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Research article

Cubic m-polar fuzzy topology with multi-criteria group decision-making

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Abstract: The concept of cubic m-polar fuzzy set (CmPFS) is a new approach to fuzzy modeling with multiple membership grades in terms of fuzzy intervals as well as multiple fuzzy numbers. We define some fundamental properties and operations of CmPFSs. We define the topological structure of CmPFSs and the idea of cubic m-polar fuzzy topology (CmPF topology) with P-order (R-order). We extend several concepts of crisp topology to CmPF topology, such as open sets, closed sets, subspaces and dense sets, as well as the interior, exterior, frontier, neighborhood, and basis of CmPF topology with P-order (R-order). A CmPF topology is a robust approach for modeling big data, data analysis, diagnosis, etc. An extension of the VIKOR method for multi-criteria group decision making with CmPF topology is designed. An application of the proposed method is presented for chronic kidney disease diagnosis and a comparative analysis of the proposed approach and existing approaches is also given.

Keywords: CmPFSs; CmPF topology with P-order (R-order); VIKOR; MCGDM
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1. Introduction

In our daily life, we deal with problems resulting from indefinite and vague information without using the appropriate modeling tools, this leads to imprecise reasoning and inexact solutions. That is why it is a quite difficult task for the decision-makers (DMs) to make reasonable and logical decisions in handling such problems. So, for such kinds of problems and difficulties, it has become particularly important to address vagueness and uncertainties. Zadeh [1] suggested an innovative idea of fuzzy set, which is an extension of a crisp set. It was eminent attainment and a milestone in the development of fuzzy set theory and fuzzy logic. To address the problems of daily life with vagueness and uncertainties in them, different models and theories have been introduced by the researchers. Later, the concept of the interval-valued fuzzy set (IVFS) was originated by Zadeh [2].

Atanassov [3, 4] suggested intuitionistic fuzzy set (IFS) theory and Pythagorean fuzzy sets (PFSs) were suggested by Yager [5, 6]. The generalization of PFSs with generalized membership grades was suggested by Yager [7], who named the generalization as follows: q-rung orthopair fuzzy sets (q-ROFSs). The idea of bipolarity was proposed by Zhang [8,9] in terms of bipolar fuzzy set (BFS). A new direct extension of fuzzy set with m degrees of membership grades was suggested by Chen *et al.* [10] and named m-polar fuzzy sets (mPFSs). Smarandache [11, 12] originated the notion of a neutrosophic set, which focuses on truthness, falsity and indeterminacy. The picture fuzzy set (PiFS) was proposed by Cuong [13]. Xu [14] developed IFS based aggregation operators for the information fusion of intuitionistic fuzzy numbers. Garg and Nancy [15] proposed linguistic single-valued neutrosophic prioritized aggregation operators and their applications.

The notion of a soft set was originated by Molodtsov [16]. The structure by merging soft sets with fuzzy sets was introduced by Cagman et al. [17]. Sometimes, it is very difficult for DMs to exactly weigh their certainty in real numbers. So, specifying their degree by the intervals is more appropriate. A hybrid model of a fuzzy set with IVFSs was suggested by Jun et al. [18] and named the cubic set (CS) along with their internal and external modes. A hybrid of the CS and mPFS was proposed by Riaz and Hashmi [19]. They developed aggregation operators for cubic m-polar fuzzy (CmPF) information aggregation. New extensions of fuzzy set, such as the linear Diophantine fuzzy sets (LDFS), linear Diophantine fuzzy soft rough sets, and spherical linear Diophantine sets [20–22], have robust features and applications in computational intelligence and information analysis. Liu et al. [23], Liu and Wang [24], and Jain et al. [25] suggested novel concepts of information aggregation for multi-criteria decision making (MCDM) problems. Fuzzy topology takes its motivation from classical analysis, and it has a vast number of applications. Chang [26] proposed the idea of fuzzy topology and the notion of intuitionistic fuzzy topology was introduced by Coker [27]. These ideas ware extended by Olgun et al. [28] to define Pythagorean fuzzy topology. Cagman et al. [29] proposed certain properties of soft topological spaces.

Saha et al. [30, 31] developed novel concepts of aggregation operators for information aggregation. Jana et al. [32, 33] proposed IFS-based Dombi and bipolar fuzzy Dombi prioritized aggregation operators in MADM. Akram et al. [34] developed MCDM with m-polar fuzzy attributes reduction algorithms. Akram et al. [35] proposed PFS-based extensions of TOPSIS and ELECTRE-I methods. Ashraf and Abdullah [36] introduced fuzzy modeling based on spherical fuzzy sine trigonometric information aggregation methods. Almagrabi [37] proposed a new approach to q-LDFSs and their operational laws with applications.

MCDM is the method that provides the ranking of the objects and also the ranking of feasible objects. The most important problem in decision analysis is how to describe the attribute values in an efficient way. It is very difficult for an individual in various situations to select an option due to the inconsistency in the data that occur because of human error or lacking information.

Many techniques have been used for the fusion of information. The word VIKOR has the abbreviation of "Vlse Kriterijiumska Optimizacija Kompromisno Resenje" and it is a very important technique in decision-making analysis. This process is widely used in decision-making analysis because of its computational comfort. It provides multiple suitable solutions for problems with unequal standards and helps DMs to achieve a neutral ending judgment. Some applications with help of VIKOR technique are discussed in Table 1.

Table 1. Some applications of the VIKOR technique.

Researchers	Benchmarks	Applications
Zhao et al. [40]	Extended VIKOR	Supplier selection
Joshi and Kumar [41]	Extended VIKOR	Supplier selection
Park et al. [42]	Extended VIKOR	Teacher performance evaluation
Shouzhen et al. [43]	Modified VIKOR	Supply chain management
Arya and Kumar [44]	VIKOR-TODIM	Management information system
Devi [45]	Extended VIKOR	Robot selection
Luo and Wang [46]	Extended VIKOR	Distance measure
Chen [47]	PF- VIKOR	Evaluating internet stock performance
Zhou and Chen [48]	Extended PF-VIKOR	Selection of Blockchain technology
Bakioglu and Atahan [49]	AHP integrated VIKOR	Prioritize risk in self-driving vehicle
Guleria and Bajaj [50]	VIKOR	Site selection for power plant
Gul [51]	VIKOR	Assessment of safety risk
Kirisci et al. [52]	Novel VIKOR	Survey on early childhood in quarantine
Dalapati and Pramanik [53]	NC-VIKOR	Selection of green supplier for cars
Pramanik et al. [54]	NC-VIKOR	Selection of green supplier for cars
Pramanik et al. [55]	VIKOR	Best option for money investment
Wang et al. [56]	VIKOR	Risk evaluation for construction project
Arya and Kumar [57]	TODIM-VIKOR	Selection of team leader in company
Joshi [58]	VIKOR	Selection of election bound country
Arya and Kumar [59]	VIKOR-TODIM	Selecting opinion polls
Khan et al. [60]	VIKOR	Selection of priority area for investment
Yue [61]	Extended VIKOR	Software reliability assessment
Meksavang [62]	Extended PiF-VIKOR	Supplier management
Singh and Kumar [63]	VIKOR	Supplier selection

Ali et al. [38, 39] proposed the idea of neutrosophic cubic sets and bipolar neutrosophic soft sets with applications in decision making.

The primary objective of this paper was to generate two different types of topological structures on cubic m-polar fuzzy sets (CmPFSs) while keeping in view the two orders of cubic sets. The concepts of CmPF topology with P-order and R-order are defined. The goals of this study are as follows: (i) to define open sets and closed sets in CmPF topology, (ii) to discuss the interior, closure, and exterior of CmPFSs in CmPF topology, (iii) to study the subspace of CmPF topology, (iv) to define the dense set, neighborhood and base of CmPF topology, (v) to develop an extension of the VIKOR method based on CmPFSs, and (vi) to develop a new multi-criteria group decision-making (MCGDM) method based on CmPF topology.

The remaining part of this paper is arranged in the following way. In Section 2, we look back to some elementary concepts like CSs, mPFSs, CmPFSs, and operations on CmPFSs. In Section 3, we describe the notion of a topological structure on CmPFSs under P-order. We also discuss some major results on CmPFs with P-order. In Section 4, we introduce the notion of a topological structure on CmPFSs under R-order. In Section 5, an extension of the VIKOR method for MCGDM with CmPF topology is introduced. An application of the proposed method for chronic kidney disease (CKD) diagnosis is presented, and a comparison analysis of the suggested approach and existing approaches is also given. The conclusion of the study is given in Section 6.

2. Preliminaries

In this section, we discuss some elementary concepts of CmPFSs.

Definition 2.1. [18] A CS $\complement$ on a universal set $\Bbbk$ is expressed as

$$\complement = \{\ell, [A^-(\ell), A^+(\ell)], \lambda(\ell) : \ell \in \Bbbk\},$$

in which $A = [A^-(\ell), A^+(\ell)]$ is an interval-valued fuzzy set and $\lambda(\ell)$ is a fuzzy set on $\Bbbk$. For simplicity, the CS $\complement = \{\ell, [A^-(\ell), A^+(\ell)], A(\ell) : \ell \in \Bbbk\}$ is denoted as $\complement =< A, \lambda >$

Definition 2.2. [10] Let $\Bbbk$ be a universal set of discourse. An mPFS on $\Bbbk$ is defined by $[0, 1]^m$, and it can be written as

$$\eth_p = \left\{\left(\ell, \mu_1(\ell), \cdots, \mu_m(\ell)\right) : \ell \in \Bbbk\right\},$$

where $\mu_1(\ell), \cdots, \mu_m(\ell)$ represents m number of membership grades (MGs) in $[0, 1]$.

Definition 2.3. [19] Let $\Bbbk$ be a universal set. A CmPFS on a universal set $\Bbbk$ is expressed as

$$\mathfrak{C} = \{(\ell, [A_1^-(\ell), A_1^+(\ell)], [A_2^-(\ell), A_2^+(\ell)], \cdots, [A_m^-(\ell), A_m^+(\ell)], A_1(\ell), A_2(\ell), \cdots, A_m(\ell)) : \ell \in \Bbbk\}$$

Here, $[A_j^-(\ell), A_j^+(\ell)]_{j=1}^m$ are fuzzy valued intervals and $(A_j(\ell))_{j=1}^m$ are fuzzy numbers. For simplicity, we can write the cubic m-polar fuzzy number (CmPFN) as

$$\mathfrak{C}_\gamma = ([A_j^-, A_j^+], A_j)_{j=1}^m$$

Definition 2.4. [18] Let $\Im_a = [F_a^-, F_a^+]$ and $\Im_b = [\daleth_b^-, \daleth_b^+]$ be any two fuzzy valued intervals . Then

1. $\Im_a \leq \Im_b \Leftrightarrow F_a^- \leq \daleth_b^-$ and $F_a^+ \leq \daleth_b^+$
2. $\Im_a \geq \Im_b \Leftrightarrow F_a^- \geq \daleth_b^-$ and $F_a^+ \geq \daleth_b^+$
3. $\Im_a = \Im_b \Leftrightarrow F_a^- = \daleth_b^-$ and $F_a^+ = \daleth_b^+$

2.1. Operations on CmPFSs

Definition 2.5. [19] Let us consider two CmPFSs on $\Bbbk$ given by

$$\mathfrak{C}^1 = \left\{\left(\lambda, [A_j^-, A_j^+], A_j\right)_{j=1}^m : \lambda \in \Bbbk\right\},$$

$$\mathfrak{C}^2 = \left\{\left(\lambda, [B_j^-, B_j^+], B_j\right)_{j=1}^m : \lambda \in \Bbbk\right\}$$

Some basic operations on these sets with P-order are defined as

1. $(\mathfrak{C}^1)^c = \left\{\left(\lambda, [1 - A_j^+, 1 - A_j^-], 1 - A_j\right)_{j=1}^m : \lambda \in \Bbbk\right\}$
2. $(\mathfrak{C}^2)^c = \left\{\left(\lambda, [1 - B_j^+, 1 - B_j^-], 1 - B_j\right)_{j=1}^m : \lambda \in \Bbbk\right\}$
3. $\mathfrak{C}^1 \subseteq_P \mathfrak{C}^2 \Leftrightarrow A_j^-(\lambda) \leq B_j^-(\lambda)$, $A_j^+(\lambda) \leq B_j^+(\lambda)$ and $A_j(\lambda) \leq B_j(\lambda)$
4. $\mathfrak{C}^1 \sqcup_P \mathfrak{C}^2 = \left\{\left(\lambda, [A_j^-(\lambda) \vee B_j^-(\lambda), A_j^+(\lambda) \vee B_j^+(\lambda)], A_j(\lambda) \vee B_j(\lambda)\right)_{j=1}^m : \lambda \in \Bbbk\right\}$
5. $\mathfrak{C}^1 \sqcap_P \mathfrak{C}^2 = \left\{\left(\lambda, [A_j^-(\lambda) \wedge B_j^-(\lambda), A_j^+(\lambda) \wedge B_j^+(\lambda)], A_j(\lambda) \wedge B_j(\lambda)\right)_{j=1}^m : \lambda \in \Bbbk\right\}$

Similarly, some basic operations on the above two CmPFSs with R-order are defined as

1. $\mathfrak{C}^1 \subseteq_R \mathfrak{C}^2 \Leftrightarrow A_j^-(\lambda) \leq B_j^-(\lambda)$, $A_j^+(\lambda) \leq B_j^+(\lambda)$ and $A_j(\lambda) \geq B_j(\lambda)$
2. $\mathfrak{C}^1 \sqcup_R \mathfrak{C}^2 = \left\{ \left(\lambda, [A_j^-(\lambda) \vee B_j^-(\lambda), A_j^+(\lambda) \vee B_j^+(\lambda)], A_j(\lambda) \wedge B_j(\lambda)\right)_{j=1}^m : \lambda \in \Bbbk \right\}$
3. $\mathfrak{C}^1 \sqcap_P \mathfrak{C}^2 = \left\{ \left(\lambda, [A_j^-(\lambda) \wedge B_j^-(\lambda), A_j^+(\lambda) \wedge B_j^+(\lambda)], A_j(\lambda) \vee B_j(\lambda)\right)_{j=1}^m : \lambda \in \Bbbk \right\}$

2.2. Some novel concepts of CmPFSs

Definition 2.6. [19] A CmPFS

$$\mathfrak{C} = \left\{ \left(\lambda, [A_j^-, A_j^+], A_j\right)_{j=1}^m : \lambda \in \Bbbk \right\}$$

on a universal set $\Bbbk$ is an internal CmPFS (ICmPFS) if $A_j^-(\lambda) \leq A_j(\lambda) \leq A_j^+(\lambda)$, for all $\lambda \in \Bbbk$ and $j = 1, 2, \cdots, m$.

