Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 13074–13103.
DOI: 10.3934/math.2022723
Received: 27 November 2021
Revised: 04 April 2022
Accepted: 06 April 2022
Published: 10 May 2022

Research article

Generalized iterated function system for common attractors in partial metric spaces

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Abstract: In this paper, we aim to obtain some new common attractors with the assistance of finite families of generalized contractive mappings, that belong to the special class of mappings defined on a partial metric space. Consequently, a variety of results for iterated function systems satisfying a different set of generalized contractive conditions are acquired. We present some examples to reinforce the results proved herein. These results generalize, unify and extend a variety of results that exist in current literature.

Keywords: common attractor; generalized iterated function system; common fixed point; generalized contraction; partial metric space
Mathematics Subject Classification: 47H04, 47H07, 47H10

1. Introduction

Iterated function system (IFS) has as a base, the mathematical foundations laid down in 1981 by Hutchinson [13]. He proved that the Hutchinson operator defined on $\mathbb{R}^k$ has a fixed point, a set in $\mathbb{R}^k$ which is closed and bounded, known as an attractor of IFS [6]. This, according to [7], may be viewed as a generalized version of the celebrated Banach's contraction principle which we state below. The importance of Banach contraction mapping principle [8] in the study of fixed point theory in metric spaces cannot be overspecialized. Its vast range of applications, which include among others, iterative methods for solving linear and nonlinear difference, differential and integral equations, attracted several researchers to intensify and extend the scope of fixed point theory in metric spaces, see for example [1, 2, 9, 11, 12, 14, 15, 17, 22, 23]. Secelean [21] studied generalized countable iterated function systems on a metric space and it was Nadler [19] who pioneered the research of fixed point theory in metric spaces involving multivalued operators.

Our primary objective in this paper is the construction of a fractal set of generalized iterated function system of a generalized contractions in a partial metric space. We observe that the

Hutchinson operator defined on a finite family of contractive mappings on a complete partial metric space is itself a generalized contractive mapping on a family of compact subsets of Y. By successive application of a generalized Hutchinson operator, a final fractal is obtained and this shall be followed by a presentation of a nontrivial example in support of the proved result.

Notations $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{R}^k$ will denote a set of natural numbers, a set of nonnegative real numbers, a set of real numbers and a set of k-tuples of real numbers respectively.

Consistent with [18], we give the following preliminary definitions and results.

Definition 1.1. Let Y be any non-empty set and $p : Y \times Y \to \mathbb{R}^+$ be a mapping. A pair (Y, p) is called a partial metric space if for all $t_1, t_2, t_3 \in Y$, the following properties hold:

(p_1) $t_1 = t_2$ if and only if $p(t_1, t_1) = p(t_1, t_2) = p(t_2, t_2)$,
(p_2) $p(t_1, t_1) \leq p(t_1, t_2)$,
(p_3) $p(t_1, t_2) = p(t_2, t_1)$,
(p_4) $p(t_1, t_2) + p(t_3, t_3) \leq p(t_1, t_3) + p(t_3, t_2)$.

The mapping p is a partial metric on a non-empty set Y.

From the definition, we see that if $p(t_1, t_2) = 0$, then properties (p_1) and (p_2) imply that $t_1 = t_2$ but in general, the converse is not true. An elementary example [3] is given by a partial metric space $(\mathbb{R}^+, p)$, with $p(t_1, t_2) = \max\{t_1, t_2\}$ for all $t_1, t_2 \in \mathbb{R}^+$.

Example 1.2. [3, 18] If $Y = \{[\phi_1, \phi_2] : \phi_1, \phi_2 \in \mathbb{R}, \phi_1 \leq \phi_2\}$, then $p([\phi_1, \phi_2], [\phi_3, \phi_4]) = \max\{\phi_2, \phi_4\} - \min\{\phi_1, \phi_3\}$ which is a partial metric p defined on Y.

Following [18], a T_0 topology τ_p on Y having as a base, a family of open p-balls $\{B_p(t_1, \varepsilon) : t_1 \in Y, \varepsilon > 0\}$ such that $B_p(t_1, \varepsilon) = \{t_2 \in Y : p(t_1, t_2) < p(t_1, t_1) + \varepsilon\}$ for all $t_1 \in Y$ and $\varepsilon > 0$, is generated by each partial metric p on Y.

Let p be a partial metric on Y. Then the mapping $p^s : Y \times Y \to \mathbb{R}^+$ defined as $p^s(t_1, t_2) = 2p(t_1, t_2) - [p(t_1, t_1) + p(t_2, t_2)]$ for all $t_1, t_2 \in Y$, is a metric on Y [4].

Definition 1.3. [15, 18] Consider a partial metric space (Y, p). Then

(i) $\{t_k\}$ is called a Cauchy sequence in Y if $\lim_{k,\eta \to +\infty} p(t_k, t_\eta)$ exists and is finite.

(ii) (Y, p) is said to be complete if every Cauchy sequence $\{t_k\}$ in Y converges to a point $t \in Y$ with respect to a topology τ_p such that $p(t, t) = \lim_{k \to +\infty} p(t_k, t)$.

(iii) A mapping $h : X \to X$ is continuous at a point $u_0 \in X$ if for each $\varepsilon > 0$, there exists $\varsigma > 0$ such that $h(B_p(u_0, \varsigma)) \subseteq B_P(hu_0, \varepsilon)$.

We shall denote by $\mathcal{CB}^p(Y)$ [3], a collection of all closed and bounded non-empty subsets of the partial metric space (Y, p).

For $\mathcal{M}, \mathcal{N} \in \mathcal{CB}^p(Y)$ and $v \in Y$, define

$$p(v, \mathcal{M}) = \inf\{p(v, \mu) : \mu \in \mathcal{M}\}, \quad \delta_p(\mathcal{M}, \mathcal{N}) = \sup\{p(\mu, \mathcal{N}) : \mu \in \mathcal{M}\}$$

and

$$\delta_p(\mathcal{N}, \mathcal{M}) = \sup\{p(\eta, \mathcal{M}) : \eta \in \mathcal{N}\}.$$

Remark 1.4. [4] For a partial metric space (Y, p) and any non-empty set $\mathcal{M}$ in (Y, p),

$$p(\mu, \mu) = p(\mu, \mathcal{M}) \text{ if and only if } \mu \in \overline{\mathcal{M}}.$$

Furthermore $\overline{\mathcal{M}} = \mathcal{M}$ if and only if $\mathcal{M}$ is closed in (Y, p).

Let (Y, p) be a partial metric space, then for $\mathcal{L}, \mathcal{M} \in \mathcal{CB}^p(Y)$, define

$$H_p(\mathcal{L}, \mathcal{M}) = \max\{\delta_p(\mathcal{L}, \mathcal{M}), \delta_p(\mathcal{M}, \mathcal{L})\}.$$

Proposition 1.5. [3] Consider a partial metric space (Y, p). Then for all $L, \mathcal{M}, \mathcal{N} \in \mathcal{CB}^p(Y)$,

(a) $H_p(\mathcal{L}, \mathcal{L}) \leq H_p(\mathcal{L}, \mathcal{M})$,

(b) $H_p(\mathcal{L}, \mathcal{M}) = H_p(\mathcal{M}, \mathcal{L})$,

(c) $H_p(\mathcal{L}, \mathcal{M}) \leq H_p(\mathcal{L}, \mathcal{N}) + H_p(\mathcal{N}, \mathcal{M}) - \inf_{\eta \in \mathcal{N}} p(\eta, \eta)$.

Corollary 1.6. [3] Consider a partial metric space (Y, p), then

$$H_p(\mathcal{L}, \mathcal{M}) = 0 \text{ implies that } \mathcal{L} = \mathcal{M}$$

for $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathcal{CB}^p(Y)$.

Based on Proposition 1.5 and Corollary 1.6, we shall refer to the mapping

$$H_p : \mathcal{CB}^p(Y) \times \mathcal{CB}^p(Y) \to [0, \infty),$$

as a partial Hausdorff metric induced by p.

Definition 1.7. [3] Let (Y, p) be a partial metric space and $C^p \subseteq Y$. Then C^p is said to be compact if every sequence $\{v_n\}$ in C^p contains a subsequence $\{v_{n_i}\}$ which converges to a point in C^p.

Note that closed and bounded subsets of an Euclidean space $\mathbb{R}^k$ are compact. Similarly, every finite set in $\mathbb{R}^k$ is compact. The half-open interval $(0, 1] \subset \mathbb{R}$ is an example of a set which is not compact since $\{1, \frac{1}{2}, \frac{1}{2^2}, ...\} \subset (0, 1]$ does not have any convergent subsequence. Similarly the set of integers, $\mathbb{Z}$ is not compact subset of $\mathbb{R}$.

Consider a partial metric space (Y, p) and denote by $C^p(Y)$ the set of all non-empty compact subsets of Y. For $\mathcal{M}, \mathcal{N} \in C^p(Y)$, let

$$H_p(\mathcal{M}, \mathcal{N}) = \max\{\sup_{\eta \in \mathcal{N}} p(\eta, \mathcal{M}), \sup_{\mu \in \mathcal{M}} p(\mu, \mathcal{N})\},$$

where $p(t, \mathcal{M}) = \inf\{p(t, \mu) : \mu \in \mathcal{M}\}$ is a measure of how far a point t is from the set $\mathcal{M}$. Such a mapping H_p is referred to as the Pompeiu-Hausdorff metric induced by the partial metric p. $(C^p(Y), H_p)$ is a complete partial metric space, provided (Y, p) is a complete partial metric space.

Lemma 1.8. Let (Y, p) be a partial metric space. Then for all $\mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N} \in C^p(Y)$, the following conditions are true:

(a) If $\mathcal{L} \subseteq \mathcal{M}$, then $\sup_{k \in \mathcal{K}} p(k, \mathcal{M}) \leq \sup_{k \in \mathcal{K}} p(k, \mathcal{L})$,

(b) $\sup_{t \in \mathcal{K} \cup \mathcal{L}} p(y, \mathcal{M}) = \max\{\sup_{k \in \mathcal{K}} p(k, \mathcal{M}), \sup_{\ell \in \mathcal{L}} p(\ell, \mathcal{M})\}$,

(c) $H_p(\mathcal{K} \cup \mathcal{L}, \mathcal{M} \cup \mathcal{N}) \leq \max\{H_p(\mathcal{K}, \mathcal{M}), H_p(\mathcal{L}, \mathcal{N})\}$.

