



Research article

Analysis of the fractional diarrhea model with Mittag-Leffler kernel

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Abstract: In this article, we have introduced the diarrhea disease dynamics in a varying population. For this purpose, a classical model of the viral disease is converted into the fractional-order model by using Atangana-Baleanu fractional-order derivatives in the Caputo sense. The existence and uniqueness of the solutions are investigated by using the contraction mapping principle. Two types of equilibrium points i.e., disease-free and endemic equilibrium are also worked out. The important parameters and the basic reproduction number are also described. Some standard results are established to prove that the disease-free equilibrium state is locally and globally asymptotically stable for the underlying continuous system. It is also shown that the system is locally asymptotically stable at the endemic equilibrium point. The current model is solved by the Mittag-Leffler kernel. The study is closed with constraints on the basic reproduction number R_0 and some concluding remarks.

Keywords: fractal fractional derivative; existence and uniqueness; stability analysis; numerical simulations

Mathematics Subject Classification: 65P99, 92D25

1. Introduction

Diarrhea is a very serious condition of someone in which feces are released frequently from the intestines in a liquid-like form. When this happens more than three times a day, a large amount of water and nutrients are discharged from the body and the patients become dehydrated. Generally, diarrhea is caused by many factors including bacteria, viruses some micro-organisms, malnutrition, allergies to certain foods, medications, and many more. The consequences of diarrhea vary from mild to very severe. The minerals and other necessary salts are drained out from the body tissue [1]. So, resultantly human body feels weak leading to death if untreated [2]. The common symptoms of diarrhea are nausea, abdominal pain, bloating, dehydration, fever, frequent stools, etc.

Diarrhea can be divided into two fundamental stages namely, the acute stage and chronic stage. If there are three or a few more loose stools per day last, no longer than two weeks can be considered an acute stage of diarrhea [3]. The chronic stage is very serious and is recognized by the different episodes of watery stools that last beyond the fourteen days (two weeks) [4]. Such classification is very important not only to study the epidemic dynamics but also necessary from a practical view. The findings of this study lead to a better understanding of the outbreak and the control of diarrhea and any other fatal disease [5]. Diarrhea is more common among children and it continuously weakens them. As a result, more than seventy-six thousand children die yearly on the planet [6]. Another reason for the contemporary research is that the maximum number of diarrhea cases are 1.74 billion reported in a single year [7]. Diarrhea, in developing countries, is even more dangerous because of food and health conditions. Every child is suffered from this disease 3.2 times on average, every year [8]. This disease is more dangerous for children than other viral diseases such as HIV, AIDS, Malaria, or Measles. Approximately, this disease causes more than 500,000 deaths each year around the world [9]. Undoubtedly, this disease is a great threat to human lives than any other viral disease. Africa is the most vulnerable to this disease due to many reasons, for instance, poverty, climate changes, lack of facilities, heavy rains, and lack of water sanitization [4]. It is now a proven fact that the main roots of the diarrhea viruses are grown in contaminated water and waste food items. Diarrhea has two types which are infectious and non-infectious diarrhea. Noninfectious diarrhea does not transmit from person to person so, it is not an epidemic. The noninfectious diarrhea only disturbs the immunity for a short time and after some time the immune system is restored. So, the disease dies out. Luckily, diarrhea is a curable and preventable disease that can be controlled by using the available antibiotics [10]. Liquid supplements such as ORS are suggested to overcome, dehydration. Ardkaew and co-authors investigated that the diarrhea disease incidence is higher in children who are below the age of 5-year in the provinces which are in Northeastern Thailand. Their research is based on the recorded cases of patients from 1999 to 2004. In linear regression models of the infectious disease epidemic, seasons, districts, and years are considered as factors to good fit the long-transformed disease incidence with generalized estimated expressions. It helped to describe the spatial co-relations between the districts [11]. The other regions showed a larger periodic mode from January to March and from the period that lasts from April to June. Their word proposed that the regional study can help the health department and other authorities in designing disease-control strategies. These policies may help the disease eradication from papulation. Adewale and co-authors discussed the diarrhea disease mathematically by considering the vaccination as an important factor [12]. They calculated the basic reproduction number R_0 and find out that the disease becomes endemic for $R_0 > 1$ and it will persist

in the population. The numerical simulations with the considered parameters revealed the graphical results, epidemic analysis in a practical scenario. The stability of the model was done by using the Routh-Hurwitz principle [13].

Diarrhea disease is a very contagious disease. Its classical mathematical model has been solved by different numerical methods. While the fractional-order model of this disease has not been formulated and the proposed method is an efficient numerical method. The existence and uniqueness of the solution are certified by using contraction mapping theory. The stability of the model at both equilibrium points is made possible by the classical Jacobian theory. The role of the basic reproductive number R_0 is investigated to describe the disease dynamics as well as for the stability of the model. Computer-aided numerical graphs demonstrate the important key physical features of the state variables. These features include the convergence towards the true steady-state, which is the main contribution.

This work has a more realistic approach. As the biological phenomenon contains memory effect and the classical epidemic models do not cover this important fact. This deficiency in the integer-order model enticed us to investigate the dynamics of the diarrhea infection in the fractional setup. Several mathematical models with various fractional operators are presented in the literature [14–19].

In the literature numerous fractional different operators are defined which can be classified on the basis of the kernel defined therein. The kernels described in these operators are singular or non-singular. The advantage of the Atangana -Baleanu fractional-order derivative is that it has a nonsingular kernel, which is very helpful in the calculation and description of disease dynamics at different stages of the virus propagation in society.

2. Preliminaries

In this section, we describe below some definitions in fractional calculus.

Definition 2.1. *Gamma function $\Gamma(\nu)$ is defined as $\Gamma(\nu) = \int_0^{\infty} \exp(-t)t^{\nu} dt$ and it converges in the right half of the complex plane $\Re(z) > 0$.*

Definition 2.2. [20] *The Caputo fractional derivative of order ν , ($k - 1 < \nu \leq k$) for $k \in \mathbb{N}$ of the function $g(s)$ is defined by ${}_s \mathcal{D}_s^{\nu} g(s) = \frac{1}{\Gamma(k-\nu)} \int_{s_0}^s (s-\zeta)^{k-\nu-1} g'(\zeta) d\zeta$.*

Definition 2.3. [21] *The Mittag Leffler function which generalizes the exponential function for fractional calculus to defined by $\mathcal{E}_{\mu,\nu}(\mathcal{Z}) = \sum_{k=0}^{\infty} \frac{\mathcal{Z}^k}{\Gamma(\nu k + 1)}$, $\nu \in \mathbb{R}^+$, $\mathcal{Z} \in \mathbb{C}$.*

3. Diarrhea mathematical model and corresponding fractional expression

Here, $S(t)$ expresses the representation of the constituents of the human population and shows that the susceptible humans who shall have the maximum probability to be affected by the diarrhea disease, $I(t)$ represent the infected humans who have convicted with diarrhea, $T(t)$ is treated class of people and the class of recovered humans is denoted by $R(t)$ where t is the time and the total population is denoted by $N(t)$. Such that $N(t) = S(t) + I(t) + T(t) + R(t)$.