Definition 2.7. [19] A CmPFS

$$\mathfrak{C} = \left\{ \left(\lambda, [A_j^-, A_j^+], A_j\right)_{j=1}^m : \lambda \in \Bbbk \right\}$$

on a universal set $\Bbbk$ is an external CmPFS (ECmPFS) if $A_j^-(\lambda) \nleq A_j(\lambda) \nleq A_j^+(\lambda)$, for some $\lambda \in \Bbbk$ or $j = 1, 2, \cdots, m$.
For simplicity, the ECmPFS is the inverse of the ICmPFS.

Definition 2.8. A CmPFS

$$\mathfrak{C} = \left\{ \left(\lambda, [A_j^-, A_j^+], A_j\right)_{j=1}^m : \lambda \in \Bbbk \right\}$$

for which $[A_j^-, A_j^+] = 0$ and $A_j(\lambda) = 0$ for all $\lambda \in \Bbbk$ and $j = 1, 2, \cdots, m$ is denoted by $\mathfrak{C}$.

Definition 2.9. A CmPFS

$$\mathfrak{C} = \left\{ \left(\lambda, [A_j^-, A_j^+], A_j\right)_{j=1}^m : \lambda \in \Bbbk \right\}$$

for which $[A_j^-, A_j^+] = 1$ and $A_j(\lambda) = 1$ for all $\lambda \in \Bbbk$ and $j = 1, 2, \cdots, m$ is denoted by ${}^g\mathfrak{C}$.

Definition 2.10. A CmPFS

$$\mathfrak{C} = \left\{ \left(\lambda, [A_j^-, A_j^+], A_j\right)_{j=1}^m : \lambda \in \Bbbk \right\}$$

for which $[A_j^-, A_j^+] = 0$ and $A_j(\lambda) = 1$ for all $\lambda \in \Bbbk$ and $j = 1, 2, \cdots, m$ is denoted by $\overline{\mathfrak{C}}$.

Definition 2.11. A CmPFS

$$\mathfrak{C} = \left\{ \left(\lambda, [A_j^-, A_j^+], A_j\right)_{j=1}^m : \lambda \in \Bbbk \right\}$$

for which $[A_j^-, A_j^+] = 1$ and $A_j(\lambda) = 0$ for all $\lambda \in \Bbbk$ and $j = 1, 2, \cdots, m$ is denoted by $\overline{{}^g\mathfrak{C}}$.

Definition 2.12. Let $\mathfrak{C}_\gamma = ([A_j^-, A_j^+], A_j)_{j=1}^m$ be a CmPFN. The score function and accuracy functions of a CmPFN are respectively defined as

$$S(\mathfrak{C}_\gamma) = \frac{\Sigma_{j=1}^m |\lambda([A_j^-, A_j^+]) - A_j|}{m} \tag{2.1}$$

and

$$\mathcal{A}(\mathfrak{C}_\gamma) = \frac{\Sigma_{j=1}^m (\lambda([A_j^-, A_j^+]) + A_j)}{2m} \tag{2.2}$$

where $\lambda([A_j^-, A_j^+]$ is the length of the fuzzy interval. Clearly, $S(\mathfrak{C}_\gamma) \in [-1, 1]$ and $\mathcal{A}(\mathfrak{C}_\gamma) \in [0, 1]$.

Let $\mathfrak{C}_\gamma^1$ and $\mathfrak{C}_\gamma^2$ be two CmPFNs. Then the ranking of CmPFNs in association with the proposed score and accuracy functions is defined as follows:

- $\mathfrak{C}_\gamma^1 < \mathfrak{C}_\gamma^2$ if $S(\mathfrak{C}_\gamma^1) < S(\mathfrak{C}_\gamma^2)$,
- If $S(\mathfrak{C}_\gamma^1) = S(\mathfrak{C}_\gamma^1)$, then $\mathfrak{C}_\gamma^1 < \mathfrak{C}_\gamma$ if $\mathcal{A}(\mathfrak{C}_\gamma^1) < \mathcal{A}(\mathfrak{C}_\gamma^2)$
- If $S(\mathfrak{C}_\gamma^1) = S(\mathfrak{C}_\gamma^1)$ and $\mathcal{A}(\mathfrak{C}_\gamma^1) = \mathcal{A}(\mathfrak{C}_\gamma^2)$, then $\mathfrak{C}_\gamma^1 = \mathfrak{C}_\gamma^2$

Definition 2.13. Let $\mathfrak{C}^1 = \langle [A_1^-, A_1^+], [A_2^-, A_2^+], \cdots, [A_m^-, A_m^+], A_1, A_2, \cdots, A_m \rangle = \langle [A_j^-, A_j^+], A_j \rangle_{j=1}^m$ and $\mathfrak{C}^2 = \langle [B_1^-, B_1^+], [B_2^-, B_2^+], \cdots, [B_m^-, B_m^+], B_1, \nu_2, \cdots, B_m \rangle = \langle [B_j^-, B_j^+], B_j \rangle_{j=1}^m$ be two CmPFSs.
The distance between two CmPFSs is defined by

$$d(\mathfrak{C}^1, \mathfrak{C}^2) = \left[\sum_{j=1}^m \left| \frac{A_j^- + A_j^+}{2} - \frac{B_j^- + B_j^+}{2} \right|^m + \sum_{j=1}^m |A_j - B_j|^m \right]^{1/m} \tag{2.3}$$

3. CmPF topology with P-order

Definition 3.1. Let $\mathbb{k}$ be a non-empty set and $\mathfrak{cmp}(\mathbb{k})$ be the collection of all CmPFSs in $\mathbb{k}$. The collection $\tau_{\mathfrak{C}p}$ containing the CmPFSs is called CmPF topology with P-order, abbreviated as CmPFT_P, if it satisfies the following properties:

1. $\mathfrak{Cp}, {}^g\mathfrak{Cp} \in \tau_{\mathfrak{C}p}$
2. If $(\mathfrak{Cp})_i \in \tau_{\mathfrak{C}p} \forall i \in \Lambda$ then $\sqcup_p (\mathfrak{Cp})_i \in \tau_{\mathfrak{C}p}$
3. If $\mathfrak{Cp}_g, \mathfrak{Cp}_y \in \tau_{\mathfrak{C}p}$ then $\mathfrak{Cp}_g \sqcap_p \mathfrak{Cp}_y \in \tau_{\mathfrak{C}p}$

Then, the pair $(\mathbb{k}, \tau_{\mathfrak{C}p})$ is called CmPF topological space with P-order, abbreviated as CmPFT_PS.

Example 3.2. Let $\mathbb{k} = \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3\}$ be a non-empty set. Then, $\mathfrak{cmp}(\mathbb{k})$ is the collection of all P-cubic mPFSs (PCmPFSs) in $\mathbb{k}$. We consider two cubic 3-polar fuzzy subsets of $\mathfrak{cmp}(\mathbb{k})$ given as

$$\begin{aligned}
\mathfrak{Cp}_g &= \Big\{ \Big(\mathbb{k}_1, [0.23, 0.46], [0.31, 0.52], [0.47, 0.65], 0.24, 0.32, 0.51\Big), \\
&\quad \Big(\mathbb{k}_2, [0.30, 0.42], [0.45, 0.56], [0.53, 0.69], 0.20, 0.41, 0.72\Big), \\
&\quad \Big(\mathbb{k}_3, [0.44, 0.63], [0.55, 0.78], [0.61, 0.83], 0.42, 0.53, 0.60\Big) \Big\} \\
\mathfrak{Cp}_y &= \Big\{ \Big(\mathbb{k}_1, [0.19, 0.36], [0.24, 0.48], [0.39, 0.52], 0.20, 0.31, 0.42\Big), \\
&\quad \Big(\mathbb{k}_2, [0.27, 0.38], [0.39, 0.52], [0.49, 0.63], 0.15, 0.30, 0.65\Big),
\end{aligned}$$

$$\left(\Bbbk_3, [0.36, 0.61], [0.50, 0.68], [0.60, 0.81], 0.39, 0.52, 0.58\right)\Big\}$$

The P-union and P-intersection results by applying Definition 2.5 to the CmPFSs $\mathfrak{Cp}_g$ and $\mathfrak{Cp}_g$ are given below in Tables 2 and 3, respectively.

Table 2. Union with P-order.

$\sqcup_p$	$\mathfrak{Cp}$	${}^g\mathfrak{Cp}$	$\mathfrak{Cp}_g$	$\mathfrak{Cp}_y$
$\mathfrak{Cp}$	$\mathfrak{Cp}$	${}^g\mathfrak{Cp}$	$\mathfrak{Cp}_g$	$\mathfrak{Cp}_y$
${}^g\mathfrak{Cp}$	${}^g\mathfrak{Cp}$	${}^g\mathfrak{Cp}$	${}^g\mathfrak{Cp}$	${}^g\mathfrak{Cp}$
$\mathfrak{Cp}_g$	$\mathfrak{Cp}_g$	${}^g\mathfrak{Cp}$	$\mathfrak{Cp}_g$	$\mathfrak{Cp}_g$
$\mathfrak{Cp}_y$	$\mathfrak{Cp}_y$	${}^g\mathfrak{Cp}$	$\mathfrak{Cp}_g$	$\mathfrak{Cp}_y$

Table 3. Intersection with P-order.

$\sqcap_p$	$\mathfrak{Cp}$	${}^g\mathfrak{Cp}$	$\mathfrak{Cp}_g$	$\mathfrak{Cp}_y$
$\mathfrak{Cp}$	$\mathfrak{Cp}$	$\mathfrak{Cp}$	$\mathfrak{Cp}$	$\mathfrak{Cp}$
${}^g\mathfrak{Cp}$	$\mathfrak{Cp}$	${}^g\mathfrak{Cp}$	$\mathfrak{Cp}_g$	$\mathfrak{Cp}_y$
$\mathfrak{Cp}_g$	$\mathfrak{Cp}$	$\mathfrak{Cp}_g$	$\mathfrak{Cp}_g$	$\mathfrak{Cp}_y$
$\mathfrak{Cp}_y$	$\mathfrak{Cp}$	$\mathfrak{Cp}_y$	$\mathfrak{Cp}_y$	$\mathfrak{Cp}_y$

Then, clearly

$$\mathfrak{T}_{\mathfrak{Cp}_g} = \left\{\mathfrak{Cp}, {}^g\mathfrak{Cp}, \mathfrak{Cp}_g, \mathfrak{Cp}_y\right\},$$

$$\mathfrak{T}_{\mathfrak{Cp}_y} = \left\{\mathfrak{Cp}, {}^g\mathfrak{Cp}, \mathfrak{Cp}_g\right\},$$

$$\mathfrak{T}_{\mathfrak{Cp}_v} = \left\{\mathfrak{Cp}, {}^g\mathfrak{Cp}, \mathfrak{Cp}_y\right\},$$

$$\mathfrak{T}_{\mathfrak{Cp}_w} = \left\{\mathfrak{Cp}, {}^g\mathfrak{Cp}\right\},$$

are cubic 3-polar fuzzy topologies with P-order.

Definition 3.3. Let $\Bbbk \neq \phi$ and $\tau_{\mathfrak{C}p} = \left\{\mathfrak{Cp}^{\Bbbk}\right\}$ where $\mathfrak{Cp}^{\Bbbk}$ are all cmPF subsets of $\Bbbk$. Then, $\tau_{\mathfrak{C}p}$ is a P-cubic m-polar fuzzy topology on $\Bbbk$ that is also the largest P-cubic m-polar fuzzy topology on $\Bbbk$, it is called P-discrete CmPF topology.

Definition 3.4. Let $\Bbbk \neq \phi$ and $\tau_{\mathfrak{C}p} = \left\{\mathfrak{Cp}, {}^g\mathfrak{Cp}\right\}$ be a collection of CmPFSs. Then, $\tau_{\mathfrak{C}p}$ is a P-cubic m-polar fuzzy topology on $\Bbbk$ that is also the smallest P-cubic m-polar fuzzy topology on $\Bbbk$, it is called P-indiscrete CmPF topology.

Definition 3.5. The members of the P-cubic m-polar fuzzy topology $\tau_{\mathfrak{C}p}$ are called P-cubic m-polar fuzzy open sets (PCmPFOSs) in $(\Bbbk, \tau_{\mathfrak{C}p})$.

Theorem 3.6. Let $(\Bbbk, \tau_{\mathfrak{C}p})$ be any P-cubic m-polar fuzzy topological space. Then

1. $\mathfrak{Cp}$ and ${}^g\mathfrak{Cp}$ are PCmPFOSs.

2. The P-union of any (finite/infinite) number of PCmPFOSs is a PCmPFOS.
3. The P-intersection of finite PCmPFOSs is a PCmPFOS.

Proof. 1. From the definition of the P-cubic m-polar fuzzy topology CmPFT_P, $\mathfrak{Cp}, {}^{g}\mathfrak{Cp} \in \tau_{\mathfrak{C}p}$. Hence $\mathfrak{Cp}$ and ${}^{g}\mathfrak{Cp}$ are PCmPFOSs.

2. Let $\left\{(\mathfrak{Cp})_i | i \in \Lambda\right\}$ be PCmPFOSs. Then, $\mathfrak{Cp})_i \in \tau_{\mathfrak{C}p}$. By the definition of CmPFT_P

$$\sqcup_p (\mathfrak{Cp})_i \in \tau_{\mathfrak{C}p}$$

Hence, $\sqcup_p (\mathfrak{Cp})_i$ represents PCmPFOSs.

3. Let $\mathfrak{Cp}_g, \mathfrak{Cp}_y, ..., \mathfrak{Cp}_\mathfrak{n}$ be PCmPOSs. Then, by the definition of $\tau_{\mathfrak{C}p}$

$$\sqcap_p (\mathfrak{Cp})_i \in \tau_{\mathfrak{C}p}$$

Hence, $\sqcap_p (\mathfrak{Cp})_i$ represents PCmPFOSs.

□

Definition 3.7. The complement of the P-cubic m-polar fuzzy open sets are called the P-cubic m-polar fuzzy closed sets (PCmPFCSs) in $(\mathbb{k}, \tau_{\mathfrak{C}p})$.

Theorem 3.8. If $(\mathbb{k}, \tau_{\mathfrak{C}p})$ is any P-cubic m-polar fuzzy topological space, then

1. $\mathfrak{Cp}$ and ${}^{g}\mathfrak{Cp}$ are PCmPFCSs.
2. The P-intersection of any number of PCmPFCSs is a PCmPFCS.
3. The P-union of finite PCmPFCSs is a PCmPFCS.

Proof. 1. $\mathfrak{Cp}$ and ${}^{g}\mathfrak{Cp}$ are PCmPFCSs. By the definition of CmPFT_P, $\mathfrak{Cp}, {}^{g}\mathfrak{Cp} \in \tau_{\mathfrak{C}p}$. Since the complement of $\mathfrak{Cp}$ is ${}^{g}\mathfrak{Cp}$, and the complement of ${}^{g}\mathfrak{Cp}$ is $\mathfrak{Cp}$. This shows that $\mathfrak{Cp}$ and ${}^{g}\mathfrak{Cp}$ are PCmPFCSs.

