R^{\mathcal{I}}(A)$.

(2) $Accuracy_R^{\mathcal{I}}(A) = |\frac{\underline{apr}_R^{\mathcal{I}}(A)}{\overline{apr}_R^{\mathcal{I}}(A)}| \leq |\frac{\underline{apr}_R^{\mathcal{J}}(A)}{\overline{apr}_R^{\mathcal{J}}(A)}| = Accuracy_R^{\mathcal{J}}(A)$.

Remark 3.3. Example 3.1 (ii) shows that the converse of (1) and (2) in Theorem 3.1 is not necessarily true in general. Take $A = \{b, c\}$; then,

(1) $Boundary_R^{\mathcal{J}}(A) = \phi \subseteq \phi = Boundary_R^{\mathcal{I}}(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

(2) $Accuracy_R^{\mathcal{I}}(A) = \frac{3}{2} < 2 = Accuracy_R^{\mathcal{J}}(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

Theorem 3.2. *Let $\phi \neq A \subseteq U$, $\mathcal{I}$ be an ideal on U and R_1, R_2 be two binary relations on U. If $R_1 \subseteq R_2$, then*

(1) $\overline{apr}_{R_1}^{\mathcal{I}}(A) \subseteq \overline{apr}_{R_2}^{\mathcal{I}}(A)$.

(2) $\underline{apr}_{R_2}^{\mathcal{I}}(A) \subseteq \underline{apr}_{R_1}^{\mathcal{I}}(A)$.

(3) $Boundary_{R_1}^{\mathcal{I}}(A) \subseteq Boundary_{R_2}^{\mathcal{I}}(A)$.

(4) $Accuracy_{R_2}^{\mathcal{I}}(A) \leq Accuracy_{R_1}^{\mathcal{I}}(A)$.

Proof.

(1) Let $x \in \overline{apr}_{R_1}^{\mathcal{I}}(A)$. Then, $< x > \breve{R}_1 \cap A \notin \mathcal{I}$. Since $< x > \breve{R}_1 \subseteq < x > \breve{R}_2$ (by Theorem 2.1 [7]). It follows that $< x > \breve{R}_2 \cap A \notin \mathcal{I}$. Thus, $x \in \overline{apr}_{R_2}^{\mathcal{I}}(A)$. Hence, $\overline{apr}_{R_1}^{\mathcal{I}}(A) \subseteq \overline{apr}_{R_2}^{\mathcal{I}}(A)$.

(2) Let $x \in \underline{apr}_{R_2}^{\mathcal{I}}(A)$. Then, $< x > \breve{R}_2 \cap A^c \in \mathcal{I}$. Since $< x > \breve{R}_1 \subseteq < x > \breve{R}_2$ (by Theorem 2.1 [7]). It follows that $< x > \breve{R}_1 \cap A^c \in \mathcal{I}$. Thus, $x \in \underline{apr}_{R_1}^{\mathcal{I}}(A)$. Hence, $\underline{apr}_{R_2}^{\mathcal{I}}(A) \subseteq \underline{apr}_{R_1}^{\mathcal{I}}(A)$.

(3) Let $x \in Boundary_{R_1}^{\mathcal{I}}(A)$. Then, $x \in \overline{apr}_{R_1}^{\mathcal{I}}(A) - \underline{apr}_{R_1}^{\mathcal{I}}(A)$. So, $x \in \overline{apr}_{R_1}^{\mathcal{I}}(A)$ and $x \in (\underline{apr}_{R_1}^{\mathcal{I}}(A))^c$. Thus, $x \in \overline{apr}_{R_2}^{\mathcal{I}}(A)$ and $x \in (\underline{apr}_{R_2}^{\mathcal{I}}(A))^c$ according to (1) and (2). Hence, $x \in Boundary_{R_2}^{\mathcal{I}}(A)$. Therefore, $Boundary_{R_1}^{\mathcal{I}}(A) \subseteq Boundary_{R_2}^{\mathcal{I}}(A)$.

(4) $Accuracy_{R_2}^{\mathcal{I}}(A) = |\frac{\underline{apr}_{R_2}^{\mathcal{I}}(A)}{\overline{apr}_{R_2}^{\mathcal{I}}(A)}| \leq |\frac{\underline{apr}_{R_1}^{\mathcal{I}}(A)}{\overline{apr}_{R_1}^{\mathcal{I}}(A)}| = Accuracy_{R_1}^{\mathcal{I}}(A)$.

The following example shows that the inclusion and less than relation in Theorem 3.2 cannot be replaced by an equality relation in general.

Example 3.3. *Let*

$$U = \{a, b, c, d\},$$

$$\mathcal{I} = \{\phi, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$$

$$R_1 = \Delta \cup \{(a, b), (b, a)\} \text{ and } R_2 = \Delta \cup \{(a, b), (b, a), (c, a), (a, c)\}$$

be two relations defined on U; thus,

$$< a > \breve{R}_1 =< b > \breve{R}_1 = \{a, b\}, < c > \breve{R}_1 = \{c\}, < d > \breve{R}_1 = \{d\},$$

$$<a>\breve{R}_2=<b>\breve{R}_2=<c>\breve{R}_2=\{a,b,c\} \text{ and } <d>\breve{R}_2=\{d\}.$$

Take

(i) $A=\{a,d\}$; *then,*

(1) $\overline{apr}^{\mathcal{I}}_{R_1}(A)=\{a,b\}\neq\{a,b,c\}=\overline{apr}^{\mathcal{I}}_{R_2}(A)$.

(2) $Accuracy^{\mathcal{I}}_{R_1}(A)=2\neq\frac{4}{3}=Accuracy^{\mathcal{I}}_{R_2}(A)$.

(ii) $A=\{b,c\}$; *then,* $\underline{apr}^{\mathcal{I}}_{R_1}(A)=\{c,d\}\neq\{d\}=\underline{apr}^{\mathcal{I}}_{R_2}(A)$.

The following theorem presents the relationships between the current approximations in Definition 3.1 and the previous ones in Definition 2.5 [11]. It shows that the present method outlined in Definition 3.1 reduces the boundary region by increasing the lower approximations and decreasing the upper approximations more effectively than Dai et al.'s method (Definition 2.5) [11]. Additionally, it shows that the current accuracy provided by Definition 3.1 is greater than the previous ones provided by Definition 2.5 [11].

Theorem 3.3. *Let $A\subseteq U$, $\mathcal{I}$ be an ideal on U and R be a similarity relation on a non-empty set U. Then,*

(1) $\overline{apr}^{\mathcal{I}}_{R}(A)\subseteq\overline{apr}_{R}(A)$.
(2) $\underline{apr}_{R}(A)\subseteq\underline{apr}^{\mathcal{I}}_{R}(A)$.
(3) $Boundary^{\mathcal{I}}_{R}(A)\subseteq Boundary_{R}(A)$.
(4) $Accuracy_{R}(A)\leq Accuracy^{\mathcal{I}}_{R}(A)$.

Proof.

(1) Let $x\in\overline{apr}^{\mathcal{I}}_{R}(A)$. Then, $<x>\breve{R}\cap A\notin\mathcal{I}$. Hence, $<x>\breve{R}\cap A\neq\phi$. Therefore, $x\in\overline{apr}_{R}(A)$. So, $\overline{apr}^{\mathcal{I}}_{R}(A)\subseteq\overline{apr}_{R}(A)$.

(2) Let $x\in\underline{apr}_{R}(A)$. Then, $<x>\breve{R}\subseteq A$. Hence, $<x>\breve{R}\cap A^c\in\mathcal{I}$. Therefore, $x\in\underline{apr}^{\mathcal{I}}_{R}(A)$. So, $\underline{apr}_{R}(A)\subseteq\underline{apr}^{\mathcal{I}}_{R}(A)$.

(3) and (4) The proof is immediately by (1) and (2).

Remark 3.4. Example 3.3 shows that the inclusion and less than relation in Theorem 3.3 cannot be replaced by an equality relation in general. Take $A=\{a,d\}$; then,

(1) $\overline{apr}^{\mathcal{I}}_{R_1}(A)=\{a,b\}\neq\{a,b,d\}=\overline{apr}_{R_1}(A)$.

(2) $\underline{apr}^{\mathcal{I}}_{R_1}(A)=U\neq\{d\}=\underline{apr}_{R_1}(A)$.

(3) $Boundary^{\mathcal{I}}_{R_1}(A)=\phi\neq\{a,b\}=Boundary_{R_1}(A)$.

(4) $Accuracy^{\mathcal{I}}_{R_1}(A)=2\neq\frac{1}{3}=Accuracy_{R_1}(A)$.

4. Second method to obtain generalized rough sets using ideals

The aim of this section is to propose the second type of the extension of rough approximations. The characteristics of these approximations are suggested. Moreover, some relationships among these approximations and the first type of approximations in the previous section are disclosed with the help of some elucidative examples. Comparisons between these approximations and the ones in [11] are presented.

Definition 4.1. Let R be a binary relation on a non-empty set U and $\mathcal{I}$ be an ideal on U. For any subset $A \subseteq U$, the second kind of generalized lower and upper approximations, boundary regions, accuracy and roughness of A using ideal and according to R are respectively defined by

$$\underline{\underline{apr}}_R^{\mathcal{I}}(A) = \{x \in A :< x > \breve{R} \cap A^c \in \mathcal{I}\}. \tag{4.1}$$

$$\overline{\overline{apr}}_R^{\mathcal{I}}(A) = A \cup \overline{apr}_R^{\mathcal{I}}(A). \tag{4.2}$$

$$\underline{Boundary}_R^{\mathcal{I}}(A) = \overline{\overline{apr}}_R^{\mathcal{I}}(A) - \underline{\underline{apr}}_R^{\mathcal{I}}(A). \tag{4.3}$$

$$\underline{Accuracy}_R^{\mathcal{I}}(A) = \frac{|\underline{\underline{apr}}_R^{\mathcal{I}}(A)|}{|\overline{\overline{apr}}_R^{\mathcal{I}}(A)}|, \overline{\overline{apr}}_R^{\mathcal{I}}(A) \neq \phi. \tag{4.4}$$

$$\underline{Roughness}_R^{\mathcal{I}}(A) = 1 - \underline{Accuracy}_R^{\mathcal{I}}(A). \tag{4.5}$$

Proposition 4.1. *Let $A \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. Then, the following properties hold:*

(1) $A \subseteq \overline{\overline{apr}}_R^{\mathcal{I}}(A)$ *equality holds if* $A = \phi$ *or* U.

(2) $A \subseteq B \Rightarrow \overline{\overline{apr}}_R^{\mathcal{I}}(A) \subseteq \overline{\overline{apr}}_R^{\mathcal{I}}(B)$.

(3) $\overline{\overline{apr}}_R^{\mathcal{I}}(A) \subseteq \overline{\overline{apr}}_R^{\mathcal{I}}(\overline{\overline{apr}}_R^{\mathcal{I}}(A))$.

(4) $\overline{\overline{apr}}_R^{\mathcal{I}}(A \cap B) \subseteq \overline{\overline{apr}}_R^{\mathcal{I}}(A) \cap \overline{\overline{apr}}_R^{\mathcal{I}}(B)$.

(5) $\overline{\overline{apr}}_R^{\mathcal{I}}(A \cup B) = \overline{\overline{apr}}_R^{\mathcal{I}}(A) \cup \overline{\overline{apr}}_R^{\mathcal{I}}(B)$.

(6) $\overline{\overline{apr}}_R^{\mathcal{I}}(A) = (\underline{\underline{apr}}_R^{\mathcal{I}}(A^c))^c$.

(7) *If* $A \in \mathcal{I}$, *then* $\overline{\overline{apr}}_R^{\mathcal{I}}(A) = A$.

(8) *If* $\mathcal{I} \subseteq \mathcal{J}$, *then* $\overline{\overline{apr}}_R^{\mathcal{J}}(A) \subseteq \overline{\overline{apr}}_R^{\mathcal{I}}(A)$.

(9) *If* $\mathcal{I} = P(U)$, *then* $\overline{\overline{apr}}_R^{\mathcal{I}}(A) = A$.

(10) $\overline{\overline{apr}}_R^{\mathcal{I} \cap \mathcal{J}}(A) = \overline{\overline{apr}}_R^{\mathcal{I}}(A) \cup \overline{\overline{apr}}_R^{\mathcal{J}}(A)$.

(11) $\overline{\overline{apr}}_R^{\mathcal{I} \vee \mathcal{J}}(A) = \overline{\overline{apr}}_R^{\mathcal{I}}(A) \cap \overline{\overline{apr}}_R^{\mathcal{J}}(A)$.

Proof. Similar to Proposition 3.1.

Proposition 4.2. *Let $A, B \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. Then, the following properties hold:*

(1) $\underline{\underline{apr}}_R^{\mathcal{I}}(A) \subseteq A$ *equality holds if* $A = \phi$ *or* U.

(2) $A \subseteq B \Rightarrow \underline{\underline{apr}}^{\mathcal{I}}_R(A) \subseteq \underline{\underline{apr}}^{\mathcal{I}}_R(B)$.

(3) $\underline{\underline{apr}}^{\mathcal{I}}_R(\underline{\underline{apr}}^{\mathcal{I}}_R(A)) \subseteq \underline{\underline{apr}}^{\mathcal{I}}_R(A)$.

(4) $\underline{\underline{apr}}^{\mathcal{I}}_R(A) \cup \underline{\underline{apr}}^{\mathcal{I}}_R(B) \subseteq \underline{\underline{apr}}^{\mathcal{I}}_R(A \cup B)$.