Proof. (a) Since $\mathcal{L} \subseteq \mathcal{M}$, for all $k \in \mathcal{K}$, we have

$$\begin{aligned} p(k, \mathcal{M}) &= \inf\{p(k, \mu) : \mu \in \mathcal{M}\} \\ &\leq \inf\{p(k, \ell) : \ell \in \mathcal{L}\} = p(k, \mathcal{L}), \end{aligned}$$

this implies that

$$\sup_{k\in\mathcal{K}} p(k, \mathcal{M}) \leq \sup_{k\in\mathcal{K}} p(k, \mathcal{L}).$$

(b) Note that

$$\begin{aligned} \sup_{t\in\mathcal{K}\cup\mathcal{L}} p(t, \mathcal{M}) &= \sup\{p(t, \mathcal{M}) : t \in \mathcal{K} \cup \mathcal{L}\} \\ &= \max\{\sup\{p(t, \mathcal{M}) : t \in \mathcal{K}\}, \sup\{p(t, \mathcal{M}) : t \in \mathcal{L}\}\} \\ &= \max\{\sup_{k\in\mathcal{K}} p(k, \mathcal{M}), \sup_{\ell\in\mathcal{L}} p(\ell, \mathcal{M})\}. \end{aligned}$$

(c) We note that

$$\begin{aligned} &\sup_{t\in\mathcal{K}\cup\mathcal{L}} p(t, \mathcal{M} \cup \mathcal{N}) \\ \leq\ & \max\{\sup_{k\in\mathcal{K}} p(k, \mathcal{M} \cup \mathcal{N}), \sup_{\ell\in\mathcal{L}} p(\ell, \mathcal{L} \cup \mathcal{N})\} \text{ (from (b))} \\ \leq\ & \max\{\sup_{k\in\mathcal{K}} p(k, \mathcal{M}), \sup_{\ell\in\mathcal{L}} p(\ell, \mathcal{N})\} \quad \text{(from (a))} \\ \leq\ & \max\left\{\max\{\sup_{k\in K} p(k, \mathcal{M}), \sup_{m\in\mathcal{M}} p(m, \mathcal{K})\}, \max\{\sup_{\ell\in\mathcal{L}} p(\ell, \mathcal{N}), \sup_{\eta\in\mathcal{N}} p(\eta, \mathcal{L})\}\right\} \\ =\ & \max\left\{H_p(\mathcal{K}, \mathcal{M}), H_p(\mathcal{L}, \mathcal{N})\right\}. \end{aligned}$$

Similarly,

$$\sup_{v\in\mathcal{N}\cup\mathcal{M}} p(v, \mathcal{L} \cup \mathcal{K}) \leq \max\left\{H_p(\mathcal{K}, \mathcal{M}), H_p(\mathcal{L}, \mathcal{N})\right\}.$$

Hence, it follows that

$$\begin{aligned} H_p(\mathcal{K} \cup \mathcal{L}, \mathcal{N} \cup \mathcal{M}) &= \max\left\{\sup_{v\in\mathcal{L}\cup\mathcal{N}} p(v, \mathcal{K} \cup \mathcal{L}), \sup_{t\in K\cup C} p(t, \mathcal{M} \cup \mathcal{N})\right\} \\ &\leq \max\left\{H_p(\mathcal{K}, \mathcal{M}), H_p(\mathcal{L}, \mathcal{N})\right\}. \quad \square \end{aligned}$$

Theorem 1.9. [18] Consider a complete partial metric space (Y, p) and let $h : Y \to Y$ be a contraction mapping such that, for any $\lambda \in [0, 1)$,

$$p(ht_1, ht_2) \leq \lambda p(t_1, t_2),$$

is true for all $t_1, t_2 \in Y$. Then there exists a unique fixed point u of h in Y and for every v_0 in Y, the sequence $\{v_0, hv_0, h^2v_0, ...\}$ converges to the fixed point u of h.

The following result shows the existence of multivalued contraction mapping with domain of sets.

Theorem 1.10. Consider a partial metric space (Y, p) and let $h : Y \to Y$ be a contraction mapping. Then

(a) h maps elements in $C^p(Y)$ to elements in $C^p(Y)$.

(b) If for any $\mathcal{M} \in C^p(Y)$,

$$h(\mathcal{M}) = \{h(t_1) : t_1 \in \mathcal{M}\},$$

then $h : C^p(Y) \to C^p(Y)$ is a contraction mapping on $(C^p(Y), H_p)$.

Proof. (a) We know that every contraction mapping is continuous. Moreover, under every continuous mapping $h : Y \to Y$, the image of a compact subset is also compact, that is, if

$$\mathcal{M} \in C^p(Y) \text{ then } h(\mathcal{M}) \in C^p(Y).$$

(b) Let $\mathcal{M}, \mathcal{N} \in C^p(Y)$. Since $h : Y \to Y$ is contraction, we obtain that

$$p\left(ht_1, h\left(\mathcal{N}\right)\right) = \inf_{t_2 \in \mathcal{N}} p\left(ht_1, ht_2\right) \leq \lambda \inf_{t_2 \in \mathcal{N}} p\left(t_1, t_2\right) = \lambda p\left(t_1, \mathcal{N}\right).$$

Also

$$p\left(ht_2, h\left(\mathcal{M}\right)\right) = \inf_{t_1 \in \mathcal{M}} p\left(ht_2, ht_1\right) \leq \lambda \inf_{t_1 \in \mathcal{M}} p\left(t_2, t_1\right) = \lambda p\left(t_2, \mathcal{M}\right).$$

Now

$$\begin{aligned} H_p\left(h\left(\mathcal{M}\right), h\left(\mathcal{N}\right)\right) &= \max\{\sup_{t_1 \in \mathcal{M}} p(ht_1, h\left(\mathcal{N}\right)), \sup_{t_2 \in \mathcal{N}} p(ht_2, h\left(\mathcal{M}\right))\} \\ &\leq \max\{\lambda \sup_{t_1 \in \mathcal{M}} p(t_1, \mathcal{N}), \lambda \sup_{t_2 \in \mathcal{N}} p(t_2, \mathcal{M})\} = \lambda H_p\left(\mathcal{M}, \mathcal{N}\right). \end{aligned}$$

Thus, h satisfies

$$H_p\left(h\left(\mathcal{M}\right), h\left(\mathcal{N}\right)\right) \leq \lambda H_p\left(\mathcal{M}, \mathcal{N}\right)$$

for all $t_1, t_2 \in C^p(Y)$, and so $h : C^p(Y) \to C^p(Y)$ is a contraction mapping. □

Theorem 1.11. Consider a partial metric space (Y, p). Let $\{h_k : k = 1, 2, ..., r\}$ be a finite collection of contraction mappings on Y with contraction constants $\lambda_1, \lambda_2, ..., \lambda_r$, respectively. Define $\Psi : C^p(Y) \to C^p(Y)$ by

$$\begin{aligned} \Psi(\mathcal{M}) &= h_1(\mathcal{M}) \cup h_2(\mathcal{M}) \cup \cdots \cup h_r(\mathcal{M}) \\ &= \cup_{k=1}^{r} h_k(\mathcal{M}), \end{aligned}$$

for each $\mathcal{M} \in C^p(Y)$. Then Ψ is a contraction mapping on $C^p(Y)$ with contraction constant $\lambda = \max\{\lambda_1, \lambda_2, ..., \lambda_r\}$.

Proof. We illustrate the claim for $r = 2$. Let $h_1, h_2 : Y \to Y$ be two contractions. We take $\mathcal{M}, \mathcal{N} \in C^p(Y)$. Using the result from Lemma 1.8 (c), we have

$$\begin{aligned} H_p(\Psi(\mathcal{M}), \Psi(\mathcal{N})) &= H_p(h_1(\mathcal{M}) \cup h_2(\mathcal{M}), h_1(\mathcal{N}) \cup h_2(\mathcal{N})) \\ &\leq \max\{H_p(h_1(\mathcal{M}), h_1(\mathcal{N})), H_p(h_2(\mathcal{M}), h_2(\mathcal{N}))\} \\ &\leq \max\{\lambda_1 H_p(\mathcal{M}, \mathcal{N})), \lambda_2 H_p(\mathcal{M}, \mathcal{N}))\} \\ &\leq \lambda H_p(\mathcal{M}, \mathcal{N}), \end{aligned}$$

where $\lambda = \max\{\lambda_1, \lambda_2\}$. □

Theorem 1.12. Consider a complete partial metric space (Y, p) and let $\{h_k : k = 1, 2, ..., r\}$ be a finite collection of contraction mappings on Y. Let a mapping on $C^p(Y)$ be defined by

$$\begin{aligned} \Psi(\mathcal{M}) &= h_1(\mathcal{M}) \cup h_2(\mathcal{M}) \cup \cdots \cup h_r(\mathcal{M}) \\ &= \cup_{k=1}^{r} h_k(\mathcal{M}), \end{aligned}$$

for each $\mathcal{M} \in C^p(Y)$. Then

(i) $\Psi : C^p(Y) \to C^p(Y)$;

(ii) Ψ has at most one fixed point $U_1 \in C^p(Y)$, this means that $U_1 = \Psi(U_1) = \cup_{k=1}^{r} h_k(U_1)$;

(iii) for any initial set $\mathcal{M}_0 \in C^p(Y)$, the sequence

$$\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Psi^2(\mathcal{M}_0), ...\}$$

of compact sets is convergent and has a fixed point of Ψ.

Proof. (i) From the definition of Ψ and Theorem 1.10 the conclusion follows immediately since each h_k is a contraction. (ii) $\Psi : C^p(Y) \to C^p(Y)$ is a contraction too, by Theorem 1.11. Thus if (Y, p) is a complete partial metric space, then $(C^p(Y), H_p)$ is complete. As a consequence, (ii) and (iii) may be deduced from Theorem 1.10. □

Definition 1.13. Let (Y, p) be a complete partial metric space. If $h_k : Y \to Y$, for each $k = 1, 2, ..., r$ are contraction mappings, then $\{Y; h_k, k = 1, 2, \cdots, r\}$ is called an iterated function system (IFS).

Definition 1.14. [20] Let $\mathcal{M} \subseteq Y$ be a non-empty compact set, then $\mathcal{M}$ is an attractor of the IFS if

(i) $\Psi(\mathcal{M}) = \mathcal{M}$ and

(ii) there exist an open set $V_1 \subseteq Y$ such that $\mathcal{M} \subseteq V_1$ and $\lim_{k\to+\infty} \Psi^k(\mathcal{N}) = \mathcal{M}$ for any compact set $\mathcal{N} \subseteq V_1$, where the limit is taken with respect to the partial Hausdorff metric.

Thus, the maximal open set V_1 such that (ii) is satisfied is referred to as a basin of attraction.

2. Generalized iIterated function system

Some results on a generalized iterated function system for multivalued mapping in a metric space may be found in [10]. In this section, we define the generalized iterated function system in the setup of partial metric spaces. We begin with the definition of a generalized contraction self-map which will be followed by some preliminary results.

Definition 2.1. Let (Y, p) be a partial metric space and $h, g : Y \to Y$ be two mappings. A pair (h, g) is called a generalized contraction if

$$p(ht_1, gt_2) \leq \lambda p(t_1, t_2)$$

for all $t_1, t_2 \in Y$, where $\lambda \in [0, 1)$.

Theorem 2.2. Let (Y, p) be a partial metric space and $h, g : Y \to Y$ be two continuous mappings. If the pair (h, g) is a generalized contraction with $\lambda \in [0, 1)$, then

(1) the elements in $C^p(Y)$ are mapped to elements in $C^p(Y)$ under h and g;

(2) if for any $U \in C^p(Y)$, the mappings $h, g : C^p(Y) \to C^p(Y)$ defined as

$$\begin{aligned} h(U) &= \{h(t_1) : t_1 \in U\} \text{ and} \\ g(U) &= \{g(t_1) : t_1 \in U\}, \end{aligned}$$

then the pair (h, g) is a generalized contraction on $(C^p(Y), H_p)$.

Proof. (1) Since h is a continuous mapping and the image of a compact subset under a continuous mapping, $h : Y \to Y$ is compact, then

$$U \in C^p(Y) \text{ implies that } h(U) \in C^p(Y).$$

Similarly, we have

$$U \in C^p(Y) \text{ implies that } g(U) \in C^p(Y).$$

(2) Let $\mathcal{M}, \mathcal{N} \in C^p(Y)$. Since the pair (h, g) is a generalized contraction, then

$$p(ht_1, gt_2) \leq \lambda p(t_1, t_2) \text{ for all } t_1, v \in Y,$$

where $\lambda \in [0, 1)$.