2. Let $\left\{(\mathfrak{Cp})_i | i \in \Lambda\right\}$ be PCmPFCSs. Then,

$$\left((\mathfrak{Cp})_i\right)^C \in \tau_{\mathfrak{C}p}$$

By the definition of CmPFT_P

$$\sqcup_p \left((\mathfrak{Cp})_i\right)^C \in \tau_{\mathfrak{C}p}$$

Hence, $\sqcup_p \left((\mathfrak{Cp})_i\right)^C$ represents PCmPFOSs but

$$\left(\sqcup_p ((\mathfrak{Cp})_i)^C\right) = \left(\sqcap_p (\mathfrak{Cp})_i\right)^C$$

So, $\sqcap_p (\mathfrak{Cp})_i$ represents PCmPFCSs.

3. Let $\mathfrak{Cp}_g, \mathfrak{Cp}_y, ..., \mathfrak{Cp}_\mathfrak{n}$ be PCmPCSs. Then, $(\mathfrak{Cp}_g)^C, (\mathfrak{Cp}_y)^C, ..., (\mathfrak{Cp}_\mathfrak{n})^C$ are PCmPFOSs. So,

$$(\mathfrak{Cp}_g)^C, (\mathfrak{Cp}_y)^C, ..., (\mathfrak{Cp}_\mathfrak{n})^C \in \tau_{\mathfrak{C}p}$$

by the definition of $\tau_{\mathfrak{C}p}$

$$\sqcap_p\Big((\mathfrak{C}\mathfrak{p})_i\Big)^C \in \tau_{\mathfrak{C}p}$$

This implies that $\sqcap_p((\mathfrak{C}\mathfrak{p})_i)^C \in \tau_{\mathfrak{C}p}$ is PCmPFOSs but

$$\Big((\sqcap_p(\mathfrak{C}\mathfrak{p})_i)^C\Big) = \Big(\sqcup_p\ ((\mathfrak{C}\mathfrak{p})_i\Big)^C$$

Hence, $\sqcup_p(\mathfrak{C}\mathfrak{p})_i$ is PCmPFOSs.

□

Definition 3.9. The PCmPFSs which include both PCmPFOSs and PCmPFCSs are called P-cubic m-polar fuzzy clopen sets in $(\Bbbk, \tau_{\mathfrak{C}p})$.

Proposition 3.10. 1. In every $\tau_{\mathfrak{C}p}$, $\mathfrak{C}\mathfrak{p}$ and ${}^g\mathfrak{C}\mathfrak{p}$ are P-cubic m-polar fuzzy clopen sets.
2. In discrete P-order CmPF topology, all cubic m-polar subsets of $\Bbbk$ are P-cubic m-polar fuzzy clopen sets.
3. In in-discrete P-order CmPF topology, only $\mathfrak{C}\mathfrak{p}$ and ${}^g\mathfrak{C}\mathfrak{p}$ are P-cubic m-polar fuzzy clopen sets.

Definition 3.11. Let $(\Bbbk, \tau_{\mathfrak{C}p_1})$ and $(\Bbbk, \tau_{\mathfrak{C}p_2})$ be two CmPFT_PSs in $\Bbbk$. These CmPFT_PSs are said to be comparable if

$$\tau_{\mathfrak{C}p_1} \subseteq_P \tau_{\mathfrak{C}p_2}$$

or

$$\tau_{\mathfrak{C}p_2} \subseteq_P \tau_{\mathfrak{C}p_1}$$

If $\tau_{\mathfrak{C}p_1} \subseteq_P \tau_{\mathfrak{C}p_2}$ then, $\tau_{\mathfrak{C}p_1}$ becomes P-cubic m-polar fuzzy coarser than $\tau_{\mathfrak{C}p_2}$. Similarly, $\tau_{\mathfrak{C}p_2}$ becomes P-cubic m-polar fuzzy finer than $\tau_{\mathfrak{C}p_1}$.

Example 3.12. Let $\Bbbk \neq \phi$; then from Example 3.2,

$$\tau_{\mathfrak{C}p_1} = \Big\{\mathfrak{C}\mathfrak{p}, {}^g\mathfrak{C}\mathfrak{p}, \mathfrak{C}\mathfrak{p}_g, \mathfrak{C}\mathfrak{p}_y\Big\}$$

and

$$\tau_{\mathfrak{C}p_2} = \Big\{\mathfrak{C}\mathfrak{p}, {}^g\mathfrak{C}\mathfrak{p}, \mathfrak{C}\mathfrak{p}_g\Big\}$$

are cubic 3-polar fuzzy topologies on $\Bbbk$. Since, $\tau_{\mathfrak{C}p_2} \subseteq_P \tau_{\mathfrak{C}p_1}$. Therefore, $\tau_{\mathfrak{C}p_2}$ becomes a P-cubic m-polar coarser than $\tau_{\mathfrak{C}p_1}$.

3.1. Sub spaces of CmPFT$_P$

Definition 3.13. Let $(\Bbbk, \mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk})$ be a CmPFT_PS. Let $\mathbb{Y} \subseteq \Bbbk$ and $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}}$ be a CmPFT_P on $\mathbb{Y}$ with PCmPFOSs that are

$$\mathfrak{C}p_\mathbb{Y} = \mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk} \sqcap_p \breve{\mathbb{Y}}$$

where $\mathfrak{C}p_\Bbbk$ are PCmPFOSs of $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk}$, $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}}$ are PCmPFOSs of $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}}$ and $\breve{\mathbb{Y}}$ is an absolute PCmPFS on $\mathbb{Y}$. Then, $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}}$ is the P-cubic m-polar fuzzy subspace of $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk}$ i.e.,

$$\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}} = \Big\{\mathfrak{C}p_\mathbb{Y} : \mathfrak{C}p_\mathbb{Y} = \mathfrak{C}p_\Bbbk \sqcap_p \breve{\mathbb{Y}},\ \mathfrak{C}p_\Bbbk \in \mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk}\Big\}$$

Example 3.14. Let $\Bbbk = \{\Bbbk_1, \Bbbk_2\}$ be a non-empty set.

$$\begin{aligned}\mathfrak{C}\mathfrak{p}_g &= \Big\{\Big(\Bbbk_1, [0.14, 0.39], [0.28, 0.42], 0.27, 0.33\Big),\\ &\quad \Big(\Bbbk_2, [0.26, 0.53], [0.43, 0.52], 0.57, 0.61\Big)\Big\}\\ \mathfrak{C}\mathfrak{p}_y &= \Big\{\Big(\Bbbk_1, [0.61, 0.86], [0.58, 0.72], 0.52, 0.64\Big),\\ &\quad \Big(\Bbbk_2, [0.37, 0.74], [0.48, 0.57], 0.81, 0.72\Big)\Big\}\end{aligned}$$

Then,

$$\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk} = \Big\{\mathfrak{C}\mathfrak{p}, {}^g\mathfrak{C}\mathfrak{p}, \mathfrak{C}\mathfrak{p}_g, \mathfrak{C}\mathfrak{p}_y\Big\}$$

is a cubic 2-polar fuzzy topology with P-order on $\Bbbk$. Now, the absolute cubic 2-polar fuzzy set on $\mathbb{Y} = \{\Bbbk_1\} \subseteq \Bbbk$ is

$$\breve{\mathbb{Y}} = \Big\{(\Bbbk_1, [1, 1], [1, 1], 1, 1)\Big\}$$

Since

$$\begin{aligned}\breve{\mathbb{Y}} \sqcap_p \mathfrak{C}\mathfrak{p} &= \mathfrak{C}\mathfrak{p} = \acute{\mathfrak{C}\mathfrak{p}},\\ \breve{\mathbb{Y}} \sqcap_p {}^g\mathfrak{C}\mathfrak{p} &= {}^g\mathfrak{C}\mathfrak{p} = \breve{\mathbb{Y}},\\ \breve{\mathbb{Y}} \sqcap_p \mathfrak{C}\mathfrak{p}_g &= \mathfrak{C}\mathfrak{p}_g = \acute{\mathfrak{C}\mathfrak{p}}_g,\\ \breve{\mathbb{Y}} \sqcap_p \mathfrak{C}\mathfrak{p}_y &= \mathfrak{C}\mathfrak{p}_y = \acute{\mathfrak{C}\mathfrak{p}}_y,\end{aligned}$$

we have

$$\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}} = \{\acute{\mathfrak{C}\mathfrak{p}}, \breve{\mathbb{Y}}, \acute{\mathfrak{C}\mathfrak{p}}_g, \acute{\mathfrak{C}\mathfrak{p}}_y\}$$

which is a cubic 2-polar fuzzy relative topology of $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk}$.

Let $(\Bbbk, \mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk})$ be a CmPFT_PS. Let $\mathbb{Y} \subseteq \Bbbk$ and $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}}$ be a CmPFT_P on $\mathbb{Y}$ with PCmPFOSs that are

$$\mathfrak{C}p_\mathbb{Y} = \mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk} \sqcap_p \mathbb{Y}$$

where $\mathfrak{C}p_\Bbbk$ denotes the PCmPFOSs of $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk}$, $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}}$ are PCmPFOSs of a $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}}$ and $\mathbb{Y}$ is any subset of PCmPFS on $\mathbb{Y}$. Then, $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}}$ is the P-cubic m-polar fuzzy subspace of $\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk}$ i.e.,

$$\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\mathbb{Y}} = \Big\{\mathfrak{C}p_\mathbb{Y} : \mathfrak{C}p_\mathbb{Y} = \mathfrak{C}p_\Bbbk \sqcap_p \mathbb{Y},\ \mathfrak{C}p_\Bbbk \in \mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk}\Big\}$$

Example 3.15. From Example 3.14

$$\mathfrak{T}_{\mathfrak{C}\mathfrak{p}_\Bbbk} = \Big\{\mathfrak{C}\mathfrak{p}, {}^g\mathfrak{C}\mathfrak{p}, \mathfrak{C}\mathfrak{p}_g, \mathfrak{C}\mathfrak{p}_y\Big\}$$

is a cubic 2-polar fuzzy topology with P-order on $\Bbbk$.
Now, take any cubic 2-polar fuzzy subset on $\Bbbk$ such that $\mathbb{Y} = \{\Bbbk_1\} \subseteq \Bbbk$ is

$$\mathbb{Y} = \Big\{(\Bbbk_1, [0.33, 0.43], [0.49, 0.68], 0.42, 0.51)\Big\}$$

Since

$$\breve{\mathbb{Y}} \sqcap_p \mathfrak{Cp} = \mathfrak{Cp} = \acute{\mathfrak{Cp}},$$

$$\breve{\mathbb{Y}} \sqcap_p {}^g\mathfrak{Cp} = {}^g\mathfrak{Cp} = \mathbb{Y},$$

$$\breve{\mathbb{Y}} \sqcap_p \mathfrak{Cp}_g = \mathfrak{Cp}_g = \acute{\mathfrak{Cp}}_g,$$

$$\breve{\mathbb{Y}} \sqcap_p \mathfrak{Cp}_y = \mathbb{Y},$$

we have

$$\mathfrak{T}_{\mathfrak{Cp}_{\mathbb{Y}}} = \{\acute{\mathfrak{Cp}}, \mathbb{Y}, \acute{\mathfrak{Cp}}_g\}$$

which is a cubic 2-polar fuzzy relative topology of $\mathfrak{T}_{\mathfrak{Cp}_{\mathbb{k}}}$.

3.2. Interior, closure, frontier and exterior of the PCmPFS

Definition 3.16. Let $(\mathbb{k}, \mathfrak{T}_{\mathfrak{C}}\mathfrak{p})$ be a CmPFT_PS and $\mathfrak{Cp} \in \mathfrak{cmp}(\mathbb{k})$ then the interior of $\mathfrak{Cp}$ is denoted as $\mathfrak{C}p^0$ and defined as the union of all open CmPF subsets contained in $\mathfrak{Cp}$. It is the greatest open cubic m-polar fuzzy set contained in $\mathfrak{Cp}$.

Example 3.17. Consider the cubic 3-polar topological space as presented in Example 3.2. Let $\mathfrak{C}p_3 \in \mathfrak{cmp}(\mathbb{k})$ given as

$$\begin{aligned} \mathfrak{Cp}_v \ = \ & \Big\{\Big(\mathbb{k}_1, [0.37, 0.49], [0.38, 0.56], [0.54, 0.69], 0.39, 0.41, 0.60\Big), \\ & \Big(\mathbb{k}_2, [0.41, 0.52], [0.48, 0.61], [0.57, 0.83], 0.38, 0.43, 0.83\Big), \\ & \Big(\mathbb{k}_3, [0.53, 0.72], [0.59, 0.91], [0.64, 0.89], 0.61, 0.82, 0.73\Big)\Big\}. \end{aligned}$$

Then,

$$\mathfrak{Cp}_v{}^0 = \mathfrak{Cp} \sqcup_p \mathfrak{Cp}_g \sqcup_p \mathfrak{Cp}_y = \mathfrak{Cp}_g.$$

Theorem 3.18. Let $(\mathbb{k}, \mathfrak{T}_{\mathfrak{C}}\mathfrak{p})$ be a CmPFT_PS and $\mathfrak{Cp} \in \mathfrak{cmp}(\mathbb{k})$. Then $\mathfrak{Cp}$ is an open CmPFS if, and only if, $(\mathfrak{C}p)^0 = \mathfrak{Cp}$.

Proof. If $\mathfrak{Cp}$ is an open CmPFS then the greatest open CmPFS contained in $\mathfrak{Cp}$ is itself $\mathfrak{Cp}$. Thus

$$(\mathfrak{C}p)^0 = \mathfrak{C}p$$

Conversely, if $(\mathfrak{C}p)^0 = \mathfrak{Cp}$ then $(\mathfrak{C}p)^0$ is an open CmPFS. This means that $\mathfrak{Cp}$ is an open CmPFS. □

Theorem 3.19. *Let* $(\mathbb{k}, \mathfrak{T}_{\mathfrak{C}}\mathfrak{p})$ *be a CmPFT_PS and* $\mathfrak{C}p_1$, $\mathfrak{C}p_2 \in \mathfrak{cmp}(\mathbb{k})$. *Then*

- $\Big((\mathfrak{C}p)^0\Big)^0 = (\mathfrak{C}p)^0$
- $\mathfrak{C}p_1 \subseteq_p \mathfrak{C}p_2 \Rightarrow (\mathfrak{C}p_1)^0 \subseteq_p (\mathfrak{C}p_2)^0$
- $\Big(\mathfrak{C}p_1 \sqcap_p \mathfrak{C}p_2\Big)^0 = (\mathfrak{C}p_1)^0 \subseteq_p (\mathfrak{C}p_2)^0$
- $\Big(\mathfrak{C}p_1 \sqcup_p \mathfrak{C}p_2\Big)^0 \supseteq_p (\mathfrak{C}p_1)^0 \sqcup_p (\mathfrak{C}p_2)^0$

Proof. The proof is obvious. □

Definition 3.20. Let $(\Bbbk, \mathfrak{T}_{\mathfrak{C}}\mathfrak{p})$ be a CmPFT_PS and $\mathfrak{C}\mathfrak{p} \in \mathfrak{cmp}(\Bbbk)$ then the closure of $\mathfrak{C}\mathfrak{p}$ is denoted as $\overline{\mathfrak{C}p}$ and is defined as the intersection of all closed cubic m-polar fuzzy supersets of $\mathfrak{C}\mathfrak{p}$. It is the smallest closed cubic m-polar fuzzy superset of $\mathfrak{C}\mathfrak{p}$.

