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Research article

Some new identities of a type of generalized numbers involving four parameters

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Abstract: This article deals with a Horadam type of generalized numbers involving four parameters. These numbers generalize several celebrated numbers in the literature such as the generalized Fibonacci, generalized Lucas, Fibonacci, Lucas, Pell, Pell-Lucas, Fermat, Fermat-Lucas, Jacobsthal, Jacobsthal-Lucas, balancing, and co-balancing numbers. We present some new identities concerned with the generalized numbers of four parameters. An explicit expression for these numbers is developed, and a mixed recurrence relation between two certain families of the generalized numbers is given, and after that, some new identities are presented and proved. A large number of identities between several celebrated numbers are obtained as special cases of our developed ones. Furthermore, some of the identities that were previously published in other articles can be deduced as special ones of our new identities.

Keywords: generalized Fibonacci and generalized Lucas numbers; Lucas and Fibonacci numbers; recurrence relation; Zeilberger's algorithm
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1. Introduction

Fibonacci and Lucas polynomials and their corresponding numbers play important parts in various disciplines. Fibonacci numbers arise in quite unexpected places. For example, they appear in nature, geography, and geometry. For some applications of these numbers, one can consult [1]. A large number of contributions were devoted to establishing formulas concerned with these polynomials and their corresponding numbers. For example, the authors in [2] found some results regarding the finite reciprocal sums of Fibonacci and Lucas polynomials. The authors in [3] derived some power sums of Fibonacci and Lucas numbers. New formulas of Fibonacci and Lucas numbers involving the golden ratio are developed in [4]. Some other contributions concerning these sequences of numbers and some of their related sequences can be found in [5–9].

Several types of sequences that generalize the standard Fibonacci and Lucas sequences are introduced and investigated in a variety of contributions by many authors. In [10], Muskat considered two classes of generalized Fibonacci and generalized Lucas numbers, and some root-finding methods are applied. In [11], Trojovskỳ considered a $k-$ generalized Fibonacci sequence and found solutions to a certain Diophantine equation. The authors in [12] developed some relationships between some numbers related to Fibonacci and Lucas numbers. In [13], a new approach to generalized Fibonacci and Lucas numbers with binomial coefficients is followed. A generalized k-Horadam sequence is investigated in [14]. In [15], the authors developed some identities concerning k-balancing and k-Lucas-balancing numbers. A type of generalized Fibonacci numbers is introduced and investigated in [16]. The authors in [17] solved a quadratic Diophantine equation involving the generalized Fibonacci numbers. In [18], some arithmetic properties and new pseudo primality results for the generalized Lucas sequences were provided. Some arithmetic properties of the generalized Lucas sequences are developed in [19]. The authors in [20] introduced and investigated distance Fibonacci Polynomials. There are interesting numbers in the literature that can be considered special ones of various generalized Fibonacci numbers and generalized Lucas numbers. For example, the celebrated numbers, namely, Jacobsthal, Jacobsthal-Lucas, balancing, and co-balancing numbers, can be extracted from some generalized sequences of numbers. For some studies regarding different celebrated sequences of numbers, one can be referred, for example, to [21–25].

From a numerical point of view, several types of Fibonacci and Lucas polynomials and their generalized and modified polynomials were utilized to find numerical solutions to some types of differential equations. For example, Abd-Elhameed and Youssri in the series of papers [26–29] solved some types of fractional differential equations utilizing Fibonacci and Lucas polynomials and some of their generalized polynomials. A Fibonacci wavelet method was utilized in [30] for solving the time-fractional bioheat transfer model. Certain modified Lucas polynomials were employed in [31] to treat some fractional optimal control problems. The authors in [32] used a certain generalized Lucas polynomial sequence to treat numerically the fractional pantograph differential equation.

In [33], Abd-Elhameed and Zeyada have considered two sets of generalized Fibonacci numbers $\left\{U_j^{c,d}\right\}_{j\geq 0}$ and generalized Lucas numbers $\left\{V_j^{c,d}\right\}_{j\geq 0}$ that can be constructed respectively by means of the following recurrence relations:

$$U_{j+2}^{c,d} = c\, U_{j+1}^{c,d} + d\, U_j^{c,d}, \quad j \geq 0, \quad U_0^{c,d} = 0,\ U_1^{c,d} = 1, \tag{1.1}$$

and

$$V_{j+2}^{c,d} = c\, V_{j+1}^{c,d} + d\, V_j^{c,d}, \quad j \geq 0, \quad V_0^{c,d} = 2,\ V_1^{c,d} = c, \tag{1.2}$$

and they developed some new identities involving the generalized Fibonacci and Lucas numbers. The famous identities of Sury and Marques were deduced as special cases, see [34,35]. Furthermore, Abd-Elhameed and Zeyada in [36] have considered another sequence of generalized numbers involving three parameters $\left\{W_j^{c,d,r}\right\}_{j\geq 0}$ that can be generated by the following recurrence relation:

$$W_{j+2}^{c,d,r} = r\, W_{j+1}^{c,d,r} + W_j^{c,d,r}, \quad j \geq 0, \quad W_0^{c,d,r} = d - r\, c,\ W_1^{c,d,r} = c, \tag{1.3}$$

and they developed some other identities involving the generalized numbers $W_j^{c,d,r}$.

An an important class of generalized numbers that generalizes the three classes of the generalized Fibonacci and Lucas numbers constructed by means of the three recurrence relations (1.1)–(1.3) is the Horadam sequence of numbers $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$ [37] that can constructed with the aid of the following recurrence relation:

$$G_{i+2}^{p,q,a,b} - p\,G_{i+1}^{p,q,a,b} - q\,G_i^{p,q,a,b} = 0, \quad G_0^{p,q,a,b} = a,\, G_1^{p,q,a,b} = b, \quad i \geq 0, \tag{1.4}$$

where p, q, a, b are arbitrary integers.

Horadam in [37] presented some basic properties of the sequence $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$, while he derived the generating function of this sequence of numbers in [38]. Some convoluted formulas concerned with these numbers are derived in [39]. In addition, Melham in [40] derived some other identities concerned with the same sequence of numbers.