(5) $\underline{\underline{apr}}^{\mathcal{I}}_R(A \cap B) = \underline{\underline{apr}}^{\mathcal{I}}_R(A) \cap \underline{\underline{apr}}^{\mathcal{I}}_R(B)$.

(6) $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = (\overline{\underline{apr}}^{\mathcal{I}}_R(A^c))^c$.

(7) *If* $A^c \in \mathcal{I}$, *then* $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = A$.

(8) *If* $\mathcal{I} \subseteq \mathcal{J}$, *then* $\underline{\underline{apr}}^{\mathcal{I}}_R(A) \subseteq \underline{\underline{apr}}^{\mathcal{J}}_R(A)$.

(9) *If* $\mathcal{I} = P(U)$, *then* $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = A$.

(10) $\underline{\underline{apr}}^{\mathcal{I}\cap\mathcal{J}}_R(A) = \underline{\underline{apr}}^{\mathcal{I}}_R(A) \cap \underline{\underline{apr}}^{\mathcal{J}}_R(A)$.

Proof. Similar to Proposition 3.2.

Remark 4.1. **(i)** Example 3.1 (i) also shows that the converse of (2), (7) and (9) in Propositions 4.1 and 4.2 is not necessarily true in general.

(a) For (2), take

(1) $A = \{a\}$ and $B = \{d\}$; then, $\overline{\overline{apr}}^{\mathcal{I}}_R(A) = \{a\} \subseteq \{a, c, d\} = \overline{\overline{apr}}^{\mathcal{I}}_R(B)$, but $A \not\subseteq B$.

(2) $A = \{a, b, c\}$ and $B = \{b, c, d\}$; then, $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = \{b\} \subseteq \{b, c, d\} = \underline{\underline{apr}}^{\mathcal{I}}_R(B)$, but $A \not\subseteq B$.

(b) For (7), take

(1) $A = \{a, c, d\}$; then, $\overline{\overline{apr}}^{\mathcal{I}}_R(A) = A$, but $A \notin \mathcal{I}$.

(2) $A = \{b\}$; then, $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = A$, but $A^c \notin \mathcal{I}$.

(c) For (9), take

(1) $A = \{a, c, d\}$; then, $\overline{\overline{apr}}^{\mathcal{I}}_R(A) = A$, but $\mathcal{I} \neq P(U)$.

(2) $A = \{b\}$; then, $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = A$, but $\mathcal{I} \neq P(U)$.

(ii) Example 3.1 (ii) also shows that the converse of (8) in Propositions 4.1 and 4.2 is not necessarily true in general. Take

(1) $A = \{a, d\}$; then, $\overline{\overline{apr}}^{\mathcal{J}}_R(A) = \{a, d\} \subseteq \{a, d\} = \overline{\overline{apr}}^{\mathcal{I}}_R(A)$, but $\mathcal{I} \not\subseteq \mathcal{J}$.

(2) $A = \{b, c\}$; then, $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = \{b, c\} \subseteq \{b, c\} = \underline{\underline{apr}}^{\mathcal{J}}_R(A)$, but $\mathcal{I} \not\subseteq \mathcal{J}$.

(iii) Example 3.1 (iii) also shows that the inclusion of (3) and (4) in Propositions 4.1 and 4.2 cannot be replaced by an equality in general.

(a) For (3), take

(1) $A = \{b, c\}$; then, $\overline{\overline{apr}}^{\mathcal{I}}_R(A) = \{a, b, c\}$ and $\overline{\overline{apr}}^{\mathcal{I}}_R(\overline{\overline{apr}}^{\mathcal{I}}_R(A)) = U$. Therefore, $\overline{\overline{apr}}^{\mathcal{I}}_R(A) = \{a, b, c\} \neq U = \overline{\overline{apr}}^{\mathcal{I}}_R(\overline{\overline{apr}}^{\mathcal{I}}_R(A))$.

(2) $A = \{a, d\}$; then, $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = \{d\}$ and $\underline{\underline{apr}}^{\mathcal{I}}_R(\underline{\underline{apr}}^{\mathcal{I}}_R(A)) = \phi$. Therefore, $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = \{d\} \neq \phi = \underline{\underline{apr}}^{\mathcal{I}}_R(\underline{\underline{apr}}^{\mathcal{I}}_R(A))$.

(b) For (4), take

(1) $A = \{a, d\}, B = \{b, c\}$ and $A \cap B = \phi$. Hence, $\overline{\overline{apr}}^{\mathcal{I}}_R(A) = U$ and $\overline{\overline{apr}}^{\mathcal{I}}_R(B) = \{a, b, c\}$. Therefore, $\overline{\overline{apr}}^{\mathcal{I}}_R(A) \cap \overline{\overline{apr}}^{\mathcal{I}}_R(B) = \{a, b, c\} \neq \phi = \overline{\overline{apr}}^{\mathcal{I}}_R(A \cap B)$.

(2) $A = \{a, d\}, B = \{b, c\}$ and $A \cup B = U$. Hence, $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = \{d\}$ and $\underline{\underline{apr}}^{\mathcal{I}}_R(B) = \phi$. Therefore, $\underline{\underline{apr}}^{\mathcal{I}}_R(A) \cup \underline{\underline{apr}}^{\mathcal{I}}_R(B) = \{d\} \neq U = \underline{\underline{apr}}^{\mathcal{I}}_R(A \cup B)$.

Remark 4.2. There are some properties that are not held or satisfied for the second type.

(i) Considering Example 3.1 (i), take

(1) $A = \{a\} \in \mathcal{I}$; then, $\overline{\overline{apr}}^{\mathcal{I}}_R(A) = A$. Hence, if $A \in \mathcal{I} \nRightarrow \overline{\overline{apr}}^{\mathcal{I}}_R(A) = \phi$.

(2) $A^c = \{a\} \in \mathcal{I}$; then, $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = A$. Hence, if $A^c \in \mathcal{I} \nRightarrow \underline{\underline{apr}}^{\mathcal{I}}_R(A) = U$.

(ii) Considering Example 3.1 (ii), take

(1) $\mathcal{J} = P(U)$ and $A = \{a, d\}$; then, $\overline{\overline{apr}}^{\mathcal{J}}_R(A) = A$. Hence, if $\mathcal{J} = P(U) \nRightarrow \overline{\overline{apr}}^{\mathcal{J}}_R(A) = \phi$.

(2) $\mathcal{J} = P(U)$ and $A = \{b, c\}$; then, $\underline{\underline{apr}}^{\mathcal{J}}_R(A) = A$. Hence, if $\mathcal{J} = P(U) \nRightarrow \underline{\underline{apr}}^{\mathcal{J}}_R(A) = U$.

(iii) Considering Example 3.2, take

(1) $A = \{c\}$; then, $\overline{\overline{apr}}^{\mathcal{I}}_R(A) = \{a, b, c\}$ and $\underline{\underline{apr}}^{\mathcal{I}}_R(\overline{\overline{apr}}^{\mathcal{I}}_R(A)) = \{c\}$. Therefore, $\overline{\overline{apr}}^{\mathcal{I}}_R(A) = \{a, b, c\} \nsubseteq \{c\} = \underline{\underline{apr}}^{\mathcal{I}}_R(\overline{\overline{apr}}^{\mathcal{I}}_R(A))$.

(2) $A = \{a, b, d\}$; then, $\underline{\underline{apr}}^{\mathcal{I}}_R(A) = \{d\}$ and $\overline{\overline{apr}}^{\mathcal{I}}_R(\underline{\underline{apr}}^{\mathcal{I}}_R(A)) = \{a, b, d\}$. Therefore, $\overline{\overline{apr}}^{\mathcal{I}}_R(\underline{\underline{apr}}^{\mathcal{I}}_R(A)) = \{a, b, d\} \nsubseteq \{d\} = \underline{\underline{apr}}^{\mathcal{I}}_R(A)$.

Theorem 4.1. *Let $A \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. If $\mathcal{I} \subseteq \mathcal{J}$, then*

(1) $\underline{Boundary}^{\mathcal{J}}_R(A) \subseteq \underline{Boundary}^{\mathcal{I}}_R(A)$.

(2) $\underline{Accuracy}^{\mathcal{I}}_R(A) \leq \underline{Accuracy}^{\mathcal{J}}_R(A)$.

Proof. Similar to the proof of Theorem 3.1.

Remark 4.3. Example 3.1 (ii) shows that the converse of (1) and (2) in Theorem 4.1 is not necessarily true in general. Take $A = \{b, c\}$; then,

(1) $\underline{Boundary}^{\mathcal{J}}_R(A) = \phi \subseteq \{a\} = \underline{Boundary}^{\mathcal{I}}_R(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

(2) $\underline{Accuracy}^{\mathcal{J}}_R(A) = 1 \leq \frac{2}{3} = \underline{Accuracy}^{\mathcal{I}}_R(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

Theorem 4.2. *Let $\phi \neq A \subseteq U$, $\mathcal{I}$ be an ideal on U and R_1, R_2 be two binary relations on U. If $R_1 \subseteq R_2$, then*

(1) $\overline{\overline{apr}}^{\mathcal{I}}_{R_1}(A) \subseteq \overline{\overline{apr}}^{\mathcal{I}}_{R_2}(A)$.

(2) $\underline{\underline{apr}}^{\mathcal{I}}_{R_2}(A) \subseteq \underline{\underline{apr}}^{\mathcal{I}}_{R_1}(A)$.

(3) $\underline{Boundary}^{\mathcal{I}}_{R_1}(A) \subseteq \underline{Boundary}^{\mathcal{I}}_{R_2}(A)$.

(4) $\underline{Accuracy}^{\mathcal{I}}_{R_2}(A) \leq \underline{Accuracy}^{\mathcal{I}}_{R_1}(A)$.

(5) $\underline{Roughness}^{\mathcal{I}}_{R_1}(A) \leq \underline{Roughness}^{\mathcal{I}}_{R_2}(A)$.

Proof. Similar to Theorem 3.2.

Remark 4.4. Example 3.3 shows that the inclusion and less than relation in Theorem 4.2 cannot be replaced by an equality relation in general. Take $A = \{a, d\}$; then,

(1) $\overline{\overline{apr}}^{\mathcal{I}}_{R_1}(A) = \{a, b, d\} \neq U = \overline{\overline{apr}}^{\mathcal{I}}_{R_2}(A)$.

(2) $\underline{Boundary}^{\mathcal{I}}_{R_1}(A) = \{b\} \neq \{b, c\} = \underline{Boundary}^{\mathcal{I}}_{R_2}(A)$.

(3) $\underline{Accuracy}^{\mathcal{I}}_{R_1}(A) = \frac{2}{3} \neq \frac{1}{2} = \underline{Accuracy}^{\mathcal{I}}_{R_2}(A)$.

(4) $\underline{Roughness}^{\mathcal{I}}_{R_1}(A) = 0.3 \neq 0.5 = \underline{Roughness}^{\mathcal{I}}_{R_2}(A)$.

The following theorem presents the relationships between the current approximations in Definitions 3.1 and 4.1.

Theorem 4.3. *Let $A \subseteq U$, $\mathcal{I}$ be an ideal on U and R be a binary relation on U. Then,*

(1) $\overline{apr}^{\mathcal{I}}_{R}(A) \subseteq \overline{\overline{apr}}^{\mathcal{I}}_{R}(A)$.

(2) $\underline{\underline{apr}}^{\mathcal{I}}_{R}(A) \subseteq \underline{apr}^{\mathcal{I}}_{R}(A)$.

(3) $Boundary^{\mathcal{I}}_{R}(A) \subseteq \underline{Boundary}^{\mathcal{I}}_{R}(A)$.

(4) $\underline{Accuracy}^{\mathcal{I}}_{R}(A) \leq Accuracy^{\mathcal{I}}_{R}(A)$.

Proof. Immediately by using the Definitions 3.1 and 4.1.

Remark 4.5. Example 3.3 shows that the inclusion and less than relation in Theorem 4.3 cannot be replaced by an equality relation in general. Take $A = \{a, c, d\}$; then,

(1) $\overline{apr}^{\mathcal{I}}_{R_1}(A) = \{a, b\} \neq U = \overline{\overline{apr}}^{\mathcal{I}}_{R_1}(A)$.

(2) $\underline{\underline{apr}}^{\mathcal{I}}_{R_1}(A) = \{a, c, d\} \neq U = \underline{apr}^{\mathcal{I}}_{R_1}(A)$.

(3) $Boundary^{\mathcal{I}}_{R_1}(A) = \phi \neq \{b\} = \underline{Boundary}^{\mathcal{I}}_{R_1}(A)$.

(4) $\underline{Accuracy}^{\mathcal{I}}_{R_1}(A) = \frac{3}{4} \lneq 2 = Accuracy^{\mathcal{I}}_{R_1}(A)$.

Comparisons between the current approximations in Definition 4.1 and the previous ones in Definition 2.5 [11] are given by the following theorem.

Theorem 4.4. *Let $A \subseteq U$, $\mathcal{I}$ be an ideal on U and R be a similarity relation on a non-empty set U. Then,*

(1) $\overline{\overline{apr}}^{\mathcal{I}}_R(A) \subseteq \overline{apr}_R(A)$.

(2) $\underline{apr}_R(A) \subseteq \underline{\underline{apr}}^{\mathcal{I}}_R(A)$.

(3) $\underline{Boundary}^{\mathcal{I}}_R(A) \subseteq Boundary_R(A)$.