Example 3.21. Consider the cubic 3-polar topological space given in Example 3.2. Then, the closed CmPFSs are $(\mathfrak{C}\mathfrak{p})^c = {}^g\mathfrak{C}\mathfrak{p}$ and $({}^g\mathfrak{C}\mathfrak{p})^c = \mathfrak{C}\mathfrak{p}$

$$
\begin{aligned}
(\mathfrak{C}\mathfrak{p}_g)^c &= \Big\{\Big(\Bbbk_1, [0.54, 0.77], [0.48, 0.69], [0.35, 0.53], 0.76, 0.68, 0.49\Big),\\
&\Big(\Bbbk_2, [0.58, 0.70], [0.44, 0.55], [0.31, 0.47], 0.80, 0.59, 0.28\Big),\\
&\Big(\Bbbk_3, [0.37, 0.56], [0.22, 0.45], [0.17, 0.39], 0.58, 0.47, 0.40\Big)\Big\}\\
(\mathfrak{C}\mathfrak{p}_y)^c &= \Big\{\Big(\Bbbk_1, [0.64, 0.81], [0.52, 0.76], [0.48, 0.61], 0.80, 0.69, 0.58\Big),\\
&\Big(\Bbbk_2, [0.62, 0.73], [0.48, 0.61], [0.37, 0.51], 0.85, 0.70, 0.35\Big),\\
&\Big(\Bbbk_3, [0.39, 0.64], [0.32, 0.50], [0.19, 0.40], 0.61, 0.48, 0.42\Big)\Big\}
\end{aligned}
$$

Let $\mathfrak{C}p_4 \in \mathfrak{cmp}(\Bbbk)$ be given as

$$
\begin{aligned}
\mathfrak{C}\mathfrak{p}_w &= \Big\{\Big(\Bbbk_1, [0.43, 0.69], [0.39, 0.54], [0.28, 0.49], 0.52, 0.49, 0.36\Big),\\
&\Big(\Bbbk_2, [0.31, 0.67], [0.40, 0.52], [0.26, 0.35], 0.48, 0.49, 0.31\Big),\\
&\Big(\Bbbk_3, [0.35, 0.43], [0.21, 0.38], [0.12, 0.36], 0.36, 0.32, 0.21\Big)\Big\}
\end{aligned}
$$

Then,

$$\overline{\mathfrak{C}\mathfrak{p}_w} = {}^g\mathfrak{C}\mathfrak{p} \sqcap_p (\mathfrak{C}\mathfrak{p}_y)^c = (\mathfrak{C}\mathfrak{p}_y)^c$$

is a closed CmPFS.

Theorem 3.22. Let $(\Bbbk, \mathfrak{T}_{\mathfrak{C}}\mathfrak{p})$ be a CmPFT_PS and $\mathfrak{C}\mathfrak{p} \in \mathfrak{cmp}(\Bbbk)$. Then, $\mathfrak{C}\mathfrak{p}$ is a closed CmPFS if, and only if, $\overline{\mathfrak{C}p} = \mathfrak{C}\mathfrak{p}$.

Proof. If $\mathfrak{C}\mathfrak{p}$ is a closed CmPFS then the smallest closed CmPFS superset of $\mathfrak{C}\mathfrak{p}$ is itself $\mathfrak{C}\mathfrak{p}$. Thus

$$\overline{\mathfrak{C}p} = \mathfrak{C}p$$

Conversely, if $\overline{\mathfrak{C}p} = \mathfrak{C}\mathfrak{p}$ then $\overline{\mathfrak{C}p}$ is a closed CmPFS. This means that $\mathfrak{C}\mathfrak{p}$ is a closed CmPFS. □

Definition 3.23. Let $\mathfrak{C}\mathfrak{p}$ be a P-cubic m-polar fuzzy subset of $(\Bbbk, \mathfrak{T}_{\mathfrak{C}}\mathfrak{p})$; then, its frontier or boundary is denoted by

$$Fr(\mathfrak{C}p) = \overline{\mathfrak{C}p} \sqcap_p \overline{\left(\mathfrak{C}p\right)^c}$$

Definition 3.24. Let $\mathfrak{Cp}$ be a P-cubic m-polar fuzzy subset of $(\Bbbk, \mathfrak{T}_{\mathfrak{C}}\mathfrak{p})$; then, its exterior is denoted by

$$Ext(\mathfrak{C}p) = \left(\overline{\mathfrak{C}p}\right)^c = \left(\mathfrak{C}p^c\right)^0$$

Example 3.25. Consider the cubic 3-polar topological space as constructed in Example 3.2 and $\mathfrak{Cp}_v$ and $\mathfrak{Cp}_w$ from Example 3.17 and Example 3.21, respectively. Then,
$(\mathfrak{Cp}_v)^0 = \mathfrak{Cp}_g$, $\overline{\mathfrak{Cp}_v} = {}^g\mathfrak{Cp}$
$Fr(\mathfrak{Cp}_v) = \mathfrak{Cp}_g$, $Ext(\mathfrak{Cp}_v) = \mathfrak{Cp}$
$(\mathfrak{Cp}_w)^0 = \mathfrak{Cp}$, $\overline{\mathfrak{Cp}_w} = (\mathfrak{Cp}_y)^c$
$Fr(\mathfrak{Cp}_w) = (\mathfrak{Cp}_y)^c$, $Ext(\mathfrak{Cp}_w) = \mathfrak{Cp}$

Remark. In the case of the CmPFT_PS, the law of contradiction $\mathfrak{Cp} \sqcap_p (\mathfrak{Cp})^c = \mathfrak{Cp}$ and law of the excluded middle $\mathfrak{Cp} \sqcup_p (\mathfrak{Cp})^c = {}^g\mathfrak{Cp}$ do not hold in general. From Example 3.17,
$\mathfrak{Cp}_v \sqcap_p (\mathfrak{Cp}_v)^c \neq \mathfrak{Cp}$
$\mathfrak{Cp}_v \sqcup_p (\mathfrak{Cp}_v)^c \neq {}^g\mathfrak{Cp}$

Theorem 3.26. Let $(\Bbbk, \mathfrak{T}_{\mathfrak{C}}\mathfrak{p})$ be a CmPFT_PS and $\mathfrak{C}p \in \mathfrak{cmp}(\Bbbk)$. Then

1. $(\mathfrak{C}p^0)^c = \overline{(\mathfrak{C}p^c)}$
2. $(\overline{\mathfrak{C}p})^c = (\mathfrak{C}p^c)^0$
3. $Ext(\mathfrak{C}p^c) = \mathfrak{C}p^0$
4. $Ext(\mathfrak{C}p) = (\mathfrak{C}p^c)^0$
5. $Ext(\mathfrak{C}p) \sqcup_p Fr(\mathfrak{C}p) \sqcup_p \mathfrak{C}p^0 \neq {}^g\mathfrak{Cp}$
6. $Fr(\mathfrak{C}p) = Fr(\mathfrak{C}p^c)$
7. $Fr(\mathfrak{C}p) \sqcap_p \mathfrak{C}p^0 \neq \mathfrak{Cp}$

Proof. 1. The proof is obvious.
2. The proof is obvious.
3. $Ext(\mathfrak{C}p^c) = \left(\overline{\mathfrak{C}p^c}\right)^c$
$Ext(\mathfrak{C}p^c) = \left((\mathfrak{C}p^c)^c\right)^0$
$Ext(\mathfrak{C}p^c) = \mathfrak{C}p^0$
4. $Ext(\mathfrak{C}p) = (\overline{\mathfrak{C}p})^c$
$Ext(\mathfrak{C}p) = (\mathfrak{C}p^c)^0$
5. $Ext(\mathfrak{C}p) \sqcup_p Fr(\mathfrak{C}p) \sqcup_p \mathfrak{C}p^0 \neq {}^g\mathfrak{Cp}$. Given Example 3.25, we can see that
$Ext(\mathfrak{C}p_3) \sqcup_p Fr(\mathfrak{C}p_3) \sqcup_p \mathfrak{C}p^0 \neq {}^g\mathfrak{Cp}$.
6. $Fr(\mathfrak{C}p^c) = \overline{(\mathfrak{C}p^c)} \sqcap_p \overline{\left((\mathfrak{C}p^c)^c\right)}\, Fr(\mathfrak{C}p^c) = \overline{(\mathfrak{C}p^c)} \sqcap_p \overline{\left(\mathfrak{C}p\right)} = Fr(\mathfrak{C}p)$
7. $Fr(\mathfrak{C}p) \sqcap_p \mathfrak{C}p^0 \neq \mathfrak{Cp}$. From Example 3.25, we can see that,
$Fr(\mathfrak{C}p_3) \sqcap_p \mathfrak{C}p^0 \neq \mathfrak{Cp}$.

□

Definition 3.27. A PCmPFS is said to be dense in a universal set $\Bbbk$ if

$$\overline{\mathfrak{Cp}} = {}^g\mathfrak{Cp}$$

From Example 3.17 $\overline{\mathfrak{Cp}_v} = {}^g\mathfrak{Cp}$. So, $\overline{\mathfrak{Cp}_v}$ is dense in $\Bbbk$.

Definition 3.28. A P-cubic m-polar number $\mathfrak{C}_\gamma = ([A_j^-, A_j^+], A_j)_{j=1}^m$ belong PCmPFS if, and only if, $A_i^-(\lambda) \le A_j^-(\lambda)$, $A_i^+(\lambda) \le A_j^+(\lambda)$ and $A_i(\lambda) \le A_j(\lambda)$ $j = 1, 2, ..., m$ and $l \in \mathbb{k}$

Let $(\mathbb{k}, \mathfrak{T}_{\mathfrak{C}}\mathfrak{p})$ be a CmPFT_PS. A PCmPFS $\acute{\mathfrak{C}}\mathfrak{p}$ of $\mathbb{k}$ which contains a PCmPF number $\mathfrak{C}_\gamma \in \mathbb{k}$ is said to be a neighborhood of $\mathfrak{C}_\gamma$ if, there exists a PCmPFOS $\mathfrak{C}\mathfrak{p}$ such that

$$\mathfrak{C}_\gamma \in \mathfrak{C}\mathfrak{p} \subseteq_p \acute{\mathfrak{C}}\mathfrak{p}.$$

Example 3.29. From Example 3.2, consider a PCmPF number

$$\mathfrak{C}_\gamma = \Big\{[0.14, 0.39], [0.28, 0.42], 0.27, 0.33\Big\}$$

which belongs to the PCmPFOS $\mathfrak{C}\mathfrak{p}_g$, which is a PCmPF subset of $\mathfrak{C}\mathfrak{p}_y$. From this, we can say that $\mathfrak{C}\mathfrak{p}_y$ is a neighborhood of $\mathfrak{C}_\gamma$.

3.3. P-cubic m-polar fuzzy basis

Definition 3.30. Let $(\mathbb{k}, \mathfrak{T}_{\mathfrak{C}}\mathfrak{p})$ be a CmPFT_PS. Then $\mathbb{B} \subseteq \mathfrak{T}_{\mathfrak{C}}\mathfrak{p}$ is said to be the P-cubic m-polar fuzzy basis for $\mathfrak{T}_{\mathfrak{C}}\mathfrak{p}$ if, for each $\mathfrak{C}\mathfrak{p} \in \mathfrak{T}_{\mathfrak{C}}\mathfrak{p}$, $\exists$ B $\in \mathbb{B}$ such that

$$\mathfrak{C}\mathfrak{p} = \sqcup_p \mathrm{B}$$

Example 3.31. From Example 3.2,

$$\tau_{\mathfrak{C}p} = \Big\{\mathfrak{C}\mathfrak{p}, {}^g\mathfrak{C}\mathfrak{p}, \mathfrak{C}\mathfrak{p}_g, \mathfrak{C}\mathfrak{p}_y\Big\}$$

is a P-cubic 3-polar fuzzy topology on $\mathbb{k}$. Then,

$$\mathbb{B} = \Big\{{}^g\mathfrak{C}\mathfrak{p}, \mathfrak{C}\mathfrak{p}_g, \mathfrak{C}\mathfrak{p}_y\Big\}$$

is a P-cubic 3-polar fuzzy basis for $\tau_{\mathfrak{C}p}$.

4. CmPF topology with R-order

Definition 4.1. Let $\mathbb{k}$ be a non-empty set and $\mathfrak{cmp}(\mathbb{k})$ be the assemblage of all CmPFSs in $\mathbb{k}$. The collection $\mathfrak{T}_{\mathfrak{Cr}}$ containing CmPFSs is called a CmPF topology with R-order, abbreviated as CmPFT_r, if it satisfies the following properties:

1. $\overline{\mathfrak{C}\mathfrak{r}}, \overline{{}^g\mathfrak{C}\mathfrak{r}} \in \mathfrak{T}_{\mathfrak{Cr}}$
2. If $(\mathfrak{C}\mathfrak{r})_i \in \mathfrak{T}_{\mathfrak{Cr}} \forall i \in \Lambda$ then $\sqcup_r (\mathfrak{C}\mathfrak{r})_i \in \mathfrak{T}_{\mathfrak{Cr}}$
3. If $\mathfrak{C}\mathfrak{r}_g, \mathfrak{C}\mathfrak{r}_y \in \mathfrak{T}_{\mathfrak{Cr}}$ then $\mathfrak{C}\mathfrak{r}_g \sqcap_r \mathfrak{C}\mathfrak{r}_y \in \mathfrak{T}_{\mathfrak{Cr}}$

Then, the pair $(\mathbb{k}, \mathfrak{T}_{\mathfrak{Cr}})$ is called the CmPF topological space with R-order, abbreviated as CmPFT_rS.

