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Research article

Comparison principle and synchronization analysis of fractional-order complex networks with parameter uncertainties and multiple time delays

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Abstract: This paper investigates the global synchronization problems of fractional-order complex dynamical networks with uncertain inner coupling and multiple time delays. In particular, both internal time delays and coupling time delays are introduced into our model. To overcome the difficulties caused by various delays and uncertainties, a generalized delayed comparison principle with fractional-order and impulsive effects is established by using the Laplace transform. Based on the Lyapunov stability theory and mixed impulsive control technologies, some new synchronization criteria for concerned complex dynamical networks are derived. In addition, the synchronization criteria are related to the impulsive interval, network topology structure, fractional-order, and control gains. The theoretical results obtained in this paper can enhance the value of previous related works. Finally, numerical simulations are presented to show the correctness of our main results.

Keywords: fractional-order; complex network; synchronization; impulsive control; uncertainty; delay
Mathematics Subject Classification: 26A33

1. Introduction

With the rapid development of information technologies, various natural and human-made complex networks have attracted increasing attention due to their wide applications in parameter identification, orbit tracking, memory filtering, and so on [1–5]. Synchronization, as one of the main dynamic behaviors of complex networks, is closely related to many practical phenomena in the real world [6]. To date, a large number of significant works for synchronization problems of various complex networks have been extensively investigated [7–10]. To our knowledge, most of the actual networks cannot achieve network synchronization only rely on their inherent structures. Then many effective control strategies such as pinning control, sliding mode control, intermittent control, impulsive control, and feedback control have been widely applied to the study of synchronization problems.

Impulsive control, as an important discrete control technique, not only greatly reduces the control cost, but also enhances the security of communications [11]. Hence, many meaningful synchronization criteria for complex dynamical networks have been obtained by virtue of impulsive control schemes. For instance, Deng et al. [12] and Liu et al. [13] derived impulsive synchronization criteria for complex dynamical networks by inputting information at several impulsive moments. Xu et al. [14] acquired sufficient conditions for asymptotic synchronization of complex dynamical networks with time-varying delays by using impulsive control techniques. The authors in [15] established synchronization criteria for complex-valued stochastic networks with time delays by designing a simple impulsive controller. Wang et al. obtained global synchronization criteria for delayed memristive dynamical systems via hybrid impulsive control [16]. It should be pointed out that all the theoretical results in [12–16] are obtained based on traditional integer-order complex dynamical systems.

Fractional-order calculus, as a significant generalization of integer-order calculus, not only adds an adjustable freedom degree compared with integer-order calculus but also has unique characteristics of infinite memory and heredity [17]. It has attracted more and more attention due to these unique properties [18–21]. To better describe the memory and cognition behaviors of neuron nodes, fractional-order calculus is introduced into neural networks to form fractional-order complex neural networks. One more essential difference between fractional-order and integer-order models is the nonlocality. Specifically, if an arbitrarily small neighborhood of t is clear, we can easily calculate the n-th order derivative $x^{(n)}(t)$ $(n \in N)$ for function $x(t)$. However, its fractional derivative ${}^{c}D_{t}^{\alpha}x(t)$ can be obtained only if the whole time interval $[0, t]$ is known. Therefore, it is more accurate to describe some real phenomena with fractional-order systems since they can make full use of the information from the initial state to the current state [22–24]. For example, the memristor called the fourth circuit element in circuit systems can be well described in fractional-order models, but it cannot be exactly described by integer-order systems [25]. Mani et al. [26] studied a fractional-order cellular neural network and its degrees of freedom for image encryption, which outstrips the existing encryption techniques based on integer-order systems. Li and Xing et al. investigated a class of fractional-order complex dynamical networks for forecasting traffic flow and obtained their optimal order in [27].

Recently, some remarkable results for the global synchronization of fractional-order complex networks have been obtained [28–31]. Xu et al. [28] dealt with the global synchronization of fractional-order dynamical networks with decentralized adaptive laws. Li et al. [29] solved global synchronization of fractional-order systems by employing an impulsive control strategy. Note that time delays were not considered in the models of [28, 29]. In fact, various time delays, such as internal delays and coupling delays, are ubiquitous in real complex networks due to finite transmission speeds and traffic congestion in the information transmission process. If the influence of time delays on complex dynamic networks is ignored, it is difficult to obtain generalized analysis results in many cases. Considering its importance, the authors [30] introduced time delays to network models and obtained the global synchronization criteria of fractional-order delayed neural networks by utilizing feedback control schemes. In [31], Li et al. obtained the global synchronization criteria for fractional-order dynamical systems with internal and coupling delays by combining the impulsive control and feedback control methods.

In reality, besides common time delays, parameter uncertainties are quite widespread in complex networks due to model inaccuracies or the changing external environment, which may destroy the stability of the systems. In engineering applications, the electric energy of the power system inevitably suffers from multiple uncertain disturbances and time delays during the transmission and use process due to the change in the external environment and the finite transmission speeds. In order to realize automatic and stable distribution of the power system, network topology identification, such as component status identification and various electrical wiring identification, should be performed first. In [32], the authors studied the problem of topology identification for fractional-order systems with uncertainties and delays, and such systems provided a good reference for topology identification of various fractional-order networks. However, for synchronization problems of fractional-order complex networks, few works have focused on both uncertainties and time delays except [33, 34]. By adopting continuous feedback control, Liang et al. [33] performed some works on the global synchronization of fractional-order complex networks with coupling delays and uncertainties. By using adaptive control, Dalir et al. [34] paid attention to the stability analysis of fractional-order chaotic systems with uncertain parameters and time delays. It is worth noting that impulsive effects were not be noticed in [33, 34]. As far as we know, there are few works concerning the global synchronization issues for fractional-order complex networks with internal time delays and coupling time delays as well as uncertain inner-coupling effects by using mixed impulsive control techniques. Integrating these real factors into fractional-order systems requires more complicated analysis and new comparison principles. Hence, it is meaningful and challenging for us to investigate this issue.

Inspired by the above analysis, this paper further investigates the global synchronization of fractional-order drive-response dynamical networks with parameter uncertainties, internal time delays and coupling time delays by means of mixed impulsive control strategies. The main contributions of this work are as follows. Firstly, a novel fractional-order impulsive comparison principle with multiple delays is established by using the Laplace transform and corresponding characteristic equations, which provides an important basis for the follow-up theoretical analysis. Secondly, compared with the fractional-order models in [28–31, 34], a more generalized model including parameter uncertainties, impulsive effects, internal delays, and coupling delays is taken into account in this paper. Complex models including multiple real factors can better simulate natural networks. Thirdly, based on fractional-order impulsive comparison principles and mixed impulsive controllers, some new sufficient conditions are obtained to achieve the global synchronization of concerned fractional-order systems.

This paper is organized as follows. In Section 2, the model description is formulated and some useful definitions, assumptions, and lemmas are given. In Section 3, theoretical results for the global synchronization of concerned fractional-order systems are obtained. Numerical examples are shown to verify our theoretical results in Section 4. The conclusion of this paper is finally presented in Section 5.