(4) $Accuracy_R(A) \leq \underline{Accuracy}^{\mathcal{I}}_R(A)$.

(5) $\underline{Roughness}^{\mathcal{I}}_R(A) \leq Roughness_R(A)$.

Proof. The proof is similar to that of Theorem 3.3.

Remark 4.6. Example 3.3 shows that the inclusion and less than relation in Theorem 4.4 cannot be replaced by an equality relation. Take $A = \{b, c, d\}$; then, $\overline{\overline{apr}}^{\mathcal{I}}_{R_1}(A) = \{b, c, d\} \neq U = \overline{apr}_{R_1}(A)$. Moreover, if $A = \{a, c, d\}$; then,

(1) $\underline{apr}_{R_1}(A) = \{c, d\} \neq \{a, c, d\} = \underline{\underline{apr}}^{\mathcal{I}}_{R_1}(A)$.

(2) $\underline{Boundary}^{\mathcal{I}}_{R_1}(A) = \{b\} \neq \{a, b\} = Boundary_{R_1}(A)$.

(3) $Accuracy_{R_1}(A) = \frac{1}{2} \lneq \frac{3}{4} = \underline{Accuracy}^{\mathcal{I}}_{R_1}(A)$.

(4) $\underline{Roughness}^{\mathcal{I}}_{R_1}(A) = \frac{1}{4} \lneq \frac{1}{2} = Roughness_{R_1}(A)$.

5. Third method to obtain generalized rough sets using ideals

This section is mainly concerned with the third type of the generalized rough approximations. The fundamental characteristics of these approximations are interjected and analyzed. Additionally, an example is introduced to show that the corresponding lower and upper approximations, boundary regions, accuracy measures and roughness measures of this type of these approximations are not monotonic. After this, the comparisons between these approximations and the approximations in Sections 3 and 4 are introduced. Finally, the relationships between these approximations and the previous ones in [11] are discussed.

Definition 5.1. Let R be a binary relation on a non-empty set U and $\mathcal{I}$ be an ideal on U. For any subset $A \subseteq U$, the third kind of generalized lower and upper approximations, boundary regions, accuracy and roughness of A using ideal and according to R are respectively defined by

$$\underline{apr'}^{\mathcal{I}}_R(A) = \cup\{< x > \breve{R} :< x > \breve{R} \cap A^c \in \mathcal{I}\}. \tag{5.1}$$

$$\overline{apr'}^{\mathcal{I}}_R(A) = (\underline{apr'}_R(A^c))^c. \tag{5.2}$$

$$Boundary'^{\mathcal{I}}_R(A) = \overline{apr'}^{\mathcal{I}}_R(A) - \underline{apr'}^{\mathcal{I}}_R(A). \tag{5.3}$$

$$Accuracy'^{\mathcal{I}}_R(A) = \frac{|\underline{apr'}^{\mathcal{I}}_R(A)|}{|\overline{apr'}^{\mathcal{I}}_R(A)}|, \overline{apr'}^{\mathcal{I}}_R(A) \neq \phi. \tag{5.4}$$

$$Roughness'^{\mathcal{I}}_R(A) = 1 - Accuracy'^{\mathcal{I}}_R(A). \tag{5.5}$$

Proposition 5.1. *Let $A, B \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. Then, the following properties hold:*

(1) $A \subseteq B \Rightarrow \underline{apr}'^{\mathcal{I}}_R(A) \subseteq \underline{apr}'^{\mathcal{I}}_R(B)$.

(2) $\underline{apr}'^{\mathcal{I}}_R(A) \cup \underline{apr}'^{\mathcal{I}}_R(B) \subseteq \underline{apr}'^{\mathcal{I}}_R(A \cup B)$.

(3) $\underline{apr}'^{\mathcal{I}}_R(A \cap B) \subseteq \underline{apr}'^{\mathcal{I}}_R(A) \cap \underline{apr}'^{\mathcal{I}}_R(B)$.

(4) $\underline{apr}'^{\mathcal{I}}_R(A) = (\overline{apr}'^{\mathcal{I}}_R(A^c))^c$.

(5) If $\mathcal{I} \subseteq \mathcal{J}$, *then* $\underline{apr}'^{\mathcal{I}}_R(A) \subseteq \underline{apr}'^{\mathcal{J}}_R(A)$.

(6) $\underline{apr}'^{\mathcal{I}\cap\mathcal{J}}_R(A) = \underline{apr}'^{\mathcal{I}}_R(A) \cap \underline{apr}'^{\mathcal{J}}_R(A)$.

Proof.

(1) Let $A \subseteq B$ and $x \in \underline{apr}'^{\mathcal{I}}_R(A)$. Then, $\exists\ y \in U$ such that $x \in < y > \breve{R} \cap A^c \in \mathcal{I}$. Hence, $x \in < y > \breve{R} \cap B^c \in \mathcal{I}$ (by $B^c \subseteq A^c$, and the properties of an ideal). Thus, $x \in \underline{apr}'^{\mathcal{I}}_R(B)$. Therefore, $\underline{apr}'^{\mathcal{I}}_R(A) \subseteq \underline{apr}'^{\mathcal{I}}_R(B)$.

(2) The proof is immediately by (1).

(3) The proof is immediately by (1).

(4) The proof is straightforward by Definition 5.1.

(5) Let $\mathcal{I} \subseteq \mathcal{J}$ and $x \in \underline{apr}'^{\mathcal{I}}_R(A)$. Then, $\exists\ y \in U$ such that $x \in < y > \breve{R} \cap A^c \in \mathcal{I} \subseteq \mathcal{J}$. So, $x \in \underline{apr}'^{\mathcal{J}}_R(A)$, and hence $\underline{apr}'^{\mathcal{I}}_R(A) \subseteq \underline{apr}'^{\mathcal{J}}_R(A)$.

(6)

$$\begin{aligned}
\underline{apr}'^{\mathcal{I}\cap\mathcal{J}}_R(A) &= \cup\{< x > \breve{R} :< x > \breve{R} \cap A^c \in \mathcal{I} \cap \mathcal{J}\} \\
&= (\cup\{< x > \breve{R} :< x > \breve{R} \cap A^c \in \mathcal{I}\})\ and\ (\cup\{< x > \breve{R} :< x > \breve{R} \cap A^c \in \mathcal{J}\}) \\
&= (\cup\{< x > \breve{R} :< x > \breve{R} \cap A^c \in \mathcal{I}\})\ \cap\ (\cup\{< x > \breve{R} :< x > \breve{R} \cap A^c \in \mathcal{J}\}) \\
&= \underline{apr}'^{\mathcal{I}}_R(A) \cap \underline{apr}'^{\mathcal{J}}_R(A).
\end{aligned}$$

Proposition 5.2. *Let $A, B \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. Then, the following properties hold:*

(1) $A \subseteq B \Rightarrow \overline{apr}'^{\mathcal{I}}_R(A) \subseteq \overline{apr}'^{\mathcal{I}}_R(B)$.

(2) $\overline{apr}'^{\mathcal{I}}_R(A \cap B) \subseteq \overline{apr}'^{\mathcal{I}}_R(A) \cap \overline{apr}'^{\mathcal{I}}_R(B)$.

(3) $\overline{apr}'^{\mathcal{I}}_R(A) \cup \overline{apr}'^{\mathcal{I}}_R(B) \subseteq \overline{apr}'^{\mathcal{I}}_R(A \cup B)$.

(4) $\overline{apr}'^{\mathcal{I}}_R(A) = (\underline{apr}'^{\mathcal{I}}_R(A^c))^c$.

(5) If $\mathcal{I} \subseteq \mathcal{J}$, *then* $\overline{apr}'^{\mathcal{J}}_R(A) \subseteq \overline{apr}'^{\mathcal{I}}_R(A)$.

(6) $\overline{apr'}_R^{\mathcal{I}\cap\mathcal{J}}(A) = \overline{apr'}_R^{\mathcal{I}}(A) \cup \overline{apr'}_R^{\mathcal{J}}(A)$.

Proof.

(1) Let $A \subseteq B$. Thus, $B^c \subseteq A^c$, and $\underline{apr'}_R^{\mathcal{I}}(B^c) \subseteq \underline{apr'}_R^{\mathcal{I}}(A^c)$ (by (1) in Proposition 5.1). So, $(\underline{apr'}_R^{\mathcal{I}}(A^c))^c \subseteq (\underline{apr'}_R^{\mathcal{I}}(B^c))^c$. Consequently, $\overline{apr'}_R^{\mathcal{I}}(A) \subseteq \overline{apr'}_R^{\mathcal{I}}(B)$.

(2) The proof is immediately by (1).

(3) The proof is immediately by (1).

(4) The proof is straightforward by Definition 5.1.

(5) Let $\mathcal{I} \subseteq \mathcal{J}$ and $x \in \overline{apr'}_R^{\mathcal{J}}(A)$. Then, $x \in (\underline{apr'}_R^{\mathcal{J}}(A^c))^c \subseteq (\underline{apr'}_R^{\mathcal{I}}(A^c))^c$, (by (5) in Proposition 5.1). Thus, $x \in (\underline{apr'}_R^{\mathcal{I}}(A^c))^c = \overline{apr'}_R^{\mathcal{I}}(A)$. Therefore, $\overline{apr'}_R^{\mathcal{J}}(A) \subseteq \overline{apr'}_R^{\mathcal{I}}(A)$.

(6)

$$\begin{aligned}
\overline{apr'}_R^{\mathcal{I}\cap\mathcal{J}}(A) &= (\underline{apr'}_R^{\mathcal{I}\cap\mathcal{J}}(A^c))^c \\
&= (\underline{apr'}_R^{\mathcal{I}}(A^c) \cap \underline{apr'}_R^{\mathcal{J}}(A^c))^c \text{ (by (6) in Proposition 5.1)} \\
&= (\underline{apr'}_R^{\mathcal{I}}(A^c))^c \cup (\underline{apr'}_R^{\mathcal{J}}(A^c))^c \\
&= \overline{apr'}_R^{\mathcal{I}}(A) \cup \overline{apr'}_R^{\mathcal{J}}(A).
\end{aligned}$$

Remark 5.1. (1) Example 3.1 (i) shows that the converse of (1) in Propositions 5.1 and 5.2 is not necessarily true in general. Take

(a) $A = \{a\}$ and $B = \{d\}$; then, $\overline{apr'}_R^{\mathcal{I}}(A) = \phi$ and $\overline{apr'}_R^{\mathcal{I}}(B) = \{a, d\}$. Therefore, $\overline{apr'}_R^{\mathcal{I}}(A) \subseteq \overline{apr'}_R^{\mathcal{I}}(B)$, but $A \nsubseteq B$.

(b) $A = \{b\}$ and $B = \{a, c, d\}$; then, $\underline{apr'}_R^{\mathcal{I}}(A) = \{b, c\}$ and $\underline{apr'}_R^{\mathcal{I}}(B) = U$. Therefore, $\underline{apr'}_R^{\mathcal{I}}(A) \subseteq \underline{apr'}_R^{\mathcal{I}}(B)$, but $A \nsubseteq B$.

(2) Example 3.1 (iii) shows that the inclusion of (2) in Propositions 5.1 and 5.2 cannot be replaced by an equality relation in general. Take $A = \{a, d\}$ and $B = \{b, c\}$, then

(a) $\overline{apr'}_R^{\mathcal{I}}(A) = U, \overline{apr'}_R^{\mathcal{I}}(B) = \{b, c\}$ and $\overline{apr'}_R^{\mathcal{I}}(A \cap B) = \phi$. Therefore, $\overline{apr'}_R^{\mathcal{I}}(A) \cap \overline{apr'}_R^{\mathcal{I}}(B) = \{b, c\} \neq \phi = \overline{apr'}_R^{\mathcal{I}}(A \cap B)$.

(b) $\underline{apr'}_R^{\mathcal{I}}(A) = A, \underline{apr'}_R^{\mathcal{I}}(B) = \phi$ and $\underline{apr'}_R^{\mathcal{I}}(A \cup B) = U$. Therefore, $\underline{apr'}_R^{\mathcal{I}}(A) \cup \underline{apr'}_R^{\mathcal{I}}(B) = A \neq U = \underline{apr'}_R^{\mathcal{I}}(A \cup B)$.