Thus, we have

$$\begin{aligned} p(ht_1, g(V)) &= \inf_{t_2 \in V} p(ht_1, gt_2) \\ &\leq \inf_{t_2 \in V} \lambda p(t_1, t_2) \\ &= \lambda p(t_1, V). \end{aligned}$$

Also

$$\begin{aligned} p(gv, h(U)) &= \inf_{t_1 \in U} p(ht_2, gt_1) \\ &\leq \inf_{t_1 \in U} \lambda p(t_2, t_1) \\ &= \lambda p(t_2, U). \end{aligned}$$

Now

$$\begin{aligned} H_p(h(U), g(V)) &= \max\{\sup_{t_1 \in U} p(ht_1, g(V)), \sup_{t_2 \in V} p(gt_2, h(U))\} \\ &\leq \max\{\sup_{t_1 \in U} \lambda p(t_1, V), \sup_{t_2 \in V} \lambda p(t_2, U)\} \\ &= \max\{\lambda \sup_{t_1 \in U} p(t_1, V), \lambda \sup_{t_2 \in B} p(t_2, U)\} \\ &= \lambda \max\{\sup_{t_1 \in U} p(t_1, V), \sup_{t_2 \in V} p(t_2, U)\} \\ &= \lambda H_p(U, V). \end{aligned}$$

Consequently,

$$H_p(h(U), g(V)) \leq \lambda H_p(U, V).$$

Hence, the pair (h, g) is a generalized contraction mapping on $(C^p(Y), H_p)$. □

Proposition 2.3. Consider a partial metric space (Y, p). Suppose that the mappings $h_k, g_k : Y \to Y$ for $k = 1, 2, \cdots, r$ are continuous and satisfy

$$p(h_k t_1, g_k t_2) \leq \lambda_k p(t_1, t_2) \text{ for all } t_1, t_2 \in Y,$$

where $\lambda_k \in [0, 1)$ for each $k \in \{1, 2, \cdots, r\}$. Then the mappings $\Psi, \Phi : C^p(Y) \to C^p(Y)$ defined as

$$\begin{aligned}\Psi(U) &= h_1(U) \cup h_2(U) \cup \cdots \cup h_r(U)\\ &= \cup_{k=1}^r h_k(U) \text{ for each } U \in C^p(Y)\end{aligned}$$

and

$$\begin{aligned}\Phi(U) &= g_1(U) \cup g_2(U) \cup \cdots \cup g_r(U)\\ &= \cup_{k=1}^r g_k(U) \text{ for each } U \in C^p(Y)\end{aligned}$$

also satisfy

$$H_p(\Psi U, \Phi V) \leq \widetilde{\lambda} H_p(U, V) \text{ for all } U, V \in C^p(Y),$$

where $\widetilde{\lambda} = \max\{\lambda_k : k \in \{1, 2, ..., r\}\}$. Then the pair (Ψ, Φ) is a generalized contraction on $C^p(Y)$.

Proof. We shall prove the result for $r = 2$. Let $h_1, h_2, g_1, g_2 : Y \to Y$ be two contractions. For $\mathcal{M}, \mathcal{N} \in C^p(Y)$ and using Lemma 1.12 (c), we have

$$\begin{aligned}H_p(\Psi(\mathcal{M}), \Phi(\mathcal{N})) &= H_p(h_1(\mathcal{M}) \cup h_2(\mathcal{M}), g_1(\mathcal{N}) \cup g_2(\mathcal{N}))\\ &\leq \max\{H_p(h_1(\mathcal{M}), g_1(\mathcal{N})), H_p(h_2(\mathcal{M}), g_2(\mathcal{N}))\}\\ &\leq \max\{\lambda_1 H_p(\mathcal{M}, \mathcal{N}), \lambda_2 H_p(\mathcal{M}, \mathcal{N})\}\\ &\leq \widetilde{\lambda} H_p(\mathcal{M}, \mathcal{N}). \quad \square\end{aligned}$$

Definition 2.4. Consider a partial metric space (Y, p) with the mappings $\Psi, \Phi : C^p(Y) \to C^p(Y)$. A pair of mappings (Ψ, Φ) is called

(1) A generalized Hutchinson contractive operator if a constant $\lambda \in [0, 1)$ exists such that for any $\mathcal{M}, \mathcal{N} \in C^p(Y)$, the following holds:

$$H_p(\Psi(\mathcal{M}), \Phi(\mathcal{N})) \leq \lambda Z_{\Psi,\Phi}(\mathcal{M}, \mathcal{N}),$$

where

$$\begin{aligned}Z_{\Psi,\Phi}(\mathcal{M}, \mathcal{N}) = \max\{&H_p(\mathcal{M}, \mathcal{N}), H_p(\mathcal{M}, \Psi(\mathcal{M})), H_p(\mathcal{N}, \Phi(\mathcal{N})),\\ &\frac{H_p(\mathcal{M}, \Phi(\mathcal{N})) + H_p(\mathcal{N}, \Psi(\mathcal{M}))}{2}\}.\end{aligned}$$

(2) A generalized rational Hutchinson contractive operator if a constant $\lambda_* \in [0, 1)$ exists such that for any $\mathcal{M}, \mathcal{N} \in C^p(Y)$, the following holds:

$$H_p(\Psi(\mathcal{M}), \Phi(\mathcal{N})) \leq \lambda_* R_{\Psi,\Phi}(\mathcal{M}, \mathcal{N}),$$

where

$$R_{\Psi,\Phi}(\mathcal{M},\mathcal{N}) = \max\left\{\frac{H_p(\mathcal{M},\Phi(\mathcal{N}))[1+H_p(\mathcal{M},\Psi(\mathcal{M}))]}{2(1+H_p(\mathcal{M},\mathcal{N}))}, \frac{H_p(\mathcal{N},\Phi(\mathcal{N}))[1+H_p(\mathcal{M},\Psi(\mathcal{M}))]}{1+H_p(\mathcal{M},\mathcal{N})}, \frac{H_p(\mathcal{M},\mathcal{N})[1+H_p(\mathcal{M},\Psi(\mathcal{M}))]}{1+H_p(\mathcal{M},\mathcal{N})}\right\}.$$

Note that if the pair (Ψ,Φ) defined as in Proposition 2.3 is generalized contraction on $C^p(Y)$, then the pair (Ψ,Φ) is a generalized Hutchinson contractive operator but the converse is not true.

Definition 2.5. Let (Y,p) be a complete partial metric space. If $h_k, g_k : Y \to Y$, $k = 1,2,...,r$ are continuous mappings such that each pair (h_k, g_k) for $k = 1,2,...,r$ is a generalized contraction, then $\{Y; (h_k, g_k), k = 1,2,\cdots,r\}$ is called a generalized iterated function system (GIFS).

Definition 2.6. Let $\mathcal{M} \subseteq Y$ be a non-empty compact set, then $\mathcal{M}$ is a common attractor of the GIFS if

(i) $\Psi(\mathcal{M}) = \Phi(\mathcal{M}) = \mathcal{M}$ and
(ii) there exists an open set $V_1 \subseteq Y$ such that $\mathcal{M} \subseteq V_1$ and $\lim_{k\to+\infty}\Psi^k(\mathcal{N}) = \lim_{k\to+\infty}\Phi^k(\mathcal{N}) = \mathcal{M}$ for any compact set $\mathcal{N} \subseteq V_1$, where the limit is taken with respect to the partial Hausdorff metric.

Thus the maximal open set V_1 such that (ii) is satisfied is referred to as a basin of common attraction.

3. Main results

In this part, we state and prove some results on the existence and uniqueness of a common attractor of generalized Hutchinson contractive operators in the setup of partial metric space. We start with the following main result.

Theorem 3.1. Let (Y,p) be a complete partial metric space and $\{Y; (h_k, g_k), k = 1,2,\cdots,r\}$ be a the generalized iterated function system. Let $\Psi,\Phi : C^p(Y) \to C^p(Y)$ be defined by

$$\Psi(\mathcal{M}) = \cup_{k=1}^{r} h_k(\mathcal{M}),$$

and

$$\Phi(\mathcal{N}) = \cup_{k=1}^{r} g_k(\mathcal{N})$$

for each $\mathcal{M},\mathcal{N} \in C^p(Y)$. If the pair (Ψ,Φ) is a generalized Hutchinson contractive operator, then Ψ and Φ have a unique common attractor $U_1 \in C^p(Y)$, that is,

$$U_1 = \Psi(U_1) = \Phi(U_1).$$

Furthermore, for an arbitrarily chosen initial set $\mathcal{M}_0 \in C^p(Y)$, the sequence

$$\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Phi\Psi(\mathcal{M}_0), \Psi\Phi\Psi(\mathcal{M}_0), ...\}$$

of compact sets converges to the common attractor U_1 of Ψ and Φ.

Proof. Choose an element $\mathcal{M}_0$ randomly in $C^p(Y)$. Define

$$\mathcal{M}_1 = \Psi(\mathcal{M}_0),\ \mathcal{M}_3 = \Psi(\mathcal{M}_2), ..., \mathcal{M}_{2k+1} = \Psi(\mathcal{M}_{2k})$$

and

$$\mathcal{M}_2 = \Phi(\mathcal{M}_1),\ \mathcal{M}_4 = \Phi(\mathcal{M}_3), ..., \mathcal{M}_{2k+2} = \Phi(\mathcal{M}_{2k+1})$$

for $k \in \{0, 1, 2, ...\}$.

Now, as the pair (Ψ, Φ) is generalized Hutchinson contractive operator, we have

$$\begin{aligned} H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}) &= H_p(\Psi(\mathcal{M}_{2k}), \Phi(\mathcal{M}_{2k+1})) \\ &\leq \lambda Z_{\Psi,\Phi}(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}), \end{aligned}$$

where

$$\begin{aligned} Z_{\Psi,\Phi}(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}) &= \max\{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}), H_p(\mathcal{M}_{2k}, \Psi(\mathcal{M}_{2k})), \\ &\quad H_p(\mathcal{M}_{2k+1}, \Phi(\mathcal{M}_{2k+1})), \\ &\quad \frac{H_p(\mathcal{M}_{2k}, \Phi(\mathcal{M}_{2k+1})) + H_p(\mathcal{M}_{2k+1}, \Psi(\mathcal{M}_{2k}))}{2}\} \\ &= \max\{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}), H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}), \\ &\quad H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}), \\ &\quad \frac{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+2}) + H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+1})}{2}\} \\ &\leq \max\{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}), H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}), \\ &\quad \frac{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}) + H_p(\mathcal{M}_{2k+1}, M_{2k+2})}{2}\} \\ &= \max\{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}), H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2})\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}) &\leq \lambda \max\{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}), H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2})\} \\ &= \lambda H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}). \end{aligned}$$

Also,

$$\begin{aligned} H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+3}) &= H_p(\mathcal{M}_{2k+3}, \mathcal{M}_{2k+2}) \\ &= H_p(\Psi(\mathcal{M}_{2k+2}), \Phi(\mathcal{M}_{2k+1})) \\ &\leq \lambda Z_{\Psi,\Phi}(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+1}), \end{aligned}$$

where

$$\begin{aligned} Z_{\Psi,\Phi}(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+1}) &= \max\{H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+1}), H_p(\mathcal{M}_{2k+2}, \Psi(\mathcal{M}_{2k+2})), \\ &\quad H_p(\mathcal{M}_{2k+1}, \Phi(\mathcal{M}_{2k+1})), \\ &\quad \frac{H_p(\mathcal{M}_{2k+2}, \Phi(\mathcal{M}_{2k+1})) + H_p(\mathcal{M}_{2k+1}, \Psi(\mathcal{M}_{2k+2}))}{2}\} \\ &= \max\{H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+1}), H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+3}), \\ &\quad H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}), \\ &\quad \frac{H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+2}) + H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+3})}{2}\} \end{aligned}$$

$$\begin{aligned} &\leq \max\{H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+1}), H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+3}), \\ &\quad \frac{H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}) + H_p(\mathcal{M}_{2k+2}, M_{2k+3})}{2}\} \\ &= \max\{H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+1}), H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+3})\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+3}) &\leq \lambda \max\{H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+1}), H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+3})\} \\ &= \lambda H_p(\mathcal{M}_{2k+2}, \mathcal{M}_{2k+1}). \end{aligned}$$

Therefore, for all $k \in \{0, 1, 2, ...\}$, we have

$$\begin{aligned} H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) &\leq \lambda H_p(\mathcal{M}_k, \mathcal{M}_{k+1}) \\ &\leq \lambda^2 H_p(\mathcal{M}_{k-1}, \mathcal{M}_k) \\ &\leq \cdots \\ &\leq \lambda^{k+1} H_p(\mathcal{M}_0, \mathcal{M}_1). \end{aligned}$$

Now, we have for $l > k$, with $k, l \in \{0, 1, 2, ...\}$,

$$\begin{aligned} H_p(\mathcal{M}_k, \mathcal{M}_l) &\leq H_p(\mathcal{M}_k, \mathcal{M}_{k+1}) + H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) + \cdots + H_p(\mathcal{M}_{l-1}, \mathcal{M}_l) \\ &\quad - \inf_{m_{k+1} \in \mathcal{M}_{k+1}} p(m_{k+1}, m_{k+1}) - \inf_{m_{k+2} \in \mathcal{M}_{k+2}} p(m_{k+2}, m_{k+2}) - \\ &\quad \cdots - \inf_{m_{k-1} \in \mathcal{M}_{k-1}} p(m_{k-1}, m_{k-1}), \\ &\leq [\lambda^k + \lambda^{k+1} + \cdots + \lambda^{l-1}] H_p(\mathcal{M}_0, \mathcal{M}_1), \\ &= \lambda^k [1 + \lambda + \lambda^2 + \cdots + \lambda^{l-k-1}] H_p(\mathcal{M}_0, \mathcal{M}_1)], \\ &\leq \frac{\lambda^k}{1-\lambda} H_p(\mathcal{M}_0, \mathcal{M}_1) \end{aligned}$$

and so $\lim_{k,l \to +\infty} H_p(\mathcal{M}_k, \mathcal{M}_l) = 0$. Thus $\{\mathcal{M}_k\}$ is a Cauchy sequence in $C^p(Y)$. Since $(C^p(Y), H_p)$ is a complete partial metric space, there exists $U_1 \in C^p(Y)$ such that $\lim_{k \to +\infty} \mathcal{M}_k = U_1$, that is, $\lim_{k \to +\infty} H_p(\mathcal{M}_k, U_1) = \lim_{k \to +\infty} H_p(M_k, \mathcal{M}_{k+1}) = H_p(U_1, U_1)$ and so, we have $\lim_{k \to +\infty} H_p(\mathcal{M}_k, U_1) = 0$.