Notations: Let I_n be an n-dimensional identity matrix. R^n ($R^{n\times n}$) denotes the n-dimensional (n×n-dimensional) real spaces with Euclidean norm $\|\cdot\|$. A^T denotes the transpose of the matrix A. For a symmetric matrix B, $\lambda_{max}(B)$ represents its maximum eigenvalue. $B \leq 0$ (or $B \geq 0$) indicates that it is a semi-negative (or semi-positive) definite matrix. $max\{a, b\}$ represents the maximum value of a and b. $\|\cdot\|_1$ and $\|\cdot\|_\infty$ represent the 1-norm and infinite norm of a matrix or a vector, respectively. diag$\{k_1, k_2, ..., k_n\}$ denotes a diagonal matrix with elements $k_1, k_2, ..., k_n$. For a real time-varying function $f(t)$, ${}^cD_t^\alpha f(t)$ represents its Caputo derivative with order α, and $I_t^\alpha f(t)$ represents its fractional integral.The Kronecker product of a matrix $A \in R^{n\times m}$ and a matrix $B \in R^{p\times q}$ is $A\otimes B \in R^{np\times mq}$, defined as [35]

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{bmatrix}.$$

2. Preliminaries and model description

In this section, some necessary definitions concerning fractional are first recalled, and then the mathematical model is introduced.

Definition 1. The fractional integral with α order for time-varying function $x(t)$ is defined by

$$I_t^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} x(\tau) d\tau,$$

where $\alpha > 0$, and $\Gamma(\cdot)$ is the gamma function.

Definition 2. The Caputo fractional derivative with α order for time-varying function $x(t)$ is defined as

$${}^{c}D_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau,$$

where $t \geq t_0$, $0 \leq n-1 < \alpha < n$ and n are positive integers. When $0 < \alpha < 1$,

$${}^{c}D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} x'(\tau) d\tau.$$

Definition 3. The Mittag-Leffler function with one-parameter is defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

where $\alpha > 0$ and $z \in C$.

Consider a general fractional-order uncertain dynamical network consisting of N nodes, which can be described by

$${}^{c}D_t^{\alpha} x_i(t) = f(t, x_i(t), x_i(t-\tau_1)) + c_1 \sum_{j=1}^{N} a_{ij}(\Upsilon + \Delta\Upsilon) x_j(t) + c_2 \sum_{j=1}^{N} b_{ij}(\Upsilon + \Delta\Upsilon) x_j(t-\tau_2), \quad (1)$$

where $i = 1, 2, ..., N$, and $0 < \alpha < 1$. $x_i(t) = (x_{1i}(t), x_{2i}(t), \cdots, x_{ni}(t))^T \in R^n$ is the state variable of the ith node. $c_1 > 0$ and $c_2 > 0$ are the non-delayed and delayed coupling strengths, respectively. $A = (a_{ij}) \in R^{N\times N}$ and $B = (b_{ij}) \in R^{N\times N}$ represent the outer coupling matrices with diffusive coupling conditions $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$ and $b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij}$, respectively. If there is a connection from the jth node to the ith node ($i \neq j$), then $a_{ij} \neq 0$ and $b_{ij} \neq 0$; otherwise, $a_{ij} = 0$ and $b_{ij} = 0$. $\Upsilon + \Delta\Upsilon \in R^{n\times n}$ denote the inner-coupling matrices, where $\Upsilon \in R^{n\times n}$ is a certain constant matrix and $\Delta\Upsilon \in R^{n\times n}$ is a coupling matrix with uncertainty. $f(t, x_i(t), x_i(t-\tau_1)) \in R^n$ is a continuous vector-valued function, that describes the local dynamics of nodes. $\tau_1 > 0$ and $\tau_2 > 0$ represent the internal delay and coupling delay, respectively.

We refer to the fractional-order dynamical system (1) as the drive network, and the following controlled fractional-order system as the response network, which is described by

$$\begin{cases} {}^cD_t^\alpha y_i(t) = f(t, y_i(t), y_i(t-\tau_1)) + c_1 \sum_{j=1}^N a_{ij}(\Upsilon + \Delta\Upsilon) y_j(t) + c_2 \sum_{j=1}^N b_{ij}(\Upsilon + \Delta\Upsilon) y_j(t-\tau_2) \\ \qquad -\sigma_i e_i(t), \ t \in [t_{k-1}, t_k), \\ y_i(t_k^+) - y_i(t_k^-) = \eta(y_i(t_k^-) - x_i(t_k^-)), k = 1, 2, 3, ... \end{cases} \tag{2}$$

where $i = 1, 2, ..., N$, and $0 < \alpha < 1$. $y_i(t) = (y_{1i}(t), y_{2i}(t), \cdots, y_{ni}(t))^T \in R^n$ denotes the state variable of node i in the response network. σ_i is the positive control gain in the impulsive intervals, and η is the impulsive gain at impulsive instants. The time sequences $\{t_k\}$ satisfy $0 \le t_0 < t_1 < t_2 < ... < t_k < ...$, and $t_k \to +\infty$ as $k \to +\infty$. Without loss of generality, it is assumed that $y_i(t_k) = y_i(t_k^+)$. The initial conditions of drive-response complex networks (1) and (2) are $x_i(t) = x_i^0(t)$ and $y_i(t) = y_i^0(t)$, respectively, $i = 1, 2, ..., N, t \in [t_0 - \hat{\tau}, t_0]$ with $\hat{\tau} = max\{\tau_1, \tau_2\}$.

Define the error variables as

$$e_i(t) = y_i(t) - x_i(t), \quad i = 1, 2, \ldots, N. \tag{3}$$

Then the error dynamical system can be derived from (1) and (2) as follows:

$$\begin{cases} {}^cD_t^\alpha e_i(t) = f(t, y_i(t), y_i(t-\tau_1)) - f(t, x_i(t), x_i(t-\tau_1)) + c_1 \sum_{j=1}^N a_{ij}(\Upsilon + \Delta\Upsilon) e_j(t) \\ \qquad + c_2 \sum_{j=1}^N b_{ij}(\Upsilon + \Delta\Upsilon) e_j(t-\tau_2) - \sigma_i e_i(t), \ t \in [t_{k-1}, t_k), \\ e_i(t_k^+) - e_i(t_k^-) = \eta e_i(t_k^-), k = 1, 2, 3, ..., \end{cases} \tag{4}$$

where the initial conditions are $e_i(t) = y_i^0(t) - x_i^0(t), t \in [t_0 - \hat{\tau}, t_0]$.

In this paper, we aim to use mixed impulsive control schemes such that fractional-order drive-response complex networks (1) and (2) achieve globally asymptotical synchronization, in the sense that

$$\lim_{t \to +\infty} \|e_i(t)\| = 0, \quad i = 1, 2, ..., N, \tag{5}$$

holds for any initial value, where $\|.\|$ refers to the Euclidean norm.

Remark 1. Compared with integer-order complex networks, fractional-order complex networks have unique properties, including but not limited to nonlocality, infinite memory, and degrees of freedom. In fact, the Itô formula and differential inequality about integer-order calculus operators fail to directly extend to fractional-order operators. Hence, most synchronization analysis techniques for integer-order delayed dynamical systems are not suitable for fractional-order systems.