Also, we described the parameters used in the model. Λ : expresses the Recruitment rate, β_1 : describes Effective contact rate, β_2 : demonstrates Saturation treatment rate, η : used for Enhancement factor, P : for Proportion of infected individuals joining either the class R or T, γ : for Rate of treated

individuals from infection, σ : for the Rate at which treated individuals move to recovered class, ω : for Rate at which recovered individuals move to susceptible class, μ : for the natural death rate in all compartments and κ : for Educational adjustment

$$\frac{dS}{dt} = \lambda + wR - \mu S - \left(\beta_1 \frac{I}{1 + KI} + \beta_2 \frac{wT}{1 + KT}\right)S, \quad (3.1)$$

$$\frac{dI}{dt} = \left(\beta_1 \frac{I}{1 + KI} + \beta_2 \frac{wT}{1 + KT}\right)S - (\mu + \gamma)I, \quad (3.2)$$

$$\frac{dT}{dt} = P\gamma I - (\mu + \sigma)T, \quad (3.3)$$

$$\frac{dR}{dt} = (1 - P)\gamma I + \sigma T - (\mu + w)R. \quad (3.4)$$

The corresponding Fractional version of the above diarrhea disease is given by

$${}^{AB}D_t^\nu S(t) = \lambda + wR - \mu S - \left(\beta_1 \frac{I}{1 + KI} + \beta_2 \frac{wT}{1 + KT}\right)S \quad \forall t \geq 0, \quad (3.5)$$

$${}^{AB}D_t^\nu I(t) = \left(\beta_1 \frac{I}{1 + KI} + \beta_2 \frac{wT}{1 + KT}\right)S - (\mu + \gamma)I \quad \forall t \geq 0, \quad (3.6)$$

$${}^{AB}D_t^\nu T(t) = P\gamma I - (\mu + \sigma)T \quad \forall t \geq 0, \quad (3.7)$$

$${}^{AB}D_t^\nu R(t) = (1 - P)\gamma I + \sigma T - (\mu + w)R \quad \forall t \geq 0. \quad (3.8)$$

Subjected to the following initial conditions

$$S = S_0 \geq 0, I = I_0 \geq 0, T = T_0 \geq 0, R = R_0.$$

4. Existence and uniqueness of the model fractional differential equations

Our goal is to find the solutions of the system (3.5)–(3.8) and unique existence of the solutions for underlying IVP (3.5)–(3.8).

So, we derived the conditions for existence and uniqueness of the solution for this system (3.5)–(3.8) one by one by fixed point theory, using Banach fixed point theorem, Lipschitz condition [22] for this we see (1) Self Mapping, (2) Contraction Mapping to understand in the simple way we say, subjected to the following initial conditions

$$Q_1(t, S, I, T, R) = \lambda + wR - \mu S - \left(\beta_1 \frac{I}{1 + KI} + \beta_2 \frac{wT}{1 + KT}\right)S \quad \forall t \geq 0,$$

$$Q_2(t, S, I, T, R) = \left(\beta_1 \frac{I}{1 + KI} + \beta_2 \frac{wT}{1 + KT}\right)S - (\mu + \gamma)I \quad \forall t \geq 0,$$

$$Q_3(t, S, I, T, R) = P\gamma I - (\mu + \sigma)T \quad \forall t \geq 0,$$

$$Q_4(t, S, I, T, R) = (1 - P)\gamma I + \sigma T - (\mu + w)R \quad \forall t \geq 0.$$

Applying the AB integral [23], above model is reduce to the following fixed point operators,

$$\mathcal{H}_S(t) = S(0) + \frac{1 - \nu}{AB(\nu)} Q_1(t, S, I, T, R) + \frac{\nu}{AB(\nu)\Gamma(\nu)} \int_0^t (t - \tau)^{\nu-1} Q_1(\tau, S, I, T, R) d\tau \quad (4.1)$$

$$\mathcal{H}_I(t) = I(0) - \frac{1-\nu}{AB(\nu)} Q_2(t, S, I, T, R) + \frac{\nu}{AB(\nu)\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} Q_2(\tau, S, I, T, R) d\tau \quad (4.2)$$

$$\mathcal{H}_T(t) = T(0) - \frac{1-\nu}{AB(\nu)} Q_3(t, S, I, T, R) + \frac{\nu}{AB(\nu)\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} Q_3(\tau, S, I, T, R) d\tau \quad (4.3)$$

$$\mathcal{H}_R(t) = R(0) - \frac{1-\nu}{AB(\nu)} Q_4(t, S, I, T, R) + \frac{\nu}{AB(\nu)\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} Q_4(\tau, S, I, T, R) d\tau \quad (4.4)$$

Here we construct the four closed balls for the above operators such that as given below, Let C be the space of all continuous functions and we consider four closed balls with the radius r and center S_0, I_0, T_0, R_0 respectively, in the space of all continuous functions such that

$$B_r(S_0) = [S, S \in C[0, \rho]; \|S - S_0\| \leq r], \quad \|S\| \leq r + S_0.$$

Similarly,

$$B_r(I_0) = [I, I \in C[0, \rho]; \|I - I_0\| \leq r], \quad \|I\| \leq r + I_0.$$

Also,

$$B_r(T_0) = [T, T \in C[0, \rho]; \|T - T_0\| \leq r], \quad \|T\| \leq (r + T_0)$$

Lastly,

$$B_r(R_0) = [R, R \in C[0, \rho]; \|R - R_0\| \leq r], \quad \|R\| \leq (r + R_0)$$

$$\mathcal{H}_S(t) = S(0) + \frac{1-\nu}{AB(\nu)} Q_1(t, S, I, T, R) + \frac{\nu}{AB(\nu)\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} Q_1(\tau, S, I, T, R) d\tau.$$

Now, for self mapping, applying norm on both sides, we have

$$\|\mathcal{H}_S(t) - S(0)\| \leq \left| \frac{1-\nu}{AB(\nu)} \right| \|Q_1(t, S, I, T, R)\| + \left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right| \int_0^t |t-\tau|^{\nu-1} \|Q_1(\tau, S, I, T, R)\| d\tau.$$

so we have,

$$\|Q_1(t, S, I, T, R)\| \leq |\lambda| + |w|(r + R_0) + [|\mu| + (|\beta_1|(r + I_0) - |\beta_2||w|(r + T_0))] (r + S_0).$$

Now, putting,

$$S_0 = C_1, I_0 = C_2, T_0 = C_3, R_0 = C_4$$

Say,

$$C = \text{Max}(C_1, C_2, C_3, C_4)$$

then

$$\|Q_1(t, S, I, T, R)\| \leq |\lambda| + [|w| + [|\mu| + (|\beta_1 - \beta_2 w|(r + C))]] (r + C),$$

and

$$\|Q_1(t, S, I, T, R)\| \leq |\lambda| + Q_1^*(r + C),$$

where

$$Q_1^* = [|w| + [|\mu| + (\beta_1 - \beta_2 w)(r + C)]]$$

and

$$\lambda > 0.$$

So,

$$\|Q_1(t, S, I, T, R)\| \leq \lambda + Q_1^*(r + C),$$

Now,

$$\|\mathcal{H}_S(t) - S(0)\| \leq \left| \frac{1-\nu}{AB(\nu)} \right| [\lambda + Q_1^*(r + C)] + \left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right| \int_0^t |t - \tau|^{\nu-1} [\lambda + Q_1^*(r + C)] d\tau,$$

$$\|\mathcal{H}_S(t) - S(0)\| \leq \left| \frac{1-\nu}{AB(\nu)} \right| [\lambda + Q_1^*(r + C)] + \left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right| \int_0^t |t - \tau|^{\nu-1} [\lambda + Q_1^*(r + C)] d\tau,$$