To show that $\Psi(U_1) = U_1$, we have

$$\begin{aligned} H_p(\Psi(U_1), U_1) &\leq H_p(\Psi(U_1), \Phi(\mathcal{M}_{2k+1})) + H_p(\Phi(\mathcal{M}_{2k+1}), U_1) \\ &\quad - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}), \\ &\leq \lambda Z_{\Psi,\Phi}(U_1, \mathcal{M}_{2k+1}) + H_p(\mathcal{M}_{2k+2}, U_1) \\ &\quad - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \end{aligned}$$

for all $k \in \{0, 1, 2, ...\}$, where

$$\begin{aligned} Z_{\Psi,\Phi}(U_1, \mathcal{M}_{2k+1}) &= \max\{H_p(U_1, \mathcal{M}_{2k+1}), H_p(U_1, \Psi(U_1)), H_p(\mathcal{M}_{2k+1}, \Phi(\mathcal{M}_{2k+1})), \\ &\quad \frac{H_p(U_1, \Phi(\mathcal{M}_{2k+1})) + H_p(\mathcal{M}_{2k+1}, \Psi(U_1))}{2}\} \end{aligned}$$

$$
\begin{aligned}
& \quad - \inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\
&= \max\{H_p(U_1,\mathcal{M}_{2k+1}), H_p(U_1,\Psi(U_1)), H_p(\mathcal{M}_{2k+1},\mathcal{M}_{2k+2}), \\
& \quad \frac{H_p(U_1,\mathcal{M}_{2k+2}) + H_p(\mathcal{M}_{2k+1},\Psi(U_1))}{2}\} \\
& \quad - \inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}).
\end{aligned}
$$

Now, we examine the following cases:
(1) If $Z_{\Psi,\Phi}(U_1,\mathcal{M}_{2k+1}) = H_p(U_1,\mathcal{M}_{2k+1})$, then

$$
\begin{aligned}
H_p(\Psi(U_1), U_1) &\leq \lambda H_p(U_1,\mathcal{M}_{2k+1}) + H_p(\mathcal{M}_{2k+2}, U_1) - \inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\
&\leq \lambda H_p(U_1,\mathcal{M}_{2k+1}) + H_p(\mathcal{M}_{2k+2}, U_1),
\end{aligned}
$$

which together with our taking the limit as $k\to+\infty$, gives

$$
H_p(\Psi(U_1), U_1) \leq \lambda H_p(U_1, U_1) + H_p(U_1, U_1),
$$

and we get $H_p(\Psi(U_1), U_1) = 0$, that is, $U_1 = \Psi(U_1)$.
(2) Provided $Z_{\Psi,\Phi}(U_1,\mathcal{M}_{2k+1}) = H_p(U_1,\Psi(U_1))$, then

$$
\begin{aligned}
H_p(\Psi(U_1), U_1) &\leq \lambda H_p(U_1,\Psi(U_1)) + H_p(\mathcal{M}_{2k+2}, U_1) - \inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\
&\leq \lambda H_p(U_1,\Psi(U_1)) + H_p(\mathcal{M}_{2k+2}, U_1),
\end{aligned}
$$

that is,

$$
H_p(\Psi(U_1), U_1) \leq \frac{1}{1-\lambda} H_p(\mathcal{M}_{2k+2}, U_1),
$$

which together with our taking the limit as $k\to+\infty$ implies that $H_p(\Psi(U_1), U_1) \leq 0$ and so $U_1 = \Psi(U_1)$.
(3) In the case of $Z_{\Psi,\Phi}(U_1,\mathcal{M}_{2k+1}) = H_p(\mathcal{M}_{2k+1},\mathcal{M}_{2k+2})$, we get

$$
\begin{aligned}
H_p(U_1,\Phi(U_1)) &\leq \lambda H_p(\mathcal{M}_{2k+1},\mathcal{M}_{2k+2}) + H_p(\mathcal{M}_{2k+2}, U_1) - \inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\
&\leq \lambda H_p(\mathcal{M}_{2k+1},\mathcal{M}_{2k+2}) + H_p(\mathcal{M}_{2k+2}, U_1),
\end{aligned}
$$

which together with our taking the limit as $k\to+\infty$ implies that $U_1 = \Psi(U_1)$.
(4) If $Z_{\Psi,\Phi}(U_1,\mathcal{M}_{2k+1}) = \dfrac{H_p(U_1,\mathcal{M}_{2k+2}) + H_p(\mathcal{M}_{2k+1},\Psi(U_1))}{2}$, then

$$
\begin{aligned}
H_p(U_1,\Psi(U_1)) &\leq \frac{\lambda}{2}[H_p(U_1,\mathcal{M}_{2k+2}) + H_p(\mathcal{M}_{2k+1},\Psi(U_1))] \\
& \quad + H_p(\mathcal{M}_{2k+2}, U_1) - \inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\
&\leq \frac{\lambda}{2}[H_p(U_1,\mathcal{M}_{2k+2}) + H_p(\mathcal{M}_{2k+1}, U_1) + H_p(U_1,\Psi(U_1)) \\
& \quad - \inf_{u\in U_1} p(u,u)] + H_p(\mathcal{M}_{2k+2}, U_1) - \inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\
&\leq \frac{\lambda}{2}[H_p(U_1,\mathcal{M}_{2k+2}) + H_p(\mathcal{M}_{2k+1}, U_1) + H_p(U_1,\Psi(U_1))]
\end{aligned}
$$

$$+H_p(\mathcal{M}_{2k+2}, U_1),$$

which together with our taking the limit as $k \to +\infty$, we get

$$H_p(U_1, \Psi(U_1)) \leq \frac{\lambda}{2} H_p(U_1, \Psi(U_1)),$$

giving us $H_p(U_1, \Psi(U_1)) = 0$, and so $U_1 = \Psi(U_1)$.

Thus, from the above cases, U_1 is the attractor of Ψ.

In a similar manner, we have

$$\begin{aligned} H_p(U_1, \Phi(U_1)) &\leq H_p(U_1, \mathcal{M}_{2k+1}) + H_p(\mathcal{M}_{2k+1}, \Phi(U_1)) \\ &\quad - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\ &= H_p(U_1, \mathcal{M}_{2k+1}) + H_p(\Psi(\mathcal{M}_{2k}), \Phi(U_1)) \\ &\quad - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{2k+1}) + \lambda Z_{\Psi,\Phi}(\mathcal{M}_{2k}, U_1) \\ &\quad - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}), \end{aligned}$$

where

$$\begin{aligned} Z_{\Psi,\Phi}(\mathcal{M}_{2k}, U_1) &= \max\{H_p(\mathcal{M}_{2k}, U_1), H_p(\mathcal{M}_{2k}, \Psi(\mathcal{M}_{2k})), H_p(U_1, \Phi(U_1)), \\ &\quad \frac{H_p(\mathcal{M}_{2k}, \Phi(U_1)) + H_p(U_1, \Psi(\mathcal{M}_{2k}))}{2}\} \\ &= \max\{H_p(\mathcal{M}_{2k}, U_1), H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}), H_p(U_1, \Phi(U_1)), \\ &\quad \frac{H_p(\mathcal{M}_{2k}, \Phi(U_1)) + H_p(U_1, \mathcal{M}_{2k+1})}{2}\}. \end{aligned}$$

Now, the following cases arise:

(1) If $Z_{\Psi,\Phi}(\mathcal{M}_{2k}, U_1) = H_p(\mathcal{M}_{2k}, U_1)$, then

$$\begin{aligned} H_p(U_1, \Phi(U_1)) &\leq H_p(U_1, \mathcal{M}_{2k+1}) + \lambda H_p(\mathcal{M}_{2k}, U_1) - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{2k+1}) + \lambda H_p(\mathcal{M}_{2k}, U_1), \end{aligned}$$

which together with our taking the limit as $k \to +\infty$, gives

$$H_p(U_1, \Phi(U_1)) \leq H_p(U_1, U_1) + \lambda H_p(U_1, U_1),$$

and we get $H_p(U_1, \Phi(U_1)) = 0$, that is, $U_1 = \Phi(U_1)$.

(2) For $Z_{\Psi,\Phi}(\mathcal{M}_{2k}, U_1) = H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})$, then

$$\begin{aligned} H_p(U_1, \Phi(U_1)) &\leq H_p(U_1, \mathcal{M}_{2k+1}) + \lambda H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}) - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{2k+1}) + \lambda H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}), \end{aligned}$$

which together with our taking the limit as $k \to +\infty$, we have

$$H_p(U_1, \Phi(U_1)) \leq H_p(U_1, U_1) + \lambda H_p(U_1, U_1),$$

which implies that $H_p(U_1, \Phi(U_1)) \leq 0$ and so $U_1 = \Phi(U_1)$.
(3) In the case of $Z_{\Psi,\Phi}(\mathcal{M}_{2k}, U_1) = H_p(U_1, \Phi(U_1))$, we get

$$\begin{aligned} H_p(U_1, \Phi(U_1)) &\leq H_p(U_1, \mathcal{M}_{2k+1}) + \lambda H_p(U_1, \Phi(U_1)) - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{2k+1}) + \lambda H_p(U_1, \Phi(U_1)), \end{aligned}$$

that is,

$$H_p(U_1, \Phi(U_1)) \leq \frac{1}{1-\lambda} H_p(U_1, \mathcal{M}_{2k+1}),$$

which together with our taking the limit as $k \to +\infty$, we can write $H_p(U_1, \Phi(U_1)) \leq 0$ and so $U_1 = \Phi(U_1)$.
(4) If $Z_{\Psi,\Phi}(\mathcal{M}_{2k}, U_1) = \dfrac{H_p(\mathcal{M}_{2k}, \Phi(U_1)) + H_p(U_1, \mathcal{M}_{2k+1})}{2}$, then

$$\begin{aligned} H_p(U_1, \Phi(U_1)) &\leq H_p(U_1, \mathcal{M}_{2k+1}) + \frac{\lambda}{2}\left[H_p(\mathcal{M}_{2k}, \Phi(U_1)) + H_p(U_1, \mathcal{M}_{2k+1})\right] \\ &\quad - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{2k+1}) + \frac{\lambda}{2}[H_p(\mathcal{M}_{2k}, \Phi(U_1)) + H_p(\mathcal{M}_{2k}, U_1) \\ &\quad + H_p(U_1, \Phi(U_1)) - \inf_{u \in U_1} p(u, u) + H_p(U_1, \mathcal{M}_{2k+1})] \\ &\quad - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \\ &\leq H_p(U_1, \mathcal{M}_{2k+1}) + \frac{\lambda}{2}[H_p(\mathcal{M}_{2k}, \Phi(U_1)) + H_p(\mathcal{M}_{2k}, U_1) \\ &\quad + H_p(U_1, \Phi(U_1)) + H_p(U_1, \mathcal{M}_{2k+1})], \end{aligned}$$

which together with our taking the as $k \to +\infty$ implies

$$\begin{aligned} H_p(U_1, \Phi(U_1)) &\leq H_p(U_1, U_1) + \frac{\lambda}{2}[H_p(U_1, \Phi(U_1)) + H_p(U_1, U_1) \\ &\quad + H_p(U_1, \Phi(U_1)) + H_p(U_1, U_1)] \\ &= \lambda H_p(U_1, \Phi(U_1)), \end{aligned}$$

giving us $H_p(U_1, \Phi(U_1)) = 0$ and so $U_1 = \Psi(U_1)$.