To derive our main results, some useful assumptions and lemmas are given, and a generalized impulsive comparison principle is proposed.

Assumption 1. [31] For all $x(t), y(t) \in R^n$, there exist positive constants θ and ψ such that the vector-valued function $f\Big(t, x(t), x(t-\tau_1)\Big)$ satisfies the following condition:

$$\begin{aligned} & \Big[x(t)-y(t)\Big]^T \Big[f\Big(t, x(t), x(t-\tau_1)\Big) - f\Big(t, y(t), y(t-\tau_1)\Big)\Big] \\ \le \;& \theta\Big[x(t)-y(t)\Big]^T\Big[x(t)-y(t)\Big] + \psi\Big[x(t-\tau_1)-y(t-\tau_1)\Big]^T\Big[x(t-\tau_1)-y(t-\tau_1)\Big]. \end{aligned} \tag{6}$$

Remark 2. In fact, Assumption 1 is very mild. It can be easily verified that many fractional-order chaotic systems satisfy Assumption 1, such as fractional-order delayed Hopfield neural networks, fractional-order delayed cellular neural networks and fractional-order delayed Chua's oscillator.

Lemma 1. [33] Let $u = (u_1, u_2, ..., u_n) \in R^n$ and $v = (v_1, v_2, ..., v_n) \in R^n$, then the following inequality

$$u^T Q v \le \frac{1}{2}\Big(\|Q\|_\infty u^T u + \|Q\|_1 v^T v\Big)$$

holds for all matrices $Q \in R^{n\times n}$.

Lemma 2. [38] If all the eigenvalues of $K + C$ satisfy $|arg(\lambda)| > \frac{\pi}{2}$ and the characteristic equation $det(\Delta(s)) = 0$ has no purely imaginary roots for all $\tau_{ij} > 0, i, j = 1, 2, ..., n$, then the zero solution of the following fractional-order delayed system

$${}^cD_t^\alpha Z(t) = CZ(t) + Z(t_\tau),\ \alpha \in (0, 1), \tag{7}$$

is globally asymptotically stable, where $C = (c_{ij}) \in R^{n\times n}$, $K = (k_{ij}) \in R^{n\times n}$, $Z(t) = (z_1(t), z_2(t), ..., z_n(t))^T$, $Z(t_\tau) = (\sum_{j=1}^n k_{1j} z_j(t-\tau_{1j}), \sum_{j=1}^n k_{2j} z_j(t-\tau_{2j}), ..., \sum_{j=1}^n k_{nj} z_j(t-\tau_{nj}))^T$, $B = (b_{ij}) = (k_{ij}e^{-s\tau_{ij}} + c_{ij}) \in R^{n\times n}, i, j = 1, 2, ..., n$ and $\Delta(s) = s^\alpha I_n - B$.

Lemma 3. [24] If $v(t)$ is a continuous function on $[t_0, +\infty)$ and satisfies the following condition

$${}^cD_t^\alpha v(t) \le \lambda v(t), \tag{8}$$

where $0 < \alpha < 1$, $\lambda \in R$ and t_0 is the initial time, then we have

$$v(t) \le v(t_0) E_\alpha[\lambda(t - t_0)^\alpha], \tag{9}$$

where $E_\alpha(.)$ is the well-known Mittag-Leffler function.

Lemma 4. [31] Assume that the nonnegative functions $x(t)$ and $y(t)$ satisfy

$$\begin{cases} {}^cD_t^\alpha x(t) \le -ax(t) + bx(t-\tau_1(t)) + cx(t-\tau_2(t)), t \ne t_k, \\ x(t_k) \le \xi_k x(t_k^-), t = t_k, k = 1, 2, 3, ..., \\ x(t) = \varphi(t), t \in [t_0 - \hat{\tau}, t_0], \end{cases} \tag{10}$$

and

$$\begin{cases} {}^cD_t^\alpha y(t) \le -ay(t) + by(t-\tau_1(t)) + cy(t-\tau_2(t)), t \ne t_k, \\ y(t) = \phi(t), t \in [t_0 - \hat{\tau}, t_0], \end{cases} \tag{11}$$

where $0 < \alpha < 1, 0 \le \tau_1(t), \tau_2(t) \le \hat{\tau}, 0 < \xi_k \le 1$, and a is an arbitrary constant. b and c are nonnegative constants. Then $\varphi(t) \le \phi(t)$ for $t_0 - \hat{\tau} \le t \le t_0$ implies that $x(t) \le y(t)$ for $t \ge t_0$.

Lemma 5. Let $u(t) \in R$ be a differentiable and nonnegative function, and it satisfies the following impulsive differential inequality:

$$\begin{cases} {}^cD_t^\alpha u(t) \leq -\beta u(t) + \gamma_1 u(t-\tau_1) + \gamma_2 u(t-\tau_2), t \in [t_{k-1}, t_k), t \geq t_0, \\ u(t_k) = \epsilon_k^2 u(t_k^-), t = t_k, k = 1,2,3,..., \\ u(t) = \zeta(t), t \in [t_0 - \hat{\tau}, t_0], \end{cases} \tag{12}$$

where $0 < \alpha < 1, 0 < \epsilon_k \leq 1$, $0 < \tau_i \leq \hat{\tau}(i = 1,2)$, and β, γ_1 and γ_2 are positive constants. $u(t_k) = u(t_k^+) = \lim_{t \to t_k^+} u(t)$ and $u(t_k^-) = \lim_{t \to t_k^-} u(t)$ exist. If $\gamma_1 + \gamma_2 < \beta sin\frac{\alpha\pi}{2}$, then $\lim_{t\to+\infty} u(t) = 0$ for all $\zeta(t) \geq 0$.

Proof. Consider the following fractional-order delayed system

$$ {}^cD_t^\alpha w(t) = -\beta w(t) + \gamma_1 w(t-\tau_1) + \gamma_2 w(t-\tau_2), \tag{13}$$

where $w(t)$ is continuous on $[t_0 - \hat{\tau}, \infty)$ and it has the same initial value with $u(t)$. Based on Lemma 4 and the inequality $0 < \epsilon_k^2 \leq 1$, we derive

$$0 \leq u(t) \leq w(t). \tag{14}$$

Taking the Laplace transformation with respect to (13) yields

$$\begin{aligned} s^\alpha w(s) - s^{\alpha-1} w(t_0) &= -\beta w(s) + \gamma_1 \int_{t_0}^{+\infty} e^{-st} w(t-\tau_1)dt + \gamma_2 \int_{t_0}^{+\infty} e^{-st} w(t-\tau_2)dt \\ &= -\beta w(s) + \gamma_1 \int_{t_0-\tau_1}^{+\infty} e^{-s(t+\tau_1)} w(t)dt + \gamma_2 \int_{t_0-\tau_2}^{+\infty} e^{-s(t+\tau_2)} w(t)dt \\ &= -\beta w(s) + \gamma_1 e^{-s\tau_1}\Big(\int_{t_0-\tau_1}^{t_0} e^{-st} w(t)dt + \int_{t_0}^{+\infty} e^{-st} w(t)dt \Big) \\ &\quad +\gamma_2 e^{-s\tau_2}\Big(\int_{t_0-\tau_2}^{t_0} e^{-st} w(t)dt + \int_{t_0}^{+\infty} e^{-st} w(t)dt \Big) \\ &= -\beta w(s) + \gamma_1 e^{-s\tau_1} w(s) + \gamma_2 e^{-s\tau_2} w(s) \\ &\quad +\gamma_1 e^{-s\tau_1} \int_{t_0-\tau_1}^{t_0} e^{-st} w(t)dt + \gamma_2 e^{-s\tau_2} \int_{t_0-\tau_2}^{t_0} e^{-st} w(t)dt. \end{aligned} \tag{15}$$