$\therefore \|S\| = (r + s_0)$

$$\|\mathcal{H}_S(t) - S(0)\| \leq \left| \frac{1-\nu}{AB(\nu)} \right| [\lambda + Q_1^*(r + C)] + \left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right| [\lambda + Q_1^*(r + C)] \int_0^t |t - \tau|^{\nu-1} d\tau,$$

$$\|\mathcal{H}_S(t) - S(0)\| \leq \left| \frac{1-\nu}{AB(\nu)} \right| [\lambda + Q_1^*(r + C)] + \left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right| [\lambda + Q_1^*(r + C)] \frac{\rho^\nu}{\nu}.$$

\therefore In both the cases $t > \tau$ and $\tau > t$ the integral $\int_0^t |t - \tau|^{\nu-1} d\tau = \frac{\rho^\nu}{\nu}$,

For self Mapping we know

$$\|\mathcal{H}_S(t) - S(0)\| \leq r$$

So,

$$\left| \frac{1-\nu}{AB(\nu)} \right| [\lambda + Q_1^{**}(r + C)] + \left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right| [\lambda + Q_1^*(r + C)] \frac{\rho^\nu}{\nu} \leq r,$$

$$\left| \frac{1-\nu}{AB(\nu)} \right| + \left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right| \frac{\rho^\nu}{\nu} \leq \frac{r}{[\lambda + Q_1^*(r + C)]},$$

$$\left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right| \frac{\rho^\nu}{\nu} \leq \frac{r}{[\lambda + Q_1^{**}(r + C)]} - \left| \frac{1-\nu}{AB(\nu)} \right|,$$

$$\rho \leq \left[\frac{\nu}{\left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right|} \left(\frac{r}{[\lambda + Q_1^*(r + C)]} - \left| \frac{1-\nu}{AB(\nu)} \right| \right) \right]^{\frac{1}{\nu}},$$

$$\rho_S \leq \left[\frac{\nu}{\left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right|} \left(\frac{r}{[\lambda + Q_1^*(r + C)]} - \left| \frac{1-\nu}{AB(\nu)} \right| \right) \right]^{\frac{1}{\nu}} \quad \left(\frac{r}{[\lambda + Q_1^*(r + C)]} > \left| \frac{1-\nu}{AB(\nu)} \right| \right). \quad (4.5)$$

Which is the condition for self mapping.

Now, for contraction mapping, we proceed as follows;

$$\mathcal{H}_S(t) = S(0) + \frac{1-\nu}{AB(\nu)} Q_1(t, S, I, T, R) + \frac{\nu}{AB(\nu)\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} Q_1(\tau, S, I, T, R) d\tau.$$

For contraction Mapping we have the two images S_1 and S_2 then we have two pre-images \mathcal{H}_{S_1} and \mathcal{H}_{S_2} , such that

$$\mathcal{H}_{S_1}(t) = S(0) + \frac{1-\nu}{AB(\nu)} Q_1(t, S_1, I, T, R) + \frac{\nu}{AB(\nu)\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} Q_1(\tau, S_1, I, T, R) d\tau, \quad (4.6)$$

$$\mathcal{H}_{S_2}(t) = S(0) + \frac{1-\nu}{AB(\nu)} Q_1(t, S_2, I, T, R) + \frac{\nu}{AB(\nu)\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} Q_1(\tau, S_2, I, T, R) d\tau, \quad (4.7)$$

$$\rho_S < \left[\frac{\nu}{\left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right|} \left(\frac{1}{Q_1^{**}} - \left| \frac{1-\nu}{AB(\nu)} \right| \right) \right]^{\frac{1}{\nu}}, \quad \left(\frac{1}{Q_1^{**}} > \left| \frac{1-\nu}{AB(\nu)} \right| \right). \quad (4.8)$$

Which is the condition for contraction mapping, where $[|\mu| + |\beta_1| + |\beta_2||w|](r+C) = Q_1^{**}$. Similarly for the

$$\rho_I \leq \left[\frac{\nu}{\left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right|} \left(\frac{r}{Q_2^*} - \left| \frac{1-\nu}{AB(\nu)} \right| \right) \right]^{\frac{1}{\nu}}, \quad \left(\frac{r}{Q_2^*} > \left| \frac{1-\nu}{AB(\nu)} \right| \right), \quad (4.9)$$

where $\|Q_2(\tau, S, I, T, R)\| \leq Q_2^*$, and which is the condition for self mapping. Similarly for I

$$\rho_I < \left[\frac{\nu}{\left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right| |Q_2'(\tau^*)|} \left(1 - \left| \frac{1-\nu}{AB(\nu)} \right| |Q_2'(t^*)| \right) \right]^{\frac{1}{\nu}}, \quad \left| \frac{1-\nu}{AB(\nu)} \right| |Q_2'(t^*)| < 1. \quad (4.10)$$

Which is the condition for contraction mapping.

(4.3) \implies the conditions for contraction and self mapping as,

$$\rho_T \leq \left[\frac{\nu}{\left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right|} \left(\frac{r}{Q_3^*} - \left| \frac{1-\nu}{AB(\nu)} \right| \right) \right]^{\frac{1}{\nu}}, \quad \frac{r}{Q_3^*} > \left| \frac{1-\nu}{AB(\nu)} \right|, \quad (4.11)$$

is the condition for self mapping

$$\rho_T < \left[\frac{\nu}{\left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right|} \left(\frac{1}{|\mu + \sigma|} - \left| \frac{1-\nu}{AB(\nu)} \right| \right) \right]^{\frac{1}{\nu}}, \quad \frac{1}{|\mu + \sigma|} > \left| \frac{1-\nu}{AB(\nu)} \right|. \quad (4.12)$$

Which is the condition for contraction mapping.

$$\rho_R \leq \left[\frac{\nu}{\left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right|} \left(\frac{r}{Q_4^*} - \left| \frac{1-\nu}{AB(\nu)} \right| \right) \right]^{\frac{1}{\nu}}, \quad \frac{r}{Q_4^*} > \left| \frac{1-\nu}{AB(\nu)} \right|, \quad (4.13)$$

is the condition for self mapping.

$$\rho_R < \left[\frac{\nu}{\left| \frac{\nu}{AB(\nu)\Gamma(\nu)} \right|} \left(\frac{1}{|(\mu + w)|} - \left| \frac{1 - \nu}{AB(\nu)} \right| \right) \right]^{\frac{1}{\nu}}, \frac{1}{|(\mu + w)|} > \left| \frac{1 - \nu}{AB(\nu)} \right|. \quad (4.14)$$

Which is the condition for contraction mapping.