This paper is concerned with deriving some new formulas concerned with the Horadam generalized sequence of numbers $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$. The presence of four parameters in the sequence of numbers implies that several celebrated numbers such as the generalized Fibonacci, generalized Lucas, Fibonacci, Lucas, Pell, Pell-Lucas, Fermat, Fermat-Lucas, Jacobsthal, Jacobsthal-Lucas, balancing, and co-balancing numbers. can be considered special ones of our considered generalized numbers of four parameters [41]. This gives a motivation to consider and investigate the generalized sequence of numbers. Another motivation to consider the generalized numbers is that many identities concerned with them generalize some identities that exist in the literature.

We can summarize the aims of this article in the following items.

- Developing a new explicit expression of these numbers.
- Deriving a recurrence relation that is satisfied between two certain generalized number sequences.
- Developing new identities between two generalized classes of numbers.
- Deducing some specific identities involving the special numbers namely, Fibonacci, Lucas, Pell, Pell-Lucas, Fermat, Fermat-Lucas, Jacobsthal, Jacobsthal-Lucas, balancing, and co-balancing numbers.

The contents of the rest of the paper are structured as follows. Section 2 introduces an account of the Horadam sequence of generalized numbers that involves four parameters. Some of their fundamental properties are given. It is also shown in this section that some of the celebrated numbers can be considered special ones of them. In Section 3, two basic theorems concerned with the generalized sequence of numbers are stated and proved. In the first theorem, we give an explicit expression to these generalized numbers, while in the second theorem, a new recurrence relation concerned with two certain families of these numbers is given. Section 4 presents the main results of this paper. Some new identities are given in this respect. Moreover, some special identities are given from the new results in this section. Section 5 is devoted to presenting other identities between some generalized and specific classes of numbers. Finally, Section 6 displays the conclusion.

2. Some fundamental properties of the sequence $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$

This section is devoted to presenting some basic formulas of the generalized sequence of numbers $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$.

It is worthy to note here that the sequence of numbers $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$ can be considered as a generalization of the three number sequences $\left\{U_i^{c,d}\right\}_{i\geq 0}, \left\{V_i^{c,d}\right\}_{i\geq 0}$ and $\left\{W_i^{c,d,r}\right\}_{i\geq 0}$ that generated respectively by the recurrence relations given in (1.1)–(1.3), for suitable choices of p, q, a and b.

It is not difficult to write the Binet's form for $G_i^{p,q,a,b}$. In fact, these numbers can be represented as

$$G_i^{p,q,a,b} = c_1\,\alpha^i + c_2\,\beta^i,$$

where c_1 and c_2 are constants to be determined, and α and β are the roots of the characteristic equation of (1.4), given by

$$y^2 - p\,y - q = 0,$$

so, we have

$$\alpha = \frac{1}{2}\left(p + \sqrt{p^2+4q}\right), \quad \beta = \frac{1}{2}\left(p - \sqrt{p^2+4q}\right).$$

It is not difficult to determine c_1 and c_2 to give

$$c_1 = \frac{b - a\beta}{\alpha - \beta}, \quad c_2 = \frac{a\,\alpha - \beta}{\alpha - \beta},$$

and this leads to the following Binet's form for $G_i^{p,q,a,b}$

$$G_i^{p,q,a,b} = \frac{\alpha^i\,(b - a\beta) + \beta^i\,(a\,\alpha\; - b)}{\alpha - \beta},$$

that is

$$G_i^{p,q,a,b} = \frac{1}{2^{i+1}\sqrt{p^2+4q}} \times \left(\left(p + \sqrt{p^2+4q}\right)^i \left(2b + a\left(-p + \sqrt{p^2+4q}\right)\right) + \left(p - \sqrt{p^2+4q}\right)^i \left(-2b + a\left(p + \sqrt{p^2+4q}\right)\right)\right).$$

This class of the generalized numbers $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$ has the advantage that several generalized and specific classes of numbers can be deduced from it. In fact, by selecting suitable choices of the involved four parameters in (1.4), these celebrated numbers can be deduced. In Table 1, we display some of these numbers that can deduced as special cases from the generalized numbers $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$. In every row of this table, we write the corresponding values of the four parameters a, b, p, q for each sequence accompanied by its corresponding recurrence relation.

It is worth mentioning that the generating function of the sequence of numbers $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$ is established by Horadam [38] as:

$$F(t) = \frac{a + t(b - ap)}{1 - pt - qt^2}.$$

Table 1. Some special numbers cases of the generalized numbers $G_n^{p,q,a,b}$.

Numbers	p	q	a	b	Recurrence relation
Generalized Fibonacci in (1.1)	c	d	0	1	$U_{n+2}^{c,d} - c\,U_{n+1}^{c,d} - d\,U_n^{c,d} = 0$
Generalized Lucas in (1.2)	c	d	2	c	$V_{n+2}^{c,d} - c\,V_{n+1}^{c,d} - d\,V_n^{c,d} = 0$
Generalized numbers in (1.3)	r	1	$d - r\,c$	c	$W_{j+2}^{c,d,r} - r\,W_{j+1}^{c,d,r} - W_j^{c,d,r} = 0$
Fibonacci	1	1	0	1	$F_n - F_{n-1} - F_{n-2} = 0$
Lucas	1	1	2	1	$L_n - L_{n-1} - L_{n-2} = 0$
Pell	2	1	0	1	$P_n - 2P_{n-1} - P_{n-2} = 0$
Pell-Lucas	2	1	2	2	$Q_n - 2Q_{n-1} - Q_{n-2} = 0$
Fermat	3	-2	0	1	$\mathcal{F}_n - 3\,\mathcal{F}_{n-1} + 2\,\mathcal{F}_{n-2} = 0$
Fermat-Lucas	3	-2	2	3	$f_n - 3\,f_{n-1} + 2\,f_{n-2} = 0$
Jacobsthal	1	2	0	1	$J_n - J_{n-1} - 2J_{n-2} = 0$
Jacobsthal-Lucas	1	2	2	1	$j_n - j_{n-1} - 2j_{n-2} = 0$
Balancing	6	-1	0	1	$B_n - 6\,B_{n-1} + B_{n-2} = 0$
Co-balancing	6	-1	2	6	$b_n - 6\,b_{n-1} + b_{n-2} = 0$

3. Two new basic formulas concerned with the numbers $G_i^{p,q,a,b}$

This section is interested in developing two new basic formulas of the generalized numbers $G_i^{p,q,a,b}$ generated by means of the recurrence relation (1.4). In the first theorem, we give a new explicit expression for these numbers. The second theorem links between the two families of numbers sequences $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$ and $\left\{G_i^{p,q,c,d}\right\}_{i\geq 0}$.