According to Lemma 2 and (15), one has

$$det(\Delta(s))w(s) = s^{\alpha-1} w(t_0) + \gamma_1 e^{-s\tau_1} \int_{t_0-\tau_1}^{t_0} e^{-st} w(t)dt + \gamma_2 e^{-s\tau_2} \int_{t_0-\tau_2}^{t_0} e^{-st} w(t)dt, \tag{16}$$

where $det(\Delta(s)) = s^\alpha + \beta - \gamma_1 e^{-s\tau_1} - \gamma_2 e^{-s\tau_2}$. Next, we prove that characteristic equation $det(\Delta(s)) = 0$ has no pure imaginary roots. Assume that $s = vi = |v|(cos\frac{\pi}{2} + isin(\pm\frac{\pi}{2}))$, where v is a real number. Based on the Euler formula $e^{i\theta} = cos\theta + isin\theta$ and De Moivre formula $(cos\theta + isin\theta)^\alpha = cos\alpha\theta + isin\alpha\theta$, substituting s into the characteristic equation, we have

$$s^\alpha + \beta - \gamma_1 e^{-s\tau_1} - \gamma_2 e^{-s\tau_2}$$

$$
\begin{aligned}
=&|v|^{\alpha}\Big(cos\frac{\pi}{2}+isin(\pm\frac{\pi}{2})\Big)^{\alpha}+\beta-\gamma_1e^{-v\tau_1 i}-\gamma_2e^{-v\tau_2 i}\\
=&|v|^{\alpha}\Big(cos\frac{\alpha\pi}{2}+isin(\pm\frac{\alpha\pi}{2})\Big)+\beta-\gamma_1\Big(cos(\tau_1v)-isin(\tau_1v)\Big)-\gamma_2\Big(cos(\tau_2v)-isin(\tau_2v)\Big)\\
=&0
\end{aligned}
\tag{17}
$$

By separating the real and imaginary parts, we obtain

$$
\begin{cases}
|v|^{\alpha}cos\frac{\alpha\pi}{2}+\beta=\gamma_1cos(\tau_1v)+\gamma_2cos(\tau_2v),\\
|v|^{\alpha}sin(\pm\frac{\alpha\pi}{2})=-\gamma_1sin(\tau_1v)-\gamma_2sin(\tau_2v).
\end{cases}
\tag{18}
$$

Equation (18) shows that

$$
|v|^{2\alpha}+2\beta|v|^{\alpha}cos\frac{\alpha\pi}{2}+\beta^2-(\gamma_1^2+\gamma_2^2+2\gamma_1\gamma_2cosv(\tau_1-\tau_2))=0. \tag{19}
$$

Let $g(x)=x^2+2\beta cos\frac{\alpha\pi}{2}x+\beta^2-(\gamma_1^2+\gamma_2^2+2\gamma_1\gamma_2cosv(\tau_1-\tau_2))$. Then $g(0)>0$, since $\gamma_1+\gamma_2<\beta sin\frac{\alpha\pi}{2}$, $0<\alpha<1$, and β,γ_1 and γ_2 are positive constants. Note that $g(x)$ is a second order polynomial, so we have $g(|v|^{\alpha})>0$. This shows that Eq (18) has no solution, that is, the above characteristic equation $det(\Delta(s))=0$ has no pure imaginary roots. Moreover, when $\gamma_1+\gamma_2<\beta sin\frac{\alpha\pi}{2}$, that is $\gamma_1+\gamma_2<\beta$, we can obtain $|arg(-\beta+\gamma_1+\gamma_2)|>\frac{\pi}{2}$. According to Lemma 2, the zero solution of system (13) is asymptotically stable and $\lim_{t\to+\infty}w(t)=0$. Then it follows from (14) that $\lim_{t\to+\infty}u(t)=0$ for all $\zeta(t)\geq0$, $0<\tau_1,\tau_2\leq\hat{\tau}$, and we complete the proof of Lemma 5. □

Remark 3. The integer-order delayed comparison principles [5, 14] cannot be directly extended to fractional-order principles. Hence, some fractional-order delayed comparison principles have been established in [17, 33, 38, 40]. For example, two fractional-order comparison principles with single delay were proposed one after another in [33, 38]. In addition, fractional-order comparison principles with multiple delays were derived in [17, 40]. The results listed above were used for the synchronization of various fractional-order systems. It should be pointed out that the comparison principles in [17,33,38,40] did not consider impulsive effects. Different from these existing works, the generalized fractional-order comparison principles obtained in this paper include both multiple delays and impulsive effects.

3. Main results

In this section, the global synchronization between the drive network (1) and the response network (2) via hybrid impulsive control is investigated, and the main results are summarized in the following theorems.

Theorem 1. When $-2<\eta<0$ and Assumption 1 is satisfied, the global synchronization between drive network (1) and response network (2) can be achieved if there exist positive scalars c_1,c_2,θ and ψ such that

$$
\gamma_b+\gamma_c<\beta_a sin\frac{\alpha\pi}{2}, \tag{20}
$$

where $\beta_a = \Big[2\lambda_{max}((H-\Theta)\otimes I_n) - c_1(\|A\|_\infty\|(\Upsilon+\Delta\Upsilon)\|_\infty + \|A\|_1\|(\Upsilon+\Delta\Upsilon)\|_1) - c_2\|B\|_\infty\|(\Upsilon+\Delta\Upsilon)\|_\infty\Big] > 0$, $\gamma_b = 2\lambda_{max}(\Psi\otimes I_n)$, $\gamma_c = c_2\|B\|_1(\Upsilon+\Delta\Upsilon)\|_1$, $\Theta = \mathrm{diag}\{\theta,\theta,...,\theta\} \in R^{N\times N}$, $\Psi = \mathrm{diag}\{\psi,\psi,...,\psi\} \in R^{N\times N}$ and $H = \mathrm{diag}\{\sigma_1,\sigma_2,...,\sigma_N\} \in R^{N\times N}$.