5. Analysis on stability

In this section, local stability for these fractional equations is investigated for the disease free point of equilibrium, and point of endemic equilibrium.

Definition 5.1. [24, 25] A point t^* is called an equilibrium point of the system

$${}_s D_s^\nu = g(s, t(s)), t(s_0) > 0 \quad g(s, t^*(s)) = 0$$

5.1. Equilibrium point

The system (3.5)–(3.8) demonstrates two equilibrium points, namely the disease free and endemic equilibrium points. We define the diarrhea free equilibrium point as

$$E_0 = (S^0, I^0, T^0, R^0) = \left(\frac{\lambda}{\mu}, 0, 0, 0 \right),$$

and diarrhea existence equilibrium point as

$$E_1 = (S^*, I^*, T^*, R^*),$$

where,

$$S^* = C + DI^*, T^* = AI^*, R^* = BI^*,$$

by defining,

$$A = \frac{P\gamma}{\mu + \sigma}, B = \frac{\gamma\mu - \mu P\gamma + \sigma\gamma}{(\mu + w)(\mu + \sigma)},$$

$$C = \frac{\lambda}{\mu}, D = \frac{w(\gamma\mu - \mu P + \sigma\gamma)}{\mu(\mu + w)(\mu + \sigma)} - \frac{\mu + \gamma}{\mu},$$

$$I^* = \frac{-(m_5 - m_2) + \sqrt{(m_5 - m_2)^2 - 4(m_4 - m_1)(m_6 - m_3)}}{2(m_4 - m_1)}.$$

The m_i 's $i = 1, 2, 3, 4, 5, 6$ are defined as,

$$m_1 = (KA\beta_1 + \beta_2 WAK)D,$$

$$m_2 = (\beta_1 + \beta_2 WA)D,$$

$$m_3 = C(\beta_1 + \beta_2 WA) + C(KA\beta_1 + \beta_2 WAK),$$

$$m_4 = K^2 A(\mu + \gamma),$$

$$m_5 = (\mu + \gamma)(K + KA),$$

$$m_6 = (\mu + \gamma).$$

Basic Reproduction number is presented as,

$$R_0 = \frac{\lambda(\mu\beta_1 + \sigma\beta_1 + P\gamma W\beta_2)}{\mu(\mu + \gamma)(\mu + \sigma)}.$$

Lemma 5.1. Consider the system with fractional order as

$${}^c D_t^\nu v(t) = g(v), \quad 0 < \nu < 1.$$

If, all the eigenvalues λ of the Jacobian matrix, $\mathcal{J}_{v^*} = \frac{\partial g}{\partial v|_{v^*}}$ fulfill $|\arg \lambda| > \nu \frac{\pi}{2}$, then v^* is locally asymptotically stable.

Theorem 5.1. The underlying system (3.5)–(3.8) at disease-free equilibrium is locally asymptotically stable if $R_0 < 1$.

$$\text{Proof. } \mathcal{J} = \begin{pmatrix} -(\mu^* + \Phi) & -\zeta_1 \frac{S(1+KI) - ISK}{(1+KI)^2} & -\zeta_2 \frac{\eta S(1+KT) - \eta TSK}{(1+KT)^2} & \omega \\ \Phi & -\zeta_2 \frac{S(1+KI) - ISK}{(1+KI)^2} - (\mu^* + \gamma^*) & -\zeta_2 \frac{\eta S(1+KT) - \eta TSK}{(1+KT)^2} & 0 \\ 0 & P\gamma^* & (-\mu^* - \sigma^*) & 0 \\ 0 & (1-P)\gamma^* & \sigma^* & (-\omega - \mu^*) \end{pmatrix}$$

where $\varphi = \left(-\zeta_1 \frac{1}{(1+KI)}\right) + -\zeta_1 \frac{\eta T}{(1+KI)}$.

Evaluating the Jacobian at E_0 .

Hence, we obtain the following matrix,

$$\mathcal{J}_{E_0} = \begin{pmatrix} -\mu^* & -\zeta_1 \frac{\lambda^*}{\mu^*} & -\zeta_2 \frac{\eta \lambda^*}{\mu^*} & \omega \\ \Phi & \zeta_1 \frac{\lambda^*}{\mu^*} - (\mu^* + \gamma^*) & \zeta_2 \frac{\eta \lambda^*}{\mu^*} & 0 \\ 0 & P\gamma^* & -(\mu^* + \sigma^*) & 0 \\ 0 & \gamma^*(1-P) & \sigma^* & (-\omega - \mu^*) \end{pmatrix}.$$

Now, we evaluate eigenvalues, by setting $|\mathcal{J}_{E_0} - \lambda^* I| = 0$.

$$|\mathcal{J}_{E_0} - \lambda^* I| = \begin{vmatrix} -\mu^* - \lambda^* & -\zeta_1 \frac{\lambda^*}{\mu^*} & -\zeta_2 \frac{\eta \lambda^*}{\mu^*} & \omega \\ \Phi & \zeta_1 \frac{\lambda^*}{\mu^*} - (\mu^* + \gamma^*) - \lambda^* & \zeta_2 \frac{\eta \lambda^*}{\mu^*} & 0 \\ 0 & P\gamma^* & -(\mu^* + \sigma^*) - \lambda^* & 0 \\ 0 & (1-P)\gamma^* & \sigma^* & -(\mu^* + \omega) - \lambda^* \end{vmatrix} = 0$$

Then, we have $\lambda^* I = -\mu^* < 0$ and $\lambda_2^* = -(\mu^* + \omega) < 0$. The remaining values can be evaluated as

$$\mathcal{H} = \begin{vmatrix} \zeta_1 \frac{\lambda^*}{\mu^*} - (\mu^* + \gamma^*) - \lambda^* & \zeta_2 \frac{\eta \lambda^*}{\mu^*} \\ P\gamma^* & -(\mu^* + \sigma^*) - \lambda^* \end{vmatrix} = 0. \text{ So, we get the following characteristic equation,}$$

$$\lambda_2^* + \left(2\mu^* + \gamma^* + \sigma^* - \zeta_1 \frac{\lambda^*}{\mu^*}\right) \lambda^* + \mu_2^* + \left(\sigma^* + \gamma^* - \zeta_1 \frac{\lambda^*}{\mu^*}\right) \mu^* + \left(\gamma^* - \zeta_1 \frac{\lambda^*}{\mu^*}\right) - \zeta_2 \frac{P\gamma^* \eta \lambda^*}{\mu^*} = 0.$$

Simply, the above equation can be put in the form as given below,

$$\mathcal{P}(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0,$$

where $a_1 = (\mu^* + \gamma^*) + (\mu^* + \sigma^*) - \zeta_1 \frac{\lambda^*}{\mu^*}$. And $a_2 = (\mu^* + \sigma^*)(\mu^* + \gamma^*)[1 - R_0]$.