Theorem 3.1. *The generalized numbers $G_i^{p,q,a,b}$ generated by the recurrence relation* (1.4) *can be explicitly expressed by the following formula:*

$$G_i^{p,q,a,b} = \sum_{n=0}^{\lfloor \frac{i}{2} \rfloor} \frac{q^n\, p^{i-2n-1}(i-2n+1)_{n-1}\,((i-2n)b + n\,a\,p)}{n!},\ i \geq 1, \tag{3.1}$$

where $\lfloor z \rfloor$ represents the well-known floor function, and $(\xi)_n$ represents the Pochhammer symbol, that is: $(\xi)_n = \frac{\Gamma(\xi+n)}{\Gamma(\xi)}$.

Proof. First, assume that

$$F_i^{p,q,a,b} = \sum_{n=0}^{\lfloor \frac{i}{2} \rfloor} \frac{q^n\, p^{i-2n-1}(i-2n+1)_{n-1}\,((i-2n)b + n\,a\,p)}{n!}. \tag{3.2}$$

It is easy to see that: $F_0^{p,q,a,b} = a$ and $F_1^{p,q,a,b} = b$, so to show that $F_i^{p,q,a,b} = G_i^{p,q,a,b}$, we need to show that it satisfies the same recurrence relation in (1.4). For this purpose, set

$$P_i^{p,q,a,b} = F_{i+2}^{p,q,a,b} - p\,F_{i+1}^{p,q,a,b} - q\,F_i^{p,q,a,b},$$

and we will show the validity of the following identity:

$$P_i^{p,q,a,b} = 0. \tag{3.3}$$

Due to (3.2), we can write

$$P_i^{p,q,a,b} = \sum_{n=0}^{\lfloor \frac{i}{2} \rfloor + 1} M_{n,i+2} - p \sum_{n=0}^{\lfloor \frac{i+1}{2} \rfloor} M_{n,i+1} - q \sum_{n=0}^{\lfloor \frac{i}{2} \rfloor} M_{n,i}, \tag{3.4}$$

where

$$M_{n,i} = \frac{q^n p^{i-2n-1} (i-2n+1)_{n-1} ((i-2n)b + nap)}{n!}. \tag{3.5}$$

To show the validity of (3.3), we consider the following two cases.

Case 1: For $i = 2j$. In such case, Formula (3.4) leads to the following formula

$$P_{2j}^{p,q,a,b} = M_{j+1,2j+2} + \sum_{n=0}^{j} \left(M_{n,2j+2} - p\, M_{n,2j+1} - q\, M_{n,2j} \right),$$

that is can be written as

$$\begin{aligned} P_{2j}^{p,q,a,b} = aq^{j+1} + \sum_{n=0}^{j} p^{-1+2j-2n} q^n \Bigg(& \frac{p^2 (b(2+2j-2n) + a(-1+n)p)(2j-n)!}{(2j-2n+2)!(n-1)!} \\ & - \frac{(2b(j-n) + anp)q(2j-n-1)!}{(2j-2n)!n!} \Bigg). \end{aligned} \tag{3.6}$$

Now, to get a closed-form for the summation that appears in (3.6), we set

$$\begin{aligned} H_{n,j} = \sum_{n=0}^{j} p^{-1+2j-2n} q^n \Bigg(& \frac{p^2 (b(2+2j-2n) + a(-1+n)p)(2j-n)!}{(2j-2n+2)!(n-1)!} \\ & - \frac{(2b(j-n) + anp)q(2j-n-1)!}{(2j-2n)!n!} \Bigg), \end{aligned}$$

and make use of the Zeilberger's algorithm through the Maple software, and in particular, the "sumrecursion command" [42] to show that $H_{n,j}$ satisfies the following recurrence relation of order one:

$$H_{n,j+1} - q\, H_{n,j} = 0, \quad H_{n,0} = -a\, q,$$

which can be easily solved to give

$$H_{n,j} = -a\, q^{1+j}. \tag{3.7}$$

Relation (3.7) along with relation (3.6) leads to

$$P_{2j}^{p,q,a,b} = 0. \tag{3.8}$$

Case 2: For $i = 2j + 1$. In such case, Formula (3.4) leads to the following formula

$$P_{2j+1}^{p,q,a,b} = \sum_{n=0}^{j+1} \left(M_{n,2j+3} - p\, M_{n,2j+2} - q\, M_{n,2j+1} \right),$$

that can be written explicitly as

$$P_{2j+1}^{p,q,a,b} = -b\, p^{2j} q + \sum_{n=0}^{j} p^{2(-1+j-n)}\, q^{1+n} \left(\frac{p^2\,(b + 2\,b\,j - 2\,b\,n + a\,n\,p)(2\,j-n)!}{(1+2j-2\,n)!n!} - \frac{(b(-1+2j-2n) + a(1+n)p)q(-1+2j-n)!}{(-1+2j-2n)!(1+n)!} \right).$$

Now, set

$$R_{n,j} = \sum_{n=0}^{j} p^{2(-1+j-n)}\, q^{1+n} \left(\frac{p^2\,(b + 2\,b\,j - 2\,b\,n + a\,n\,p)(2\,j-n)!}{(1+2j-2\,n)!n!} - \frac{(b(-1+2j-2n) + a(1+n)p)q(-1+2j-n)!}{(-1+2j-2n)!(1+n)!} \right).$$

Making use of Zeilberger's algorithm again leads to the following recurrence relation:

$$R_{n,j+1} = p^2\, R_{n,j}, \quad R_{n,0} = b\, q,$$

that can be easily solved to give

$$R_{n,j} = b p^{2j} q,$$

and accordingly,

$$P_{2j+1}^{p,q,a,b} = 0. \tag{3.9}$$

The two Formulas (3.8) and (3.9) show the desired result of Theorem 3.1.

□

Now, the following theorem is useful in the sequel. In this theorem, we will give a recurrence relation between the two classes of numbers $\left\{G_i^{p,q,a,b}\right\}_{i\geq 0}$ and $\left\{G_i^{p,q,c,d}\right\}_{i\geq 0}$.