Proof. Consider the following candidate function

$$V(t) = \sum_{i=1}^{N} e_i^T(t)e_i(t). \tag{21}$$

When $t \in [t_{k-1}, t_k)$, the fractional derivative of $V(t)$ with respect to (4) is

$$\begin{aligned} {}^cD_t^\alpha V(t) &\le 2\sum_{i=1}^{N} e_i^T(t)\,{}^cD_t^\alpha e_i(t) \\ &= 2\sum_{i=1}^{N} e_i^T(t)\Big[f(t,y_i(t),y_i(t-\tau_1)) - f(t,x_i(t),x_i(t-\tau_1)) + c_1\sum_{j=1}^{N} a_{ij}(\Upsilon+\Delta\Upsilon)e_j(t) \\ &\quad + c_2\sum_{j=1}^{N} b_{ij}(\Upsilon+\Delta\Upsilon)e_j(t-\tau_2) - \sigma_i e_i(t)\Big]. \end{aligned} \tag{22}$$

By Assumption 1, we can obtain

$$\sum_{i=1}^{N} e_i^T(t)\Big[f(t,y_i(t),y_i(t-\tau_1)) - f(t,x_i(t),x_i(t-\tau_1))\Big] \le \sum_{i=1}^{N}\theta e_i^T(t)e_i(t) + \sum_{i=1}^{N}\psi e_i^T(t-\tau_1)e_i(t-\tau_1). \tag{23}$$

Substituting inequality (23) into (22), we have

$$\begin{aligned} {}^cD_t^\alpha V(t) &\le 2\Big[\sum_{i=1}^{N}\theta e_i^T(t)e_i(t) + \sum_{i=1}^{N}\psi e_i^T(t-\tau_1)e_i(t-\tau_1) + c_1\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)a_{ij}(\Upsilon+\Delta\Upsilon)e_j(t) \\ &\quad + c_2\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)b_{ij}(\Upsilon+\Delta\Upsilon)e_j(t-\tau_2) - \sum_{i=1}^{N}\sigma_i e_i^T(t)e_i(t)\Big]. \end{aligned} \tag{24}$$

Denote $e(t) = [e_1^T(t), e_2^T(t), ..., e_N^T(t)]^T$; by Lemma 1, we obtain

$$\begin{aligned} {}^cD_t^\alpha V(t) &\le 2\Big[e^T(t)(\Theta\otimes I_n)e(t) + e^T(t-\tau_1)(\Psi\otimes I_n)e(t-\tau_1) + c_1e^T(t)(A\otimes(\Upsilon+\Delta\Upsilon))e(t) \\ &\quad + c_2e^T(t)(B\otimes(\Upsilon+\Delta\Upsilon))e(t-\tau_2) - e^T(t)(H\otimes I_n)e(t)\Big] \\ &\le 2\Big[e^T(t)(\Theta\otimes I_n)e(t) + e^T(t-\tau_1)(\Psi\otimes I_n)e(t-\tau_1) - e^T(t)(H\otimes I_n)e(t) \\ &\quad + \frac{c_1}{2}e^T(t)(\|(A\otimes(\Upsilon+\Delta\Upsilon))\|_\infty + \|(A\otimes(\Upsilon+\Delta\Upsilon))\|_1)e(t) \\ &\quad + \frac{c_2}{2}e^T(t)\|(B\otimes(\Upsilon+\Delta\Upsilon))\|_\infty e(t) + \frac{c_2}{2}e^T(t-\tau_2)\|(B\otimes(\Upsilon+\Delta\Upsilon))\|_1 e(t-\tau_2)\Big] \\ &\le -\Big[2\lambda_{max}((H-\Theta)\otimes I_n) - c_1(\|A\|_\infty\|(\Upsilon+\Delta\Upsilon)\|_\infty + \|A\|_1\|(\Upsilon+\Delta\Upsilon)\|_1) \end{aligned}$$

$$
\begin{aligned}
& -c_2\|B\|_\infty\|(\Upsilon+\Delta\Upsilon)\|_\infty\Big]V(t)+2\lambda_m(\Psi\otimes I_n)V(t-\tau_1)+c_2\|B\|_1\|(\Upsilon+\Delta\Upsilon)\|_1V(t-\tau_2) \\
= \ & -\beta_aV(t)+\gamma_bV(t-\tau_1)+\gamma_cV(t-\tau_2),
\end{aligned} \tag{25}
$$

where $\beta_a = \Big[2\lambda_{max}((H-\Theta)\otimes I_n) - c_1(\|A\|_\infty\|(\Upsilon+\Delta\Upsilon)\|_\infty + \|A\|_1\|(\Upsilon+\Delta\Upsilon)\|_1) - c_2\|B\|_\infty\|(\Upsilon+\Delta\Upsilon)\|_\infty\Big]$, $\gamma_b = 2\lambda_{max}(\Psi\otimes I_n)$ and $\gamma_c = c_2\|B\|_1\|(\Upsilon+\Delta\Upsilon)\|_1$.

When $t = t_k$, we have

$$
\begin{aligned}
V(t_k) &= \sum_{i=1}^{N} e_i^T(t_k)e_i(t_k) = \sum_{i=1}^{N} e_i^T(t_k^-)(1+\eta)^2e_i(t_k^-) \\
&= \varrho^2V(t_k^-),
\end{aligned} \tag{26}
$$

where $\varrho = |1+\eta|$. When $-2<\eta<0$, one can easily obtain $0<\varrho<1$. Using Lemma 5, if $\gamma_b+\gamma_c<\beta_a sin\frac{\alpha\pi}{2}$, it follows from (25) and (26) that $V(t)\to 0$ as $t\to+\infty$. Obviously, when $V(t) = \sum_{i=1}^{N} e_i^T(t)e_i(t)\to 0$, we can obtain $\|e_i(t)\|\to 0$ for $i = 1,2,...,N$, which implies that complex dynamical networks (1) and (2) can achieve globally asymptotical synchronization. □

Remark 4. Note that there have been some studies on synchronization issues of various fractional-order systems [28–31,34]. Compared to these existing results, the model in this paper considers internal delays and coupling delays in addition to parameter uncertainty, which makes our results extend the previous related works.

Remark 5. In [29], pinning impulsive control schemes for the global synchronization of fractional-order complex dynamical systems were considered based on the generalized Barbalat's Lemma. In [41], the authors considered the synchronization of fractional-order chaotic dynamical systems with a single delay via impulsive control. However, the time delay was ignored in [29] and a single delay was considered in [41]. Obviously, the impulsive control methods used in [29,41] cannot be extended for our network model since it includes multiple time delays. To overcome the difficulties caused by various delays, a generalized fractional-order comparison principle with multiple time delays is established by using the Laplace transform and a mixed impulsive control scheme is used. The mixed impulsive control used in this paper cannot be replaced by pure impulsive control. In fact, if only impulsive control is used instead of mixed impulsive control, it is easy to find that the parameter β_a in condition (20) will be negative. Namely, under pure impulsive control, the condition in Theorem 1 cannot be guaranteed since the parameters γ_b and γ_c are positive in most cases.