If $R_0 < 1$, then $a_2 > 0$ and if $R_0 < 1$ and $\zeta_1 \frac{\lambda^*}{\mu^*(\mu^*+1)} < 1$ implies that $a_1 > 0$. hence, the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$. \square

Theorem 5.2. *The underlying system (3.5)–(3.8) at disease free equilibrium is globally asymptotically stable if $R_0 < 1$.*

Proof. For the global stability, we use the method of Castillo-Chavez, as followings;

$$a_1 : \frac{dX}{dt} = F^*(X, Z), a_2 : \frac{dZ}{dt} = G^*(X, Z), G^*(X, 0) = 0. \quad (5.1)$$

Here, $X = (S, R)$, $Z = (I, T)$ with $X \in R_2^+$ representing the uninfected individuals and $Z \in R_2^+$ representing infected individuals including the latent and infectious individuals. Let us denote the disease-free equilibrium point by

$$E_0 = (N_0, 0), N_0 = \left(\frac{\lambda}{\mu^*}, 0 \right). \quad (5.2)$$

A point $(E_0) = (N_0, 0)$ is said to be globally asymptotically stable equilibrium point for the given equations if the following axioms are hold: $b_1 : E_0$ is globally asymptotically stable for $\frac{dX}{dt} = F^*(X, 0)$, $b_2 : G^*(X, Z) \geq 0$, $(X, Z) \in \pi$, here $G^*(X, Z) = AZ - G^*(X, Z)$, $A = D_Z G^*(N_0, 0)$ which is called Metzler matrix. So we will write this matrix A as $F^* - V$ and π which is given in the above equation.

So, we have: $\frac{dX}{dt} = F^*(X, Z) = \begin{bmatrix} \lambda + \omega R - (\varphi + \mu^*)S \\ (1 - P)\gamma^*I + \sigma T - (\mu^* + \omega)R \end{bmatrix}$, where, $\varphi = \left(\zeta_1 \frac{1}{(1+KT)} + \zeta_2 \frac{\eta T}{(1+KT)} \right)$,

$F^*(X, Z) = \begin{bmatrix} \lambda - \mu^*S \\ 0 \end{bmatrix}$, $\frac{dZ}{dt} = G^*(X, Z) = \begin{bmatrix} \Phi S - (\mu^* + \gamma^*)I \\ P\gamma^*I - (\mu^* + \sigma)T \end{bmatrix}$, $G^*(X, 0) = 0$. Hence, $N_0 = \left(\frac{\lambda}{\mu^*}, 0 \right)$

is known as globally asymptotically stable equilibrium point of $\frac{dX}{dt} = F^*(X, 0)$, which is, $b_1 : \frac{dX}{dt} =$

$$F^*(X, 0) = \begin{bmatrix} \lambda - \mu^*S \\ 0 \end{bmatrix}.$$

By solving this for the condition b_2 , we attain the following form, $A = F^* - V = \begin{bmatrix} \zeta_1 \frac{\lambda}{\mu^*} - (\mu^* + \gamma^*) - \mu^*S & \zeta_2 \frac{\eta \lambda}{\mu^*} \\ P\gamma^* & -(\mu^* + \sigma) \end{bmatrix}$, and AZ is given by: $AZ =$

$\begin{bmatrix} \zeta_1 \frac{\lambda}{\mu^*} - (\mu^* + \gamma^*) & \zeta_2 \frac{\eta \lambda}{\mu^*} \\ P\gamma^* & -(\mu^* + \sigma) \end{bmatrix} \begin{bmatrix} I \\ T \end{bmatrix} = \begin{bmatrix} \zeta_1 \frac{\lambda}{\mu^*} I - (\mu^* + \gamma^*)I + \zeta_2 \frac{\eta \lambda T}{\mu^*} \\ P\gamma^* I - (\mu^* + \sigma)T \end{bmatrix}$, So, $G^*(X, Z) =$

$$AZ - G^*(X, Z) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} G^*(X, Z)_1 \\ G^*(X, Z)_2 \end{bmatrix}.$$

Given $G^*(X, Z)_1 = 0$, also $G^*(X, Z)_2 = 0$, we get $G^*(X, Z) \geq 0$ for $(X, Z) \in \pi$. So, the disease free equilibrium point E_0 is said to be globally asymptotically stable point if, $R_0 < 1$. \square

Theorem 5.3. *The fractional epidemic system (3.5)–(3.8) is locally-asymptotically stable at E^1 if $R_0 > 1$.*

Proof. Now, by solving this Jacobian matrix at E^1 , we get $J_{E^1} = \begin{bmatrix} -(\nu + A_1) & -A_2 & -A_3 & -\omega \\ A_1 & A_2 - A_4 & A_3 & 0 \\ 0 & P\gamma & -A_5 & 0 \\ 0 & (1 - P)\gamma & \theta & -A_6 \end{bmatrix}$,

where $A_1 = \left(\beta_1 \frac{I^*}{1+KI^*} + \beta_2 \frac{\eta T^*}{1+KT^*}\right)$, $A_2 = \beta_1 \frac{S^*(1+KI^*) - IS^*K}{(1+KI^*)^2}$, $A_3 = \beta_2 \left(\frac{S^*(1+KT^*) - \eta T^*S^*K}{(1+KT^*)^2}\right)$, $A_4 = (\nu + \phi)$, $A_5 = (\nu + \theta)$, $A_6 = (\nu + \omega)$.

The eigenvalues of the Jacobian matrix above are, $|J_{E^1} - \lambda I| = \begin{vmatrix} -(\nu + A_1) - \lambda & -A_2 & -A_3 & -\omega \\ A_1 & A_2 - A_4 - \lambda & A_3 & 0 \\ 0 & P\gamma & -A_5 - \lambda & 0 \\ 0 & (1 - P)\gamma & \theta & -A_6 - \lambda \end{vmatrix} = 0$.

This will have the following type of the characteristic equation $-(\nu + A_1 + \lambda)[-(A_6 + \lambda)](\lambda^2 + a_1\lambda + a_2) = 0$. where

$$\begin{aligned} a_1 = & -k\nu(\phi + \nu)(\nu + \theta)(P\phi + \nu + \theta)(\nu + \theta) - (\nu + \theta)(\nu(\phi + \nu)(\nu + \theta)) \\ & + (\nu(\nu + \theta) + \phi(2\nu - P\nu + 2\theta))\omega - kP\phi\Lambda(\nu + \omega)\beta_1 \\ & + P\phi\eta(-\eta\nu(\nu + \theta) + \phi((-2 + P)\nu \\ & - 2\theta)\omega + (k\Lambda - \nu)(\nu + \theta)(\nu + \omega)\beta_2 \end{aligned}$$

and $a_2 = -(\nu + \theta)(\nu + \omega)[1 - R_0]$.