Example 4.2. Let $\mathbb{k} = \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3\}$ be a non-empty set. Then, $\mathfrak{cmp}(\mathbb{k})$ is the collection of all R-cubic m-polar fuzzy sets (RCmPFSs) in $\mathbb{k}$. We consider two cubic 3-polar fuzzy subsets of $\mathfrak{cmp}(\mathbb{k})$, given as

$$\mathfrak{C}\mathfrak{r}_g = \Big\{\Big(\mathbb{k}_1, [0.19, 0.39], [0.31, 0.48], [0.38, 0.61], 0.8, 0.7, 0.6\Big),$$

$$
\begin{aligned}
&\Big(\Bbbk_2, [0.21, 0.40], [0.34, 0.53], [0.40, 0.70], 0.7, 0.6, 0.5\Big),\\
&\Big(\Bbbk_3, [0.24, 0.52], [0.40, 0.62], [0.47, 0.81], 0.6, 0.5, 0.4\Big)\Big\}\\
\mathfrak{Cr}_y \quad = \quad &\Big\{\Big(\Bbbk_1, [0.27, 0.41], [0.32, 0.50], [0.39, 0.65], 0.7, 0.6, 0.5\Big),\\
&\Big(\Bbbk_2, [0.24, 0.49], [0.39, 0.58], [0.42, 0.73], 0.6, 0.5, 0.4\Big),\\
&\Big(\Bbbk_3, [0.26, 0.58], [0.44, 0.70], [0.51, 0.89], 0.5, 0.4, 0.3\Big)\Big\}.
\end{aligned}
$$

The R-union and R-intersection results obtained by applying Definition 2.5 to the CmPFSs $\mathfrak{Cr}_g$ and $\mathfrak{Cy}_g$ are given below in Tables 4 and 5, respectively.

Table 4. Union with R-order.

$\sqcup_r$	$\overline{\mathfrak{Cr}}$	$\overline{{}^g\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$
$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{{}^g\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$
$\overline{{}^g\mathfrak{Cr}}$	$\overline{{}^g\mathfrak{Cr}}$	$\overline{{}^g\mathfrak{Cr}}$	$\overline{{}^g\mathfrak{Cr}}$	$\overline{{}^g\mathfrak{Cr}}$
$\mathfrak{Cr}_g$	$\mathfrak{Cr}_g$	$\overline{{}^g\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$
$\mathfrak{Cr}_y$	$\mathfrak{Cr}_y$	$\overline{{}^g\mathfrak{Cr}}$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_y$

Table 5. Intersection with R-order.

$\sqcap_r$	$\overline{\mathfrak{Cr}}$	$\overline{{}^g\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$
$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$
$\overline{{}^g\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{{}^g\mathfrak{Cr}}$	$\mathfrak{Cg}_g$	$\mathfrak{Cr}_y$
$\mathfrak{Cr}_g$	$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_g$
$\mathfrak{Cr}_y$	$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$

Then, clearly

$$\mathfrak{T}_{\mathfrak{Cr}_g} = \Big\{\overline{\mathfrak{Cr}}, \overline{{}^g\mathfrak{Cr}}, \mathfrak{Cr}_g, \mathfrak{Cr}_y\Big\},$$

$$\mathfrak{T}_{\mathfrak{Cr}_y} = \Big\{\overline{\mathfrak{Cr}}, \overline{{}^g\mathfrak{Cr}}, \mathfrak{Cr}_g\Big\},$$

$$\mathfrak{T}_{\mathfrak{Cr}_v} = \Big\{\overline{\mathfrak{Cr}}, \overline{{}^g\mathfrak{Cr}}, \mathfrak{Cr}_y\Big\},$$

$$\mathfrak{T}_{\mathfrak{Cr}_w} = \Big\{\overline{\mathfrak{Cr}}, \overline{{}^g\mathfrak{Cr}}\Big\},$$

are cubic 3-polar fuzzy topologies with R-order.

Example 4.3. Let $\Bbbk = \{\Bbbk_1, \Bbbk_2\}$ be a non-empty set. Then, $\mathfrak{cmp}(\Bbbk)$ is the collection of all RCmPFSs in $\Bbbk$. We consider five cubic 3-polar fuzzy subsets of $\mathfrak{cmp}(\Bbbk)$, given as

$$\begin{aligned}
\mathfrak{Cr}_g &= \Big\{\Big(\Bbbk_1, [0.20, 0.41], [0.33, 0.50], [0.44, 0.60], 0.9, 0.8, 0.7\Big),\\
&\quad \Big(\Bbbk_2, [0.24, 0.42], [0.35, 0.60], [0.45, 0.65], 0.8, 0.7, 0.6\Big)\Big\},\\
\mathfrak{Cr}_y &= \Big\{\Big(\Bbbk_1, [0, 0], [0, 0], [0, 0], 0.9, 0.8, 0.7\Big),\\
&\quad \Big(\Bbbk_2, [0, 0], [0, 0], [0, 0], 0.8, 0.7, 0.6\Big)\Big\},\\
\mathfrak{Cr}_v &= \Big\{\Big(\Bbbk_1, [1, 1], [1, 1], [1, 1], 0.9, 0.8, 0.7\Big),\\
&\quad \Big(\Bbbk_2, [1, 1], [1, 1], [1, 1], 0.8, 0.7, 0.6\Big)\Big\},\\
\mathfrak{Cr}_w &= \Big\{\Big(\Bbbk_1, [0.20, 0.41], [0.33, 0.50], [0.44, 0.60]0, 0, 0\Big),\\
&\quad \Big(\Bbbk_2, [0.24, 0.42], [0.35, 0.60], [0.45, 0.65], 0, 0, 0\Big)\Big\},\\
\mathfrak{Cr} &= \Big\{\Big(\Bbbk_1, [0.20, 0.41], [0.33, 0.50], [0.44, 0.60]1, 1, 1\Big),\\
&\quad \Big(\Bbbk_2, [0.24, 0.42], [0.35, 0.60], [0.45, 0.65], 1, 1, 1\Big)\Big\}.
\end{aligned}$$

The R-union and R-intersection results obtained by applying Definition 2.5 to these CmPFSs are given below in Tables 6 and 7, respectively.

Table 6. Union with R-order.

$\sqcup_r$	$\mathfrak{Cr}$	${}^{g}\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_{\mathfrak{r}_y}$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}$
$\mathfrak{Cr}$	$\mathfrak{Cr}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}_w$
${}^{g}\mathfrak{Cr}$	$\overline{{}^{g}\mathfrak{Cr}}$	${}^{g}\mathfrak{Cr}$	${}^{g}\mathfrak{Cr}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_v$	$\overline{{}^{g}\mathfrak{Cr}}$	${}^{g}\mathfrak{Cr}$
$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}$	${}^{g}\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}$
$\overline{{}^{g}\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$
$\mathfrak{Cr}_g$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_g$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}_g$
$\mathfrak{Cr}_y$	$\mathfrak{Cr}$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_{\mathfrak{r}_y}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_v$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_g$
$\mathfrak{Cr}_v$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_v$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_v$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_v$
$\mathfrak{Cr}_w$	$\mathfrak{Cr}_w$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_w$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}_w$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}_w$
$\mathfrak{Cr}$	$\mathfrak{Cr}_w$	${}^{g}\mathfrak{Cr}$	$\mathfrak{Cr}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}$

Table 7. Intersection with R-order.

$\sqcap_r$	$\mathfrak{Cr}$	${}^{g}\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cp}_{\mathfrak{r}_y}$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}$
$\mathfrak{Cr}$	$\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$
${}^{g}\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	${}^{g}\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	${}^{g}\mathfrak{Cr}$	$\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	${}^{g}\mathfrak{Cr}$	$\mathfrak{Cr}$	$\mathfrak{Cr}$
$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$
$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}$	${}^{g}\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\overline{{}^{g}\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}$
$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}$
$\mathfrak{Cr}_y$	$\mathfrak{Cr}_y$	$\overline{\mathfrak{Cr}}$	$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_y$	$\overline{\mathfrak{Cr}}$
$\mathfrak{Cr}_v$	$\mathfrak{Cr}_y$	${}^{g}\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_v$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}$
$\mathfrak{Cr}_w$	$\mathfrak{Cr}$	$\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_y$	$\mathfrak{Cr}_g$	$\mathfrak{Cr}_w$	$\mathfrak{Cr}$
$\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}$	$\mathfrak{Cr}$	$\overline{\mathfrak{Cr}}$	$\mathfrak{Cr}$	$\mathfrak{Cr}$	$\mathfrak{Cr}$

Then clearly,

$$\mathfrak{T}_{\mathfrak{Cr}} = \left\{\mathfrak{Cr}, {}^{g}\mathfrak{Cr}, \overline{\mathfrak{Cr}}, \overline{{}^{g}\mathfrak{Cr}}, \mathfrak{Cr}_g, \mathfrak{Cp}_{\mathfrak{r}_y}, \mathfrak{Cr}_v, \mathfrak{Cr}_w, \mathfrak{Cr}\right\},$$

$$\mathfrak{T}_{\mathfrak{Cr}} = \left\{\mathfrak{Cr}, {}^{g}\mathfrak{Cr}, \overline{\mathfrak{Cr}}, \overline{{}^{g}\mathfrak{Cr}}\right\},$$

are cubic 3-polar fuzzy topologies with R-order.

Definition 4.4. Let $\mathbb{k} \neq \phi$ and $\mathfrak{T}_{\mathfrak{Cr}} = \left\{\mathfrak{Cr}^{\mathbb{k}}\right\}$, where $\mathfrak{Cr}^{\mathbb{k}}$ denotes all of the CmPFSs of $\mathbb{k}$. Then, $\mathfrak{T}_{\mathfrak{Cr}}$ is a R-cubic m-polar fuzzy topology on $\mathbb{k}$ that is also the largest R-cubic m-polar fuzzy topology on $\mathbb{k}$; it is called an R-discrete CmPF topology.

Definition 4.5. Let $\mathbb{k} \neq \phi$ and $\mathfrak{T}_{\mathfrak{Cr}} = \left\{\overline{\mathfrak{Cr}}, \overline{{}^{g}\mathfrak{Cr}}\right\}$ be the collection of CmPFSs. Then, $\mathfrak{T}_{\mathfrak{Cr}}$ is a R-cubic m-polar fuzzy topology on $\mathbb{k}$ that is also the smallest R-cubic m-polar fuzzy topology on $\mathbb{k}$; it is called an R-indiscrete cubic m-polar fuzzy topology.

Definition 4.6. The members of the RCmPF topology $\mathfrak{T}_{\mathfrak{Cr}}$ are called RCmPFOSs in $(\mathbb{k}, \mathfrak{T}_{\mathfrak{Cr}})$.

Theorem 4.7. If $(\mathbb{k}, \mathfrak{T}_{\mathfrak{Cr}})$ is any R-cubic m-polar fuzzy topological space, then

1. $\overline{\mathfrak{Cr}}$ and $\overline{{}^{g}\mathfrak{Cr}}$ are RCmPFOSs.
2. The R-union of any (finite/infinite) number of RCmPFOSs is an RCmPFOS.
3. The R-intersection of finite RCmPFOSs is an RCmPFOS.

Proof. The proof is the same as that for PCmPFOSs. □

Definition 4.8. The complement of RCmPFOSs are called RCmPFCSs in $(\mathbb{k}, \mathfrak{T}_{\mathfrak{Cr}})$.

Theorem 4.9. If $(\mathbb{k}, \mathfrak{T}_{\mathfrak{Cr}})$ is any R-cubic m-polar fuzzy topological space, then

1. $\overline{\mathfrak{Cr}}$ and $\overline{{}^{g}\mathfrak{Cr}}$ are RCmPFCSs.
2. The R-intersection of any (finite/infinite) number of RCmPFCSs is an RCmPFCS.
3. The R-union of finite RCmPFCSs is an RCmPFCS.

Proof. The proof is the same as that for PCmPFCSs. □

Definition 4.10. Let $(\Bbbk, \mathfrak{T}_{\mathfrak{Cr}_g})$ and $(\Bbbk, \mathfrak{T}_{\mathfrak{Cr}_y})$ be two CmPFT_rSs in $\Bbbk$. Two CmPFT_rSs are said to be comparable if,

$$\mathfrak{T}_{\mathfrak{Cr}_g} \subseteq_R \mathfrak{T}_{\mathfrak{Cr}_y} \text{ or } \mathfrak{T}_{\mathfrak{Cr}_y} \subseteq_R \mathfrak{T}_{\mathfrak{Cr}_g}.$$

Furthermore if, $\mathfrak{T}_{\mathfrak{Cr}_g} \subseteq_R \mathfrak{T}_{\mathfrak{Cr}_y}$, then $\mathfrak{T}_{\mathfrak{Cr}_g}$ becomes R-cubic m-polar fuzzy coarser than $\mathfrak{T}_{\mathfrak{Cr}_y}$. Similarly, $\mathfrak{T}_{\mathfrak{Cr}_y}$ becomes R-cubic m-polar fuzzy finer than $\mathfrak{T}_{\mathfrak{Cr}_g}$.

Example 4.11. Let $\Bbbk \neq \phi$; then from Example 4.3,

$$\mathfrak{T}_{\mathfrak{Cr}_g} = \left\{\mathfrak{Cr}, {}^g\mathfrak{Cr}, \overline{\mathfrak{Cr}}, \overline{{}^g\mathfrak{Cr}}, \mathfrak{Cr}_g, \mathfrak{Cp}_{\mathfrak{r}_y}, \mathfrak{Cr}_v, \mathfrak{Cr}_w, \mathfrak{Cr}\right\},$$

$$\mathfrak{T}_{\mathfrak{Cr}_y} = \left\{\mathfrak{Cr}, {}^g\mathfrak{Cr}, \overline{\mathfrak{Cr}}, \overline{{}^g\mathfrak{Cr}}\right\},$$

are cubic 3-polar fuzzy topologies on $\Bbbk$. Since $\mathfrak{T}_{\mathfrak{Cr}_y} \subseteq_R \mathfrak{T}_{\mathfrak{Cr}_g}$. So that, $\mathfrak{T}_{\mathfrak{Cr}_y}$ is R-cubic m-polar coarser then $\mathfrak{T}_{\mathfrak{Cr}_g}$.