Theorem 2. When control gain $\eta>0$ or $\eta<-2$ and Assumption 1 is satisfied, the global synchronization between drive network (1) and response network (2) can be achieved if scalars $l_1>1, l_2>1$ and $\epsilon>1$ exist such that

$$
\epsilon\varrho^2E_\alpha[-\mu(t_k-t_{k-1})^\alpha]<1, \tag{27}
$$

where $\mu = \beta_a-\gamma_bl_1-\gamma_cl_2>0, \beta_a = \Big[2\lambda_{max}((H-\Theta)\otimes I_n) - c_1(\|A\|_\infty\|(\Upsilon+\Delta\Upsilon)\|_\infty + \|A\|_1\|(\Upsilon+\Delta\Upsilon)\|_1) - c_2\|B\|_\infty\|(\Upsilon+\Delta\Upsilon)\|_\infty\Big]>0$, $\gamma_b = 2\lambda_{max}(\Psi\otimes I_n)$, $\gamma_c = c_2\|B\|_1(\Upsilon+\Delta\Upsilon)\|_1$, $\Theta = \text{diag}\{\theta,\theta,...,\theta\}\in R^{N\times N}$, $\Psi = \text{diag}\{\psi,\psi,...,\psi\}\in R^{N\times N}$ and $H = \text{diag}\{\sigma_1,\sigma_2,...,\sigma_N\}\in R^{N\times N}$.

Proof. Consider the following candidate function

$$V(t)=\sum_{i=1}^{N}e_i^T(t)e_i(t). \tag{28}$$

When $t\in[t_{k-1},t_k)$, using the similar proof of (22)–(25) in Theorem 1, one has

$${}^cD_t^\alpha V(t)\le-\beta_aV(t)+\gamma_bV(t-\tau_1)+\gamma_cV(t-\tau_2), \tag{29}$$

whenever $e_i(t)$ satisfies the following conditions

$$V(t-\tau_1)\le l_1V(t), V(t-\tau_2)\le l_2V(t), \tag{30}$$

for $l_1>1$ and $l_2>1$, one can obtain from (29) and (30) that

$${}^cD_t^\alpha V(t)\le-\mu V(t), \tag{31}$$

where $\mu=\beta_a-\gamma_bl_1-\gamma_cl_2$. It follows from (31) and Lemma 3 that

$$V(t)\le V(t_{k-1})E_\alpha[-\mu(t-t_{k-1})^\alpha],\ t\in[t_{k-1},t_k). \tag{32}$$

When $t=t_k$, we obtain

$$\begin{aligned}V(t_k) &= \sum_{i=1}^{N}e_i^T(t_k)e_i(t_k)=\sum_{i=1}^{N}e_i^T(t_k^-)(1+\eta)^2e_i(t_k^-)\\ &= \varrho^2V(t_k^-)\end{aligned} \tag{33}$$

where $\varrho=|1+\eta|$.

For $t\in[t_0,t_1)$, it follows from (32) that

$$V(t)\le V(t_0)E_\alpha[-\mu(t-t_0)^\alpha]. \tag{34}$$

Combining (33) with (34), we have

$$V(t_1)=\varrho^2V(t_1^-)\le\varrho^2V(t_0)E_\alpha[-\mu(t_1-t_0)^\alpha]. \tag{35}$$

For $t\in[t_1,t_2)$, we obtain

$$\begin{aligned}V(t)\le& V(t_1)E_\alpha[-\mu(t-t_1)^\alpha]\\ \le&\varrho^2V(t_0)E_\alpha[-\mu(t_1-t_0)^\alpha]E_\alpha[-\mu(t-t_1)^\alpha],\end{aligned} \tag{36}$$

and

$$V(t_2)=\varrho^2V(t_2^-)\le V(t_0)\Big\{\varrho^2E_\alpha[-\mu(t_1-t_0)^\alpha]\Big\}\Big\{\varrho^2E_\alpha[-\mu(t_2-t_1)^\alpha]\Big\}. \tag{37}$$

Similarly, for $t\in[t_2,t_3)$, one can get

$$V(t)\ \le\ V(t_2)E_\alpha[-\mu(t-t_2)^\alpha]$$

$$\leq \quad V(t_0)\Big\{\varrho^2 E_\alpha[-\mu(t_1-t_0)^\alpha]\Big\}\Big\{\varrho^2 E_\alpha[-\mu(t_2-t_1)^\alpha]\Big\}E_\alpha[-\mu(t-t_2)^\alpha], \tag{38}$$

and

$$V(t_3)=\varrho^2 V(t_3^-)\leq V(t_0)\Big\{\varrho^2 E_\alpha[-\mu(t_1-t_0)^\alpha]\Big\}\Big\{\varrho^2 E_\alpha[-\mu(t_2-t_1)^\alpha]\Big\}\Big\{\varrho^2 E_\alpha[-\mu(t_3-t_2)^\alpha]\Big\}. \tag{39}$$

Repeating the above reasoning process, for $t\in[t_{k-1},t_k)$, it follows from the conditions of Theorem 2 that

$$\begin{aligned} V(t) &\leq V(t_0)\Big\{\varrho^2 E_\alpha[-\mu(t_1-t_0)^\alpha]\Big\}\Big\{\varrho^2 E_\alpha[-\mu(t_2-t_1)^\alpha]\Big\}\Big\{\varrho^2 E_\alpha[-\mu(t_3-t_2)^\alpha]\Big\} \\ &\quad \times ...\times\Big\{\varrho^2 E_\alpha[-\mu(t_{k-1}-t_{k-2})^\alpha]\Big\}\Big\{E_\alpha[-\mu(t-t_{k-1})^\alpha]\Big\} \\ &\leq V(t_0)\frac{1}{\epsilon^{k-1}}\Big\{E_\alpha[-\mu(t-t_{k-1})^\alpha]\Big\} \end{aligned} \tag{40}$$

Because of $\epsilon>1$, wa can obtain that $V(t)\to 0$ as $k\to+\infty$, which leads to $||e_i(t)||\to 0$ for $i=1,2,...,N$. This shows that global synchronization between complex dynamical networks (1) and (2) can be realized. □

According to the proof process of Theorem 2, it is not difficult to obtain the following corollary.

Corollary 1. When $-2<\eta<0$ and Assumption 1 is satisfied, the global synchronization between drive network (1) and controlled response network (2) can be achieved if scalars $l_1>1, l_2>1$ and $\epsilon>1$ exist such that

$$\epsilon\varrho^2 E_\alpha[-\mu(t_k-t_{k-1})^\alpha]<1, \tag{41}$$

where $\mu=\beta_a-\gamma_b l_1-\gamma_c l_2<0, \beta_a=\Big[2\lambda_{max}((H-\Theta)\otimes I_n)-c_1(||A||_\infty||(\Upsilon+\Delta\Upsilon)||_\infty+||A||_1||(\Upsilon+\Delta\Upsilon)||_1)-c_2||B||_\infty||(\Upsilon+\Delta\Upsilon)||_\infty\Big]>0, \gamma_b=2\lambda_{max}(\Psi\otimes I_n), \gamma_c=c_2||B||_1(\Upsilon+\Delta\Upsilon)||_1, \Theta=diag\{\theta,\theta,...,\theta\}\in R^{N\times N}, \Psi=diag\{\psi,\psi,...,\psi\}\in R^{N\times N}$ and $H=diag\{\sigma_1,\sigma_2,...,\sigma_N\}\in R^{N\times N}$.