We can see that $\lambda_1 = -U$, $\lambda_2 = -(\nu + A_1)$, $(a_1, a_2) > 0$. Then, we have all the results of this characteristic equation also we have -ve real parts. Therefore, the endemic equilibrium point of this Equation is known locally asymptotically stable if $R_0 > 1$. \square

6. Analysis of the model with Mittag-Leffler kernel

Here we let the some discussed problems with the derivative using Atangana and Baleanu derivative as:

$${}^{\mathcal{AB}}D_{t^*}^\nu S(t^*) = \lambda + wR - \mu S - \left(\beta_1 \frac{I}{1+KI} + \beta_2 \frac{wT}{1+KT}\right)S, \quad \forall t^* \geq 0, \quad (6.1)$$

$${}^{\mathcal{AB}}D_{t^*}^\nu I(t^*) = \left(\beta_1 \frac{I}{1+KI} + \beta_2 \frac{wT}{1+KT}\right)S - (\mu + \gamma)I, \quad \forall t^* \geq 0, \quad (6.2)$$

$${}^{\mathcal{AB}}D_{t^*}^\nu T(t^*) = P\gamma I - (\mu + \sigma)T, \quad \forall t^* \geq 0, \quad (6.3)$$

$${}^{\mathcal{AB}}D_{t^*}^\nu R(t^*) = (1 - P)\gamma I + \sigma T - (\mu + w)R, \quad \forall t^* \geq 0. \quad (6.4)$$

For simplicity, we define

$$Q_1(t^*, S, I, T, R) = \lambda + wR - \mu S - \left(\beta_1 \frac{I}{1+KI} + \beta_2 \frac{wT}{1+KT}\right)S, \quad \forall t^* \geq 0,$$

$$Q_2(t^*, S, I, T, R) = \left(\beta_1 \frac{I}{1+KI} + \beta_2 \frac{wT}{1+KT}\right)S - (\mu + \gamma)I, \quad \forall t^* \geq 0,$$

$$Q_3(t^*, S, I, T, R) = P\gamma I - (\mu + \sigma)T, \quad \forall t^* \geq 0,$$

$$Q_4(t^*, S, I, T, R) = (1 - P)\gamma I + \sigma T - (\mu + w)R, \quad \forall t^* \geq 0.$$

Then, we get

$$\frac{\mathcal{AB}(\nu)}{1 - \nu} \frac{d}{dt} \int_0^{t^*} S(\xi) E_\alpha \left(\frac{-\nu}{1 - \nu} (t^* - \xi)^\nu \right) d\xi = Q_1(t^*, S, I, T, R),$$

$$\frac{\mathcal{AB}(\nu)}{1 - \nu} \frac{d}{dt} \int_0^{t^*} I(\xi) E_\alpha \left(\frac{-\nu}{1 - \nu} (t^* - \xi)^\nu \right) d\xi = Q_2(t^*, S, I, T, R),$$

$$\frac{\mathcal{AB}(\nu)}{1 - \nu} \frac{d}{dt} \int_0^{t^*} T(\xi) E_\alpha \left(\frac{-\nu}{1 - \nu} (t^* - \xi)^\nu \right) d\xi = Q_3(t^*, S, I, T, R),$$

$$\frac{\mathcal{AB}(\nu)}{1 - \nu} \frac{d}{dt} \int_0^{t^*} R(\xi) E_\alpha \left(\frac{-\nu}{1 - \nu} (t^* - \xi)^\nu \right) d\xi = Q_4(t^*, S, I, T, R),$$

taking \mathcal{AB} derivative,

$$S(t^* - S(0)) = \frac{1 - \nu}{\mathcal{AB}(\nu)} Q_1(t^*, S, I, R) + \frac{\nu}{\mathcal{AB}(\nu)\Gamma(\nu)} \int_0^{t^*} (t^* - \xi)^{\nu-1} Q_1(\xi, S, I, T, R) d\xi$$

$$I(t^*) - I(0) = \frac{1 - \nu}{\mathcal{AB}(\nu)} Q_2(t^*, S, I, R) + \frac{\nu}{\mathcal{AB}(\nu)\Gamma(\nu)} \int_0^{t^*} (t^* - \xi)^{\nu-1} Q_2(\xi, S, I, T, R) d\xi$$

$$T(t^*) - T(0) = \frac{1 - \nu}{\mathcal{AB}(\nu)} Q_3(t^*, S, I, R) + \frac{\nu}{\mathcal{AB}(\nu)\Gamma(\nu)} \int_0^{t^*} (t^* - \xi)^{\nu-1} Q_3(\xi, S, I, T, R) d\xi$$

$$R(t^*) - R(0) = \frac{1 - \nu}{\mathcal{AB}(\nu)} Q_4(t^*, S, I, R) + \frac{\nu}{\mathcal{AB}(\nu)\Gamma(\nu)} \int_0^{t^*} (t^* - \xi)^{\nu-1} Q_4(\xi, S, I, T, R) d\xi$$

Now discretized the equations on t_{p+1}^* such that:

$$S^{p+1} = S^0 + \frac{1 - \nu}{\mathcal{AB}(\nu)} Q_1(t_{p+1}^*, S^p, I^p, T^p, R^p)$$

$$+ \frac{\nu}{\mathcal{AB}(\nu)\Gamma(\nu)} \int_0^{t_{p+1}^*} (t_{p+1}^* - \xi)^{\nu-1} Q_1(\xi, S, I, T, R) d\xi$$

$$I^{p+1} = I^0 + \frac{1 - \nu}{\mathcal{AB}(\nu)} Q_2(t_{p+1}^*, S, I^p, T^p, R^p)$$

$$+ \frac{\nu}{\mathcal{AB}(\nu)\Gamma(\nu)} \int_0^{t_{p+1}^*} (t_{p+1}^* - \xi)^{\nu-1} Q_2(\xi, S, I, T, R) d\xi$$

$$T^{p+1} = T^0 + \frac{1 - \nu}{\mathcal{AB}(\nu)} Q_3(t_{p+1}^*, S^p, I^p, T^p, R^p)$$

$$+ \frac{\nu}{\mathcal{AB}(\nu)\Gamma(\nu)} \int_0^{t_{p+1}^*} (t_{p+1}^* - \xi)^{\nu-1} Q_3(\xi, S, I, T, R) d\xi$$

$$R^{p+1} = R^0 + \frac{1 - \nu}{\mathcal{AB}(\nu)} Q_4(t_{p+1}^*, S^p, I^p, T^p, R^p)$$

$$+ \frac{\nu}{\mathcal{AB}(\nu)\Gamma(\nu)} \int_0^{t_{p+1}^*} (t_{p+1}^* - \xi)^{\nu-1} Q_4(\xi, S, I, T, R) d\xi$$

we get

$$\begin{aligned}
 S^{p+1} &= S^0 + \frac{1-\nu}{\mathcal{AB}(\nu)} Q_1(t_{p+1}^*, S^p, I^p, T^p, R^p) \\
 &+ \frac{\nu}{\mathcal{AB}(\nu)} \sum_{k=0}^p \left[\frac{h^\nu Q_1(t_k^*, S^p, I^p, R^p)}{\Gamma(\nu+2)} ((p+1-k)^\nu (p-k+2+\nu) \right. \\
 &\quad \left. -(p-k)^\nu (p-k+2+2\nu)) \right] \\
 &- \frac{\nu}{\mathcal{AB}(\nu)} \sum_{k=0}^p \left[\frac{h^\nu Q_1(t_{k-1}^*, S^{p-1}, I^{p-1}, T^{p-1}, R^{p-1})}{\Gamma(\nu+2)} ((p+1-k)^{\nu+1} \right. \\
 &\quad \left. -(p-k)^\nu (p-k+1+\nu)) \right]
 \end{aligned}$$