Theorem 3.2. *For $q \neq 0$ and $a^2 q + a\,b\,p - b^2 \neq 0$, the following recurrence relation holds for every non-negative integer i:*

$$G_{i+1}^{p,q,c,d} = G_i^{p,q,a,b} + \left(\frac{p}{q} + \frac{-b\,d + b\,c\,p + a\,c\,q}{-b^2 + a\,b\,p + a^2\,q} \right) G_{i+1}^{p,q,a,b} + \left(-\frac{1}{q} + \frac{-b\,c + a\,d}{-b^2 + a\,b\,p + a^2\,q} \right) G_{i+2}^{p,q,a,b}. \tag{3.10}$$

Proof. To show the validity of the recurrence relation (3.10), we prove that

$$L_i = G_i^{p,q,a,b} + M\, G_{i+1}^{p,q,a,b} + R\, G_{i+2}^{p,q,a,b} - G_{i+1}^{p,q,c,d} = 0, \tag{3.11}$$

where

$$M = \frac{p}{q} + \frac{-b\,d + b\,c\,p + a\,c\,q}{-b^2 + a\,b\,p + a^2\,q}, \quad R = -\frac{1}{q} + \frac{-b\,c + a\,d}{-b^2 + a\,b\,p + a^2\,q}.$$

We are going to prove that

$$L_{2i} = 0, \quad \text{and} \quad L_{2i+1} = 0.$$

The proofs of the two identities are similar, We will prove that $L_{2i+1} = 0$.
In virtue of the explicit representation of $G_i^{p,q,a,b}$ in (3.1), we can write L_{2i+1} in the form

$$L_{2i+1} = \sum_{m=0}^{i} S_{m,i} + M \sum_{m=0}^{i} \bar{S}_{m,i} + R \sum_{m=0}^{i+1} S_{m,i+1} - \sum_{m=0}^{i} H_{m,i}, \tag{3.12}$$

where

$$\begin{aligned}
S_{m,i} =& \frac{q^m\,p^{2\,i-2\,m-1}(2\,i-2\,m+1)_{m-1}\,\left(2\,(i-m)\,b + m\,a\,p\right)}{m!},\\
\bar{S}_{m,i} =& \frac{q^m\,p^{2\,i-2\,m}(2\,i-2\,m+2)_{m-1}\left((2\,i-2\,m+1)b + m\,a\,p\right)}{m!},\\
H_{m,i} =& \frac{q^m\,p^{2\,i-2\,m}(2\,i-2\,m+2)_{m-1}\left((2\,i-2\,m+1)d + m\,c\,p\right)}{m!}.
\end{aligned}$$

Relation (3.12) can be written alternatively as

$$L_{2i+1} = R\,S_{i+1,i+1} + \sum_{m=0}^{i} T_{m,i}, \tag{3.13}$$

and

$$T_{m,i} = S_{m,i} + M\,\bar{S}_{m,i} + R\,S_{m,i+1} - H_{m,i}.$$

It can be shown that

$$\sum_{m=0}^{i} T_{m,i} = -aq^{i+1}\left(\frac{a\,d - b\,c}{a^2\,q + a\,bp - b^2} - \frac{1}{q}\right) = -R\,S_{i+1,i+1},$$

and accordingly

$$L_{2i+1} = 0.$$

Theorem 3.2 is now proved. □

4. New identities involving the generalized numbers $G_i^{p,q,a,b}$

This section is confined to presenting new identities involving the generalized numbers $G_i^{p,q,a,b}$. Some identities involving the generalized Fibonacci, generalized Lucas, Fibonacci, Lucas, Pell, Pell-Lucas, Fermat, Fermat-Lucas, Jacobsthal, Jacobsthal-Lucas, balancing, and co-balancing numbers are also deduced as special cases.

Theorem 4.1. *Let k be a non-negative integer, and let $x \in R^* = R - \{0\}$. For $-b\,d + a\,c\,q + b\,c\,p \neq 0$, one has*

$$\begin{aligned} x^{k+1}\, G_{k+1}^{p,q,a,b} =& a + \frac{1}{-b\,d + a\,c\,q + b\,c\,p} \sum_{i=0}^{k} x^i \Big((a\,d - b\,c + x(-b\,d + a\,c\,q + b\,c\,p)) G_{i+1}^{p,q,a,b} \\ & + \left(b^2 - a^2 q - a\,b\,p\right) G_i^{p,q,c,d} \Big). \end{aligned} \tag{4.1}$$

Proof. We will prove the result by induction. For $k = 0$, it is easy to see that the left-hand side equals the right-hand side which is equal to $(b\,x)$. Now assume the validity of (4.1), and then to complete the proof, we have to prove the following identity:

$$\begin{aligned} x^{k+2}\, G_{k+2}^{p,q,a,b} =& a + \frac{1}{-b\,d + a\,c\,q + b\,c\,p} \sum_{i=0}^{k+1} x^i \Big((a\,d - b\,c + x(-b\,d + a\,c\,q + b\,c\,p)) G_{i+1}^{p,q,a,b} \\ & + \left(b^2 - a^2 q - a\,b\,p\right) G_i^{p,q,c,d} \Big). \end{aligned} \tag{4.2}$$

It is clear that each side of relation (4.2) is a polynomial of degree $(k + 2)$. Now, let

$$\begin{aligned} M(x) =& a + \frac{1}{-b\,d + a\,c\,q + b\,c\,p} \sum_{i=0}^{k+1} x^i \Big((a\,d - b\,c + x(-b\,d + a\,c\,q + b\,c\,p)) G_{i+1}^{p,q,a,b} \\ & + \left(b^2 - a^2 q - a\,b\,p\right) G_i^{p,q,c,d} \Big). \end{aligned}$$