4.1. Sub spaces of CmPFT_r

Definition 4.12. Let $(\Bbbk, \mathfrak{T}_{\mathfrak{Cr}_\Bbbk})$ be a CmPFT_rS. Let $\mathbb{Y} \subseteq \Bbbk$ and $\mathfrak{T}_{\mathfrak{Cr}_\mathbb{Y}}$ be a CmPFT_r on $\mathbb{Y}$ with the following RCmPFOSs:

$$\mathfrak{C}r_\mathbb{Y} = \mathfrak{T}_{\mathfrak{Cr}_\Bbbk} \sqcap_R \breve{\mathbb{Y}}$$

where $\mathfrak{C}r_\Bbbk$ are RCmPFOSs of $\mathfrak{T}_{\mathfrak{Cr}_\Bbbk}$, $\mathfrak{T}_{\mathfrak{Cr}_\mathbb{Y}}$ represents the RCmPFOSs of $\mathfrak{T}_{\mathfrak{Cr}_\mathbb{Y}}$, and $\breve{\mathbb{Y}}$ is an absolute RCmPFS on $\mathbb{Y}$. Then, $\mathfrak{T}_{\mathfrak{Cr}_\mathbb{Y}}$ is the R-cubic m-polar fuzzy subspace of $\mathfrak{T}_{\mathfrak{Cr}_\Bbbk}$ i.e.,

$$\mathfrak{T}_{\mathfrak{Cr}_\mathbb{Y}} = \left\{\mathfrak{C}r_\mathbb{Y} : \mathfrak{C}r_\mathbb{Y} = \mathfrak{C}r_\Bbbk \sqcap_R \breve{\mathbb{Y}},\ \mathfrak{C}r_\Bbbk \in \mathfrak{T}_{\mathfrak{Cr}_\Bbbk}\right\}$$

Example 4.13. Let $\Bbbk = \{\Bbbk_1\}$ be a non-empty set. From Example 4.3,
$\mathfrak{T}_{\mathfrak{Cr}} = \left\{\mathfrak{Cr}, {}^g\mathfrak{Cr}, \overline{\mathfrak{Cr}}, \overline{{}^g\mathfrak{Cr}}, \mathfrak{Cr}_g, \mathfrak{Cr}_{\mathfrak{r}_y}, \mathfrak{Cr}_v, \mathfrak{Cr}_w, \mathfrak{Cr}\right\}$ is a cubic 3-polar fuzzy topology with R-order on $\Bbbk$.
Now, the absolute cubic 3-polar fuzzy set on $\mathbb{Y} = \{\Bbbk_1\} \subseteq \Bbbk$ is

$$\breve{\mathbb{Y}} = \left\{(\Bbbk_1, [1,1], [1,1], [1,1], 1, 1, 1)\right\}$$

Since

$$\breve{\mathbb{Y}} \sqcap_R \mathfrak{Cr} = \overline{\mathfrak{Cr}} = \overline{\acute{\mathfrak{Cr}}},$$

$$\breve{\mathbb{Y}} \sqcap_R {}^g\mathfrak{Cr} = {}^g\mathfrak{Cr} = \breve{\mathbb{Y}},$$

$$\breve{\mathbb{Y}} \sqcap_R \overline{\mathfrak{Cr}} = \overline{\mathfrak{Cr}} = \overline{\acute{\mathfrak{Cr}}},$$

$$\breve{\mathbb{Y}} \sqcap_R \overline{{}^g\mathfrak{Cr}} = {}^g\mathfrak{Cr} = \breve{\mathbb{Y}},$$

$$\breve{\mathbb{Y}} \sqcap_R \mathfrak{Cr}_g = \mathfrak{Cr} = \acute{\mathfrak{Cr}},$$

$$\breve{\mathbb{Y}} \sqcap_R \mathfrak{Cr}_y = \overline{\mathfrak{Cr}} = \overline{\acute{\mathfrak{Cr}}},$$

$$\breve{\mathbb{Y}} \sqcap_R \mathfrak{Cr}_v = {}^g\mathfrak{Cr} = \breve{\mathbb{Y}},$$

$$\breve{\mathbb{Y}} \sqcap_R \mathfrak{Cr}_w = \mathfrak{Cr} = \acute{\mathfrak{Cr}},$$

$$\breve{\mathbb{Y}} \sqcap_R \mathfrak{Cr} = \mathfrak{Cr} = \acute{\mathfrak{Cr}},$$

it follows that

$$\mathfrak{T}_{\mathfrak{Cr}_{\mathbb{Y}}} = \{\overline{\acute{\mathfrak{Cr}}}, \breve{\mathbb{Y}}, \acute{\mathfrak{Cr}}\}$$

is a cubic 3-polar fuzzy relative topology of $\mathfrak{T}_{\mathfrak{Cr}_{\Bbbk}}$.

Let $(\Bbbk, \mathfrak{T}_{\mathfrak{Cr}_{\Bbbk}})$ be a CmPFT_rS. Let $\mathbb{Y} \subseteq \Bbbk$ and $\mathfrak{T}_{\mathfrak{Cr}_{\mathbb{Y}}}$ be a CmPFT_r on $\mathbb{Y}$ with the following RCmPFOSs:

$$\mathfrak{C}r_{\mathbb{Y}} = \mathfrak{T}_{\mathfrak{Cr}_{\Bbbk}} \sqcap_R \mathbb{Y}$$

where $\mathfrak{C}r_{\Bbbk}$ represents the RCmPFOSs of $\mathfrak{T}_{\mathfrak{Cr}_{\Bbbk}}$, $\mathfrak{T}_{\mathfrak{Cr}_{\mathbb{Y}}}$ are RCmPFOSs of $\mathfrak{T}_{\mathfrak{C}\mathfrak{R}_{\mathbb{Y}}}$ and $\mathbb{Y}$ is any subset of an RCmPFS on $\mathbb{Y}$. Then, $\mathfrak{T}_{\mathfrak{Cr}_{\mathbb{Y}}}$ is an R-cubic m-polar fuzzy subspace of $\mathfrak{T}_{\mathfrak{Cr}_{\Bbbk}}$ i.e.,

$$\mathfrak{T}_{\mathfrak{Cr}_{\mathbb{Y}}} = \left\{\mathfrak{C}r_{\mathbb{Y}} : \mathfrak{C}r_{\mathbb{Y}} = \mathfrak{C}r_{\Bbbk} \sqcap_R \mathbb{Y},\ \mathfrak{C}r_{\Bbbk} \in \mathfrak{T}_{\mathfrak{Cr}_{\Bbbk}}\right\}$$

Example 4.14. Let $\Bbbk = \{\Bbbk_1\}$ be a non-empty set. From Example 4.3

$$\mathfrak{T}_{\mathfrak{Cr}} = \left\{\mathfrak{Cr}, {}^{g}\mathfrak{Cr}, \overline{\mathfrak{Cr}}, \overline{{}^{g}\mathfrak{Cr}}, \mathfrak{Cr}_g, \mathfrak{Cr}_{\mathfrak{r}_y}, \mathfrak{Cr}_v, \mathfrak{Cr}_w, \mathfrak{Cr}\right\}$$

is a cubic 3-polar fuzzy topology with R-order on $\Bbbk$.
Now, the absolute cubic 3-polar fuzzy set on $\mathbb{Y} = \{\Bbbk_1\} \subseteq \Bbbk$ is

$$\mathbb{Y} = \left\{(\Bbbk_1, [0.31, 0.52], [0.43, 0.62], [0.54, 0.70], 0.8, 0.7, 0.6)\right\}$$

Since

$$\begin{aligned}
\breve{\mathbb{Y}} \sqcap_R \mathfrak{Cr} &= \left\{(\Bbbk_1, [0,0], [0,0], [0,0], 0.8, 0.7, 0.6)\right\} = \acute{\mathfrak{Cr}},\\
\mathbb{Y} \sqcap_R {}^{g}\mathfrak{Cr} &= \left\{(\Bbbk_1, [0.31, 0.52], [0.43, 0.62], [0.54, 0.70], 1, 1, 1)\right\} = \acute{\mathfrak{Cr}},\\
\mathbb{Y} \sqcap_R \overline{\mathfrak{Cr}} &= \left\{(\Bbbk_1, [0,0], [0,0], [0,0], 1, 1, 1)\right\} = \overline{\acute{\mathfrak{Cr}}},\\
\mathbb{Y} \sqcap_R \overline{{}^{g}\mathfrak{Cr}} &= \left\{(\Bbbk_1, [0.31, 0.52], [0.43, 0.62], [0.54, 0.70], 0.8, 0.7, 0.6)\right\} = \mathbb{Y},\\
\mathbb{Y} \sqcap_R \mathfrak{Cr}_g &= \left\{(\Bbbk_1, [0.2, 0.41], [0.33, 0.50], [0.44, 0.60], 0.9, 0.8, 0.7)\right\} = \acute{\mathfrak{Cr}}_g,\\
\mathbb{Y} \sqcap_R \mathfrak{Cr}_y &= \left\{(\Bbbk_1, [0,0], [0,0], [0,0], 0.9, 0.8, 0.7)\right\} = \acute{\mathfrak{Cr}}_y,\\
\mathbb{Y} \sqcap_R \mathfrak{Cr}_v &= \left\{(\Bbbk_1, [0.31, 0.52], [0.43, 0.62], [0.54, 0.70], 0.9, 0.8, 0.7)\right\} = \acute{\mathfrak{Cr}}_v,\\
\mathbb{Y} \sqcap_R \mathfrak{Cr}_w &= \left\{(\Bbbk_1, [0.2, 0.41], [0.33, 0.50], [0.44, 0.60], 0.8, 0.7, 0.6)\right\} = \acute{\mathfrak{Cr}}_w,\\
\mathbb{Y} \sqcap_R \mathfrak{Cr} &= \left\{(\Bbbk_1, [0.2, 0.41], [0.33, 0.50], [0.44, 0.60], 1, 1, 1)\right\} = \acute{\mathfrak{Cr}},
\end{aligned}$$

it follows that

$$\mathfrak{T}_{\mathfrak{Cr}_{\mathbb{Y}}} = \left\{\overline{\acute{\mathfrak{Cr}}}, \mathbb{Y}, \acute{\mathfrak{Cr}}_g, \acute{\mathfrak{Cr}}_y, \acute{\mathfrak{Cr}}_v, \acute{\mathfrak{Cr}}_w, \acute{\mathfrak{Cr}}, \acute{\mathfrak{Cr}}, \acute{\mathfrak{Cr}}\right\}$$

is a cubic 3-polar fuzzy relative topology of $\mathfrak{T}_{\mathfrak{Cr}_{\Bbbk}}$.

4.2. Interior, closure, frontier and exterior of RCmPFSs

Definition 4.15. Let $(\mathbb{k}, \mathfrak{T}_{\mathfrak{C}}\mathfrak{r})$ be a CmPFT_rS and $\mathfrak{C}r \in \mathfrak{cmp}(\mathbb{k})$ then the interior of $\mathfrak{Cr}$ is denoted as $\mathfrak{Cr}$ and is defined as the union of all open cubic m-polar fuzzy subsets contained in $\mathfrak{Cr}$. It is the greatest open CmPFS contained in $\mathfrak{Cr}$.

Example 4.16. Consider the cubic 3-polar topological space as constructed in Example 4.3. Let $\mathfrak{C}r_6 \in \mathfrak{cmp}(\mathbb{k})$: given as

$\mathfrak{C}r_6 = \Big\{(\mathbb{k}_1, [0.33, 0.50], [0.42, 0.69], [0.52, 0.73], 0.8, 0.7, 0.6),$

$(\mathbb{k}_2, [0.38, 0.49], [0.51, 0.64], [0.63, 0.82], 0.7, 0.6, 0.5)\Big\}.$

So,

$$\mathfrak{C}r_6^0 = \mathfrak{Cr} \sqcup_R \mathfrak{Cr}_g \sqcup_R \mathfrak{Cr}_y \sqcup_R \mathfrak{Cr} = \mathfrak{Cr}_g$$

is open CmPFS.

Theorem 4.17. Let $(\mathbb{k}, \mathfrak{T}_{\mathfrak{C}}\mathfrak{r})$ be a CmPFT_rS and $\mathfrak{Cr} \in \mathfrak{cmp}(\mathbb{k})$. Then, $\mathfrak{Cr}$ is an open CmPFS if, and only if, $(\mathfrak{C}r)^0 = \mathfrak{Cr}$.

Proof. The proof is obvious. □

Definition 4.18. Let $(\mathbb{k}, \mathfrak{T}_{\mathfrak{C}}\mathfrak{r})$ be a CmPFT_rS and $\mathfrak{Cr} \in \mathfrak{cmp}(\mathbb{k})$ then the closure of $\mathfrak{Cr}$ is denoted as $\overline{\mathfrak{C}r}$ and is defined as the intersection of all closed cubic m-polar fuzzy supersets of $\mathfrak{Cr}$. It is the smallest closed cubic m-polar fuzzy superset of $\mathfrak{Cr}$.

Example 4.19. Consider the cubic 3-polar topological space constructed in Example 4.3. Then, the closed CmPFSs are

$(\mathfrak{Cp})^c = {}^g\mathfrak{Cp}$, $({}^g\mathfrak{Cp})^c = \mathfrak{Cp}$, $(\overline{\mathfrak{Cr}})^c = {}^g\overline{\mathfrak{Cr}}$, $(\overline{\mathfrak{Cr}})^c = \overline{\mathfrak{Cr}}$

$$\begin{aligned}
(\mathfrak{Cr}_g)^c &= \Big\{\Big(\mathbb{k}_1, [0.59, 0.8], [0.0.50, 0.67], [0.40, 0.56], 0.1, 0.2, 0.3\Big),\\
&\quad \Big(\mathbb{k}_2, [0.58, 0.76], [0.40, 0.65], [0.35, 0.55], 0.2, 0.3, 0.4\Big)\Big\}\\
(\mathfrak{Cr}_y)^c &= \Big\{\Big(\mathbb{k}_1, [1, 1], [1, 1], [1, 1], 0.1, 0.2, 0.3\Big),\\
&\quad \Big(\mathbb{k}_2, [1, 1], [1, 1], [1, 1], 0.2, 0.3, 0.4\Big)\Big\}\\
(\mathfrak{Cr}_v)^c &= \Big\{\Big(\mathbb{k}_1, [0, 0], [0, 0], [0, 0], 0.1, 0.2, 0.3\Big),\\
&\quad \Big(\mathbb{k}_2, [0, 0], [0, 0], [0, 0], 0.2, 0.3, 0.4\Big)\Big\}\\
(\mathfrak{Cr}_w)^c &= \Big\{\Big(\mathbb{k}_1, [0.59, 0.8], [0.0.50, 0.67], [0.40, 0.56], 1, 1, 1\Big),\\
&\quad \Big(\mathbb{k}_2, [0.58, 0.76], [0.40, 0.65], [0.35, 0.55], 1, 1, 1\Big)\Big\}\\
(\mathfrak{Cr}_w)^c &= \Big\{\Big(\mathbb{k}_1, [0.59, 0.8], [0.50, 0.67], [0.40, 0.56], 0, 0, 0\Big),\\
&\quad \Big(\mathbb{k}_2, [0.58, 0.76], [0.40, 0.65], [0.35, 0.55], 0, 0, 0\Big)\Big\}
\end{aligned}$$

Let $\mathfrak{Cr}_7 \in \mathfrak{cmp}(\Bbbk)$ be given as

$$\mathfrak{C}r_7 = \Big\{\Big(\Bbbk_1, [0.43, 0.76], [0.47, 0.53], [0.29, 0.42], 0.2, 0.3, 0.4\Big), \Big(\Bbbk_2, [0.49, 0.72], [0.38, 0.58], [0.30, 0.49], 0.3, 0.4, 0.5\Big)\Big\}$$

Then,

$$\overline{\mathfrak{Cr}} = (\overline{\mathfrak{Cr}})^c \sqcap_R (\mathfrak{Cr}_g)^c \sqcap_R (\mathfrak{Cr}_y)^c \sqcap_R (\mathfrak{Cr})^c = (\mathfrak{Cr}_g)^c$$

is closed a CmPFS.