(3)

Example 5.1. *Let $U = \{a, b, c, d\}, \mathcal{I} = \{\phi, \{a\}\}$ and $R = \{(a, a), (a, c), (a, d), (b, a), (b, b), (b, c), (d, a), (d, b)\}$ be a binary relation defined on U; thus, $< a > \breve{R} =< c > \breve{R} = U, < b > \breve{R} = \{a, b, c\}$ and $< d > \breve{R} = \{a, c, d\}$. This example shows that the inclusion of (3) in Propositions 5.1 and 5.2 cannot be replaced by an equality relation in general. Take*

(a) *$A = \{a, c, d\}, B = \{a, b, c\}$ and $A \cap B = \{a, c\}$; then, $\underline{apr'}_R^{\mathcal{I}}(A) = A, \underline{apr'}_R^{\mathcal{I}}(B) = B$ and $\underline{apr'}_R^{\mathcal{I}}(A \cap B) = \phi$. Therefore, $\underline{apr'}_R^{\mathcal{I}}(A) \cap \underline{apr'}_R^{\mathcal{I}}(B) = \{a, c\} \neq \phi = \underline{apr'}_R^{\mathcal{I}}(A \cap B)$.*

(b) *$A = \{b\}, B = \{d\}$ and $A \cup B = \{b, d\}$; then, $\overline{apr'}^{\mathcal{I}}_R(A) = A, \overline{apr'}^{\mathcal{I}}_R(B) = B$ and $\overline{apr'}^{\mathcal{I}}_R(A \cup B) = U$. Therefore, $\overline{apr'}^{\mathcal{I}}_R(A) \cup \overline{apr'}^{\mathcal{I}}_R(B) = \{b, d\} \neq U = \overline{apr'}^{\mathcal{I}}_R(A \cup B)$.*

(4) Example 3.1 (ii) shows that the converse of (5) in Propositions 5.1 and 5.2 is not necessarily true in general. Take

(a) $A = \{a, d\}$; then, $\overline{apr'}^{\mathcal{I}}_R(A) = \{a, d\}$ and $\overline{apr'}^{\mathcal{J}}_R(A) = \{d\}$. Therefore, $\overline{apr'}^{\mathcal{J}}_R(A) \subseteq \overline{apr'}^{\mathcal{I}}_R(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

(b) $A = \{b, c\}$; then, $\underline{apr'}^{\mathcal{I}}_R(A) = \{b, c\}$ and $\underline{apr'}^{\mathcal{J}}_R(A) = \{a, b, c\}$. Therefore, $\underline{apr'}^{\mathcal{I}}_R(A) \subseteq \underline{apr'}^{\mathcal{J}}_R(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

Remark 5.2. There are some properties that are not held or satisfied for the third type.

(i) Considering Example 3.1 (i), take

(1) $A = \{a\}$; then, $\overline{apr'}^{\mathcal{I}}_R(A) = \phi$. Hence, $A \nsubseteq \overline{apr'}^{\mathcal{I}}_R(A)$.

(2) $A = \{b, c, d\}$; then, $\underline{apr'}^{\mathcal{I}}_R(A) = U$. Hence, $\underline{apr'}^{\mathcal{I}}_R(A) \nsubseteq A$.

(3) $A = U$; then, $\overline{apr'}^{\mathcal{I}}_R(U) = \{a, d\}$. Hence, $\overline{apr'}^{\mathcal{I}}_R(U) \neq U$.

(4) $A = \phi$; then, $\underline{apr'}^{\mathcal{I}}_R(\phi) = \{b, c\}$. Hence, $\underline{apr'}^{\mathcal{I}}_R(\phi) \neq \phi$.

(ii)

Example 5.2. *Let $U = \{a, b, c, d\}, \mathcal{I} = \{\phi, \{a\}\}$ and $R = \{(a, a)\}$ be a binary relation defined on U; thus, $< a > \breve{R} = \{a\}$ and $< b > \breve{R} =< c > \breve{R} =< d > \breve{R} = \phi$. Take*

(1) $A = U$; then, $\underline{apr'}^{\mathcal{I}}_R(U) = \{a\}$. Hence, $\underline{apr'}^{\mathcal{I}}_R(U) \neq U$.

(2) $A = \phi$; then, $\overline{apr'}^{\mathcal{I}}_R(\phi) = \{b, c, d\}$. Hence, $\overline{apr'}^{\mathcal{I}}_R(\phi) \neq \phi$.

(iii) Considering Example 5.2, take

(1) $A = \{b, c, d\}$; then, $A^c \in \mathcal{I}$ and $\underline{apr'}^{\mathcal{I}}_R(A) = \{a\}$. Hence, if $A^c \in \mathcal{I} \nRightarrow \underline{apr'}^{\mathcal{I}}_R(A) = U$ or A.

(2) $A = \{a\} \in \mathcal{I}$; then, $\overline{apr'}^{\mathcal{I}}_R(A) = \{b, c, d\}$. Hence, if $A \in \mathcal{I} \nRightarrow \overline{apr'}^{\mathcal{I}}_R(A) = \phi$ or A.

(3) $A = \{b, c, d\}$ and $\mathcal{I} = P(U)$; then, $\underline{apr'}^{\mathcal{I}}_R(A) = \{a\}$. Hence, if $\mathcal{I} = P(U) \nRightarrow \underline{apr'}^{\mathcal{I}}_R(A) = U$, or A.

(4) $A = \{a\}$ and $\mathcal{I} = P(U)$; then, $\overline{apr'}^{\mathcal{I}}_R(A) = \{b, c, d\}$. Hence, if $\mathcal{I} = P(U) \nRightarrow \overline{apr'}^{\mathcal{I}}_R(A) = \phi$, or A.

Theorem 5.1. *Let $A \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. If $\mathcal{I} \subseteq \mathcal{J}$, then*

(1) $Boundary'^{\mathcal{J}}_R(A) \subseteq Boundary'^{\mathcal{I}}_R(A)$.

(2) $Accuracy'^{\mathcal{I}}_R(A) \leq Accuracy'^{\mathcal{J}}_R(A)$.

Proof. Similar to Theorem 3.1.

Remark 5.3. Example 3.1 (ii) shows that the converse of (1) and (2) in Theorem 5.1 is not necessarily true in general. Take $A = \{b, c\}$; then,

(1) $Boundary'^{\mathcal{J}}_{R}(A) = \{d\} \subseteq \{d\} = Boundary'^{\mathcal{I}}_{R}(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

(2) $Accuracy'^{\mathcal{I}}_{R}(A) = \frac{2}{3} < 1 = Accuracy'^{\mathcal{J}}_{R}(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

The following example shows that the corresponding lower and upper approximations, boundary regions, accuracy measures and roughness measures for the third type do not have monotonicity.

Example 5.3. *Let $U = \{a, b, c, d, e, f, g\}, \mathcal{I} = \{\phi, \{a\}\}$ and R_1, R_2 be two relations on U where*

$$R_1 = \Delta \cup \{(a, c), (c, a), (c, g), (d, f), (e, g), (f, d), (g, c), (g, e)\},$$

$$R_2 = \Delta \cup \{(a, c), (a, d), (a, e), (b, f), (c, a), (c, g), (d, a), (d, f), (e, a), (e, g), (f, b), (f, d), (g, c), (g, e)\}.$$

Thus, $< a > \breve{R}_1 = \{a, c, g\}, < b > \breve{R}_1 = \{b\}, < c > \breve{R}_1 =< g > \breve{R}_1 = \{a, c, e, g\}, < d > \breve{R}_1 =< f > \breve{R}_1 = \{d, f\}, < e > \breve{R}_1 = \{c, e, g\}, < a > \breve{R}_2 = \{a, c, d, e, f, g\}, < b > \breve{R}_2 = \{b, d, f\}, < c > \breve{R}_2 =< e > \breve{R}_2 = \{a, c, d, e, g\}, < d > \breve{R}_2 = \{a, b, c, d, e, f\}, < f > \breve{R}_2 = \{a, b, d, f\}$ and $< g > \breve{R}_2 = \{a, c, e, g\}$. Take

(1) $A = \{a, b, c, d, e, f\}$; then, $\underline{apr'}^{\mathcal{I}}_{R_1}(A) = \{b, d, f\}$ and $\underline{apr'}^{\mathcal{I}}_{R_2}(A) = A$. Therefore, $\underline{apr'}^{\mathcal{I}}_{R_1}(A) \nsupseteq \underline{apr'}^{\mathcal{I}}_{R_2}(A)$.

(2) $A = \{g\}$; then, $\overline{apr'}^{\mathcal{I}}_{R_1}(A) = \{a, c, e, g\}$ and $\overline{apr'}^{\mathcal{I}}_{R_2}(A) = \{g\}$. Therefore, $\overline{apr'}^{\mathcal{I}}_{R_1}(A) \nsubseteq \overline{apr'}^{\mathcal{I}}_{R_2}(A)$.

(3) $A = \{a, b, c, d, e, f\}$; then, $\underline{apr'}^{\mathcal{I}}_{R_1}(A) = \{b, d, f\}, \overline{apr'}^{\mathcal{I}}_{R_1}(A) = U, \underline{apr'}^{\mathcal{I}}_{R_2}(A) = A$ and $\overline{apr'}^{\mathcal{I}}_{R_2}(A) = U$. Therefore,

(a) $Boundary'^{\mathcal{I}}_{R_1}(A) = \{a, c, e, g\} \nsubseteq \{g\} = Boundary'^{\mathcal{I}}_{R_2}(A)$.

(b) $Accuracy'^{\mathcal{I}}_{R_1}(A) = \frac{3}{7} < \frac{6}{7} = Accuracy'^{\mathcal{I}}_{R_2}(A)$.

(c) $Roughness'^{\mathcal{I}}_{R_1}(A) = \frac{4}{7} > \frac{1}{7} = Roughness'^{\mathcal{I}}_{R_2}(A)$.

Although, $R_1 \subseteq R_2$.

Theorem 5.2. *Let $A \subseteq U$, $\mathcal{I}$ be an ideal on U and R be a reflexive relation on U. Then,*

(1) $\underline{\underline{apr}}^{\mathcal{I}}_{R}(A) \subseteq \underline{apr}^{\mathcal{I}}_{R}(A) \subseteq \underline{apr'}^{\mathcal{I}}_{R}(A) \subseteq A \subseteq \overline{apr'}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr}^{\mathcal{I}}_{R}(A) \subseteq \overline{\overline{apr}}^{\mathcal{I}}_{R}(A)$.

(2) $Boundary'^{\mathcal{I}}_{R}(A) \subseteq Boundary^{\mathcal{I}}_{R}(A) \subseteq \underline{Boundary}^{\mathcal{I}}_{R}(A)$.

(3) $\underline{Accuracy}^{\mathcal{I}}_{R}(A) \leq Accuracy^{\mathcal{I}}_{R}(A) \leq Accuracy'^{\mathcal{I}}_{R}(A)$.

Proof. We prove (1) only and the others are straightforward from (1). By Theorem 4.3, we have $\underline{\underline{apr}}^{\mathcal{I}}_{R}(A) \subseteq \underline{apr}^{\mathcal{I}}_{R}(A)$. To prove $\underline{apr}^{\mathcal{I}}_{R}(A) \subseteq \underline{apr'}^{\mathcal{I}}_{R}(A)$, let $x \in \underline{apr}^{\mathcal{I}}_{R}(A)$; then, $< x > \breve{R} \cap A^c \in \mathcal{I}$. Hence, $< x > \breve{R} \subseteq \underline{apr'}^{\mathcal{I}}_{R}(A)$. Since R is a reflexive relation, it follows that $x \in< x > \breve{R} \subseteq \underline{apr'}^{\mathcal{I}}_{R}(A)$. Therefore, $x \in \underline{apr'}^{\mathcal{I}}_{R}(A)$. Since R is reflexive. It follows that $\underline{apr'}^{\mathcal{I}}_{R}(A) \subseteq A \subseteq \overline{apr'}^{\mathcal{I}}_{R}(A)$ is straightforward from Definition 5.1. To prove $\overline{apr'}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr}^{\mathcal{I}}_{R}(A)$, let $x \in \overline{apr'}^{\mathcal{I}}_{R}(A) = (\underline{apr'}_{R}(A^c))^c$; then, $x \notin \underline{apr'}_{R}(A^c)$. Hence, by Definition 5.1, we get $< x > \breve{R} \cap A \notin \mathcal{I}$. It follows that $x \in \overline{apr}^{\mathcal{I}}_{R}(A)$. By Theorem 4.3, we have $\overline{apr}^{\mathcal{I}}_{R}(A) \subseteq \overline{\overline{apr}}^{\mathcal{I}}_{R}(A)$.

Remark 5.4. Example 3.1 (iii) shows that the inclusion and less than relation in Theorem 5.2 cannot be replaced by an equality relation in general. Take $A = \{b, c\}$; then, $\overline{apr'}^{\mathcal{I}}_{R}(A) = \{b, c\} \subsetneq \{a, b, c\} = \overline{apr}^{\mathcal{I}}_{R}(A)$. Moreover, take $A = \{a, d\}$, then

(1) $\underline{apr}^{\mathcal{I}}_{R}(A) = \{d\} \subsetneq \{a, d\} = \underline{apr'}^{\mathcal{I}}_{R}(A)$.

(2) $Boundary'^{\mathcal{I}}_{R}(A) = \{b, c\} \subsetneq \{a, b, c\} = Boundary^{\mathcal{I}}_{R}(A)$.

(3) $Accuracy^{\mathcal{I}}_{R}(A) = \frac{1}{4} \lneq \frac{1}{2} = Accuracy'^{\mathcal{I}}_{R}(A)$.

The following theorem shows that the current approximations in Definition 5.1 constitute an extension and a generalization of the previous Definition 2.6 [11].

Theorem 5.3. *Let $A \subseteq U$, $\mathcal{I}$ be an ideal on U and R be a similarity relation on a non-empty set U. Then,*

(1) $\overline{apr'}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr'}_{R}(A)$.

(2) $\underline{apr'}_{R}(A) \subseteq \underline{apr'}^{\mathcal{I}}_{R}(A)$.

(3) $Boundary'^{\mathcal{I}}_{R}(A) \subseteq Boundary'_{R}(A)$.

(4) $Accuracy'_{R}(A) \leq Accuracy'^{\mathcal{I}}_{R}(A)$.

Proof. The proof is similar to that of Theorem 3.3.