Now to show (4.2), it suffices to show the validity of the following formula

$$M(x) = x^{k+2}\, G_{k+2}^{p,q,a,b}.$$

The polynomial $M(x)$ can be written in the form

$$\begin{aligned} & M(x) = a + \frac{1}{-b\,d + a\,c\,q + b\,c\,p} \times \\ & \sum_{i=0}^{k} x^i \Big((a\,d - b\,c + x(-b\,d + a\,c\,q + b\,c\,p))\, G_{i+1}^{p,q,a,b} + \left(b^2 - a^2 q - a\,b\,p\right) G_i^{p,q,c,d} \Big) \\ & + \frac{1}{-b\,d + b\,c\,p + a\,c\,q} x^{k+1} \Big((-b\,(c + d\,x - c\,p\,x) + a(d + c\,q\,x))\, G_{k+1}^{p,q,a,b} + \left(b^2 - a\,b\,p - a^2 q\right) G_{k+1}^{p,q,c,d} \Big). \end{aligned} \tag{4.3}$$

Making use of the inductive step leads to

$$\begin{aligned} M(x) =& x^{k+1}\, G_{k+1}^{p,q,a,b} + \frac{1}{-bd + bcp + acq} x^{k+1} \times \\ & \Big((-b(c + dx - cpx) + a(d + cqx))\, G_{k+2}^{p,q,a,b} + \left(b^2 - abp - a^2 q\right) G_{k+1}^{p,q,c,d} \Big). \end{aligned} \tag{4.4}$$

If we insert the mixed recurrence relation (3.10) written in the form

$$G_{k+1}^{p,q,c,d} = G_k^{p,q,a,b} + \left(\frac{p}{q} + \frac{-b\,d + b\,c\,p + a\,c\,q}{-b^2 + a\,b\,p + a^2\,q} \right) G_{k+1}^{p,q,a,b} + \left(-\frac{1}{q} + \frac{-b\,c + a\,d}{-b^2 + a\,b\,p + a^2\,q} \right) G_{k+2}^{p,q,a,b}, \tag{4.5}$$

into relation (4.4), then after collecting the similar terms, the following identity can be obtained:

$$M(x) = x^{k+1}\left(\frac{-b^2 + a\,q(a + c\,q\,x) + b(a\,p + (-d + c\,p)q\,x)}{q(-b\,d + b\,c\,p + a\,c\,q)}\,G_{k+2}^{p,q,a,b} + \frac{p\left(b^2 - a\,b\,p - a^2\,q\right)}{q(-b\,d + b\,c\,p + a\,c\,q)}\,G_{k+1}^{p,q,a,b}\right.$$
$$\left.\frac{b^2 - a\,b\,p - a^2\,q}{-b\,d + b\,c\,p + a\,c\,q}\,G_k^{p,q,a,b}\right). \tag{4.6}$$

In virtue of the recurrence relation (1.4), and doing some simplifications enable one to reduce $M(x)$ in the form

$$M(x) = x^{k+2}\,G_{k+2}^{p,q,a,b},$$

and hence the following formula is obtained

$$x^{k+1}\,G_{k+1}^{p,q,a,b} = a + \frac{1}{-b\,d + a\,c\,q + b\,c\,p}\sum_{i=0}^{k} x^i\left((a\,d - b\,c + x(-b\,d + a\,c\,q + b\,c\,p))G_{i+1}^{p,q,a,b}\right.$$
$$\left. + \left(b^2 - a^2 q - a\,b\,p\right)G_i^{p,q,c,d}\right).$$

This completes the proof of Theorem 4.1. □

Several special formulas involving some generalized Fibonacci and generalized Lucas numbers can be deduced as special cases of Theorem 4.1. The following corollaries display some identities between the generalized numbers of four parameters and the generalized Fibonacci and generalized Lucas numbers that are generated respectively by the two recurrence relations in (1.1) and (1.2).

Corollary 4.1. *Let k be a non-negative integer, and $x \in R^*$. For $c\,p \neq d$, one has*

$$x^{k+1}\,U_{k+1}^{p,q} = \frac{1}{-d + c\,p}\sum_{i=0}^{k} x^i\left(G_i^{p,q,c,d} + (-c + (-d + c\,p)x)U_{i+1}^{p,q}\right). \tag{4.7}$$

Proof. The substitution by $a = 0, b = 1$ in Identity (4.1) yields relation (4.7). □

Corollary 4.2. *Let k be a non-negative integer, and $x \in R^*$. Then for $b \neq 0$, one has*

$$x^{k+1}\,G_{k+1}^{p,q,a,b} = a - \frac{1}{b}\sum_{i=0}^{k} x^i\left(\left(b^2 - a\,b\,p - a^2 q\right)U_i^{p,q} + (a - b\,x)\,U_{i+1}^{p,q}\right). \tag{4.8}$$

Proof. The substitution by $c = 0, d = 1$ in Identity (4.1) yields relation (4.8). □

Corollary 4.3. *Let k be a non-negative integer, and $x \in R^*$. Then for $c(p^2 + 2\,q) \neq d\,p$, one has*

$$x^{k+1}\,V_{k+1}^{p,q} = 2 + \frac{1}{-d\,p + c\,(p^2 + 2\,q)}\times$$
$$\sum_{i=0}^{k} x^i\left(\left(-p^2 - 4\,q\right)G_i^{p,q,c,d} + \left(2d - c\,p + \left(-d\,p + c\,p^2 + 2\,c\,q\right)x\right)V_{i+1}^{p,q}\right). \tag{4.9}$$

Proof. The substitution by $a = 2, b = p$ in Identity (4.1) yields relation (4.9). □

Corollary 4.4. *Let k be a non-negative integer, and $x \in R^*$. Then for $b\,p + 2\,a\,q \neq 0$, one has*

$$x^{k+1}\,G_{k+1}^{p,q,a,b} = a + \frac{1}{b\,p + 2\,a\,q} \times \sum_{i=0}^{k} x^i \left(\left(b^2 - a\,b\,p - a^2\,q\right) V_i^{p,q} + (-2\,b + a\,p + (b\,p + 2\,a\,q)\,x)\,G_{i+1}^{p,q,a,b} \right). \tag{4.10}$$

Proof. The substitution by $c = 2, d = p$ in Identity (4.1) yields relation (4.10). □

Taking into consideration the special number sequences mentioned in Table 1, the following corollaries provide some of the special formulas.