Theorem 4.20. Let $(\Bbbk, \mathfrak{T}_{\mathfrak{C}}\mathfrak{r})$ be a CmPFT_rS and $\mathfrak{Cr} \in \mathfrak{cmp}(\Bbbk)$. Then, $\mathfrak{Cr}$ is a closed CmPFS $\Leftrightarrow$ $\overline{\mathfrak{C}r} = \mathfrak{Cr}$.

Proof. The proof is obvious. □

Definition 4.21. Let $\mathfrak{Cr}$ be an R-cubic m-polar fuzzy subset of $(\Bbbk, \mathfrak{T}_{\mathfrak{C}}\mathfrak{r})$; then, its frontier or boundary is denoted by

$$Fr(\mathfrak{C}r) = \overline{\mathfrak{C}r} \sqcap_R \overline{\left(\mathfrak{C}r\right)^c}$$

Definition 4.22. Let $\mathfrak{Cr}$ be an R-cubic m-polar fuzzy subset of $(\Bbbk, \mathfrak{T}_{\mathfrak{C}}\mathfrak{r})$; then, its exterior is denoted by

$$Ext(\mathfrak{C}r) = \left(\overline{\mathfrak{C}r}\right)^c = \left(\mathfrak{C}r^c\right)^0$$

Example 4.23. Consider the cubic 3-polar topological space given in Example 4.3 and $\mathfrak{Cr}$ from Example 4.19. Then
$\mathfrak{C}r_7^0 = (\mathfrak{Cr}_g)^c$, $\overline{\mathfrak{C}r_7} = (\mathfrak{Cr}_g)^c$
$Fr(\mathfrak{C}r_7) = (\mathfrak{Cr}_g)^c$, $Ext(\mathfrak{C}r_7) = (\mathfrak{Cr}_g)^c$

Remark. For a CmPFT_rS, the law of contradiction, $\mathfrak{Cr} \sqcap_p (\mathfrak{Cr})^c = \mathfrak{Cr}$, and the law of excluded middle, $\mathfrak{Cr} \sqcup_R (\mathfrak{Cr})^c = {}^g\mathfrak{Cr}$ do not hold in general. From Example 4.19
$\mathfrak{Cr} \sqcap_R (\mathfrak{Cr})^c \neq \mathfrak{Cr}$
$\mathfrak{Cr} \sqcup_R (\mathfrak{Cr})^c \neq {}^g\mathfrak{Cr}$

Definition 4.24. An RCmPFS is said to be dense in a universal set $\Bbbk$ if

$$\overline{\mathfrak{Cr}} = {}^g\mathfrak{Cr}$$

Definition 4.25. An RCmPF number $\mathfrak{C}_\gamma = ([A_j^-, A_j^+], A_j)_{j=1}^m$ belong RCmPFS if, and only if,
$A_i^-(\lambda) \leq A_j^-(\lambda)$, $A_i^+(\lambda) \leq A_j^+(\lambda)$ and $A_i(\lambda) \geq A_j(\lambda)$ $j = 1, 2, ..., m$ and $l \in \Bbbk$
Let $(\Bbbk, \mathfrak{T}_{\mathfrak{C}}\mathfrak{r})$ be a CmPFT_rS. An RCmPFS $\acute{\mathfrak{Cr}}$ of $\Bbbk$ which contains an RCmPF number $\mathfrak{C}_\gamma \in \Bbbk$ is said to be a neighborhood of $\mathfrak{C}_\gamma$ if, there exists an RCmPFOS $\mathfrak{Cr}$ containing $\mathfrak{C}_\gamma$, such that

$$\mathfrak{C}_\gamma \in \mathfrak{Cr} \subseteq_R \acute{\mathfrak{Cr}}$$

Example 4.26. From Example 4.3, an R-cubic m-polar fuzzy number

$$\mathfrak{C}_{\gamma} = \Big\{\Bbbk_1, [0.20, 0.41], [0.33, 0.50], [0.44, 0.60], 0.9, 0.8, 0.7\Big\}$$

belongs to the RPCmPFOS $\mathfrak{Cr}_g$ which is an R-cubic m-polar fuzzy subset of $\mathfrak{Cr}_v$. From this, we can say that $\mathfrak{Cr}_v$ is a neighborhood of $\mathfrak{C}_{\gamma}$.

4.3. R-Cubic m-polar fuzzy basis

Definition 4.27. Let $(\Bbbk, \mathfrak{T}_{\mathfrak{C}}\mathfrak{r})$ be a CmPFT_rS. Then, $\mathbb{B} \subseteq \mathfrak{T}_{\mathfrak{C}}\mathfrak{r}$ is said to be the R-cubic m-polar fuzzy basis for $\mathfrak{T}_{\mathfrak{C}}\mathfrak{r}$ if, for each $\mathfrak{Cr} \in \mathfrak{T}_{\mathfrak{C}}\mathfrak{r}$, $\exists$ B $\in \mathbb{B}$ such that

$$\mathfrak{Cr} = \sqcup_R \mathrm{B}$$

Example 4.28. From Example 4.3,

$$\mathfrak{T}_{\mathfrak{Cr}} = \Big\{\mathfrak{Cr}, {}^{g}\mathfrak{Cr}, \overline{\mathfrak{Cr}}, \overline{{}^{g}\mathfrak{Cr}}, \mathfrak{Cr}_g, \mathfrak{Cp}_{\mathfrak{r}_y}, \mathfrak{Cr}_v, \mathfrak{Cr}_w, \mathfrak{Cr}\Big\}$$

is an R-cubic 3-polar fuzzy topology on $\Bbbk$. Then,

$$\mathbb{B} = \Big\{{}^{g}\mathfrak{Cr}, \overline{{}^{g}\mathfrak{Cr}}, \mathfrak{Cr}_g, \mathfrak{Cp}_{\mathfrak{r}_y}, \mathfrak{Cr}_v, \mathfrak{Cr}_w, \mathfrak{Cr}\Big\}$$

is the R-cubic 3-polar fuzzy basis for $\mathfrak{T}_{\mathfrak{Cr}}$.

5. Extension of VIKOR method to CmPFSs

As a sample, in this section, we first discuss several types of CKD by providing a brief but comprehensive overview of this fatal disease, including its types and symptoms, and then use the established VIKOR methodology to diagnose those who are affected.

Case Study

Kidneys have very important positions for human beings. They work as channels for your blood, eliminating waste, poisons, and surplus liquids. They also help to manage circulatory strain and synthetic compounds in the blood, keep bones sound and animate red platelet creation. If you have CKD, your kidneys have been damaged for more than a few months. Diseased kidneys also do not channel blood properly, which can prompt an assortment of genuine human concerns. There are five stages to CKD. The phases are determined by the results of an eGFR test and how successfully your kidneys filter waste and excess fluid from your blood. Kidney disease worsens as the stages progress and your kidneys become less effective. The stages of CKD[*,†,‡] are summarized below.

Stage 1 of CKD

An eGFR of 90 or higher indicates that your kidneys are healthy, but you might have other symptoms of kidney damage. Protein in your urine or physical harm to your kidneys could be signs of kidney damage. When the kidneys function at a 90 or higher eGFR, there are usually no symptoms. Here are

[*] https://www.freseniuskidneycare.com/kidney-disease/stages
[†] https://www.healthline.com/health/ckd-stages
[‡] https://www.cdc.gov/kidneydisease/prevention-risk.html

some things one can do to help slow the harm to the kidneys in Stage 1. In the case of diabetes, keep the blood sugar under control and maintain the blood pressure. Consume a nutritious diet. Do not use tobacco or smoke. Engage in physical activity for 30 minutes five days a week and maintain a healthy weight.

Stage 2 of CKD

If you have Stage 2 CKD kidney problem, it is slight, and corresponds to an eGFR between 60 and 89. Almost all of the time, an eGFR between 60 and 89 indicates that your kidneys are healthy and operating correctly. However, if you do have Stage 2 kidney disease, you have some other symptoms of kidney damage even if your eGFR is normal. You may still be symptom-free at this point. Or the symptoms are general, such as fatigue, appetite loss, sleep issues and weakness.

Stage 3 of CKD

If your eGFR is between 30 and 59, you have Stage 3 CKD. An eGFR of 30 to 59 implies that your kidneys have been harmed and are not operating correctly. There are two stages in Stage 3; if your eGFR is between 45 and 59, you are at Stage 3a and if your eGFR is between 30 and 44, you are in Stage 3b. The kidneys are not filtering waste, poisons, or fluids efficiently, and they are starting to pile up. Many persons with kidney disease in Stage 3 do not show any noticeable symptoms. However, if symptoms exist, they may include hand and foot swelling, pain in the back, and more or less urination than usual.

Stage 4 of CKD

You have Stage 4 CKD if your eGFR is between 15 and 29. An eGFR between 15 and 30 indicates that your kidneys are relatively or badly damaged and are not operating normally. Stage 4 kidney disease must be taken seriously because it is the final step before kidney failure. Many people with Stage 4 CKD experience symptoms such as swelling of the hands and feet, back pain and more or less urination than usual. At Stage 4, you will most likely experience medical complications as waste piles up in your kidneys and your body fails to function properly, such as excessively high blood pressure, anemia and a disease of the bones.

Stage 5 of CKD

You have Stage 5 CKD if your eGFR is below 15. An eGFR less than 15 indicates that the kidneys are nearing failure or have failed completely. When your kidneys fail, waste accumulates in your blood, making you very sick. The following are some of the symptoms of kidney failure: itching, cramps in the muscles, throwing up, swelling of the hands and feet, back pain, more or less urination than usual, breathing difficulties and sleeping problems. Once your kidneys fail, you will need dialysis or a kidney transplant to survive.

We show, in this part, how the VIKOR might be applied to CmPFSs. Right away, we will apply VIKOR to the CmPFS and later apply it to deal with an issue from life sciences. We start by expounding the recommended strategy stage by stage as described below. The linguistic terms for weighing choices are given in Table 8.

Algorithm (VIKOR)

Suppose there are $\jmath$ number of DMs, that is, $D_1, D_2, \cdots, D_\jmath$, subject to '$\imath$' number of criteria $\mu_1, \mu_2, \cdots, \mu_\imath$ and 'λ' number of alternatives $\nu_1, \nu_2, \cdots, \nu_\lambda$.

Step 1. In the first step: the DMs have to allocate the preference weights to the criteria. Let ω_{jk} be the weight allocated by j^{th} DM to the k^{th} criteria. We set the weights in the matrix form $\wp = [\omega_{jk}]_{\jmath\times\imath}$ for convenience.

Table 8. Linguistic terms for weighing choices.

Linguistic terms	Fuzzy weights
Stage 0: Healthy kidney ($\mathcal{S}_0$)	0.10
Stage 1: Beginning of kidney damage ($\mathcal{S}_1$)	0.30
Stage 2: Moderate kidney damage ($\mathcal{S}_2$)	0.50
Stage 3: Severe kidney damage ($\mathcal{S}_3$)	0.70
Stage 4: Kidney failure ($\mathcal{S}_4$)	0.90

Step 2. The weights assigned by DMs must be normalized. Suppose that the weights ω_{jk} for the criteria are not normal, so they must be normalized by utilizing the formula $\bar{\omega}_{jk} = \frac{\omega_{jk}}{\sqrt{\sum_i \omega_{jk}^2}}$. Then, the weights are gathered as $\mathbb{W} = (\omega_1, \omega_2, \cdots, \omega_\iota)$, where $\omega_i = \frac{1}{\jmath} \frac{\sum_i \bar{\omega}_{jk}}{\sum_j \bar{\omega}_{jk}}$.
Step 3. Every model of a DM is a PCmPF matrix $D_x = (\zeta_{jk}^x)_{\lambda \times \iota}$, $x = 1, 2, \cdots, \jmath$, where ζ_{jk}^x is the value that is allotted by the DM X to the criteria K corresponding the alternative J.
Step 4. Compute the PCmPFS decision matrix by taking the average. The matrix that is obtained can be named $\mathbb{A} = (\zeta_{jk})_{\lambda \times \iota}$.
Step 5. Construct the PCmPF weighted matrix to be $\mathbb{B} = (\varsigma_{jk})_{\lambda \times \iota}$, where $\varsigma_{jk} = \omega_k \zeta_{jk}$.
Step 6. Positive ideal solutions (PIS) and negative ideal solutions (NIS) with P-order (R-order) for CmPFSs are respectively obtained by using the relations given as
PCmpF-PIS: $\zeta_k^+ = \max_P^j \varsigma_{jk}$ or $\zeta_k^+ = \max_R^j \varsigma_{jk}$
PCmPF-NIS: $\zeta_k^- = \min_P^j \varsigma_{jk}$ or $\zeta_k^- = \min_R^j \varsigma_{jk}$.
Step 7. To find the strategic value of VIKOR, the utility value Si, regret value Ri and compromise value Qi are calculated by using following formula

$$
\begin{aligned}
Si &= \Sigma_{j=1}^m \omega_j \Big(\frac{d(\ddot{\zeta}_j^+, \ddot{\zeta}_j)}{d(\ddot{\zeta}_j^+, \ddot{\zeta}_j^-)} \Big) \\
Ri &= \max_{j=1}^m \omega_j \Big(\frac{d(\ddot{\zeta}_j^+, \ddot{\zeta}_j)}{d(\ddot{\zeta}_j^+, \ddot{\zeta}_j^-)} \Big) \\
Qi &= \chi \Big(\frac{Si - S^-}{S^+ - S^-} \Big) + (1 - \chi) \Big(\frac{Ri - R^-}{R^+ - R^-} \Big)
\end{aligned}
$$

Here, $S^+ = \vee_i Si$, $S^- = \wedge_i Si$, $R^+ = \vee_i Ri$ and $R^- = \wedge_i R$. The parameter χ is coefficient of decision analysis. If, in decision making, the majority selects the compromise solution, then we take $\chi > 0.5$, where $\chi > 0.5$ denotes a veto and, for agreement, $\chi = 0.5$
Note that the distance between two CmPFNs,
$\mathfrak{C}^1 = \langle [A_1^-, A_1^+], [A_2^-, A_2^+], \cdots, [A_m^-, A_m^+], A_1, A_2, \cdots, A_m \rangle = \langle [A_j^-, A_j^+], A_j \rangle_{j=1}^m$, and
$\mathfrak{C}^2 = \langle [B_1^-, B_1^+], [B_2^-, B_2^+], \cdots, [B_m^-, B_m^+], B_1, \nu_2, \cdots, B_m \rangle = \langle [B_j^-, B_j^+], B_j \rangle_{j=1}^m$, is defined as

$$
d(\mathfrak{C}^1, \mathfrak{C}^2) = \left[\sum_{j=1}^m \left| \frac{A_j^- + A_j^+}{2} - \frac{B_j^- + B_j^+}{2} \right|^m + \sum_{j=1}^m |A_j - B_j|^m \right]^{1/m}
$$

Step 8. We rank Si, Ri and Qi by arranging them in ascending order. The alternative ϱ_a considered as a compromise solution if it holds the highest ranking (minimum value) and satisfies the following two conditions at the same time.
C-1 If ϱ_{a_1} and ϱ_{a_1} are the top two alternatives having minimum values in Qi, then
$Q(\varrho_{a_2}) - Q(\varrho_{a_1}) \geq \frac{1}{\imath-1}$
where $\imath$ is the number of criteria.
C-2 The alternative ϱ_{a_1} must also be supreme ranked by at least one of the Si or Ri.
If the above two conditions are not satisfied at a time, then we have multiple compromise solutions. In this case, the conditions are given as follows:

- If *C-1* is satisfied, then ϱ_{a_1} and ϱ_{a_2} are both compromise solutions.
- If *C-1* is not satisfied and

$$Q(\varrho_{a_k}) - Q(\varrho_{a_1}) < \frac{1}{\imath - 1}$$

then, $\varrho_{a_1}, \varrho_{a_2}, ..., \varrho_{a_k}$ should act as multiple compromise solutions.