Remark 5.5. Example 3.3 shows that the inclusion and less than relation in Theorem 5.3 cannot be replaced by an equality relation in general. Take $A = \{a, c, d\}$; then,

(1) $\overline{apr'}^{\mathcal{I}}_{R_1}(A) = \{a, b\} \neq U = \overline{apr}_{R_1}(A)$.
(2) $\underline{apr'}_{R_1}(A) = \{c, d\} \neq U = \underline{apr'}^{\mathcal{I}}_{R_1}(A)$.
(3) $Boundary'^{\mathcal{I}}_{R_1}(A) = \phi \neq \{a, b\} = Boundary'_{R_1}(A)$.
(4) $Accuracy'_{R_1}(A) = \frac{1}{2} \lneq 2 = Accuracy'^{\mathcal{I}}_{R_1}(A)$.

6. Fourth method to obtain generalized rough sets using ideals

The objective of this section is to define the fourth type of the generalized rough approximations. The basic properties of these approximations are constructed and established. Moreover, the comparisons between these approximations and the approximations in Sections 3, 4 and 5 are illustrated. At the end of this section, the relationships between these approximations and the approximations in [11] are presented.

Definition 6.1. Let R be a binary relation on a non-empty set U and $\mathcal{I}$ be an ideal on U. For any subset $A \subseteq U$, the fourth kind of generalized upper and lower approximations, boundary regions, accuracy and roughness of A using ideal and according to R are respectively defined by

$$\overline{apr''}_{R}(A) = \cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I}\}. \tag{6.1}$$

$$\underline{apr''}_{R}(A) = (\overline{apr''}_{R}(A^c))^c. \tag{6.2}$$

$$Boundary''^{\mathcal{I}}_{R}(A) = \overline{apr''}^{\mathcal{I}}_{R}(A) - \underline{apr''}^{\mathcal{I}}_{R}(A). \tag{6.3}$$

$$Accuracy''^{\mathcal{I}}_{R}(A) = \frac{|\underline{apr''}^{\mathcal{I}}_{R}(A)|}{|\overline{apr''}^{\mathcal{I}}_{R}(A)}|, \overline{apr''}^{\mathcal{I}}_{R}(A) \neq \phi. \tag{6.4}$$

$$Roughness''^{\mathcal{I}}_{R}(A) = 1 - Accuracy''^{\mathcal{I}}_{R}(A). \tag{6.5}$$

Proposition 6.1. *Let $A, B \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. Then, the following properties hold:*

(1) $\overline{apr''}^{\mathcal{I}}_{R}(\phi) = \phi$.
(2) $A \subseteq B \Rightarrow \overline{apr''}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr''}^{\mathcal{I}}_{R}(B)$.
(3) $\overline{apr''}^{\mathcal{I}}_{R}(A \cap B) \subseteq \overline{apr''}^{\mathcal{I}}_{R}(A) \cap \overline{apr''}^{\mathcal{I}}_{R}(B)$.
(4) $\overline{apr''}^{\mathcal{I}}_{R}(A \cup B) = \overline{apr''}^{\mathcal{I}}_{R}(A) \cup \overline{apr''}^{\mathcal{I}}_{R}(B)$.
(5) $\overline{apr''}^{\mathcal{I}}_{R}(A) = (\underline{apr''}^{\mathcal{I}}_{R}(A^c))^c$.
(6) If $A \in \mathcal{I}$, *then* $\overline{apr''}^{\mathcal{I}}_{R}(A) = \phi$.
(7) If $\mathcal{I} \subseteq \mathcal{J}$, *then* $\overline{apr''}^{\mathcal{J}}_{R}(A) \subseteq \overline{apr''}^{\mathcal{I}}_{R}(A)$.
(8) If $\mathcal{I} = P(U)$, *then* $\overline{apr''}^{\mathcal{I}}_{R}(A) = \phi$.
(9) $\overline{apr''}^{\mathcal{I}\cap\mathcal{J}}_{R}(A) = \overline{apr''}^{\mathcal{I}}_{R}(A) \cup \overline{apr''}^{\mathcal{J}}_{R}(A)$.
(10) $\overline{apr''}^{\mathcal{I}\vee\mathcal{J}}_{R}(A) = \overline{apr''}^{\mathcal{I}}_{R}(A) \cap \overline{apr''}^{\mathcal{J}}_{R}(A)$.

Proof.

(1) $\overline{apr''}^{\mathcal{I}}_{R}(\phi) = \cup\{< x > \breve{R} :< x > \breve{R} \cap \phi \notin \mathcal{I}\} = \phi$.
(2) Let $A \subseteq B$ and $x \in \overline{apr''}^{\mathcal{I}}_{R}(A)$. Then, $\exists\ y \in U$ such that $x \in< y > \breve{R}$ and $< y > \breve{R} \cap A \notin \mathcal{I}$. Thus, $< y > \breve{R} \cap B \notin \mathcal{I}$. So, $x \in \overline{apr''}^{\mathcal{I}}_{R}(B)$. Consequently, $\overline{apr''}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr''}^{\mathcal{I}}_{R}(B)$.
(3) The proof is immediately by (2).
(4)

$$\begin{aligned}\overline{apr''}^{\mathcal{I}}_{R}(A \cup B) &= \cup\{< x > \breve{R} :< x > \breve{R} \cap (A \cup B) \notin \mathcal{I}\}.\\ &= (\cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I}\}) \cup (\cup\{< x > \breve{R} :< x > \breve{R} \cap B \notin \mathcal{I}\}).\\ &= (\cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I}\})\ or\ (\cup\{< x > \breve{R} :< x > \breve{R} \cap B \notin \mathcal{I}\}).\\ &= \overline{apr''}^{\mathcal{I}}_{R}(A) \cup \overline{apr''}^{\mathcal{I}}_{R}(B).\end{aligned}$$

(5)

$$\begin{aligned}(\underline{apr''}^{\mathcal{I}}_{R}(A^c))^c &= ((\overline{apr''}^{\mathcal{I}}_{R}(A))^c)^c.\\ &= \overline{apr''}^{\mathcal{I}}_{R}(A).\end{aligned}$$

(6) The proof is straightforward by Definition 6.1.

(7) Let $\mathcal{I} \subseteq \mathcal{J}, x \in \overline{apr''}^{\mathcal{J}}_R(A)$. Then, $\exists\ y \in U$ such that $x \in < y > \breve{R}$ and $< y > \breve{R} \cap A \notin \mathcal{J}$. Thus, $< y > \breve{R} \cap A \notin \mathcal{I}$ as $\mathcal{I} \subseteq \mathcal{J}$. So, $x \in \overline{apr''}^{\mathcal{I}}_R(A)$. Hence, $\overline{apr''}^{\mathcal{J}}_R(A) \subseteq \overline{apr''}^{\mathcal{I}}_R(A)$.
(8) The proof is straightforward by Definition 6.1.
(9)

$$\begin{aligned}\overline{apr''}^{\mathcal{I}\cap\mathcal{J}}_R(A) &= \cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I} \cap \mathcal{J}\}\\ &= (\cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I}\})\ or\ (\cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{J}\})\\ &= (\cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I}\}) \cup (\cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{J}\})\\ &= \overline{apr''}^{\mathcal{I}}_R(A) \cup \overline{apr''}^{\mathcal{J}}_R(A).\end{aligned}$$

(10)

$$\begin{aligned}\overline{apr''}^{\mathcal{I}\vee\mathcal{J}}_R(A) &= \cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I} \vee \mathcal{J}\}\\ &= \cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I} \cup \mathcal{J}\}\\ &= (\cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I}\})\ and\ (\cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{J}\})\\ &= (\cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{I}\}) \cap (\cup\{< x > \breve{R} :< x > \breve{R} \cap A \notin \mathcal{J}\})\\ &= \overline{apr''}^{\mathcal{I}}_R(A) \cap \overline{apr''}^{\mathcal{J}}_R(A).\end{aligned}$$

Proposition 6.2. *Let $A, B \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. Then, the following properties hold:*

(1) $\underline{apr''}^{\mathcal{I}}_R(U) = U$.
(2) $A \subseteq B \Rightarrow \underline{apr''}^{\mathcal{I}}_R(A) \subseteq \underline{apr''}^{\mathcal{I}}_R(B)$.
(3) $\underline{apr''}^{\mathcal{I}}_R(A) \cup \underline{apr''}^{\mathcal{I}}_R(B) \subseteq \underline{apr''}^{\mathcal{I}}_R(A \cup B)$.
(4) $\underline{apr''}^{\mathcal{I}}_R(A \cap B) = \underline{apr''}^{\mathcal{I}}_R(A) \cap \underline{apr''}^{\mathcal{I}}_R(B)$.
(5) $\underline{apr''}^{\mathcal{I}}_R(A) = (\overline{apr''}^{\mathcal{I}}_R(A^c))^c$.
(6) If $A^c \in \mathcal{I}$, *then* $\underline{apr''}^{\mathcal{I}}_R(A) = U$.
(7) If $\mathcal{I} \subseteq \mathcal{J}$, *then* $\underline{apr''}^{\mathcal{I}}_R(A) \subseteq \underline{apr''}^{\mathcal{J}}_R(A)$.
(8) If $\mathcal{I} = P(U)$, *then* $\underline{apr''}^{\mathcal{I}}_R(A) = U$.
(9) $\underline{apr''}^{\mathcal{I}\cap\mathcal{J}}_R(A) = \underline{apr''}^{\mathcal{I}}_R(A) \cap \underline{apr''}^{\mathcal{J}}_R(A)$.
(10) $\underline{apr''}^{\mathcal{I}\vee\mathcal{J}}_R(A) = \underline{apr''}^{\mathcal{I}}_R(A) \cup \underline{apr''}^{\mathcal{J}}_R(A)$.

Proof.

(1) $\underline{apr''}^{\mathcal{I}}_R(U) = (\overline{apr''}^{\mathcal{I}}_R(\phi))^c = \phi^c = U$ by (1) in Proposition 6.1.
(2) Let $A \subseteq B$. Thus, $B^c \subseteq A^c$ and $\overline{apr''}^{\mathcal{I}}_R(B^c) \subseteq \overline{apr''}^{\mathcal{I}}_R(A^c)$ (by (2) in Proposition 6.1). Then, $(\overline{apr''}^{\mathcal{I}}_R(A^c))^c \subseteq (\overline{apr''}^{\mathcal{I}}_R(B^c))^c$. So, $\underline{apr''}^{\mathcal{I}}_R(A) \subseteq \underline{apr''}^{\mathcal{I}}_R(B)$.
(3) The proof is immediately by (2).

(4)

$$\begin{aligned}\underline{apr''}_R^{\mathcal{I}}(A \cap B) &= (\overline{apr''}_R^{\mathcal{I}}(A \cap B)^c)^c \\ &= (\overline{apr''}_R^{\mathcal{I}}(A^c \cup B^c))^c \\ &= (\overline{apr''}_R^{\mathcal{I}}(A^c) \cup \overline{apr''}_R^{\mathcal{I}}(B^c))^c \text{ (by (4) in Proposition 6.1)} \\ &= (\overline{apr''}_R^{\mathcal{I}}(A^c))^c \cap (\overline{apr''}_R^{\mathcal{I}}(B^c))^c \\ &= \underline{apr''}_R^{\mathcal{I}}(A) \cap \underline{apr''}_R^{\mathcal{I}}(B).\end{aligned}$$

(5) The proof is straightforward by Definition 6.1.

(6) Let $A^c \in \mathcal{I}$; then, $\underline{apr''}_R^{\mathcal{I}}(A) = (\overline{apr''}_R^{\mathcal{I}}(A^c))^c = (\phi)^c = U$ according to Proposition 6.1 (6).

(7) Let $\mathcal{I} \subseteq \mathcal{J}$. Then, $\overline{apr''}_R^{\mathcal{J}}(A^c) \subseteq \overline{apr''}_R^{\mathcal{I}}(A^c)$ according to Proposition 6.1 (7). Thus, $(\overline{apr''}_R^{\mathcal{I}}(A^c))^c \subseteq (\overline{apr''}_R^{\mathcal{J}}(A^c))^c$. Hence, $\underline{apr''}_R^{\mathcal{I}}(A) \subseteq \underline{apr''}_R^{\mathcal{J}}(A)$.

(8) Let $\mathcal{I} = P(U)$; then, $\underline{apr''}_R^{\mathcal{I}}(A) = (\overline{apr''}_R^{\mathcal{I}}(A^c))^c = (\phi)^c = U$ according to Proposition 6.1 (8).

(9)

$$\begin{aligned}\underline{apr''}_R^{\mathcal{I}\cap\mathcal{J}}(A) &= (\overline{apr''}_R^{\mathcal{I}\cap\mathcal{J}}(A^c))^c \\ &= (\overline{apr''}_R^{\mathcal{I}}(A^c) \cup \overline{apr''}_R^{\mathcal{J}}(A^c))^c \text{ (by (9) in Proposition 6.1)} \\ &= (\overline{apr''}_R^{\mathcal{I}}(A^c))^c \cap (\overline{apr''}_R^{\mathcal{J}}(A^c))^c \\ &= \underline{apr''}_R^{\mathcal{I}}(A) \cap \underline{apr''}_R^{\mathcal{J}}(A).\end{aligned}$$

(10)

$$\begin{aligned}\underline{apr''}_R^{\mathcal{I}\vee\mathcal{J}}(A) &= (\overline{apr''}_R^{\mathcal{I}\vee\mathcal{J}}(A^c))^c \\ &= (\overline{apr''}_R^{\mathcal{I}}(A^c) \cap \overline{apr''}_R^{\mathcal{J}}(A^c))^c \text{ by (10) in Proposition 6.1)} \\ &= (\overline{apr''}_R^{\mathcal{I}}(A^c))^c \cup (\overline{apr''}_R^{\mathcal{J}}(A^c))^c \\ &= \underline{apr''}_R^{\mathcal{I}}(A) \cup \underline{apr''}_R^{\mathcal{J}}(A).\end{aligned}$$

Remark 6.1. (1) Example 3.1 (i) shows that the converse of (2) in Propositions 6.1 and 6.2 is not necessarily true in general. Take

(a) $A = \{a\}$ and $B = \{d\}$; then, $\overline{apr''}_R^{\mathcal{I}}(A) = \phi$ and $\overline{apr''}_R^{\mathcal{I}}(B) = U$. Therefore, $\overline{apr''}_R^{\mathcal{I}}(A) \subseteq \overline{apr''}_R^{\mathcal{I}}(B)$, but $A \nsubseteq B$.