Corollary 4.5. *For the generalized classes of Fibonacci and Lucas numbers that generated respectively by the two recurrence relations* (1.1) *and* (1.2)*, the following two identities hold if k is a non-negative integer, $x \in R^*$, and $p \neq 0$*

$$x^{k+1}\,U_{k+1}^{p,q} = \frac{1}{p} \sum_{i=0}^{k} x^i \left((-2 + px)\,U_{i+1}^{p,q} + V_i^{p,q} \right), \tag{4.11}$$

$$x^{k+1}\,V_{k+1}^{a,b} = 2 - \frac{1}{p} \sum_{i=0}^{k} x^i \left((2 - px)\,V_{i+1}^{p,q} + \left(-p^2 - 4q\right)\,U_i^{p,q} \right). \tag{4.12}$$

Proof. Identity (4.11) is a direct consequence of Identity (4.7) for the special case corresponding to $c = 2$ and $d = p$, while Identity (4.12) is a direct consequence of Identity (4.10) for the special case corresponding to $c = 0$ and $d = 1$. □

Remark 4.1. *The Identity* (4.11) *coincides with that obtained in [33].*

Remark 4.2. *Since the Fibonacci, Pell, Fermat, Jacobsthal and balancing numbers are special ones of the generalized Fibonacci numbers $U_k^{p,q}$, and the Lucas, Pell-Lucas, Fermat-Lucas, Jacobsthal-Lucas and co-balancing numbers are special ones of the generalized Lucas numbers $V_k^{p,q}$, so the two Identities* (4.11) *and* (4.12) *lead to some interesting identities concerning these celebrated numbers. The following corollaries display these identities.*

Corollary 4.6. *For Fibonacci and Lucas numbers, the following two identities hold for every non-negative integer k, and every $x \in R^*$:*

$$x^{k+1}\,F_{k+1} = \sum_{i=0}^{k} x^i\,\left(L_i + (-2 + x)\,F_{1+i}\right), \tag{4.13}$$

$$x^{k+1}\,L_{k+1} = 2 - \sum_{i=0}^{k} x^i\,\left(-5\,F_i + (2 - x)L_{1+i}\right). \tag{4.14}$$

Proof. If we set $a = 1$ and $b = 1$ in (4.11) and (4.12) respectively, the two Identities (4.13) and (4.14) can be obtained. □

Corollary 4.7. *For Pell and Pell-Lucas numbers, the following two identities hold for every non-negative integer k, and every $x \in R^*$:*

$$x^{k+1} P_{k+1} = \frac{1}{2} \sum_{i=0}^{k} x^i \left(Q_i + (-2 + 2x) P_{i+1}\right), \tag{4.15}$$

$$x^{k+1} Q_{k+1} = 2 - \frac{1}{2} \sum_{i=0}^{k} x^i (-8 P_i + (2 - 2x) Q_{i+1}). \tag{4.16}$$

Proof. If we set $a = 2$ and $b = 1$ in (4.11) and (4.12) respectively, the two Identities (4.15) and (4.16) can be obtained. □

Corollary 4.8. *For Fermat and Fermat-Lucas numbers, the following two identities hold for every non-negative integer k, and every $x \in R^*$:*

$$x^{k+1} \mathcal{F}_{k+1} = \frac{1}{3} \sum_{i=0}^{k} x^i \left((-2 + 3x)\mathcal{F}_{i+1} + f_i\right), \tag{4.17}$$

$$x^{k+1} f_{k+1} = 2 - \frac{1}{3} \sum_{i=0}^{k} x^i \left((2 - 3x) f_{i+1} - \mathcal{F}_i\right). \tag{4.18}$$

Proof. If we set $a = 3$ and $b = -2$ in (4.11) and (4.12) respectively, the two Identities (4.17) and (4.18) can be obtained. □

Corollary 4.9. *For Jacobsthal and Jacobsthal-Lucas numbers, the following two identities hold for every non-negative integer k, and every $x \in R^*$:*

$$x^{k+1} \mathcal{J}_{k+1} = \sum_{i=0}^{k} x^i \left((-2 + x)\mathcal{J}_{i+1} + j_i\right), \tag{4.19}$$

$$x^{k+1} j_{k+1} = 2 - \sum_{i=0}^{k} x^i \left((2 - x) j_{i+1} - 9 J_i\right). \tag{4.20}$$

Proof. If we set $a = 1$ and $b = 2$ in (4.11) and (4.12) respectively, the two Identities (4.19) and (4.20) can be obtained. □

Corollary 4.10. *For balancing and co-balancing numbers, the following two identities hold for every non-negative integer k, and every $x \in R^*$:*

$$x^{k+1} B_{k+1} = \frac{1}{6} \sum_{i=0}^{k} x^i \left((-2 + 6x) B_{i+1} + b_i\right), \tag{4.21}$$

$$x^{k+1} b_k = 2 - \frac{1}{6} \sum_{i=0}^{k} x^i \left((2 - 6x) b_{i+1} - 32 B_i\right). \tag{4.22}$$

Proof. If we set $a = 6$ and $b = -1$ in (4.11) and (4.12) respectively, the two Identities (4.21) and (4.22) can be obtained. □

5. Some other identities involving the generalized numbers

This section concentrates on introducing some other identities concerned with the Horadam generalized numbers $G_k^{p,q,a,b}$ and some of their special numbers.

Theorem 5.1. *For every non-negative integer k, every $x \in R^*$, and for $b\,c\,p + a\,c\,q \neq b\,d$, one has*

$$(k+1)\,x^{k+1}\,G_{k+1}^{p,q,a,b} = \sum_{i=0}^{k} x^i \left(\left(\frac{(-b\,c + a\,d)i}{-bd + b\,c\,p + a\,c\,q} + (i+1)x\right) G_{i+1}^{,p,q,a,b} + \frac{i\left(b^2 - abp - a^2 q\right)}{-b\,d + b\,c\,p + a\,c\,q}\,G_i^{p,q,c,d}\right). \tag{5.1}$$

Proof. If we differentiate both sides of (4.1) with respect to x, then Identity (5.1) can be obtained. □

Taking into consideration the special numbers of the generalized numbers $G_{i+1}^{p,q,a,b}$ that can be deduced by choosing the four parameters p, q, a and b suitably, then some identities can be deduced are direct consequences of Theorem 5.1. The following theorem exhibits these identities.