5.1. Numerical example

Step 1. Suppose $\mathbb{D} = \{D_i : i = 1, \cdots, 4\}$ is the set of medical experts, $\mathbb{P} = \{\varrho_j : j = 1, \cdots, 4\}$ is the set of patients and $\mathbb{C} = \{\xi_k : k = 1, \cdots, 4\}$ for a set of criteria, where ξ_1 =vomitting, ξ_2 = nausea, ξ_3 = loss of appetite and ξ_4 = fatigue and weakness.
Step 2. The matrix $\wp$ of the weighted criteria is

$$\wp = [\omega_{ij}]_{4\times 4}$$

$$\wp = \begin{pmatrix} \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_0 & \mathcal{S}_1 \\ \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_1 & \mathcal{S}_2 \\ \mathcal{S}_1 & \mathcal{S}_0 & \mathcal{S}_2 & \mathcal{S}_3 \\ \mathcal{S}_4 & \mathcal{S}_2 & \mathcal{S}_0 & \mathcal{S}_1 \end{pmatrix}$$

$$\wp = \begin{pmatrix} 0.50 & 0.70 & 0.10 & 0.30 \\ 0.70 & 0.90 & 0.30 & 0.50 \\ 0.30 & 0.10 & 0.50 & 0.70 \\ 0.90 & 0.50 & 0.10 & 0.30 \end{pmatrix}$$

where ω_{ij} represents the weights assigned by the DMs to criteria.
Step 3. The normalized weighted matrix appears to be

$$\wp = \begin{pmatrix} 0.39 & 0.56 & 0.16 & 0.31 \\ 0.54 & 0.72 & 0.50 & 0.52 \\ 0.23 & 0.08 & 0.83 & 0.72 \\ 0.70 & 0.40 & 0.16 & 0.31 \end{pmatrix}$$

and thus the weight vectors comes out to be $\mathbb{W} = \{0.26, 0.24, 0.23, 0.26\}$

Step 4. We consider that the four experts give the following four assessment PCmPFS matrices.

$$D_1 = \begin{pmatrix} [0.11, 0.23],[0.20, 0.31],0.12,0.19 & [0.43, 0.57],[0.63, 0.74],0.45,0.61 & [0.44, 0.53],[0.50, 0.64],0.40,0.59 & [0.19, 0.24],[0.21, 0.36],0.20,0.31 \\ [0.19, 0.26],[0.24, 0.42],0.14,0.20 & [0.21, 0.32],[0.42, 0.53],0.24,0.47 & [0.27, 0.32],[0.31, 0.42],0.23,0.37 & [0.23, 0.30],[0.27, 0.38],0.28,0.36 \\ [0.29, 0.39],[0.41, 0.53],0.37,0.40 & [0.32, 0.46],[0.51, 0.67],0.39,0.56 & [0.31, 0.47],[0.38, 0.52],0.38,0.43 & [0.32, 0.40],[0.29, 0.41],0.39,0.40 \\ [0.21, 0.34],[0.32, 0.49],0.23,0.33 & [0.27, 0.34],[0.46, 0.59],0.28,0.49 & [0.17, 0.24],[0.20, 0.32],0.19,0.29 & [0.42, 0.51],[0.36, 0.58],0.48,0.57 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} [0.21, 0.29],[0.31, 0.36],0.18,0.35 & [0.45, 0.64],[0.67, 0.78],0.50,0.68 & [0.48, 0.54],[0.58, 0.66],0.49,0.61 & [0.20, 0.29],[0.29, 0.38],0.28,0.35 \\ [0.24, 0.30],[0.29, 0.46],0.19,0.36 & [0.30, 0.44],[0.51, 0.62],0.33,0.54 & [0.33, 0.40],[0.39, 0.50],0.36,0.42 & [0.27, 0.32],[0.31, 0.40],0.32,0.40 \\ [0.42, 0.53],[0.48, 0.61],0.41,0.52 & [0.40, 0.54],[0.59, 0.70],0.44,0.61 & [0.39, 0.46],[0.47, 0.59],0.41,0.50 & [0.36, 0.42],[0.38, 0.52],0.42,0.51 \\ [0.30, 0.41],[0.38, 0.52],0.38,0.44 & [0.32, 0.47],[0.52, 0.64],0.38,0.50 & [0.20, 0.29],[0.31, 0.38],0.23,0.32 & [0.47, 0.53],[0.41, 0.59],0.50,0.63 \end{pmatrix}$$

$$D_3 = \begin{pmatrix} [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 \\ [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 \\ [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 \\ [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 & [1.00, 1.00],[1.00, 1.00],1.00,1.00 \end{pmatrix}$$

$$D_4 = \begin{pmatrix} [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 \\ [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 \\ [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 \\ [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 & [0.00, 0.00],[0.00, 0.00],0.00,0.00 \end{pmatrix}$$

The aggregated matrix is

$$\mathbb{A} = \begin{pmatrix} [0.33, 0.63],[0.37, 0.41],0.32,0.38 & [0.47, 0.55],[0.57, 0.63],0.48,0.57 & [0.48, 0.51],[0.52, 0.51],0.47,0.55 & [0.35, 0.38],[0.37, 0.44],0.37,0.42 \\ [0.55, 0.39],[0.38, 0.72],0.33,0.39 & [0.37, 0.44],[0.48, 0.53],0.39,0.50 & [0.40, 0.43],[0.42, 0.48],0.39,0.44 & [0.37, 0.40],[0.39, 0.44],0.40,0.44 \\ [0.42, 0.48],[0.47, 0.53],0.44,0.48 & [0.43, 0.50],[0.52, 0.59],0.45,0.54 & [0.42, 0.48],[0.46, 0.52],0.44,0.48 & [0.42, 0.45],[0.41, 0.48],0.45,0.47 \\ [0.37, 0.43],[0.42, 0.50],0.40,0.44 & [0.39, 0.45],[0.49, 0.55],0.41,0.49 & [0.34, 0.38],[0.37, 0.42],0.35,0.40 & [0.47, 0.51],[0.35, 0.54],0.49,0.55 \end{pmatrix}$$

$\mathbb{A} = [\dot{\zeta}_{jk}]_{4\times4}$

Step 5. The weighted PCmPFS matrix is

$$\mathbb{B} = \begin{pmatrix} [0.09, 0.22],[0.11, 0.12],0.09,0.11 & [0.14, 0.17],[0.18, 0.21],0.14,0.18 & [0.13, 0.15],[0.15, 0.17],0.13,0.16 & [0.10, 0.11],[0.12, 0.14],0.11,0.12 \\ [0.10, 0.12],[0.11, 0.28],0.09,0.12 & [0.10, 0.13],[0.14, 0.16],0.11,0.15 & [0.11, 0.12],[0.11, 0.13],0.10,0.12 & [0.11, 0.12],[0.12, 0.13],0.12,0.13 \\ [0.13, 0.15],[0.15, 0.17],0.13,0.15 & [0.12, 0.15],[0.16, 0.19],0.13,0.17 & [0.11, 0.13],[0.13, 0.15],0.12,0.13 & [0.13, 0.14],[0.12, 0.15],0.14,0.15 \\ [0.11, 0.13],[0.13, 0.16],0.12,0.13 & [0.11, 0.13],[0.14, 0.17],0.11,0.14 & [0.34, 0.10],[0.10, 0.11],0.09,0.11 & [0.15, 0.16],[0.10, 0.18],0.16,0.18 \end{pmatrix}$$

$\mathbb{B} = [\ddot{\zeta}_{jk}]_{4\times4}$

Step 6. Thus, PCmP-PIS and PCmP-NIS, respectively are

$$\begin{aligned} PCmP-PIS &= \{\dot{\zeta}_1^+, \dot{\zeta}_2^+, \dot{\zeta}_3^+, \dot{\zeta}_4^+\} \\ PCmP-PIS &= \Big\{[0.13, 0.22], [0.15, 0.28], 0.13, 0.15\ ,\ [0.14, 0.17], [0.18, 0.21], 0.14, 0.18\ , \\ &= [0.34, 0.15], [0.15, 0.17], 0.13, 0.16\ ,\ [0.15, 0.16], [0.12, 0.18], 0.16, 0.18\Big\} \end{aligned}$$

$$\begin{aligned} PCmP-NIS &= \{\dot{\zeta}_1^-, \dot{\zeta}_2^-, \dot{\zeta}_-^+, \dot{\zeta}_4^-\} \\ PCmP-NIS &= \Big\{[0.09, 0.12], [0.11, 0.12], 0.01, 0.11\ ,\ [0.10, 0.13], [0.14, 0.16], 0.11, 0.14\ , \\ &= [0.11, 0.10], [0.10, 0.11], 0.09, 0.11\ ,\ [0.10, 0.11], [0.10, 0.13], 0.11, 0.12\Big\} \end{aligned}$$

Step 7. By selecting $\chi = 0.5$, we calculated the values of Si, Ri and Qi for each alternative by making use of following formula

$$
\begin{aligned}
Si &= \Sigma_{j=1}^{4}\omega_j\left(\frac{d(\ddot{\zeta}_j^{+},\ddot{\zeta}_j)}{d(\ddot{\zeta}_j^{+},\ddot{\zeta}_j^{-})}\right)\\
Ri &= \max_{j=1}^{4}\omega_j\left(\frac{d(\ddot{\zeta}_j^{+},\ddot{\zeta}_j)}{d(\ddot{\zeta}_j^{+},\ddot{\zeta}_j^{-})}\right)\\
Qi &= \chi\left(\frac{Si-S^{-}}{S^{+}-S^{-}}\right)+1-\chi\left(\frac{Ri-R^{-}}{R^{+}-R^{-}}\right)
\end{aligned}
$$

They are given in Table 9.

Table 9. Values of Si, Ri and Qi for each alternative.

Alternatives	Si	Ri	Qi
ϱ_1	0.6217	0.2487	0.6894
ϱ_1	0.8013	0.2258	0.8271
ϱ_1	0.5164	0.1836	0.000
ϱ_1	0.5350	0.2209	0.3192

Step 8. The ranking of choices are as follows:
By Qi : $\varrho_3 < \varrho_4 < \varrho_1 < \varrho_2$
By Si : $\varrho_3 < \varrho_4 < \varrho_1 < \varrho_2$
By Ri : $\varrho_3 < \varrho_4 < \varrho_2 < \varrho_1$
Since, $Q(\varrho_4) - Q(\varrho_3) \ngeq \frac{1}{3}$
So, by $Q(\varrho_4) - Q(\varrho_3) > \frac{1}{3}$, we infer that both ϱ_4 and ϱ_3 serve as multiple compromise solutions.

Comparative analysis:
The advantages of using a CmPFS are described in Table 10.

Table 10. Advantages of the CmPFS.

Fuzzy models	Advantages and limitations
Cubic set (CS) (Jun *et. al.* [18])	It describes information in terms of a fuzzy interval and a fuzzy number. It can not handel multi-polarity.
m-Polar fuzzy set (mPFS) (Chen et al. [10])	It describes the multi-polarity of objects with *m* grades. It can not deal with fuzzy intervals.
Cubic m-polar fuzzy set (Riaz and Hashmi [19])	A strong hybrid model for the CS and mPFS to address the cubic environment and multi-polarity of objects.

To comparatively analyze the method of the ranking of alternatives and an optimal alternative, we solve the above problem by applying the TOPSIS approach to the same data. The first six steps of TOPSIS is the same as VIKOR. In Step 7, we find the the closeness of each alternative from the

PCmP-PIS and PCmP-NIS. In Step 8, we find the relative closeness of each alternative. In Step 9, we rank the alternatives.

Step 7 and 8. The distance of each alternative from the PCmP-PIS and PCmP-NIS and their relative closeness are given in Table 11.

Table 11. Distance and coefficient of closeness of each patient.

Alternatives	d_i^+	d_i^-	C_i^*
ϱ_1	0.1114	0.0781	0.4123
ϱ_2	0.1274	0.0445	0.2590
ϱ_3	0.0958	0.0788	0.4566
ϱ_4	0.0917	0.0907	0.4973

Step 9. Thus, the ranking of each patient is

$$\varrho_4 > \varrho_3 > \varrho_1 > \varrho_2$$

This ranking shows that Patient ϱ_4 is in a more critical situation. So this is optimal decision for the VIKOR and TOPSIS approaches.

6. Conclusions

When applied in MCDM techniques, a CmPFS is an effective model for coping with uncertain information. A CS is a two-component system that can describe information in terms of a fuzzy interval and a fuzzy number. Alternatively, an mPFS describes multi-polarity with m degrees. To take advantages of a CS and mPFS, we focused on a hybrid model of CmPFS and introduced the idea of a topological structure of CmPFSs and CmPF topology with P-order (R-order). We defined certain concepts of a CmPF topology such as, open sets, closed sets, subspaces and dense sets, as well as the interior, exterior, frontier, neighborhood, and basis with P-order (R-order). A CmPF topology is a robust approach for modeling big data, data analysis, and diagnosis, etc. An extension of the VIKOR method for MCGDM with a CmPF topology was designed and its application to CKD diagnosis is presented. A comparative analysis of the proposed approach and TOPSIS method was also performed to seek the optimal decision.

Conflict of interest

The authors declare that they have no conflict of interest.

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