(b) $A = \{b\}$ and $B = \{a, c, d\}$; then, $\underline{apr''}_R^{\mathcal{I}}(A) = \phi$ and $\underline{apr''}_R^{\mathcal{I}}(B) = U$. Therefore, $\underline{apr''}_R^{\mathcal{I}}(A) \subseteq \underline{apr''}_R^{\mathcal{I}}(B)$, but $A \nsubseteq B$.

(2) Example 3.1 (ii) shows that the converse of (6)–(8) in Propositions 6.1 and 6.2 is not necessarily true in general.

(i) For (6), take

(a) $A = \{a, d\}$; then, $\overline{apr''}^{\mathcal{J}}_{R}(A) = \phi$. Therefore, $\overline{apr''}^{\mathcal{J}}_{R}(A) = \phi$, but $A \notin \mathcal{J}$.

(b) $A = \{b, c\}$; then, $\underline{apr''}^{\mathcal{J}}_{R}(A) = U$. Therefore, $\underline{apr''}^{\mathcal{J}}_{R}(A) = U$, but $A^c \notin \mathcal{J}$.

(ii) For (7), take

(a) $A = \{a, d\}$; then, $\overline{apr''}^{\mathcal{I}}_{R}(A) = \{a\}$ and $\overline{apr''}^{\mathcal{J}}_{R}(A) = \phi$. Therefore, $\overline{apr''}^{\mathcal{J}}_{R}(A) \subseteq \overline{apr''}^{\mathcal{I}}_{R}(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

(b) $A = \{b, c\}$; then, $\underline{apr''}^{\mathcal{I}}_{R}(A) = \{b, c, d\}$ and $\underline{apr''}^{\mathcal{J}}_{R}(A) = U$. Therefore, $\underline{apr''}^{\mathcal{I}}_{R}(A) \subseteq \underline{apr''}^{\mathcal{J}}_{R}(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

(iii) For (8), take

(a) $A = \{a, d\}$; then, $\overline{apr''}^{\mathcal{J}}_{R}(A) = \phi$, but $\mathcal{J} \neq P(U)$.

(b) $A = \{b, c\}$; then, $\underline{apr''}^{\mathcal{J}}_{R}(A) = U$, but $\mathcal{J} \neq P(U)$.

(3) Example 3.1 (iii) shows that the inclusion of (3) in Propositions 6.1 and 6.2 cannot be replaced by an equality relation in general. Take $A = \{a, d\}$ and $B = \{b, c\}$; then,

(a) $\overline{apr''}^{\mathcal{I}}_{R}(A) = \overline{apr''}^{\mathcal{I}}_{R}(B) = U$ and $\overline{apr''}^{\mathcal{I}}_{R}(A \cap B) = \phi$. Therefore, $\overline{apr''}^{\mathcal{I}}_{R}(A) \cap \overline{apr''}^{\mathcal{I}}_{R}(B) = U \neq \phi = \overline{apr'}^{\mathcal{I}}_{R}(A \cap B)$.

(b) $\underline{apr''}^{\mathcal{I}}_{R}(A) = \underline{apr''}^{\mathcal{I}}_{R}(B) = \phi$ and $\underline{apr''}^{\mathcal{I}}_{R}(A \cup B) = U$. Therefore, $\underline{apr''}^{\mathcal{I}}_{R}(A) \cup \underline{apr''}^{\mathcal{I}}_{R}(B) = \phi \neq U = \underline{apr''}^{\mathcal{I}}_{R}(A \cup B)$.

Remark 6.2. There are some properties that are not held or satisfied for the fourth type.

(i) Considering Example 3.1 (i), take

(1) $A = \{a\}$; then, $\overline{apr''}^{\mathcal{I}}_{R}(A) = \phi$. Hence, $A \nsubseteq \overline{apr''}^{\mathcal{I}}_{R}(A)$.

(2) $A = \{b, c, d\}$; then, $\underline{apr''}^{\mathcal{I}}_{R}(A) = U$. Hence, $\underline{apr''}^{\mathcal{I}}_{R}(A) \nsubseteq A$.

(ii) Considering Example 3.1 (ii), take

(1) $A = U$; then, $\overline{apr''}^{\mathcal{I}}_{R}(U) = \{a, b, c\}$. Hence, $\overline{apr''}^{\mathcal{I}}_{R}(U) \neq U$.

(2) $A = \phi$; then, $\underline{apr''}^{\mathcal{I}}_{R}(\phi) = \{d\}$. Hence, $\underline{apr''}^{\mathcal{I}}_{R}(\phi) \neq \phi$.

Theorem 6.1. *Let $A \subseteq U$, $\mathcal{I}, \mathcal{J}$ be two ideals on U and R be a binary relation on U. If $\mathcal{I} \subseteq \mathcal{J}$, then*

(1) $Boundary''^{\mathcal{J}}_{R}(A) \subseteq Boundary''^{\mathcal{I}}_{R}(A)$.

(2) $Accuracy''^{\mathcal{I}}_{R}(A) \leq Accuracy''^{\mathcal{J}}_{R}(A)$.

Proof. Similar to Theorem 3.1.

Remark 6.3. Example 3.1 (ii) shows that the converse of (1) and (2) in Theorem 6.1 is not necessarily true in general. Take $A = \{b, c\}$; then,

(1) $Boundary''^{\mathcal{J}}_{R}(A) = \phi \subseteq \phi = Boundary''^{\mathcal{I}}_{R}(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

(2) $Accuracy''^{\mathcal{I}}_{R}(A) = \frac{3}{2} < 2 = Accuracy''^{\mathcal{J}}_{R}(A)$, but $\mathcal{I} \nsubseteq \mathcal{J}$.

Theorem 6.2. *Let $\phi \neq A \subseteq U$, $\mathcal{I}$ be an ideal on U and R_1, R_2 be two binary relations on U. If $R_1 \subseteq R_2$, then*

(1) $\overline{apr''}^{\mathcal{I}}_{R_1}(A) \subseteq \overline{apr''}^{\mathcal{I}}_{R_2}(A)$.

(2) $\underline{apr''}^{\mathcal{I}}_{R_2}(A) \subseteq \underline{apr''}^{\mathcal{I}}_{R_1}(A)$.

(3) $Boundary''^{\mathcal{I}}_{R_1}(A) \subseteq Boundary''^{\mathcal{I}}_{R_2}(A)$.

(4) $Accuracy''^{\mathcal{I}}_{R_2}(A) \leq Accuracy''^{\mathcal{I}}_{R_1}(A)$.

Proof.

(1) Let $x \in \overline{apr''}^{\mathcal{I}}_{R_1}(A)$. Then, $\exists\ y \in U$ such that $x \in < y > \breve{R}_1 \cap A \notin \mathcal{I}$. Since $< y > \breve{R}_1 \subseteq < y > \breve{R}_2$ (by Theorem 2.1 [7]), it follows that $x \in < y > \breve{R}_2 \cap A \notin \mathcal{I}$. Thus, $x \in \overline{apr''}^{\mathcal{I}}_{R_2}(A)$. Hence, $\overline{apr''}^{\mathcal{I}}_{R_1}(A) \subseteq \overline{apr''}^{\mathcal{I}}_{R_2}(A)$.

(2) $x \in \underline{apr''}^{\mathcal{I}}_{R_2}(A) = (\overline{apr''}^{\mathcal{I}}_{R_2}(A^c))^c \subseteq (\overline{apr''}^{\mathcal{I}}_{R_1}(A^c))^c$ (according to (1)) $= \underline{apr''}^{\mathcal{I}}_{R_1}(A)$.

(3) Let $x \in Boundary''^{\mathcal{I}}_{R_1}(A)$. Then, $x \in \overline{apr''}^{\mathcal{I}}_{R_1}(A) - \underline{apr''}^{\mathcal{I}}_{R_1}(A)$. So, $x \in \overline{apr''}^{\mathcal{I}}_{R_1}(A)$ and $x \in (\underline{apr''}^{\mathcal{I}}_{R_1}(A))^c$. Thus, $x \in \overline{apr''}^{\mathcal{I}}_{R_2}(A)$ and $x \in (\underline{apr''}^{\mathcal{I}}_{R_2}(A))^c$ according to (1) and (2). Hence, $x \in Boundary''^{\mathcal{I}}_{R_2}(A)$. Therefore, $Boundary''^{\mathcal{I}}_{R_1}(A) \subseteq Boundary''^{\mathcal{I}}_{R_2}(A)$.

(4) $Accuracy''^{\mathcal{I}}_{R_2}(A) = \left|\frac{\underline{apr''}^{\mathcal{I}}_{R_2}(A)}{\overline{apr''}^{\mathcal{I}}_{R_2}(A)}\right| \leq \left|\frac{\underline{apr''}^{\mathcal{I}}_{R_1}(A)}{\overline{apr''}^{\mathcal{I}}_{R_1}(A)}\right| = Accuracy''^{\mathcal{I}}_{R_1}(A)$.

Remark 6.4. Example 3.3 shows that the inclusion and less than relation in Theorem 6.2 cannot be replaced by an equality relation in general. Take

(i) $A = \{a, d\}$; then,

(1) $\overline{apr''}^{\mathcal{I}}_{R_1}(A) = \{a, b\} \neq \{a, b, c\} = \overline{apr''}^{\mathcal{I}}_{R_2}(A)$.

(2) $Accuracy''^{\mathcal{I}}_{R_1}(A) = 2 \neq \frac{4}{3} = Accuracy''^{\mathcal{I}}_{R_2}(A)$.

(ii) $A = \{b, c\}$; then, $\underline{apr''}^{\mathcal{I}}_{R_1}(A) = \{c, d\} \neq \{d\} = \underline{apr''}^{\mathcal{I}}_{R_2}(A)$.

Theorem 6.3. *Let $A \subseteq U$, $\mathcal{I}$ be an ideal on U and R be a reflexive relation on U. Then,*

(1) $\underline{apr''}^{\mathcal{I}}_{R}(A) \subseteq \underline{apr}^{\mathcal{I}}_{R}(A) \subseteq \underline{apr'}^{\mathcal{I}}_{R}(A) \subseteq A \subseteq \overline{apr'}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr''}^{\mathcal{I}}_{R}(A)$.

(2) $Boundary'^{\mathcal{I}}_{R}(A) \subseteq Boundary^{\mathcal{I}}_{R}(A) \subseteq Boundary''^{\mathcal{I}}_{R}(A)$.

(3) $Accuracy''^{\mathcal{I}}_{R}(A) \leq Accuracy^{\mathcal{I}}_{R}(A) \leq Accuracy'^{\mathcal{I}}_{R}(A)$.

Proof. By Theorem 5.2, we have $\underline{apr}^{\mathcal{I}}_{R}(A) \subseteq \underline{apr'}^{\mathcal{I}}_{R}(A) \subseteq A \subseteq \overline{apr'}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr}^{\mathcal{I}}_{R}(A)$. To prove $\underline{apr''}^{\mathcal{I}}_{R}(A) \subseteq \underline{apr}^{\mathcal{I}}_{R}(A)$, let $x \in \underline{apr''}^{\mathcal{I}}_{R}(A) = \overline{apr''}^{\mathcal{I}}_{R}(A^c)^c$. Then, $x \notin \overline{apr''}^{\mathcal{I}}_{R}(A^c)$. Thus, by Definition 6.1, $< x > \breve{R} \cap A^c \in \mathcal{I}$. It follows that $< x > \breve{R} \subseteq \underline{apr}^{\mathcal{I}}_{R}(A)$. Since R is a reflexive relation, it follows that $x \in < x > \breve{R} \subseteq \underline{apr}^{\mathcal{I}}_{R}(A)$. Therefore, $x \in \underline{apr}^{\mathcal{I}}_{R}(A)$. To prove $\overline{apr}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr''}^{\mathcal{I}}_{R}(A)$, let $x \in \overline{apr}^{\mathcal{I}}_{R}(A)$, then

$< x > \breve{R} \cap A \notin \mathcal{I}$. It follows that $< x > \breve{R} \subseteq \overline{apr''}^{\mathcal{I}}_{R}(A)$. Since R is a reflexive relation, it follows that $x \in < x > \breve{R} \subseteq \overline{apr''}^{\mathcal{I}}_{R}(A)$. Therefore, $x \in \overline{apr''}^{\mathcal{I}}_{R}(A)$.

Remark 6.5. Example 3.1 (iii) shows that the inclusion and less than relation in Theorem 6.3 cannot be replaced by an equality relation in general. Take $A = \{b, c\}$; then, $\overline{apr}^{\mathcal{I}}_{R}(A) = \{a, b, c\} \subsetneq U = \overline{apr''}^{\mathcal{I}}_{R}(A)$. Moreover, take $A = \{a, d\}$; then,

(1) $\underline{apr''}^{\mathcal{I}}_{R}(A) = \phi \subsetneq \{d\} = \underline{apr}^{\mathcal{I}}_{R}(A)$.