Thus $U_1 = \Psi(U_1) = \Phi(U_1)$, which means that U_1 is the common attractor of Ψ and Φ.

Now, to establish the uniqueness of the common attractor, let U_2 be another common attractor of Ψ and Φ. Since the pair (Ψ, Φ) is generalized Hutchinson contractive operator, we have

$$\begin{aligned} H_p(U_1, U_2) &= H_p(\Psi(U_1), \Phi(U_2)) \\ &\leq \lambda \max\{H_p(U_1, U_2), H_p(U_1, \Psi(U_1)), H_p(U_2, \Phi(U_2)), \\ &\quad \frac{H_p(U_1, \Phi(U_2)) + H_p(U_2, \Psi(U_1))}{2}\} \\ &= \lambda \max\{H_p(U_1, U_2), H_p(U_1, U_1), H_p(U_2, U_2), \\ &\quad \frac{H_p(U_1, U_2) + H_p(U_2, U_1)}{2}\} \end{aligned}$$

$$\leq \lambda H_p(U_1, U_2),$$

and so $(1-\lambda)H_p(U_1, U_2) \leq 0$, that is, $H_p(U_1, U_2) = 0$ and hence $U_1 = U_2$. Thus $U_1 \in C^p(Y)$ is a unique common attractor of Ψ and Φ. □

Theorem 3.2. (**Generalized Collage**) Let (Y, p) be a complete partial metric space. For a given generalized iterated function system $\{Y; h_1, h_2..., h_r; g_1, g_2, ..., g_r\}$ which have contractive constant $\lambda \in [0, 1)$ and for a given $\varepsilon \geq 0$, if for any $\mathcal{M} \in C^p(Y)$, we have either

$$H_p(\mathcal{M}, \Psi(\mathcal{M})) \leq \varepsilon,$$

or

$$H_p(\mathcal{M}, \Phi(\mathcal{M})) \leq \varepsilon,$$

where $\Psi(\mathcal{M}) = \cup_{k=1}^{r} h_k(\mathcal{M})$ and $\Phi(\mathcal{M}) = \cup_{k=1}^{r} g_k(\mathcal{M})$. Then,

$$H_p(\mathcal{M}, U_1) \leq \frac{\varepsilon}{1-\lambda},$$

where $U_1 \in C^p(Y)$ is a common attractor of Ψ and Φ.

Proof. It follows from Proposition 2.3 that the mappings $\Psi, \Phi : C^p(Y) \to C^p(Y)$ satisfy

$$H_p(\Psi(\mathcal{U}), \Phi(\mathcal{V})) \leq \lambda H_p(\mathcal{U}, \mathcal{V}) \text{ for all } \mathcal{U}, \mathcal{V} \in C^p(Y).$$

From Theorem 3.1, there exists a unique common attractor $U_1 \in C^p(Y)$ of mappings Ψ and Φ, that is, $U_1 = \Psi(U_1) = \Phi(U_1)$.

In addition, for any $\mathcal{N}_0 \in C^p(Y)$, a sequence $\{\mathcal{N}_k\}$ defined by $\mathcal{N}_{2k+1} = \Psi(\mathcal{N}_{2k})$ and $\mathcal{N}_{2k+2} = \Phi(\mathcal{N}_{2k+1})$ for $k = 0, 1, 2, ...$, we have

$$\lim_{k\to+\infty} H_p(\Psi(\mathcal{N}_{2k}), U_1) = \lim_{k\to+\infty} H_p(\Phi(\mathcal{N}_{2k+1}), U_1) = 0.$$

Assume that $H_p(\mathcal{M}, \Psi(\mathcal{M})) \leq \varepsilon$ for any $\mathcal{M} \in C^p(Y)$, one can write

$$\begin{aligned} H_p(\mathcal{M}, U_1) &\leq H_p(\mathcal{M}, \Psi(\mathcal{M})) + H_p(\Psi(\mathcal{M}), \Phi(U_1)) - \inf_{m\in\Psi(\mathcal{M})} p(m, m) \\ &\leq \varepsilon + \lambda H_p(\mathcal{M}, U_1), \end{aligned}$$

which further implies that

$$H_p(\mathcal{M}, U_1) \leq \frac{\varepsilon}{1-\lambda}.$$

In a similar manner, by assuming that $H_p(\mathcal{M}, \Phi(\mathcal{M})) \leq \varepsilon$ for any $\mathcal{M} \in C^p(Y)$. Then,

$$\begin{aligned} H_p(\mathcal{M}, U_1) &\leq H_p(\mathcal{M}, \Phi(\mathcal{M})) + H_p(\Phi(\mathcal{M}), \Psi(U_1)) - \inf_{m\in\Phi(\mathcal{M})} p(m, m) \\ &\leq \varepsilon + \lambda H_p(\mathcal{M}, U_1), \end{aligned}$$

implies

$$H_p(\mathcal{M}, U_1) \leq \frac{\varepsilon}{1-\lambda}. \quad \square$$

Remark 3.3. If we take in Theorem 3.1, $\mathcal{S}^p(Y)$ the collection of all singleton subsets of the given space Y, then $\mathcal{S}^p(Y) \subseteq C^p(Y)$. Furthermore, if we take a pair of mappings $(h_k, g_k) = (h, g)$ for each k, where $h = h_1$ and $g = g_1$ then the pair of operators (Ψ, Φ) becomes

$$(\Psi(y_1), \Phi(y_2)) = (h(y_1), g(y_2)).$$

Consequently, the following common fixed point result is obtained.

Corollary 3.4. Suppose $\{Y; (h_k, g_k), k = 1, 2, \cdots, r\}$ is a generalized iterated function system defined in a complete partial metric space (Y, p) and define a pair of mappings $h, g : Y \to Y$ as in Remark 3.3. If some $\lambda \in [0, 1)$ exists such that for any $y_1, y_2 \in Y$, the following condition holds:

$$p(hy_1, gy_2) \leq \lambda Z_{h,g}(y_1, y_2),$$

where

$$Z_{h,g}(y_1, y_2) = \max\{p(y_1, y_2), p(y_1, hy_1), p(y, gy_2), \frac{p(y_1, gy_2) + p(y, hy_1)}{2}\}.$$

Then h and g have a unique common fixed point $u \in Y$. Furthermore, for any $u_0 \in Y$, the sequence $\{u_0, hu_0, ghu_0, hghu_0, \cdots\}$ converges to the common fixed point of h and g.

Corollary 3.5. Let $\{Y; (h_k, g_k), k = 1, 2, \cdots, r\}$ be a generalized iterated function system defined in a complete partial metric space (Y, p) and (h_k, g_k) for $k = 1, 2, ..., r$ be a pair of generalized contractive self-mappings on Y. Then the pair $(\Psi, \Phi) : C^p(Y) \to C^p(Y)$ defined in Theorem 3.1 has at most one common attractor in $C^p(Y)$. Furthermore, for any initial set $\mathcal{M}_0 \in C^p(Y)$, the sequence $\{\mathcal{M}_0, \Phi\Psi(\mathcal{M}_0), \Psi\Phi\Psi(\mathcal{M}_0), \cdots\}$ of compact sets has a limit point which is the common attractor of Ψ and Φ.

The following example shows the validity of Corollary 3.5.

Example 3.6. Let $Y = [0, 10]$ be endowed with the partial metric $p : Y \times Y \to \mathbb{R}^+$ defined by

$$p(y_1, y_2) = \frac{1}{2}\max\{y_1, y_2\} + \frac{1}{4}|y_1 - y_2| \text{ for all } y_1, y_2 \in Y.$$

Define $h_1, h_2 : Y \to Y$ as

$$\begin{aligned} h_1(y) &= \frac{10 - y}{3} \text{ for all } y \in Y, \\ h_2(y) &= \frac{16 - y}{4} \text{ for all } y \in Y \end{aligned}$$

and $g_1, g_2 : Y \to Y$ as

$$\begin{aligned} g_1(y) &= \frac{15 - y}{3} \text{ for all } y \in Y, \\ g_2(y) &= \frac{y_2 + 4}{4} \text{ for all } y \in Y. \end{aligned}$$

Now, for $y_1, y_2 \in Y$, we have

$$p(h_1(y_1), g_1(y_2)) = \frac{1}{2}\max\left\{\frac{10 - y_1}{3}, \frac{15 - y_2}{3}\right\} + \frac{1}{4}\left|\frac{10 - y_1}{3} - \frac{15 - y_2}{3}\right|$$

$$
\begin{aligned}
&= \frac{1}{3}\left[\frac{1}{2}\max\{10-y_1,15-y_2\}+\frac{1}{4}\left|(10-y_1)-(15+y_2)\right|\right]\\
&\leq \lambda_1 p\left(y_1,y_2\right),
\end{aligned}
$$

where $\lambda_1 = \dfrac{1}{3}$.
Also, for $y_1, y_2 \in Y$, we have

$$
\begin{aligned}
p\left(h_2\left(y_1\right), g_2\left(y_2\right)\right) &= \frac{1}{2}\max\left\{\frac{16-y_1}{4},\frac{y_2+4}{4}\right\}+\frac{1}{4}\left|\frac{16-y_1}{4}-\frac{y_2+4}{4}\right|\\
&= \frac{1}{4}\left[\frac{1}{2}\max\{16-y_1,y_2+4\}+\frac{1}{4}\left|(16-y_1)-(y_2+4)\right|\right]\\
&\leq \lambda_2 p\left(y_1,y_2\right),
\end{aligned}
$$

where $\lambda_2 = \dfrac{1}{4}$.

Consider the generalized iterated function system $\{Y;\left(h_1,g_1\right),\left(h_2,g_2\right)\}$ with the mappings $\Psi,\Phi : C^p\left(Y\right) \to C^p\left(Y\right)$ given as

$$
\left(\Psi,\Phi\right)\left(U\right) = \left(h_1,g_1\right)\left(U\right) \cup \left(h_2,g_2\right)\left(U\right) \text{ for all } U \in C^p\left(Y\right).
$$

By Proposition 2.3, for $\mathcal{M},\mathcal{N} \in C^p\left(Y\right)$, we have

$$
H_p\left(\Psi\left(\mathcal{M}\right),\Phi\left(\mathcal{N}\right)\right) \leq \lambda^* H_p\left(\mathcal{M},\mathcal{N}\right),
$$

where $\lambda^* = \max\left\{\dfrac{1}{3},\dfrac{1}{4}\right\} = \dfrac{1}{3}$.

Thus, all conditions of Corollary 3.5 are satisfied. Moreover, for any initial set $\mathcal{M}_0 \in C^p(Y)$, the sequence

$$
\{\mathcal{M}_0,\Psi\left(\mathcal{M}_0\right),\Phi\Psi\left(\mathcal{M}_0\right),\Psi\Phi\Psi\left(\mathcal{M}_0\right),\cdots\}
$$

of compact sets is convergent and has a limit point which is the common attractor of Ψ and Φ. □

The following result shows the existence of unique common attractor of generalized rational Hutchinson contractive operators in partial metric space.