Corollary 5.1. *For the generalized Fibonacci and generalized Lucas numbers, the following two identities hold for every non-negative integer k, every $x \in R^*$, and for $p \neq 0$:*

$$(k+1)\,x^{k+1}\,U_{k+1}^{p,q} = \sum_{i=0}^{k} x^i \left(\left(-\frac{2i}{p} + (1+i)x\right) U_{i+1}^{p,q} + \frac{i}{p}\,V_i^{p,q}\right), \tag{5.2}$$

$$(k+1)\,x^{k+1}\,V_{k+1}^{p,q} = \sum_{i=0}^{k} x^i \left(\frac{i\left(p^2 + 4\,q\right)}{p}\,U_i^{p,q} + \left(-\frac{2i}{p} + (1+i)x\right) V_{i+1}^{p,q}\right). \tag{5.3}$$

Corollary 5.2. *For Fibonacci and Lucas numbers, the following two identities hold for every non-negative integer k, every $x \in R^*$:*

$$(k+1)\,x^{k+1}\,F_{k+1} = \sum_{i=0}^{k} x^i\,\left((-2i + (1+i)x)\,F_{i+1} + i\,L_i\right), \tag{5.4}$$

$$(k+1)\,x^{k+1}\,L_{k+1} = \sum_{i=0}^{k} x^i\,\left((-2\,i + (i+1)x)\,L_{1+i} + 5\,i\,F_i\right). \tag{5.5}$$

Corollary 5.3. *For Pell and Pell-Lucas numbers, the following two identities hold for every non-negative integer k, every $x \in R^*$:*

$$(k+1)\,x^{k+1}\,P_{k+1} = \sum_{i=0}^{k} x^i \left((-i + (1+i)x)\,P_{i+1} + \frac{1}{2}\,i\,Q_i\right), \tag{5.6}$$

$$(k+1)\,x^{k+1}\,Q_{k+1} = \sum_{i=0}^{k} x^i\,\left((-i + (1+i)\,x)\,Q_{i+1} + 4\,i\,P_i\right). \tag{5.7}$$

Corollary 5.4. *For Fermat and Fermat-Lucas numbers, the following two identities hold for every non-negative integer k, every $x \in R^*$:*

$$(k+1)\,x^{k+1}\,\mathcal{F}_{k+1} = \sum_{i=0}^{k} x^i \left(\left(-\frac{2\,i}{3} + (i+1)x\right) \mathcal{F}_{i+1} + \frac{1}{3}\,i\,f_i\right), \tag{5.8}$$

$$(k+1)\,x^{k+1}\,f_{k+1} = \sum_{i=0}^{k} x^i \left(\left(-\frac{2\,i}{3} + (i+1)x\right) f_{i+1} + \frac{1}{3}\,i\,F_i\right). \tag{5.9}$$

Corollary 5.5. *For Jacobsthal and Jacobsthal-Lucas numbers, the following two identities hold for every non-negative integer k, every $x \in R^*$:*

$$(k+1)\,x^{k+1}\,J_{k+1} = \sum_{i=0}^{k} x^i\,\left(i\,j_i + (-2\,i + (i+1)\,x)\,J_{i+1}\right), \tag{5.10}$$

$$(k+1)\,x^{k+1}\,j_{k+1} = \sum_{i=0}^{k} x^i\,\left((-2\,i + (i+1)\,x)\,j_{i+1} + 9\,i\,J_i\right). \tag{5.11}$$

Corollary 5.6. *For balancing and co-balancing numbers, the following two identities hold for every non-negative integer k, every $x \in R^*$:*

$$(k+1)\,x^{k+1}\,B_{k+1} = \sum_{i=0}^{k} x^i\,\left(\left(-\frac{i}{3} + (i+1)x\right) B_{i+1} + \frac{1}{6}\,i\,b_i\right), \tag{5.12}$$

$$(k+1)\,x^{k+1}\,b_i = \sum_{i=0}^{k} x^i\,\left(\left(-\frac{i}{3} + (i+1)\,x\right) b_{i+1} + \frac{16}{3}\,i\,B_i\right). \tag{5.13}$$

Remark 5.1. *All the above identities in Sections 4 and 5 can be generalized if both sides of Eq* (4.1) *is differentiated $r-$ times. The following theorem is a generalization of Theorem 4.1.*

Theorem 5.2. *For all non-negative integer k, r, every $x \in R^*$, and for $b\,c\,p + a\,c\,q \neq b\,d$, the following identity is valid:*

$$\begin{aligned}(k-r+2)_r\;x^{k+1}\;G_{k+1}^{p,q,a,b} =& \sum_{i=0}^{k} x^i \left(\left(\frac{(-b\,c + a\,d)(1+i-r)}{-b\,d + b\,c\,p + a\,c\,q} + (1+i)x\right)(2+i-r)_{r-1}\,G_{i+1}^{p,q,a,b}\right.\\ &\left. + \frac{\left(b^2 - a\,b\,p - a^2\,q\right)(1+i-r)_r}{-b\,d + b\,c\,p + a\,cq}\;\;G_i^{p,q,c,d}\right).\end{aligned} \tag{5.14}$$

Proof. Differentiating both sides of (4.1) with respect to x yields the following identity:

$$\begin{aligned}(k-r+2)_r\,x^{k-r+1}\,U_{k+1}^{p,q,a,b} =& \sum_{i=0}^{k}\left(x^{i-r}\,\xi\,(i+1-r)_r\,G_{i+1}^{p,q,a,b}\right.\\ &\left. +\;x^{i-r+1}\,(i-r+2)_r\,G_{i+1}^{p,q,a,b} + \gamma\,x^{i-r}\,(i-r+1)_r\,V_i^{p,q,c,d}\right).\end{aligned} \tag{5.15}$$

with

$$\xi = \frac{a\,d - b\,c}{-b\,d + a\,c\,q + b\,c\,p}, \quad \eta = \frac{b^2 - a^2 q - a\,bp}{-b\,d + b\,c\,p + a\,c\,q},$$

that can be written after some simplifications in the form

$$\begin{aligned}(k-r+2)_r\;x^{k+1}\;G_{k+1}^{p,q,a,b} =& \sum_{i=0}^{k} x^i \left(\left(\frac{(-bc + ad)(1+i-r)}{-bd + bcp + acq} + (1+i)x\right)(i-r+2)_{r-1}\;G_{i+1}^{p,q,a,b}\right.\\ &\left. + \frac{\left(b^2 - abp - a^2q\right)(1+i-r)_r}{-bd + bcp + acq}\;\;G_i^{p,q,c,d}\right).\end{aligned}$$

□

In the following, we write the generalizations of Corollaries 5.1–5.6 which are special cases of the general result in Theorem 5.2. They can be deduced by selecting the six parameters p, q, a, b, c and d that appears in Identity (5.14).