(2) $Boundary^{\mathcal{I}}_{R}(A) = \{a, b, c\} \subsetneq U = Boundary''^{\mathcal{I}}_{R}(A)$.

(3) $Accuracy''^{\mathcal{I}}_{R}(A) = 0 \lneq \frac{1}{4} = Accuracy^{\mathcal{I}}_{R}(A)$.

Remark 6.6. From the above results, it is noted that there are different methods to approximate the sets. The best of these methods is the third type explained in Section 5, as the boundary regions in this case are more effectively reduced (or canceled) by increasing the lower approximations and decreasing the upper approximations as compared to the other types in the other sections. Moreover, the accuracy is higher than the other types.

Theorem 6.4. *Let $A \subseteq U$, $\mathcal{I}$ be an ideal on U and R be a similarity relation on a non-empty set U. Then,*

(1) $\overline{apr''}^{\mathcal{I}}_{R}(A) \subseteq \overline{apr''}_{R}(A)$.

(2) $\underline{apr''}_{R}(A) \subseteq \underline{apr''}^{\mathcal{I}}_{R}(A)$.

(3) $Boundary''^{\mathcal{I}}_{R}(A) \subseteq Boundary''_{R}(A)$.

(4) $Accuracy''_{R}(A) \leq Accuracy''^{\mathcal{I}}_{R}(A)$.

Proof. The proof is similar to that of Theorem 3.3.

Remark 6.7. Example 3.3 shows that the inclusion and less than relation in Theorem 6.4 cannot be replaced by an equality relation in general. Take $A = \{a, c, d\}$; then,

(1) $\overline{apr''}^{\mathcal{I}}_{R_1}(A) = \{a, b, d\} \neq U = \overline{apr''}_{R_1}(A)$.

(2) $\underline{apr''}_{R_1}(A) = \{c, d\} \neq U = \underline{apr''}^{\mathcal{I}}_{R_1}(A)$.

(3) $Boundary''^{\mathcal{I}}_{R_1}(A) = \phi \neq \{a, b\} = Boundary''_{R_1}(A)$.

(4) $Accuracy''_{R_1}(A) = \frac{1}{2} \lneq \frac{4}{3} = Accuracy''^{\mathcal{I}}_{R_1}(A)$.

Tables 1 and 2 summarize the differences among the properties of the proposed four methods.

Table 1. Comparison between the first and second methods according to the properties in Definition 2.3. $\surd$ means that the property holds, while X denotes that the property does not hold.

	The first method	The second method
$\mathcal{L}_1$	$\surd$	$\surd$
$\mathcal{L}_2$	$\surd$	$\surd$
$\mathcal{L}_3$	X	$\surd$
$\mathcal{L}_4$	X	$\surd$
$\mathcal{L}_5$	$\surd$	$\surd$
$\mathcal{L}_6$	$\surd$	$\surd$
$\mathcal{L}_7$	$\surd$	$\surd$
$\mathcal{L}_8$	X	X
$\mathcal{L}_9$	X	X
$\mathcal{U}_1$	$\surd$	$\surd$
$\mathcal{U}_2$	X	$\surd$
$\mathcal{U}_3$	$\surd$	$\surd$
$\mathcal{U}_4$	X	$\surd$
$\mathcal{U}_5$	$\surd$	$\surd$
$\mathcal{U}_6$	$\surd$	$\surd$
$\mathcal{U}_7$	$\surd$	$\surd$
$\mathcal{U}_8$	X	X
$\mathcal{U}_9$	X	X

Table 2. Comparison between the third and fourth methods according to the properties in Definition 2.3.

	The third method	The fourth method
$\mathcal{L}_1$	$\surd$	$\surd$
$\mathcal{L}_2$	X	$\surd$
$\mathcal{L}_3$	X	X
$\mathcal{L}_4$	X	X
$\mathcal{L}_5$	X	$\surd$
$\mathcal{L}_6$	$\surd$	$\surd$
$\mathcal{L}_1$	$\surd$	$\surd$
$\mathcal{U}_1$	$\surd$	$\surd$
$\mathcal{U}_2$	X	X
$\mathcal{U}_3$	X	$\surd$
$\mathcal{U}_4$	X	X
$\mathcal{U}_5$	X	$\surd$
$\mathcal{U}_6$	$\surd$	$\surd$
$\mathcal{U}_7$	$\surd$	$\surd$

7. Medical applications

The central goal of this section is to apply the suggested techniques to real-life problems, especially in the domain of medical diagnosis, where more precise decisions are needed. Therefore, the features of the proposed approximations in terms of ideals are scrutinized for two different medical applications. These applications prove the adequacy of the generalization of rough sets using ideals to treat and model a lot of real-life issues. It is shown that the application of ideals to the rough set theory helps to remove the uncertainty and vagueness in data.

Example 7.1. *Medical application: Decision-making for COVID-19*

The purpose of this example is to demonstrate the significance of the current approximations in obtaining the best tools to identify the decisive factors of infections for COVID-19 in humans. The information in Table 3 was collected by the World Health Organization as well as through medical groups specializing in COVID-19 [15]. It was taken from 1000 patients and reduced to 10 patients because the attributes in rows (objects) are identical. So, the set of objects is

$$U = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\}.$$

The most common symptoms (set of attributes) of COVID-19 are as follows:
{Difficulty breathing = a_1, Chest pain = a_2, Headache = a_3, Dry cough = a_4, High Temperature = a_5, Loss of smell or taste = a_6} and Decision COVID-19 = $\{d\}$, as shown in Table 3.

Table 3. Decisive information data set.

Patients	Serious symptoms		Most common symptoms				Decision COVID-19
	a_1	a_2	a_3	a_4	a_5	a_6	
$\{p_1\}$	yes	yes	no	yes	yes	yes	yes
$\{p_2\}$	yes	yes	yes	yes	yes	yes	yes
$\{p_3\}$	yes	yes	no	yes	no	yes	no
$\{p_4\}$	yes	yes	no	no	no	no	no
$\{p_5\}$	yes	yes	no	yes	no	no	no
$\{p_6\}$	yes	no	yes	yes	yes	no	yes
$\{p_7\}$	no	no	no	yes	yes	no	yes
$\{p_8\}$	no	no	no	yes	yes	no	no
$\{p_9\}$	no	no	no	no	no	yes	yes
$\{p_{10}\}$	no	no	yes	yes	yes	no	yes

From Table 3, the symptoms are given as follows:

$$\mathcal{V}(p_1) = \{a_1, a_2, a_4, a_5, a_6\}, \mathcal{V}(p_2) = \{a_1, a_2, a_3, a_4, a_5, a_6\}, \mathcal{V}(p_3) = \{a_1, a_2, a_4, a_6\}, \mathcal{V}(p_4) = \{a_1, a_2\},$$

$$\mathcal{V}(p_5) = \{a_1, a_2, a_4\}, \mathcal{V}(p_6) = \{a_1, a_3, a_4, a_5\}, \mathcal{V}(p_7) = \mathcal{V}(p_8) = \{a_4, a_5\}, \mathcal{V}(p_9) = \{a_6\}$$

and

$$\mathcal{V}(p_{10}) = \{a_3, a_4, a_5\}.$$

Hence, the similarity relation is given as follows: $p_i \mathcal{R} p_j \Leftrightarrow \mathcal{V}(p_i) \cap \mathcal{V}(p_j) \neq \phi$. Consequently,

$$\mathcal{R} = \Delta \cup \{(p_1,p_2),(p_1,p_3),(p_1,p_4),(p_1,p_5),(p_1,p_6),(p_1,p_7),(p_1,p_8),(p_1,p_9),(p_1,p_{10}),$$
$$(p_2,p_1),(p_2,p_3),(p_2,p_4),(p_2,p_5),(p_2,p_6),(p_2,p_7),(p_2,p_8),(p_2,p_9),(p_2,p_{10}),(p_3,p_1),$$
$$(p_3,p_2),(p_3,p_4),(p_3,p_5),(p_3,p_6),(p_3,p_7),(p_3,p_8),(p_3,p_9),(p_3,p_{10}),(p_4,p_1),(p_4,p_2),$$
$$(p_4,p_3),(p_4,p_5),(p_4,p_6),(p_5,p_1),(p_5,p_2),(p_5,p_3),(p_5,p_4),(p_5,p_6),(p_5,p_7),(p_5,p_8),$$
$$(p_5,p_{10}),(p_6,p_1),(p_6,p_2),(p_6,p_3),(p_6,p_4),(p_6,p_5),(p_6,p_7),(p_6,p_8),(p_6,p_{10}),(p_7,p_1),$$
$$(p_7,p_2),(p_7,p_3),(p_7,p_5),(p_7,p_6),(p_7,p_8),(p_7,p_{10}),(p_8,p_1),(p_8,p_2),(p_8,p_3),(p_8,p_5),$$
$$(p_8,p_6),(p_8,p_7),(p_8,p_{10}),(p_9,p_1),(p_9,p_2),(p_9,p_3),(p_{10},p_1),(p_{10},p_2),(p_{10},p_3),(p_{10},p_5),$$
$$(p_{10},p_6),(p_{10},p_7),(p_{10},p_8)\};$$

thus,

$$< p_1 > \breve{\mathcal{R}} =< p_2 > \breve{\mathcal{R}} =< p_3 > \breve{\mathcal{R}} =< p_4 > \breve{\mathcal{R}} =< p_5 > \breve{\mathcal{R}} =< p_6 > \breve{\mathcal{R}}$$
$$=< p_7 > \breve{\mathcal{R}} =< p_8 > \breve{\mathcal{R}} =< p_9 > \breve{\mathcal{R}} =< p_{10} > \breve{\mathcal{R}} = U.$$

Let

$$\mathcal{I} = \{\phi,\{p_3\},\{p_4\},\{p_5\},\{p_8\},\{p_3,p_4\},\{p_3,p_5\},\{p_3,p_8\},\{p_4,p_5\},\{p_4,p_8\},\{p_5,p_8\},$$
$$\{p_3,p_4,p_5\},\{p_3,p_4,p_8\},\{p_3,p_5,p_8\},\{p_4,p_5,p_8\},\{p_3,p_4,p_5,p_8\}\}.$$

Hence, Table 3 represents a decision system; thus, the patients with confirmed infections for COVID-19 are surely $A = \{p_1, p_2, p_6, p_7, p_9, p_{10}\}$. Then,

(1) By the previous approximations [11], the first (second/third) kind of lower and upper approximations, boundary regions and accuracy of A are respectively ϕ, U, U and 0. This means that the patients p_1, p_2, p_6, p_7, p_9 and p_{10} are not infected with COVID-19, which contradicts the decision system in Table 3. Therefore, we are unable to decide whether the patient is infected with COVID-19 and this produced vagueness in the medical diagnosis decision-making process. Consequently, Dai et al.'s methods [11] are not suitable for obtaining an accurate decision.
(2) According to the proposed second type, the lower and upper approximations, boundary regions and accuracy of A are respectively $A, U, \{p_3, p_4, p_5, p_8\}$ and $\frac{6}{10}$. This means that the patients p_1, p_2, p_6, p_7, p_9 and p_{10} are surely infected with COVID-19 according to the present technique which is consistent with Table 3. Accordingly, the vagueness is reduced in the data and the accuracy measure is increased.

Example 7.2. *Medical application: Decision-making for a heart attack problem*

In this example the proposed methods are applied to decision-making for heart attacks. The data set in Table 4 was obtained from Al-Azhar University's cardiology department [10] (Hospital of Sayed Glal University, Cairo, Egypt). Table 4 represents the set of objects (patients) as

$$U = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}\}.$$

It was reduced to $\{p_1, p_2, p_3, p_4, p_5, p_8, p_9\}$ because the attributes in rows (objects) are identical. The study included patients with different symptoms, i.e., the set of attributes = {Breathlessness = a_1, Orthopnea = a_2, Paroxysmal nocturnal dyspnea = a_3, Reduced exercise tolerance = a_4, Ankle swelling = a_5} and decision of heart attacks is confirmed or ruled out = $\{d\}$ as shown in Table 4.

Table 4. Decision information data set.

Patients	a_1	a_2	a_3	a_4	a_5	d
$\{p_1\}$	yes	yes	yes	yes	no	yes
$\{p_2\}$	no	no	no	yes	yes	no
$\{p_3\}$	yes	yes	yes	yes	yes	yes
$\{p_4\}$	no	no	no	yes	no	no
$\{p_5\}$	yes	no	no	yes	yes	no
$\{p_8\}$	yes	yes	no	yes	yes	yes
$\{p_9\}$	yes	no	yes	yes	no	yes

Therefore, from Table 4, the symptoms are given as follows: $\mathcal{V}(p_1) = \{a_1, a_2, a_3, a_4\}, \mathcal{V}(p_2) = \{a_4, a_5\}, \mathcal{V}(p_3) = \{a_1, a_2, a_3, a_4, a_5\}, \mathcal{V}(p_4) = \{a_4\}, \mathcal{V}(p_5) = \{a_1, a_4, a_5\}, \mathcal{V}(p_8) = \{a_1, a_2, a_4, a_5\}$ and $\mathcal{V}(p_9) = \{a_1, a_3, a_4\}$.