Theorem 3.7. Consider a complete partial metric space (Y,p) and the generalized iterated function system given as $\{Y;(h_k,g_k),k=1,2,\cdots,r\}$. Let $\Psi,\Phi : C^p(Y) \to C^p(Y)$ be defined by

$$
\Psi(\mathcal{M}) = \cup_{k=1}^r h_k(\mathcal{M})
$$

and

$$
\Phi(\mathcal{N}) = \cup_{k=1}^r g_k(\mathcal{N}),
$$

for each $\mathcal{M},\mathcal{N} \in C^p(Y)$. If the pair (Ψ,Φ) is generalized rational Hutchinson contractive operator, then Ψ and Φ have a unique common attractor $U_1 \in C^p(Y)$, that is,

$$
U_1 = \Psi\left(U_1\right) = \Phi\left(U_1\right).
$$

Furthermore, for arbitrarily chosen initial set $\mathcal{M}_0 \in C^p(Y)$, the sequence

$$\{\mathcal{M}_0, \Psi(\mathcal{M}_0), \Phi\Psi(\mathcal{M}_0), \Psi\Phi\Psi(\mathcal{M}_0), \cdots\}$$

of compact sets converges to a common attractor U_1.
Proof. Let $\mathcal{M}_0$ be arbitrarily chosen in $C^p(Y)$. Define

$$\mathcal{M}_1 = \Psi(\mathcal{M}_0),\ \mathcal{M}_3 = \Psi(\mathcal{M}_2), \cdots, \mathcal{M}_{2k+1} = \Psi(\mathcal{M}_{2k})$$

and

$$\mathcal{M}_2 = \Phi(\mathcal{M}_1),\ \mathcal{M}_4 = \Phi(\mathcal{M}_3), \cdots, \mathcal{M}_{2k+2} = \Phi(\mathcal{M}_{2k+1})$$

for $k \in \{0, 1, 2, \cdots\}$.

Now, since the pair (Ψ, Φ) is a generalized rational Hutchinson contractive operator, we have

$$\begin{aligned} H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}) &= H_p(\Psi(\mathcal{M}_{2k}), \Phi(\mathcal{M}_{2k+1})) \\ &\leq \lambda_* R_{\Psi,\Phi}(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}) \end{aligned}$$

for $k \in \{0, 1, 2, \cdots\}$, where

$$\begin{aligned} R_{\Psi,\Phi}(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}) &= \max\left\{\frac{H_p(\mathcal{M}_{2k}, \Phi(\mathcal{M}_{2k+1}))[1 + H_p(\mathcal{M}_{2k}, \Psi(\mathcal{M}_{2k}))]}{2(1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}))},\right. \\ &\quad \frac{H_p(\mathcal{M}_{2k+1}, \Phi(\mathcal{M}_{2k+1}))[1 + H_p(\mathcal{M}_{2k}, \Psi(\mathcal{M}_{2k}))]}{1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})}, \\ &\quad \left.\frac{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})[1 + H_p(\mathcal{M}_{2k}, \Psi(\mathcal{M}_{2k}))]}{1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})}\right\} \\ &= \max\left\{\frac{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+2})[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{2(1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}))},\right. \\ &\quad \frac{H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2})[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})}, \\ &\quad \left.\frac{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})}\right\} \\ &= \max\{\frac{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+2})}{2}, H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}), \\ &\quad H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})\} \\ &= \max\{\frac{H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+2})}{2}, H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}) &\leq \frac{\lambda_*}{2}[H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}) + H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}) \\ &\quad - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1})] \\ &\leq \frac{\lambda_*}{2}[H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1}) + H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2})], \end{aligned}$$

that is,

$$H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}) \leq \frac{\lambda_*}{2-\lambda_*} H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})$$

and for $\eta_* = \dfrac{\lambda_*}{2-\lambda_*} < 1$, we have

$$H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2}) \leq \eta_* H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})$$

for all $k \in \{0, 1, 2, ...\}$. Therefore for $k < l$, with $k, l \in \{0, 1, 2, ...\}$

$$\begin{aligned} H_p(\mathcal{M}_k, \mathcal{M}_l) &\leq H_p(\mathcal{M}_k, \mathcal{M}_{k+1}) + H_p(\mathcal{M}_{k+1}, \mathcal{M}_{k+2}) + \cdots + H_p(\mathcal{M}_{l-1}, \mathcal{M}_l) \\ &\quad - \inf_{m_{k+1} \in \mathcal{M}_{k+1}} p(m_{k+1}, m_{k+1}) - \inf_{m_{k+2} \in \mathcal{M}_{k+2}} p(m_{k+2}, m_{k+2}) - \\ &\quad \cdots - \inf_{m_{l-1} \in \mathcal{M}_{l-1}} p(m_{l-1}, m_{l-1}), \\ &\leq \eta_*^k H_p(\mathcal{M}_0, \mathcal{M}_1) + \eta_*^{k+1} H_p(\mathcal{M}_0, \mathcal{M}_1) + \cdots + \eta_*^{l-1} H_p(\mathcal{M}_0, \mathcal{M}_1), \\ &\leq [\eta_*^k + \eta_*^{k+1} + \cdots + \eta_*^{l-1}] H_p(\mathcal{M}_0, M_1), \\ &\leq \eta_*^k [1 + \eta_* + \eta_*^2 + \cdots + \eta_*^{l-k-1}] H_p(\mathcal{M}_0, \mathcal{M}_1), \\ &\leq \frac{\eta_*^k}{1-\eta_*} H_p(\mathcal{M}_0, \mathcal{M}_1). \end{aligned}$$

By convergence towards 0 from the right hand side, we get $H_p(\mathcal{M}_k, \mathcal{M}_l) \to 0$ as $k, l \to +\infty$. Therefore $\{\mathcal{M}_k\}$ is a Cauchy sequence in $C^p(Y)$. But $(C^p(Y), H_p)$ is complete, we have $\mathcal{M}_k \to U_1$ as $k \to +\infty$ for some $U_1 \in C^p(Y)$, in other words, $\lim_{k\to+\infty} H_p(\mathcal{M}_k, U_1) = \lim_{k\to+\infty} H_p(\mathcal{M}_k, \mathcal{M}_{k+1}) = H_p(U_1, U_1)$ and we have $\lim_{k\to+\infty} H_p(\mathcal{M}_k, U_1) = 0$.

To prove that U_1 is a common attractor of Ψ and Φ, we have

$$\begin{aligned} H_p(\Psi(U_1), U_1) &\leq H_p(\Psi(U_1), \Phi(\mathcal{M}_{2k+1})) + H_p(\Phi(\mathcal{M}_{2k+1}), U_1) \\ &\quad - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}), \\ &\leq \lambda_* R_{\Psi,\Phi}(U_1, \mathcal{M}_{2k+1}) + H_p(\mathcal{M}_{2k+2}, U_1) \\ &\quad - \inf_{m_{2k+1} \in \mathcal{M}_{2k+1}} p(m_{2k+1}, m_{2k+1}) \end{aligned}$$

for all $k \in \{0, 1, 2, ...\}$, where

$$\begin{aligned} R_{\Psi,\Phi}(U_1, \mathcal{M}_{2k+1}) &= \max\left\{ \frac{H_p(U_1, \Phi(\mathcal{M}_{2k+1}))[1 + H_p(U_1, \Psi(U_1))]}{2(1 + H_p(U_1, \mathcal{M}_{2k+1}))}, \right. \\ &\qquad \frac{H_p(\mathcal{M}_{2k+1}, \Phi(\mathcal{M}_{2k+1}))[1 + H_p(U_1, \Psi(U_1))]}{1 + H_p(U_1, \mathcal{M}_{2k+1})}, \\ &\qquad \left. \frac{H_p(\mathcal{M}_{2k+1}, U_1)[1 + H_p(U_1, \Psi(U_1))]}{1 + H_p(U_1, \mathcal{M}_{2k+1})} \right\} \\ &= \max\left\{ \frac{H_p(\mathcal{M}_{2k+2}, U_1)[1 + H_p(U_1, \Psi(U_1))]}{2(1 + H_p(U_1, \mathcal{M}_{2k+1}))}, \right. \\ &\qquad \frac{H_p(\mathcal{M}_{2k+1}, \mathcal{M}_{2k+2})[1 + H_p(U_1, \Psi(U_1))]}{1 + H_p(U_1, \mathcal{M}_{2k+1})}, \end{aligned}$$

$$\left.\frac{H_p(U_1,\mathcal{M}_{2k+1})[1+H_p(U_1,\Psi(U_1))]}{1+H_p(U_1,\mathcal{M}_{2k+1})}\right\}.$$

Consider the following three cases:
(1) If $R_{\Psi,\Phi}(U_1,\mathcal{M}_{2k+1})=\frac{H_p(U_1,\mathcal{M}_{2k+2})[1+H_p(U_1,\Psi(U_1))]}{2(1+H_p(U_1,\mathcal{M}_{2k+1}))}$, then we have

$$\begin{aligned}H_p(\Psi(U_1),U_1) &\leq \frac{\lambda_* H_p(U_1,\mathcal{M}_{2k+2})[1+H_p(U_1,\Psi(U_1))]}{2(1+H_p(U_1,\mathcal{M}_{2k+1}))}+H_p(\mathcal{M}_{2k+2},U_1)\\ &\quad -\inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1},m_{2k+1}),\end{aligned}$$

which together with our taking the limit as $k\to+\infty$, we get $H_p(U_1,\Psi(U_1))\leq 0$ and so $U_1=\Psi(U_1)$.
(2) If $R_{\Psi,\Phi}(U_1,\mathcal{M}_{2k+1})=\frac{H_p(\mathcal{M}_{2k+1},\mathcal{M}_{2k+2})[1+H_p(U_1,\Psi(U_1))]}{1+H_p(U_1,\mathcal{M}_{2k+1})}$, we have

$$\begin{aligned}H_p(\Psi(U_1),U_1) &\leq \lambda_*\frac{H_p(\mathcal{M}_{2k+1},\mathcal{M}_{2k+2})[1+H_p(U_1,\Psi(U_1))]}{1+H_p(U_1,\mathcal{M}_{2k+1})}+H_p(\mathcal{M}_{2k+2},U_1)\\ &\quad -\inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1},m_{2k+1})\\ &\leq \lambda_*\frac{H_p(\mathcal{M}_{2k+1},\mathcal{M}_{2k+2})[1+H_p(U_1,\Psi(U_1))]}{1+H_p(U_1,\mathcal{M}_{2k+1})}+H_p(\mathcal{M}_{2k+2},U_1)),\end{aligned}$$

which together with our taking the limit as $k\to+\infty$, we get $H_p(U_1,\Psi(U_1))\leq 0$ and thus $U_1=\Psi(U_1)$.
(3) In case of $R_{\Psi,\Phi}(U_1,\mathcal{M}_{2k+1})=\frac{H_p(\mathcal{M}_{2k+1},U_1)[1+H_p(U_1,\Psi(U_1))]}{1+H_p(U_1,\mathcal{M}_{2k+1})}$, we obtain

$$\begin{aligned}H_p(U_1,\Psi(U_1)) &\leq \lambda_*\frac{H_p(\mathcal{M}_{2k+1},U_1)[1+H_p(U_1,\Psi(U_1))]}{1+H_p(U_1,\mathcal{M}_{2k+1})}+H_p(\mathcal{M}_{2k+2},U_1)\\ &\quad -\inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1},m_{2k+1})\\ &\leq \frac{\lambda_* H_p(\mathcal{M}_{2k+1},U_1)[1+H_p(U_1,\Psi(U_1))]}{1+H_p(U_1,\mathcal{M}_{2k+1})}+H_p(\mathcal{M}_{2k+2},U_1),\end{aligned}$$

which together with our taking the limit as $k\to+\infty$, we get

$$H_p(U_1,\Psi(U_1)),$$

that is, $U_1=\Psi(U_1)$.