$$\begin{aligned}
 I^{p+1} &= I^0 + \frac{1-\nu}{\mathcal{AB}(\nu)} Q_2(t_{p+1}^*, S^p, I^p, T^p, R^p) \\
 &+ \frac{\nu}{\mathcal{AB}(\nu)} \sum_{k=0}^p \left[\frac{h^\nu Q_2(t_k^*, S^p, I^p, R^p)}{\Gamma(\nu+2)} ((p+1-k)^\nu (p-k+2+\nu) \right. \\
 &\quad \left. -(p-k)^\nu (p-k+2+2\nu)) \right] \\
 &- \frac{\nu}{\mathcal{AB}(\nu)} \sum_{k=0}^p \left[\frac{h^\nu Q_2(t_{k-1}^*, S^{p-1}, I^{p-1}, T^{p-1}, R^{p-1})}{\Gamma(\nu+2)} ((p+1-k)^{\nu+1} \right. \\
 &\quad \left. -(p-k)^\nu (p-k+1+\nu)) \right]
 \end{aligned}$$

$$\begin{aligned}
 T^{p+1} &= T^0 + \frac{1-\nu}{\mathcal{AB}(\nu)} Q_3(t_{p+1}^*, S^p, I^p, T^p, R^p) \\
 &+ \frac{\nu}{\mathcal{AB}(\nu)} \sum_{k=0}^p \left[\frac{h^\nu Q_3(t_k^*, S^p, I^p, R^p)}{\Gamma(\nu+2)} ((p+1-k)^\nu (p-k+2+\nu) \right. \\
 &\quad \left. -(p-k)^\nu (p-k+2+2\nu)) \right] \\
 &- \frac{\nu}{\mathcal{AB}(\nu)} \sum_{k=0}^p \left[\frac{h^\nu Q_3(t_{k-1}^*, S^{p-1}, I^{p-1}, T^{p-1}, R^{p-1})}{\Gamma(\nu+2)} ((p+1-k)^{\nu+1} \right. \\
 &\quad \left. -(p-k)^\nu (p-k+1+\nu)) \right]
 \end{aligned}$$

$$\begin{aligned}
 R^{p+1} &= R^0 + \frac{1-\nu}{\mathcal{AB}(\nu)} Q_4(t_{p+1}^*, S^p, I^p, T^p, R^p) \\
 &+ \frac{\nu}{\mathcal{AB}(\nu)} \sum_{k=0}^p \left[\frac{h^\nu Q_4(t_k^*, S^p, I^p, R^p)}{\Gamma(\nu+2)} ((p+1-k)^\nu (p-k+2+\nu) \right. \\
 &\quad \left. -(p-k)^\nu (p-k+2+2\nu)) \right] \\
 &- \frac{\nu}{\mathcal{AB}(\nu)} \sum_{k=0}^p \left[\frac{h^\nu Q_4(t_{k-1}^*, S^{p-1}, I^{p-1}, T^{p-1}, R^{p-1})}{\Gamma(\nu+2)} ((p+1-k)^{\nu+1} \right. \\
 &\quad \left. -(p-k)^\nu (p-k+1+\nu)) \right]
 \end{aligned}$$

7. Numerical and quantitative representations

In this portion, we will elaborate the numerical graphs of the state variables in the underlying model, with the help of proposed method. The simulated graphs are plotted against the disease free and endemic equilibrium points. The set of parametric values for both the fixed points are mentioned as, For disease free: $\beta_1 = 1.00093$, $\beta_2 = 1.0031$, $\sigma = 0.9$, $P = 0.04$, $\lambda = 0.5$, $w = 0.8$, $\nu = 0.5$, $K = 0.012$. For endemic: $\beta_1 = 2.00093$, $\beta_2 = 2.0031$, $\sigma = 0.9$, $P = 0.04$, $\lambda = 0.5$, $w = 0.8$, $\nu = 0.5$, $K = 0.012$.

It is observed that all the parametric values meet the criteria of the required stated of equilibrium. Figure 1 demonstrates the path followed by the different graphs by the susceptible individuals. Four graphs are sketched against the different values of fractional order parameter ν , it is noticeable that each graph converges towards true fixed point of the diarrhea model. The basic difference between these graphs is that, they possess the different rate of convergence to attain the steady state. Similarly, the graphs in Figures 2–4 reveal the convergence towards the equilibrium points with different rate of convergence.

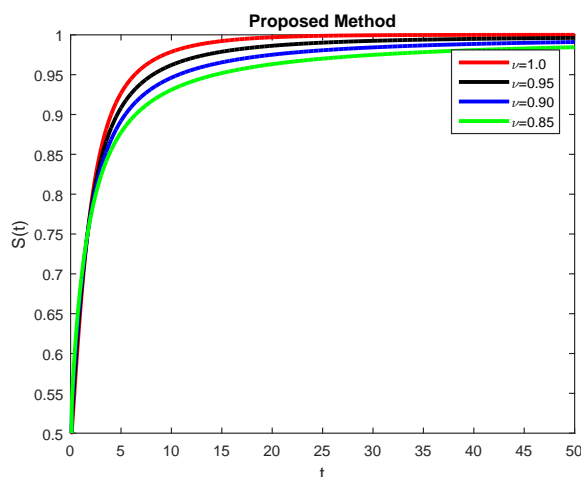


Figure 1. Numerical simulations of $S(t)$ at DFE for various values of ν .

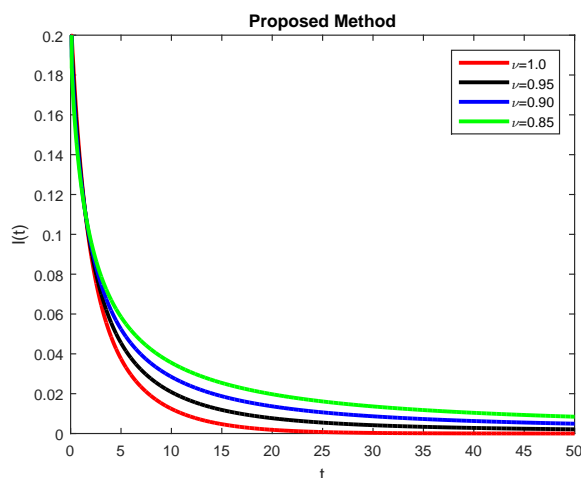


Figure 2. Numerical simulations of $I(t)$ at DFE for various values of ν .

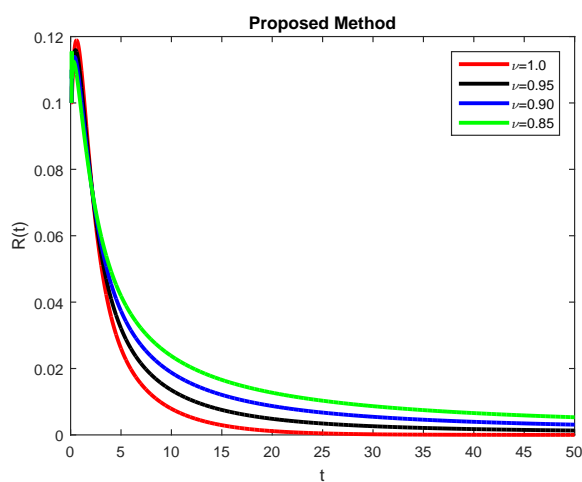


Figure 3. Numerical simulations of $R(t)$ at DFE for various values of ν .