Remark 6. Theorems 1 and 2 as well as Corollary 1 are still true for $\alpha=1$. However, these results cannot be generalized to a more general case $\alpha>1$ since Lemma 3 is not true for $\alpha>1$ and we will study it in the future.

Remark 7. According to different control demands, the impulsive intervals t_k-t_{k-1} and other parameters can be determined flexibly by the following rules. (i) If impulsive gain η satisfies $-2<\eta<0$ and condition (20) holds, then impulsive intervals are arbitrary. (ii) If impulsive gain η satisfies $-2<\eta<0$, then impulsive intervals t_k-t_{k-1} can be obtained by $\epsilon\varrho^2E_\alpha[-\mu(t_k-t_{k-1})^\alpha]<1$, where $\mu=\beta_a-\gamma_b l_1-\gamma_c l_2<0$, $l_1>1$, $l_2>1$ and $\epsilon>1$. (iii) If impulsive gain η satisfies $\eta>0$ or $\eta<-2$, then impulsive intervals t_k-t_{k-1} can be obtained by $\epsilon\varrho^2E_\alpha[-\mu(t_k-t_{k-1})^\alpha]<1$, where $\mu=\beta_a-\gamma_b l_1-\gamma_c l_2>0$, $l_1>1$, $l_2>1$ and $\epsilon>1$. Obviously, the greater ϵ is, the faster the synchronization speed is.

Remark 8. If $0<\alpha<1$, then the nonnegative function $E_\alpha(\mu(t-t_0)^\alpha)$ is monotonically nonincreasing and $0\leq E_\alpha(\mu(t-t_0)^\alpha)\leq 1$ for $t\geq t_0$ and $\mu\leq 0$. On the other hand, the non-negative function $E_\alpha(\mu(t-t_0)^\alpha)$ is monotonically non-decreasing and $E_\alpha(\mu(t-t_0)^\alpha)\geq 1$ for $t\geq t_0$ and $\mu\geq 0$. Based on these properties , it is not difficult to verify the condition of Theorem 2 in numerical simulations.

4. Numerical simulations

In this section, some numerical examples are presented to illustrate the effectiveness of the derived theoretical results by the following delayed neural networks.

$$f(t, x(t), x(t-\tau_1)) = -C_1 x(t) + A_1 \tanh(x(t)) + B_1 \tanh(x(t-\tau_1)), \tag{42}$$

where

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 2 & -0.1 \\ -5 & 4.5 \end{bmatrix}, B_1 = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{bmatrix},$$

and $\tau_1 = 1$. By simple computation, one can obtain that $\theta = 7.9146$ and $\psi = 2.0047$ such that Assumption 1 holds. The fractional-order response dynamical networks consisting of 15 nodes are given as follows.

$$\begin{cases} {}^cD_t^\alpha y_i(t) = f(t, y_i(t), y_i(t-\tau_1)) + c_1 \sum\limits_{j=1}^{15} a_{ij}(\Upsilon + \Delta\Upsilon) y_j(t) + c_2 \sum\limits_{j=1}^{15} b_{ij}(\Upsilon + \Delta\Upsilon) y_j(t-\tau_2) \\ \qquad\qquad -\sigma_i e_i(t), \ t \in [t_{k-1}, t_k), \\ y_i(t_k^+) - y_i(t_k^-) = \eta(y_i(t_k^-) - x_i(t_k^-)), \ k = 1, 2, 3, ..., \end{cases} \tag{43}$$

where $i = 1, 2, ..., 15$, $\alpha = 0.98$, $\tau_2 = 0.2$, $c_1 = 0.2$ and $c_2 = 0.5$.

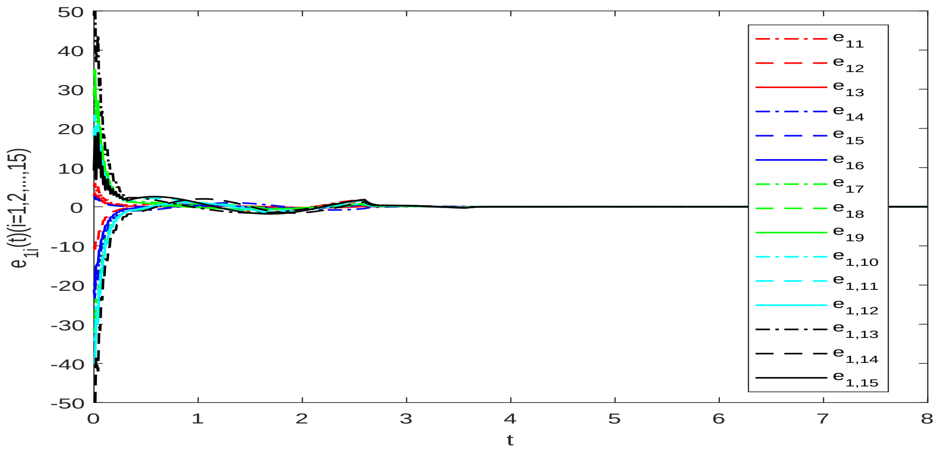
The inner-coupling matrices are chosen as

$$\Upsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Delta\Upsilon = \kappa \begin{bmatrix} \mathbb{R}_{[0\ 1]} & 0 \\ 0 & \mathbb{R}_{[0\ 1]} \end{bmatrix}.$$

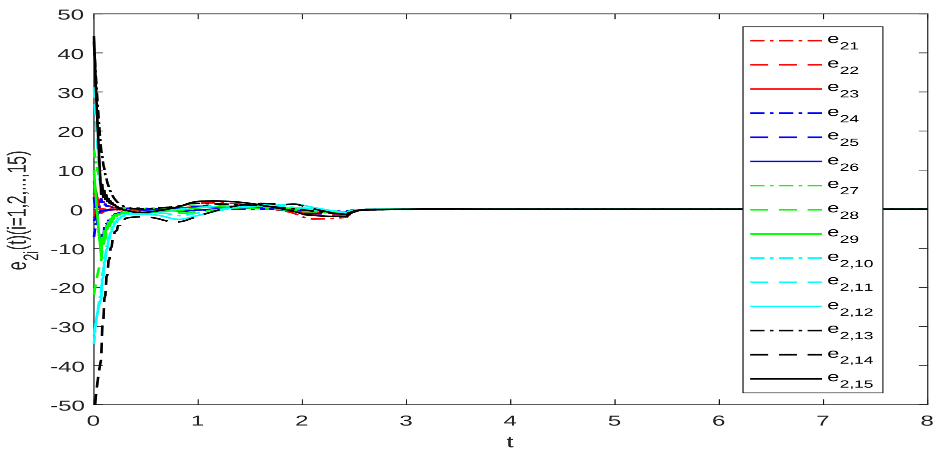
κ is a constant coefficient to be set later and $\mathbb{R}_{[0\ 1]}$ represents a random number between 0 and 1. The outer coupling matrices A and B are set as

$$A = B = \begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -3
\end{bmatrix},$$