In a similar manner, one can obtain

$$\begin{aligned}H_p(U_1,\Phi(U_1)) &\leq H_p(U_1,\mathcal{M}_{2k+1})+H_p(\mathcal{M}_{2k+1},\Phi(U_1))-\inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1},m_{2k+1})\\ &= H_p(U_1,\mathcal{M}_{2k+1})+H_p(\Psi(\mathcal{M}_{2k}),\Phi(U_1))-\inf_{m_{2k+1}\in\mathcal{M}_{2k+1}} p(m_{2k+1},m_{2k+1})\\ &\leq H_p(U_1,\mathcal{M}_{2k+1})+\lambda_* R_{\Psi,\Phi}(\mathcal{M}_{2k},\Phi(U_1)),\end{aligned}$$

where

$$R_{\Psi,\Phi}(\mathcal{M}_{2k},U_1) = \max\left\{\frac{H_p(\mathcal{M}_{2k},\Phi(U_1))[1+H_p(\mathcal{M}_{2k},\Psi(\mathcal{M}_{2k}))]}{2(1+H_p(\mathcal{M}_{2k},U_1))},\right.$$

$$
\begin{aligned}
& \frac{H_p(U_1, \Phi(U_1))[1 + H_p(\mathcal{M}_{2k}, \Psi(\mathcal{M}_{2k}))]}{1 + H_p(\mathcal{M}_{2k}, U_1)}, \\
& \left.\frac{H_p(\mathcal{M}_{2k}, U_1))[1 + H_p(\mathcal{M}_{2k}, \Psi(\mathcal{M}_{2k}))]}{1 + H_p(\mathcal{M}_{2k}, U_1)}\right\} \\
= \; & \max\left\{\frac{H_p(\mathcal{M}_{2k}, \Phi(U_1))[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{2(1 + H_p(\mathcal{M}_{2k}, U_1))},\right. \\
& \frac{H_p(U_1, \Phi(U_1))[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{1 + H_p(\mathcal{M}_{2k}, U_1)}, \\
& \left.\frac{H_p(\mathcal{M}_{2k}, U_1)[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{1 + H_p(\mathcal{M}_{2k}, U_1)}\right\}.
\end{aligned}
$$

Again, we have the following three cases:
(1) If $R_{\Psi,\Phi}(\mathcal{M}_{2k}, U_1) = \frac{H_p(\mathcal{M}_{2k}, \Phi(U_1))[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{2(1 + H_p(\mathcal{M}_{2k}, U_1))}$, then

$$
H_p(U_1, \Phi(U_1)) \leq H_p(U_1, \mathcal{M}_{2k+1}) + \lambda_* \left\{\frac{H_p(\mathcal{M}_{2k}, \Phi(U_1))[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{2(1 + H_p(\mathcal{M}_{2k}, U_1))}\right\}.
$$

Which together with our taking the limit as $k \to +\infty$, we get

$$
H_p(U_1, \Phi(U_1)) \leq H_p(U_1, U_1) + \frac{\lambda_*}{2}\left\{\frac{H_p(U_1, \Phi(U_1))[1 + H_p(U_1, U_1)]}{(1 + H_p(U_1, U_1))}\right\},
$$

that is,

$$
\left(1 - \frac{\lambda_*}{2}\right) H_p(U_1, \Phi(U_1)) \leq 0,
$$

thus, $U_1 = \Phi(U_1)$.
(2) If $R_{\Psi,\Phi}(\mathcal{M}_{2k}, U_1) = \frac{H_p(U_1, \Phi(U_1))[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{1 + H_p(\mathcal{M}_{2k}, U_1)}$, then

$$
H_p(U_1, \Phi(U_1)) \leq H_p(U_1, \mathcal{M}_{2k+1}) + \lambda_* \left\{\frac{H_p(U_1, \Phi(U_1))[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{1 + H_p(\mathcal{M}_{2k}, U_1)}\right\},
$$

which together with our taking the limit as $k \to +\infty$, we get

$$
(1 - \lambda_*) H_p(U_1, \Phi(U_1)) \leq 0,
$$

which implies that $U_1 = \Phi(U_1)$.

(3) If $R_{\Psi,\Phi}(\mathcal{M}_{2k}, U_1) = \frac{H_p(\mathcal{M}_{2k}, U_1)[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{1 + H_p(\mathcal{M}_{2k}, U_1)}$ then

$$
H_p(U_1, \Phi(U_1)) \leq H_p(U_1, \mathcal{M}_{2k}) + \lambda_* \left\{\frac{H_p(\mathcal{M}_{2k}, U_1)[1 + H_p(\mathcal{M}_{2k}, \mathcal{M}_{2k+1})]}{1 + H_p(\mathcal{M}_{2k}, U_1)}\right\},
$$

which together with our taking the limit as $k \to +\infty$, we get $H_p(U_1, \Phi(U_1)) \leq 0$, which gives $U_1 = \Phi(U_1)$.

Thus U_1 is a common attractor of the mappings Ψ and Φ.

For the uniqueness, assume that U_1 and U_2 are distinct common attractors of Ψ and Φ. Since the pair (Ψ, Φ) is generalized rational Hutchinson contractive operator, we obtain that

$$\begin{aligned} H_p(U_1, U_2) &= H_p(\Psi(U_1), \Phi(U_2)) \\ &\leq \lambda_* \max\left\{ \frac{H_p(U_1, \Phi(U_2))[1 + H_p(U_1, \Psi(U_1))]}{2(1 + H_p(U_1, U_2))}, \right. \\ &\quad \frac{H_p(U_2, \Phi(U_2))[1 + H_p(U_1, \Psi(U_1))]}{1 + H_p(U_1, U_2)}, \\ &\quad \left. \frac{H_p(U_1, U_2)[1 + H_p(U_1, \Psi(U_1))]}{1 + H_p(U_1, U_2)} \right\} \\ &= \lambda_* \max\left\{ \frac{H_p(U_1, U_2)[1 + H_p(U_1, U_1)]}{2(1 + H_p(U_1, U_2))}, \right. \\ &\quad \left. \frac{H_p(U_2, U_2)[1 + H_p(U_1, U_1)]}{1 + H_p(U_1, U_2)}, \frac{H_p(U_1, U_2)[1 + H_p(U_1, U_1)]}{1 + H_p(U_1, U_2)} \right\} \\ &\leq \lambda_* H_p(U_1, U_2), \end{aligned}$$

and so $(1-\lambda_*)H_p(U_1, U_2) \leq 0$, which implies that $H_p(U_1, U_2) = 0$ and hence $U_1 = U_2$. Thus $U_1 \in C^p(Y)$ is a unique common attractor of Ψ and Φ. □

Corollary 3.8. Consider a generalized iterated function system $\{Y; h_k, g_k, k = 1, 2, \cdots, r\}$ on a complete partial metric space (Y, p) and the mappings $h, g : Y \to Y$ as given in Remark 3.3. If there exists $\lambda_* \in [0, 1)$ such that for any $y_1, y_2 \in Y$, the following condition holds:

$$p(hy_1, gy_2) \leq \lambda_* R_{h,g}(y_1, y_2),$$

where

$$\begin{aligned} R_{h,g}(y_1, y_2) &= \max\left\{ \frac{p(y_1, gy_2)[1 + p(y_1, hy_1)]}{2(1 + p(y_1, y_2))}, \frac{p(y_2, gy_2)[1 + p(y_1, hy_1)]}{1 + p(y_1, y_2)}, \right. \\ &\quad \left. \frac{p(y_1, y_2)[1 + p(y_1, hy_1)]}{1 + p(y_1, y_2)} \right\}. \end{aligned}$$

Then a unique common fixed point for h and g exists. Furthermore, for any initial choice of $v_0 \in Y$, the sequence $\{v_0, hv_0, ghv_0, hghv_0, ...\}$ converges to the common fixed point of h and g.

4. Well-posedness

Now, we investigate the well-posedness of attractor-based problems of generalized Hutchinson contractive operators pair and generalized rational Hutchinson contractive operators pair given in Definitions 2.4 and 2.5, respectively, in the framework of Hausdorff partial metric spaces. Some useful results of well-posedness of fixed point problems are appearing in [16].

First we define the well-posedness of common attractor-based problem.

Definition 4.1. A common attractor-based problem of a pair of mappings $\Psi, \Phi : C^p(Y) \to C^p(Y)$ is said to be well-posed if the pair (Ψ, Φ) has a unique common attractor $\Lambda_* \in C^p(Y)$ and for any sequence

$\{\Lambda_k\}$ in $C^p(Y)$ such that $\lim_{k\to+\infty} H_p(\Psi(\Lambda_k), \Lambda_k) = 0$ and $\lim_{k\to+\infty} H_p(\Phi(\Lambda_k), \Lambda_k) = 0$, then $\lim_{k\to+\infty} H_p(\Lambda_k, \Lambda_*) = H_p(\Lambda_*, \Lambda_*)$, that is, $\lim_{k\to+\infty} \Lambda_k = \Lambda_*$.

The following result shows the well-posedness of common attractor-based problem of a generalized Hutchinson contractive operators.

Theorem 4.2. Let (Y, p) be a complete partial metric space and $\Psi, \Phi : C^p(Y) \to C^p(Y)$ be defined as in Theorem 3.1. Then the pair (Ψ, Φ) has a well-posed common attractor-based problem.

Proof. From Theorem 3.1, it follows that the mappings Ψ and Φ have a unique common attractor (say) B_*.

Let a sequence $\{B_k\}$ in $C^p(Y)$ be such that $\lim_{k\to+\infty} H_p(\Psi(B_k), B_k) = 0$ and $\lim_{k\to+\infty} H_p(\Phi(B_k), B_k) = 0$. We want to show that $B_* = \lim_{k\to+\infty} B_k$. As the pair (Ψ, Φ) is generalized Hutchinson contractive operator, so that

$$\begin{aligned} H_p(B_*, B_k) &\leq H_p(\Psi(B_*), \Psi(B_k)) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \lambda Z_\Psi(B_*, B_k) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k), \end{aligned}$$

where

$$\begin{aligned} Z_{\Psi,\Phi}(B_*, B_k) &= \max\Big\{H_p(B_*, B_k), H_p(B_*, \Psi(B_*)), H_p(B_k, \Phi(B_k)), \\ &\quad \frac{H_p(B_*, \Phi(B_k) + H_p(B_k, \Psi(B_*))}{2}\Big\}. \end{aligned}$$

Then we have the following cases:

(i) If $Z_{\Psi,\Phi}(B_*, B_k) = H_p(B_*, B_k)$, then

$$H_p(B_*, B_k) \leq \lambda H_p(B_*, B_k) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k),$$

which further implies

$$(1 - \lambda)\, H_p(B_*, B_k) \leq H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k),$$

that is,

$$H_p(B_*, B_k) \leq \frac{1}{1-\lambda}[H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k)].$$

As $k \to +\infty$, we have

$$\lim_{k\to+\infty} H_p(B_*, B_k) \leq \frac{1}{1-\lambda} \lim_{k\to+\infty} [H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k)],$$

this implies that $\lim_{k\to+\infty} B_k = B_*$.

(ii) In case of $Z_{\Psi,\Phi}(B_*, B_k) = H_p(B_*, \Phi(B_*))$, we have

$$H_p(B_*, B_k) \leq \lambda H_p(B_*, \Phi(B_*)) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k).$$

As $k \to +\infty$, we have

$$\lim_{k\to+\infty} H_p(B_*, B_k) \leq \lambda \lim_{k\to+\infty} [H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k)].$$

Thus $\lim_{k\to+\infty} B_k = B_*$.

(iii) If $Z_{\Psi,\Phi}(B_*, B_k) = H_p(B_*, \Psi(B_k))$, then

$$\begin{aligned} H_p(B_*, B_k) &\leq \lambda H_p(B_*, \Psi(B_k)) + H_p(B_k, \Psi(B_k)) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \lambda[H_p(B_*, B_k) + H_p(B_k, \Psi(B_k)) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k)] \\ &\quad + H_p(B_k, \Psi(B_k)) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k), \end{aligned}$$

which further implies that

$$\begin{aligned} H_p(B_*, B_k) &\leq \frac{\lambda}{1-\lambda}[H_p(B_k, \Psi(B_k)) - \inf_{b_k \in B_k} p(b_k, b_k)] \\ &\quad + \frac{1}{1-\lambda}[H_p(B_k, \Psi(B_k)) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k)] \end{aligned}$$

As $k \to +\infty$, we have that $\lim_{k \to +\infty} B_k = B_*$.

(iv) Finally, if $Z_{\Psi,\Phi}(B_*, B_k) = \dfrac{H_p(B_*, \Phi(B_k) + H_p(B_k, \Psi(B_*))}{2}$, then we have

$$\begin{aligned} H_p(B_*, B_k) &\leq \frac{\lambda}{2}[H_p(B_*, \Phi(B_k) + H_p(B_k, \Psi(B_*))] \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \frac{\lambda}{2}[H_p(B_*, B_k) + H_p(B_k, \Psi(B_k)) - \inf_{b_k \in B_k} p(b_k, b_k) + H_p(B_k, B_*)] \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k), \end{aligned}$$

which gives

$$\begin{aligned} H_p(B_*, B_k) &\leq \frac{\lambda}{2(1-\lambda)}[H_p(B_k, \Psi(B_k)) - \inf_{b_k \in B_k} p(b_k, b_k)] \\ &\quad + \frac{1}{1-\lambda}[H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k)] \end{aligned}$$

and by $k \to +\infty$, we obtain

$$\lim_{k \to +\infty} H_p(B_*, B_k) \leq 0,$$

which implies that $\lim_{k \to +\infty} B_k = B_*$. □

The following result shows the well-posedness of common attractor-based problem of a generalized rational Hutchinson contractive operators.