Hence, the following binary relation is obtained: $p_i \mathcal{R} p_j \Leftrightarrow \mathcal{V}(p_i) \subsetneq \mathcal{V}(p_j)$. Consequently,

$$\mathcal{R} = \{(p_1, p_3), (p_2, p_3), (p_2, p_5), (p_2, p_8), (p_4, p_1), (p_4, p_2), (p_4, p_3), (p_4, p_5), (p_4, p_8),$$

$$(p_4, p_9), (p_5, p_3), (p_5, p_8), (p_8, p_3), (p_9, p_1), (p_9, p_3)\};$$

thus,

$$< p_1 > \breve{\mathcal{R}} =< p_2 > \breve{\mathcal{R}} =< p_3 > \breve{\mathcal{R}} =< p_5 > \breve{\mathcal{R}} =< p_8 > \breve{\mathcal{R}}$$

$$=< p_9 > \breve{\mathcal{R}} = \{p_1, p_2, p_3, p_5, p_8, p_9\} \text{ and } < p_4 > \breve{\mathcal{R}} = \phi.$$

Let

$$\mathcal{I} = \{\phi, \{p_2\}, \{p_4\}, \{p_5\}, \{p_2, p_4\}, \{p_2, p_5\}, \{p_4, p_5\}, \{p_2, p_4, p_5\}\}.$$

Thus, Table 4 represents a decision system and the patients with confirmed heart attacks were surely $A = \{p_1, p_3, p_8, p_9\}$. Thus, we respectively computed the approximations, the boundary and the accuracy measure of A to be as follows:

(1) The first (second/third) kind in Dai et al.'s approach [11] yielded $\{p_4\}, \{p_1, p_2, p_3, p_5, p_8, p_9\}$, $\{p_1, p_2, p_3, p_5, p_8, p_9\}$ and $\frac{1}{6}$ which means that A is a rough set according to the Dai technique. Further, the patients p_4 and p_6 experienced heart attacks, which contradicts the decision system in Table 4. Therefore, we are unable to decide whether the patient has experienced a heart attack.

(2) The second kind in the present approach yield $A, \{p_1, p_2, p_3, p_5, p_8, p_9\}, \{p_2, p_5\}$ and $\frac{4}{6}$. This means that the patients $\{p_1, p_3, p_8, p_9\}$ surely experienced heart attack according to the proposed technique which is consistent with Table 4. Additionally, the boundary region is reduced and the accuracy measure is increased.

Remark 7.1. It should be noted that

(1) The relation is identified according to the viewpoint of the system's experts.

(2) Dai et al.'s approximations [11] satisfy some properties of the basic properties of the rough set when the relation is a binary relation. Meanwhile, the reminders of the properties are achieved only if the relation is a similarity relation.
(3) The present techniques extend the applicability of rough sets. The similarity relations do not always hold for many real-life applications; consequently this restriction limits the application of this set.

8. Conclusions

This work combines two disciplines, namely, rough set theory and ideals. Rough set theory deals with vagueness and imperfect knowledge by using the lower and upper approximations, whereas an ideal is a fundamental concept in topological spaces that plays an important role in the study of the generalization of rough sets. Since the advent of the ideals, several research papers with interesting results in different respects has come to existence. In view of the recent applications of ideals in rough set theory, it seems very natural to extend the interesting concept of the rough set further by using ideals as done here. So, in this study different methods dependent on ideals and the maximal right neighborhood which was generated by binary relations, were proposed to approximate the sets. The use of ideals made the boundary region smaller; consequently, the accuracy measure was higher than that achieved through the use of Dai et al.'s approximations [11], which depended only on the maximal neighborhood generated by similarity relations. Hence, the present approach was a generalization of Dai et al.'s approach [11]. The basic properties of the current methods were studied. More importantly, it was proved for three of the current methods that the corresponding lower and upper approximations, boundary regions, accuracy measures and roughness measures were monotonic. Moreover, to add strength and make the current work vivid two medical applications were proposed to illustrate the main idea of the present results. The present techniques were successful and powerful techniques to reduce the boundary region and improve the accuracy measure. They allowed the medical staff to decide the impact factors of COVID-19 infections and heart attacks. They handled any imperfect data for symptoms of the diseases and this automatically made the diagnosis of patients easy and accurate. Consequently, this can help the medical staff to make a precise decision about the diagnosis of patients.

In upcoming works, we will

(1) Study new types of approximations resulting from neighborhoods and ideals.
(2) Search how these approximations can be applied to model real-life issues.
(3) Investigate the concepts and results presented here to generalize rough multisets using multisets ideals.

Acknowledgments

The author extends her appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through a research groups program under grant (R. G. P. 2/144/43). She would also like to express her sincere thanks to the editor and anonymous reviewers for their valuable comments and suggestions that have helped immensely in improving the quality of the paper.

Conflict of interest

This work does not have any conflicts of interest.

References

1. E. A. Abo-Tabl, A comparison of two kinds of definitions of rough approximations based on a similarity relation, *Inform. Sci.*, **181** (2011), 2587–2596. https://doi.org/10.1016/j.ins.2011.01.007
2. E. A. Abo-Tabl, M. K. El-Bably, Rough topological structure based on reflexivity with some applications, *AIMS Math.*, **7** (2022), 9911–9925. https://doi.org/10.3934/math.2022553
3. H. M. Abu-Doniaa, Comparison between different kinds of approximations by using a family of binary relations, *Knowl. Based Syst.*, **21** (2008), 911–919. https://doi.org/10.1016/j.knosys.2008.03.046
4. H. M. Abu-Doniaa, Multi knowledge based rough approximations and applications, *Knowl. Based Syst.*, **26** (2012) 20–29. https://doi.org/10.1016/j.knosys.2011.06.010
5. A. A. Allam, M. Y. Bakeir, E. A. Abo-Tabl, New approach for basic rough set concepts, In: D. Ślęzak, G. Wang, M. Szczuka, I. Düntsch, Y. Yao, *International workshop on rough sets, fuzzy sets, data mining, and granular computing*, Lecture Notes in Computer Science, Springer, **3641** (2005), 64–73. https://doi.org/10.1007/11548669_7
6. A. A. Allam, M. Y. Bakeir, E. A. Abo-Tabl, New approach for closure spaces by relations, *Acta Math. Acad. Paedagog. Nyregyhziensis*, **22** (2006), 285–304.
7. T. M. Al-shami, Maximal rough neighborhoods with a medical application, *J. Ambient Intell. Human. Comput.*, 2022. https://doi.org/10.1007/s12652-022-03858-1
8. T. M. Al-shami, Topological approach to generate new rough set models, *Complex Intell. Syst.*, 2022. https://doi.org/10.1007/s40747-022-00704-x
9. T. M. Al-shami, H. Işık, A. S. Nawar, R. A. Hosny, Some topological approaches for generalized rough sets via ideals, *Math. Prob. Eng.*, **2021** (2021), 5642982. https://doi.org/10.1155/2021/5642982
10. A. A. Azzam, A. M. Khalil, S. G. Li, Medical applications via minimal topological structure, *J. Intell. Fuzzy Syst.*, **39** (2020), 4723–4730. https://doi.org/10.3233/JIFS-200651
11. J. Dai, S. Gao, G. Zheng, Generalized rough set models determined by multiple neighborhoods generated from a similarity relation, *Soft Comput.*, **13** (2018), 2081–2094. https://doi.org/10.1007/s00500-017-2672-x
12. J. Dai, W. Wang, Q. Xu, H. Tian, Uncertainty measurement for interval-valued decision systems based on extended conditional entropy, *Knowl Based Syst.*, **27** (2012), 443–450. https://doi.org/10.1016/j.knosys.2011.10.013
13. J. Dai, Q. Xu, Approximations and uncertainty measures in incomplete information systems, *Inform. Sci.*, **198** (2012), 62–80. https://doi.org/10.1016/j.ins.2012.02.032
14. M. K. El-Bably, T. M. Al-shami, Different kinds of generalized rough sets based on neighborhoods with a medical application, *Int. J. Biomath.*, **14** (2021), 2150086. https://doi.org/10.1142/S1793524521500868

15. M. A. El Safty, S. Al-Zahrani, Topological modeling for symptom reduction of Corona virus, *Punjab Univ. J. Math.*, **53** (2021), 47–59.
16. M. Hosny, Topological approach for rough sets by using J-nearly concepts via ideals, *Filomat*, **34** (2020), 273–286. https://doi.org/10.2298/FIL2002273H
17. M. Hosny, Idealization of j-approximation spaces, *Filomat*, **34** (2020), 287–301. https://doi.org/10.2298/FIL2002287H
18. M. Hosny, Rough sets theory via new topological notions based on ideals and applications, *AIMS Math.*, **7** (2021), 869–902. https://doi.org/10.3934/math.2022052
19. R. A. Hosny, B. A. Asaad, A. A. Azzam, T. M. Al-shami, Various topologies generated from E_j-neighbourhoods via ideals, *Complexity*, **2021** (2021), 4149368. https://doi.org/10.1155/2021/4149368
20. D. Jankovic, T. R. Hamlet, New topologies from old via ideals, *Amer. Math. Monthly*, **97** (1990), 295–310. https://doi.org/10.1080/00029890.1990.11995593
21. J. Järinen, Approximations and rough sets based on tolerances, In: W. Ziarko, Y. Yao, *Rough sets and current trends in computing*, Lecture Notes in Computer Science, Springer, **2005** (2001), 182–189. https://doi.org/10.1007/3-540-45554-X_21
22. A. Kandil, S. A. El-Sheikh, M. Hosny, M. Raafat, Bi-ideal approximation spaces and their applications, *Soft Comput.*, **24** (2020), 12989–3001. https://doi.org/10.1007/s00500-020-04720-2
23. M. Kondo, On the structure of generalized rough sets, *Inform. Sci.*, **176** (2005), 589–600. https://doi.org/10.1016/j.ins.2005.01.001
24. A. M. Kozae, On topology expansions by ideals and applications, *Chaos Soliton. Fract.*, **13** (2002), 55–60. https://doi.org/10.1016/S0960-0779(00)00224-1
25. M. Kryszkiewicz, Rough set approach to incomplete information systems, *Inform. Sci.*, **112** (1998), 39–49. https://doi.org/10.1016/S0020-0255(98)10019-1
26. K. Kuratowski, *Topology*, Vol. I, New York: Academic Press, 1966.
27. A. S. Nawar, M. A. El-Gayar, M. K. El-Bably, R. A. Hosny, $\theta\beta$-ideal approximation spaces and their applications, *AIMS Math.*, **7** (2021), 2479–2497. https://doi.org/10.3934/math.2022139
28. J. Nieminen, Rough tolerance equality and tolerance black boxes, *Fund. Inform.*, **11** (1988), 289–296. https://doi.org/10.3233/FI-1988-11306
29. Z. Pawlak, Rough sets, *Int. J. Inform. Comput. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
30. Z. Pawlak, Rough concept analysis, *Bull. Pol. Acad. Sci. Math.*, **33**(1985), 9–10.
31. D. Pei, Z. Xu, Transformation of rough set models, *Knowl.-Based Syst.*, **20** (2007), 745–751. https://doi.org/10.1016/j.knosys.2006.10.006
32. J. A. Pomykala, About tolerance and similarity relations in information systems, In: J. Alpigini, J. Peters, A. Skowron, N. Zhong, *Rough sets and current trends in computing*, Lecture Notes in Computer Science, Springer, **2475** (2002), 175–182. https://doi.org/10.1007/3-540-45813-1_22
33. A. Skowron, J. Stepaniuk, Tolerance approximation spaces, *Fund. Inform.*, **27** (1996), 245–253. https://doi.org/10.3233/FI-1996-272311

34. A. Skowron, D. Vanderpooten, A generalized definition of rough approximations based on similarity, *IEEE T. Knowl. Data En.*, **12** (2000), 331–336. https://doi.org/10.1109/69.842271
35. R. Slowinski, J. Stefanowski, Rough-set reasoning about uncertain data, *Fund. Inform.*, **27** (1996), 229–243. https://doi.org/10.3233/FI-1996-272310
36. R. Vaidynathaswamy, The localization theory in set topology, *Proc. Indian Acad. Sci.*, **20** (1945), 51–61. https://doi.org/10.1007/BF03048958
37. Y. Y. Yao, Two views of the theory of rough sets in finite universes, *Int. J. Approx. Reason.*, **15** (1996), 291–317. https://doi.org/10.1016/S0888-613X(96)00071-0
38. Y. Y. Yao, Generalized rough set models, In: L. Polkowski, A. Skowron, *Rough sets in knowledge discovery*, Heidelberg: Physica Verlag, 1998, 286–318.
39. Y. Y. Yao, Constructive and algebraic methods of the theory of rough sets, *Inform. Sci.*, **109** (1998), 21–47. https://doi.org/10.1016/S0020-0255(98)00012-7
40. Y. Y. Yao, On generalized Pawlak approximation operators, In: L. Polkowski, A. Skowron, *Rough sets and current trends in computing*, Lecture Notes in Computer Science, Springer, **1424** (1998), 298–307. https://doi.org/10.1007/3-540-69115-4_41
41. Y. Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, *Inform. Sci.*, **111** (1998), 239–259. https://doi.org/10.1016/S0020-0255(98)10006-3
42. X. Zhang, J. Dai, Y. Yu, On the union and intersection operations of rough sets based on various approximation spaces, *Inform. Sci.*, **292** (2015), 214–229. https://doi.org/10.1016/j.ins.2014.09.007
43. W. Zhu, Generalized rough sets based on relations, *Inform. Sci.*, **177** (2007), 4997–5011. https://doi.org/10.1016/j.ins.2007.05.037