Corollary 5.7. *For all non-negative integer k, r, every $x \in R^*$, and for $p \neq 0$, the following two identities are valid:*

$$\begin{aligned}(k-r+2)_r\, x^{k+1}\, U_{k+1}^{p,q} &= \sum_{i=0}^{k} x^i \left(\left(-\frac{2(1+i-r)}{p} + (1+i)x\right)(2+i-r)_{r-1}\, U_{i+1}^{p,q}\right. \\ &\quad \left. + \frac{(1+i-r)_r}{p} V_i^{p,q}\right),\end{aligned} \tag{5.16}$$

$$\begin{aligned}(k-r+2)_r\, x^{k+1}\, V_{k+1}^{p,q} &= \sum_{i=0}^{k} x^i \left(\left(\frac{(-bc+ad)(1+i-r)}{-bd+bcp+acq} + (1+i)x\right)(2+i-r)_{r-1} V_{i+1}^{p,q}\right. \\ &\quad \left. + \frac{\left(b^2 - a\,b\,p - a^2 q\right)(1+i-r)_r}{-bd+b\,c\,p+acq}\, U_i^{p,q}\right).\end{aligned} \tag{5.17}$$

Corollary 5.8. *For Fibonacci and Lucas numbers, the following two identities hold for all non-negative integers k and r, every $x \in R^*$:*

$$\begin{aligned}(k-r+2)_r\, x^{k+1}\, F_{k+1} &= \sum_{i=0}^{k} x^i\, ((-2(1+i-r) + (1+i)x)(2+i-r)_{r-1}\, F_{i+1} \\ &\quad + (1+i-r)_r\, L_i),\end{aligned} \tag{5.18}$$

$$(k-r+2)_r\, x^{k+1} L_{k+1} = \sum_{i=0}^{k} x^i\, (5(1+i-r)_r\, F_i + (-2(1+i-r) \tag{5.19}$$

$$+ (1+i)x)(2+i-r)_{r-1}\, L_{1+i}). \tag{5.20}$$

Corollary 5.9. *For Pell and Pell-Lucas numbers, the following two identities hold for all non-negative integers k and r, every $x \in R^*$:*

$$\begin{aligned}x^{k+1}\,(2+k-r)_r\, P_{k+1} &= \sum_{i=0}^{k} x^i\, ((-1-i+r+(1+i)x)\,(2+i-r)_{r-1} P_{i+1} \\ &\quad \left. + \frac{1}{2}(1+i-r)_r\, Q_i, \right),\end{aligned} \tag{5.21}$$

$$\begin{aligned}(k-r+2)_r\, x^{k+1}\, Q_{k+1} &= \sum_{i=0}^{k} x^i\, ((-1-i+r+(1+i)\,x)(2+i-r)_{r-1}\, Q_{i+1} \\ &\quad + 4(1+i-r)_r\, P_i).\end{aligned} \tag{5.22}$$

Corollary 5.10. *For Fermat and Fermat-Lucas numbers, the following two identities hold for all non-negative integers k and r, every $x \in R^*$:*

$$(k-r+2)_r\, x^{k+1}\, \mathcal{F}_{k+1} = \sum_{i=0}^{k} x^i \left(\left(\frac{1}{3}(-2)(1+i-r)+(1+i)x\right)(2+i-r)_{r-1}\, \mathcal{F}_{i+1} + \frac{1}{3}(1+i-r)_r\, f_i\right), \tag{5.23}$$

$$(k-r+2)_r\, x^{k+1}\, f_{k+1} = \sum_{i=0}^{k} x^i \left(\left(-\frac{2}{3}(1+i-r)+(1+i)x\right)(2+i-r)_{r-1}\, f_{i+1}) \tag{5.24}$$

$$+ \frac{1}{3}(1+i-r)_r\, \mathcal{F}_i\right). \tag{5.25}$$

Corollary 5.11. *For Jacobsthal and Jacobsthal-Lucas numbers, the following two identities hold for all non-negative integers k and r, every $x \in R^*$:*

$$(k-r+2)_r x^{k+1}\, J_{k+1} = \sum_{i=0}^{k} x^i\, ((-2(1+i-r)+(1+i)x)(2+i-r)_{r-1}\, J_{i+1} + (1+i-r)_r j_i), \tag{5.26}$$

$$(k-r+2)_r\, x^{k+1}\, j_{k+1} = \sum_{i=0}^{k} x^i\, ((-2(1+i-r)+(1+i)x)(2+i-r)_{r-1}\, j_{i+1} + 9(1+i-r)_r\, J_i). \tag{5.27}$$

Corollary 5.12. *For balancing and co-balancing numbers, the following two identities hold for all non-negative integers k and r, every $x \in R^*$:*

$$(k-r+2)_r\, x^{k+1}\, B_{k+1} = \sum_{i=0}^{k} x^i \left(\left(\frac{1}{3}(-1-i+r)+(1+i)x\right)(2+i-r)_{r-1}\, B_{i+1} + \frac{1}{6}(1+i-r)_r\, b_i\right), \tag{5.28}$$

$$(k-r+2)_r\, x^{k+1}\, b_{k+1} = \sum_{i=0}^{k} x^i \left(\left(\frac{1}{3}(-1-i+r)+(1+i)x\right)(2+i-r)_{r-1}\, b_{i+1} + \frac{16}{3}(1+i-r)_r\, B_i\right). \tag{5.29}$$

6. Conclusions

In this paper, we have investigated Horadam generalized numbers involving four parameters that generalize some generalized and specific types of Fibonacci and Lucas numbers. A new identity involving six parameters was derived. Some of the interesting identities involving generalized and specific numbers such as Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, balancing, and co-balancing numbers were extracted as special cases. Some well-known identities in the literature can be extracted from our identities as special cases by selecting suitably the involved parameters. In future work, we aim to perform some other studies regarding these generalized numbers. In addition, we aim to study other types of generalized Fibonacci and Lucas numbers.

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Conflict of interest

The authors declare that they have no competing interests.

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