Theorem 4.3. Consider a complete partial metric space (Y, p) with $\Psi, \Phi : C^p(Y) \to C^p(Y)$ defined as in Theorem 3.7. Then the pair (Ψ, Φ) has a well-posed common attractor-based problem.

Proof. From Theorem 3.7, it follows that the mappings Ψ and Φ have a unique common attractor (say) B_*.

Let a sequence $\{B_k\}$ in $C^p(Y)$ be such that $\lim_{k\to+\infty} H(\Psi(B_k), B_k) = 0$ and $\lim_{k\to+\infty} H(\Phi(B_k), B_k) = 0$. We want to show that $B_* = \lim_{k\to+\infty} B_k$. As the pair (Ψ, Φ) is generalized rational Hutchinson contractive operator, so that

$$\begin{aligned} H_p(B_k, B_*) &\leq H_p(\Psi(B_k), \Psi(B_*)) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \lambda_* R_{\Psi,\Phi}(B_k, B_*) + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k), \end{aligned}$$

where

$$\begin{aligned} R_{\Psi,\Phi}(B_k, B_*) = \max\Bigg\{ & \frac{H_p(B_k, \Phi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{2(1 + H_p(B_k, B_*))}, \\ & \frac{H_p(B_*, \Phi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)}, \\ & \frac{H_p(B_*, B_k)[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)}\Bigg\}. \end{aligned}$$

We consider the following three cases:

(i) For $R_{\Psi,\Phi}(B_k, B_*) = \dfrac{H_p(B_k, \Phi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{2(1 + H_p(B_k, B_*))}$, we have

$$\begin{aligned} H_p(B_k, B_*) &\leq \lambda_* \frac{H_p(B_k, \Phi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{2(1 + H_p(B_k, B_*))} \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k) \\ &\leq \lambda_* H_p(B_k, B_*)[1 + H_p(B_k, \Psi(B_k))] \\ &\quad + H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k), \end{aligned}$$

which implies that

$$H_p(B_k, B_*) - \lambda_* H_p(B_k, B_*)[1 + H_p(\Psi(B_k), B_k)] \leq H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k)$$

and so

$$H_p(B_k, B_*) \leq \frac{1}{1 - \lambda_*[1 + H_p(\Psi(B_k), B_k)]}[H_p(\Psi(B_k), B_k) - \inf_{\beta_k\in\Psi(B_k)} p(\beta_k, \beta_k)].$$

And taking the limit as $k \to +\infty$ gives

$$\lim_{k\to+\infty} H_p(B_k, B_*) \leq 0,$$

which implies that $\lim_{k\to+\infty} B_k = B_*$.

(ii) If $R_{\Psi,\Phi}(B_k, B_*) = \dfrac{H_p(B_*, \Phi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)}$, then

$$H_p(B_k, B_*) \leq \lambda_* \left(\frac{H_p(B_*, \Phi(B_*))[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)} \right)$$

$$
\begin{aligned}
&+H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k) \\
= \ & H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k).
\end{aligned}
$$

And taking the limit as $k \to +\infty$, we have

$$
\lim_{k \to +\infty} H_p(B_k, B_*) \leq 0,
$$

which implies that $\lim_{k \to +\infty} B_k = B_*$.

(iii) Finally if $R_{\Psi,\Phi}(B_k, B_*) = \dfrac{H_p(B_*, B_k)[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)}$, then

$$
\begin{aligned}
H_p(B_k, B_*) \ \leq \ & \lambda_* \frac{H_p(B_*, B_k)[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)} \\
& +H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k),
\end{aligned}
$$

that is,

$$
\begin{aligned}
& H_p(B_k, B_*)[1 - \lambda_* \frac{[1 + H_p(B_k, \Psi(B_k))]}{1 + H_p(B_k, B_*)}] \\
\leq \ & H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k),
\end{aligned}
$$

which further implies

$$
\begin{aligned}
& H_p(B_k, B_*)[1 - \lambda_*[1 + H_p(B_k, \Psi(B_k))]] \\
\leq \ & H_p(\Psi(B_k), B_k) - \inf_{\beta_k \in \Psi(B_k)} p(\beta_k, \beta_k).
\end{aligned}
$$

On taking the limit as $k \to +\infty$, gives $\lim_{k \to +\infty} H_p(B_k, B_*) \leq 0$ implies that $\lim_{k \to +\infty} B_k = B_*$. Thus the proof is complete. □

5. Application to functional equations

In this section, we are applying our obtained results to solve a functional equation arising in the dynamic programming.

Let W_1 and W_2 be two Banach spaces with $U \subseteq W_1$ and $V \subseteq W_2$. Suppose that

$$
\kappa\colon U \times V \longrightarrow U, \quad g_1, g_2\colon U \times V \longrightarrow \mathbb{R}, \quad h_1, h_2\colon U \times V \times \mathbb{R} \longrightarrow \mathbb{R}.
$$

If we consider U and V as the state and decision spaces respectively, then the problem of dynamic programming reduces to the problem of solving the functional equations:

$$
q_1(x) = \sup_{y \in V}\{g_1(x, y) + h_1(x, y, q_1(\kappa(x, y)))\}, \text{ for } x \in U \tag{5.1}
$$

$$
q_2(x) = \sup_{y \in V}\{g_1(x, y) + h_2(x, y, q_2(\kappa(x, y)))\}, \text{ for } x \in U. \tag{5.2}
$$

The Eqs (5.1) and (5.2) can be reformulated as

$$q_1(x) = \sup_{y\in V}\{g_2(x,y) + h_1(x,y,q_1(\kappa(x,y)))\} - b, \text{ for } x \in U \tag{5.3}$$

$$q_2(x) = \sup_{y\in V}\{g_2(x,y) + h_2(x,y,q_2(\kappa(x,y)))\} - b, \text{ for } x \in U, \tag{5.4}$$

where $b > 0$.

We study the existence and uniqueness of the bounded solution of the functional Eqs (5.3) and (5.4) arising in dynamic programming in the setup of partial metric spaces.

Let $B(U)$ denotes the set of all bounded real valued functions on U. For an arbitrary $\eta \in B(U)$, define $\|\eta\| = \sup_{t\in U} |\eta(t)|$. Then $(B(U), \|\cdot\|)$ is a Banach space. Now consider

$$p_B(\eta,\xi) = \sup_{t\in U} |\eta(t) - \xi(t)| + b,$$

where $\eta, \xi \in B(U)$. Then p_B is a partial metric on $B(U)$ (see also [5]).
Assume that:
(D_1) : g_1, g_2, h_1 and h_2 are bounded and continuous.
(D_2) : For $x \in U$, $\eta \in B(U)$ and $b > 0$, take $\Psi, \Phi : B(U) \to B(U)$ as

$$\Psi\eta(x) = \sup_{y\in V}\{g_2(x,y) + h_1(x,y,\eta(\kappa(x,y)))\} - b, \text{ for } x \in U, \tag{5.5}$$

$$\Phi\eta(x) = \sup_{y\in V}\{g_2(x,y) + h_2(x,y,\eta(\kappa(x,y)))\} - b, \text{ for } x \in U. \tag{5.6}$$

Moreover, for every $(x,y) \in U \times V$, $\eta, \xi \in B(U)$ and $t \in U$ implies

$$|h_1(x,y,\eta(t)) - h_2(x,y,\xi(t))| \le \lambda Z_{\Psi,\Phi}(\eta(t), \xi(t)) - 2b, \tag{5.7}$$

where

$$Z_{\Psi,\Phi}(\eta(t), \xi(t)) = \max\Big\{p_B(\eta(t), \xi(t)), p_B(\eta(t), \Psi\eta(t)), p_B(\xi(t), \Phi\xi(t)), \frac{p_B(\eta(t), \Phi\xi(t)) + p_B(\xi(t), \Psi\eta(t))}{2}\Big\}.$$

Theorem 5.1. Assume that the conditions (D_1) and (D_2) hold. Then, the functional Eqs (5.3) and (5.4) have a unique common and bounded solution in $B(U)$.

Proof. Note that $(B(U), p_B)$ is a complete partial metric space. By (D_1), Ψ and Φ are self-mappings of $B(U)$. By (5.5) and (5.6) in (D_2), it follows that for any $\eta, \xi \in B(U)$ and $b > 0$, choose $x \in U$ and $y_1, y_2 \in V$ such that

$$\Psi\eta < g_2(x,y_1) + h_1(x,y_1,\eta(\kappa(x,y_1))) \tag{5.8}$$

$$\Phi\xi < g_2(x,y_2) + h_2(x,y_2,\xi(\kappa(x,y_2))), \tag{5.9}$$

which further implies that

$$\Psi\eta \ge g_2(x,y_2) + h_1(x,y_2,\eta(\kappa(x,y_2))) - b \tag{5.10}$$

$$\Phi\xi \ge g_2(x,y_1) + h_2(x,y_1,\xi(\kappa(x,y_1))) - b. \tag{5.11}$$

From (5.8) and (5.11) together with (5.7) implies

$$\begin{aligned}\Psi\eta(t)-\Phi\xi(t) &< h_1(x,y_1,\eta(\kappa(x,y_1)))-h_2(x,y_1,\xi(\kappa(x,y_1)))+b\\ &\leq |h_1(x,y_1,\eta(\kappa(x,y_1)))-h_2(x,y_1,\xi(\kappa(x,y_1)))|+b\\ &\leq \lambda Z_{\Psi,\Phi}(\eta(t),\xi(t))-b.\end{aligned} \tag{5.12}$$

From (5.9) and (5.10) together with (5.7) implies

$$\begin{aligned}\Phi\xi(t)-\Psi\eta(t) &< h_2(x,y_2,\xi(\kappa(x,y_2)))-h_1(x,y_2,\eta(\kappa(x,y_2)))+b\\ &\leq |h_1(x,y_2,\eta(\kappa(x,y_2)))-h_2(x,y_2,\xi(\kappa(x,y_2)))|+b\\ &\leq \lambda Z_{\Psi,\Phi}(\eta(t),\xi(t))-b.\end{aligned} \tag{5.13}$$

From (5.12) and (5.13), we get

$$|\Psi\eta(t)-\Phi\xi(t)|+b \leq \lambda Z_{\Psi,\Phi}(\eta(t),\xi(t)). \tag{5.14}$$

The inequality (5.14) implies that

$$p_B(\Psi\eta(t),\Phi\xi(t)) \leq \lambda Z_{\Psi,\Phi}(\eta(t),\xi(t)), \tag{5.15}$$

where

$$\begin{aligned}Z_{\Psi,\Phi}(\eta(t),\xi(t)) &= \max\{p_B(\eta(t),\xi(t)), p_B(\eta(t),\Psi\eta(t)), p_B(\xi(t),\Phi\xi(t)),\\ &\quad \frac{p_B(\eta(t),\Phi\xi(t))+p_B(\xi(t),\Psi\eta(t))}{2}\}.\end{aligned}$$

Therefore, all conditions of Corollary 3.4 hold. Thus, there exists a common fixed point of Ψ and Φ, that is, $\eta^* \in B(U)$, where $\eta^*(t)$ is a common solution of functional Eqs (5.3) and (5.4). □

6. Conclusions

The results of this paper broadened the scope of iterated function system and fixed point theory of pair of mappings by incorporating the generalized contraction approaches. We obtained unique common attractors with the assistance of finite families of generalized contractive mappings, that belong to the special class of mappings defined on a partial metric space. The well-posedness of these obtained results is also established. The ideas in this work, being discussed in the setting of partial metric spaces, are completely fundamental. Hence, they can be made better, when presented in the extended generalized metric spaces, like dislocated metric, semi metric, b-metric spaces, G-metric spaces and some other pseudo-metric or quasi metric spaces.

Conflict of interest

The authors declare that they have no competing interests.

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