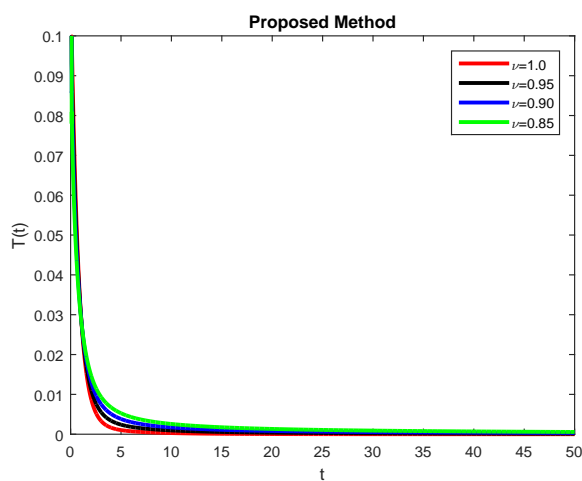


Figure 4. Numerical simulations of $T(t)$ at DFE for various values of ν .

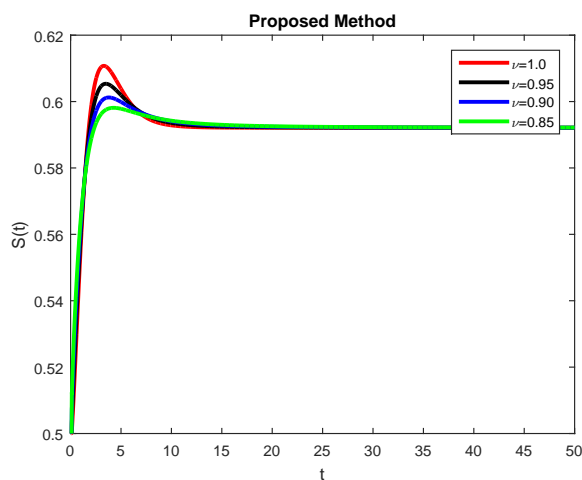


Figure 5. Numerical simulations of $S(t)$ at EE for various values of ν .

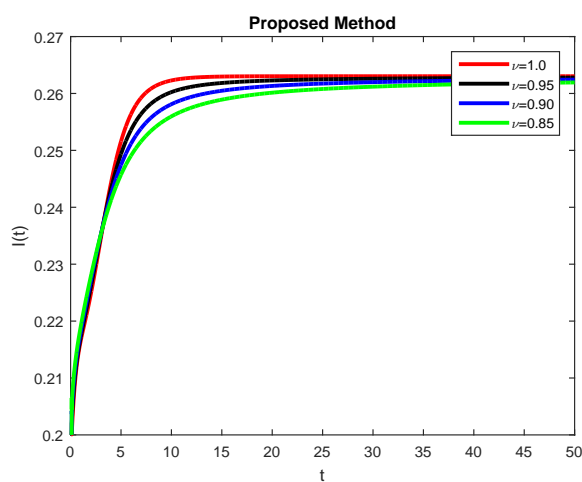


Figure 6. Numerical simulations of $I(t)$ at EE for various values of ν .

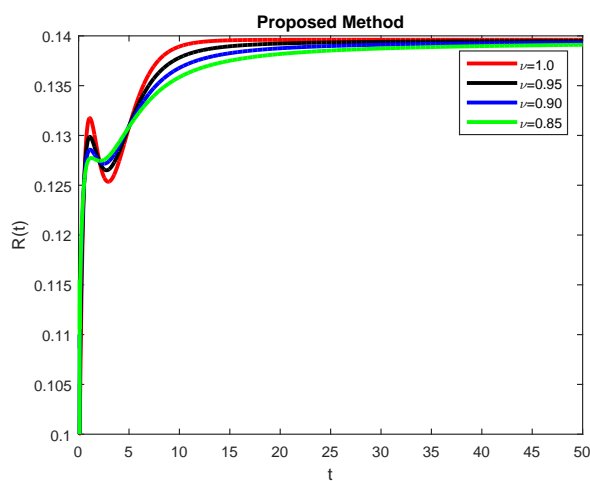


Figure 7. Numerical simulations of $R(t)$ at EE for various values of ν .

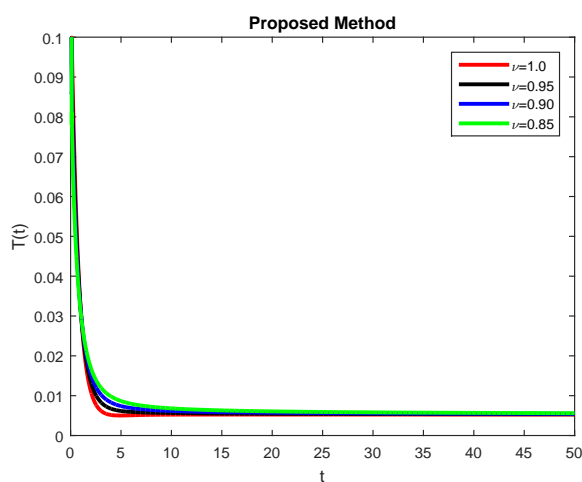


Figure 8. Numerical simulations of $T(t)$ at EE for various values of ν .

Moreover, the Figures 5–7 explain the graphs of susceptible, infected, treated and recovered individuals at endemic equilibrium point. Four values of fractional order parameters are chosen to draw the graphs as shown in the figures. It is clear that every graph in any of the figure attains the steady state value by following a specific trajectory depending upon the value of ν . The rate of convergence has a direct relation with the fractional parameter ν i.e., the greater the value of ν , faster the rate of convergence and vice versa.

8. Conclusions

In this work, a classical contagious diarrhea model is converted to the fractional order diarrhea model. For the conversion, Atangana Baleanu fractional differential operators in Caputo-sense are applied. After modification, the extended model is extended model is solved by using the method of Mittag-Leffler Kernel. But, before solving the underlying model, the existence and uniqueness of the solution is established. Computer aided numerical graphs are plotted to study the behavior of the proposed method. All the graphs exhibit the convergence towards the exact steady state with a specific rate of convergence depending upon the value ν . The rate of convergence is directly proportional to the value of ν , which is the order of the fractional derivative. It is proved that the disease free and endemic steady states are locally and globally stable. Furthermore, the role of basic reproduction number in disease dynamics and stability of the equilibrium states is investigated and it is concluded that the equilibrium points are locally and globally stable according to the values of R_0 . If the value of R_0 is less than one local and global stability is attained and similar for endemic state. As a future perspective this work may be applied to modify and solve the various nonlinear systems.

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Conflict of interest

The authors declare no conflict of interest.

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