The modified predictor-corrector technique in [42] has been applied to deal with the numerical simulations by the MATLAB toolbox. Set $\kappa = 1.5$, $\sigma_i = 20$, $\eta = -0.6$ and $t_k - t_{k-1} = 0.1$. By calculation, we can obtain $\|A\|_1 = 5$, $\|A\|_\infty = 6$, $\lambda_{max}((H - \Theta) \otimes I_n) = 12.0854$, $1 \leq \|\Upsilon + \Delta\Upsilon\|_1 \leq 2.5$, $1 \leq \|\Upsilon + \Delta\Upsilon\|_\infty \leq 2.5$, $\gamma_b + \gamma_c \leq 10.2594$ and $\beta_a \sin \frac{\alpha\pi}{2} \geq 11.1653$. It is obvious that the parameters above satisfy the conditions of Theorem 1. Under the proposed hybrid impulsive control, Figure 1 (a) displays the error $e_{1i}(t) = y_{1i}(t) - x_{1i}(t)(i = 1, 2, ..., 15)$ between drive networks and response networks with randomly selected initial values. Obviously, the error $e_{1i}(t)(i = 1, 2, ..., 15)$ gradually converges to zero with time evolution. Similarly, Figure 1 (b) displays that the error $e_{2i}(t)(i = 1, 2, ..., 15)$ gradually converges to zero over time. According to the definition of the global synchronization in Eq (5), $e_i(t) \to 0$ as $e_{1i}(t) \to 0$ and $e_{2i}(t) \to 0$, which shows that the controlled response networks are synchronized with the drive networks. Hence, the theoretical result obtained in Theorem 1 has been proven right by this example.



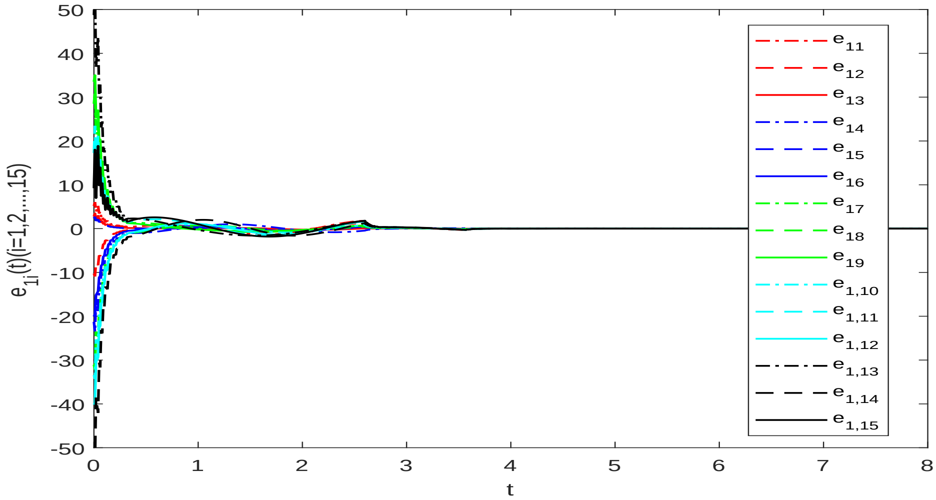
(a)



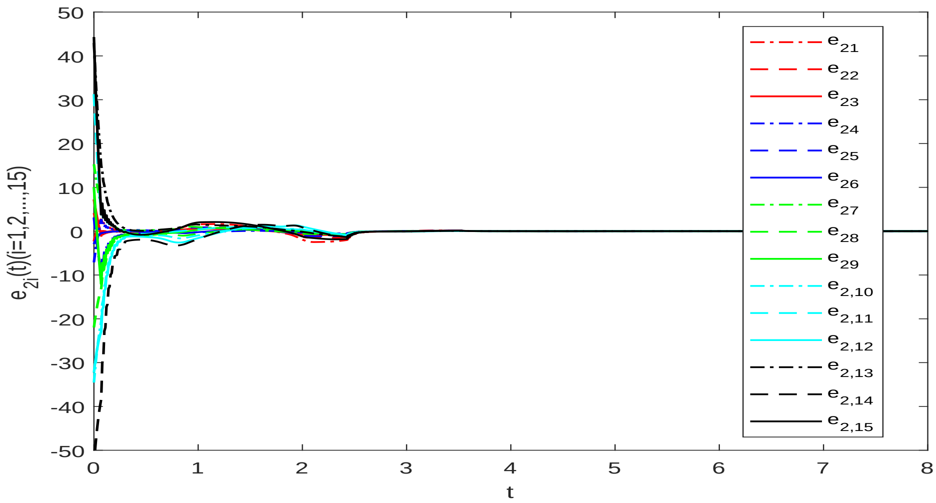
(b)

Figure 1. Time evolution of $e_i(t)$ in drive-response networks (43) under control parameters $\sigma_i = 25$, $\eta = 0.1$ and $t_k - t_{k-1} = 1$. (a) e_{1i}, (b) e_{2i}.

If we set $\kappa = 1.5$, $\sigma_i = 25$, $\eta = 0.1$, $t_k - t_{k-1} = 1$, and let $\epsilon = 1.1, l_1 = l_2 = 2.5$. By simple calculation, one can obtain $\varrho = 1.1$, $\lambda_{max}((H - \Theta) \otimes I_n) = 17.0854$, and $\mu = \beta_a - \gamma_b l_1 - \gamma_c l_2 \geq 0.6520$. Using the series expansion of the Mittag-Leffler function, we can future obtain $\epsilon \varrho^2 E_\alpha[-\mu(t_k - t_{k-1})^\alpha] \leq 0.6930$; then, the conditions of Theorem 2 are satisfied. For randomly selected initial values and all the network nodes, Figure 2 (a) and (b) display that the synchronization error $e_{1i}(t)(i = 1, 2, ..., 15)$ and $e_{2i}(t)(i = 1, 2, ..., 15)$ between drive networks and response networks gradually converge to zero with time evolution. Hence, $e_i(t) \to 0$ as $e_{1i}(t) \to 0$ and $e_{2i}(t) \to 0$, which shows that the global synchronization goal between response networks and drive networks is achieved under suitable control parameters. Hence, the correctness of the theoretical results in Theorem 2 is verified.



(a)



(b)

Figure 2. Time evolution of $e_i(t)$ in drive-response networks (43) under control parameters $\sigma_i = 25$, $\eta = 0.1$ and $t_k - t_{k-1} = 1$. (a) e_{1i}, (b) e_{2i}.

5. Conclusions

In this paper, we discussed global synchronization problems of drive-response complex dynamical networks with uncertain effects, nondelayed couplings and delayed couplings. A generalized fractional-order impulsive comparison principle with multiple delays was established to overcome the difficulties caused by various delays. Mixed impulsive controllers including impulsive control and feedback control schemes have been considered in our work. Based on the mixed impulsive control strategies and the proposed comparison principle, some new sufficient conditions for global synchronization of concerned fractional-order complex networks were derived. Finally, some numerical examples were performed to verify the effectiveness of our theoretical results. The order is assumed to be $0 < \alpha < 1$ in this paper, and we will continue to investigate its generalization version for $\alpha > 1$. Besides, how to realize the adaptation of the control gains could be also considered in the future.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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