AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 12952–12961.
DOI: 10.3934/math.2022717
Received: 07 January 2022
Revised: 06 April 2022
Accepted: 24 April 2022
Published: 07 May 2022

Research article

Parameter estimation for partially observed stochastic differential equations driven by fractional Brownian motion

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Abstract: This paper is concerned with parameter estimation for partially observed stochastic differential equations driven by fractional Brownian motion. Firstly, the state estimation equation is given and the parameter estimator is derived. Then, the strong consistency and asymptotic normality of the maximum likelihood estimator are derived by applying the strong law of large numbers for continuous martingales and the central limit theorem for stochastic integrals with respect to Gaussian martingales. Finally, an example is provided to verify the results.

Keywords: parameter estimation; partially observed stochastic differential equations; fractional Brownian motion; strong consistency; asymptotic normality
Mathematics Subject Classification: 60H10, 62F12

1. Introduction

Almost all systems are affected by noise and possess certain random characteristics. Therefore, it is reasonable and necessary to use random systems to model actual systems. When modeling or optimizing a stochastic system, due to the complexity of the internal structure and the uncertainty of the external environment, system parameters are unknown. It is necessary to use theoretical tools to estimate the system parameters. In the last few decades, some authors have studied the parameter estimation problem for stochastic models driven by Brownian motion [2, 9, 18]. For example, Ding et al. [4] implemented a least squares algorithm for parameter estimation for stochastic dynamical systems with ARMA noise using the model equivalence. Ji et al. [8] investigated the use of a hierarchical least squares parameter estimation algorithm for two-input Hammerstein finite impulse response systems. Shen et al. [21] analyzed parameter estimation for the discretely observed Vasicek model with small fractional Lévy noise. Wang et al. [22] developed a recursive parameter estimation algorithm for multivariate output-error systems. Wei and Shu [23] studied the existence, consistency and asymptotic normality of the maximum likelihood estimator for the nonlinear stochastic differential equation. Wei [25] used a least squares method to obtain the estimators of a stochastic Lotka-Volterra model driven by small α-stable noises and discussed the consistency and asymptotic distribution of the estimators. Long memory processes have been widely applied in various fields, such as finance, hydrology and network traffic analysis. The fractional Brownian motion, which is a suitable generalization of Brownian motion, is one of the simplest stochastic processes exhibiting long-range dependence. When a long-memory model is used to describe some phenomena, it is important to identify the parameters in the model. Therefore, some authors investigated the parameter estimation problem for stochastic models driven by fractional Brownian motion [10, 16, 17, 26]. For example, Dai et al. [3] derived the Girsanov formula for the stochastic differential equation driven by fractional Brownian motion and used maximum likelihood estimation to estimate the parameters. Hu et al. [6] discussed the strong consistency of the least squares estimator for the fractional stochastic differential system. Prakasa Rao [19] studied parameter estimation for models governed by a stochastic differential equation driven by mixed fractional Brownian motion with Gaussian random effects based on discrete observations.

When controlling a system and due to some reasons, the state of a system cannot be directly obtained or the cost of obtaining the system state is high, it is necessary to estimate the state of the system by using some algorithms. In the last few decades, some authors have investigated the state estimation problem for stochastic systems [1, 5, 11, 12]. When parameters and state are unknown simultaneously, it is necessary to combine the use of theory and algorithms to estimate the parameters and state. For example, for a system that is observed partially, Imani and Braga-Neto [7] presented a framework for the simultaneous estimation of the state and parameters of partially observed Boolean dynamical systems. Onsy et al. [14] studied the parameter estimation problem for the Ornstein-Uhlenbeck process with long-memory noise. Rathinam and Yu [20] discussed state and parameter estimation from the perspective of exact partial state observation in stochastic reaction networks. Wei [24] analyzed state and parameter estimation for nonlinear stochastic systems by using extended Kalman filtering.

Although the parameter estimation problem for stochastic differential equations has been studied by many authors, there is minimal literature on parameter estimation for partially observed stochastic differential equations driven by fractional Brownian motion. In this paper, we investigate this topic. We give the state estimation equation and obtain the parameter estimator. We prove the strong consistency and asymptotic normality of the maximum likelihood estimator by applying the strong law of large numbers for continuous martingales and the central limit theorem for stochastic integrals with respect to Gaussian martingales.

The paper is organized as follows. In Section 2, we give some assumptions and definitions and derive the state estimation equation and maximum likelihood estimator. In Section 3, we derive the strong consistency and asymptotic normality of the estimator. In Section 4, an example is provided. The conclusion is given in Section 5.

2. Problem formulation and preliminaries

Let $(\Omega, F, \mathbb{P})$ be a basic probability space equipped with a right continuous and increasing family of σ-algebras $\{F_t\}_{t\geq 0}$.

Here, we consider the following partially observed stochastic differential equations driven by fractional Brownian motion:

$$\begin{cases} dY_t = \theta f(X_t)dt + g(X_t)dW_t^H \\ dX_t = m(X_t)dt + n(X_t)dV_t, \quad t \in [0, T], \\ Y_0 = \xi, X_0 = \eta, \end{cases} \tag{2.1}$$

where $W^H, H \in (\frac{1}{2}, 1)$ represents the fractional Brownian motion, V denotes standard Brownian motion independent of W^H 0 and θ is an unknown parameter to be estimated on the observation $\{Y_t, 0 \le t \le T\}$. We assume that the conditional distribution of ξ and η is some fixed π_0.

Firstly, we introduce some assumptions below.

Assumption 1. $|f(x)| + |g(x)| \le K(1 + |X|)$ *for all* $t \in [0, T]$ *where* $K > 0$ *is constant.*

Assumption 2. $|f(x) - f(y)| + |g(x) - g(y)| \le K_1(|x - y|)$ *for all* $t \in [0, T]$ *where* $K_1 > 0$ *is constant.*

Assumption 3. $|m(x)| + |n(x)| \le K_2(1 + |X|)$ *for all* $t \in [0, T]$ *where* $K_2 > 0$ *is constant.*

Assumption 4. $|m(x) - m(y)| + |n(x) - n(y)| \le K_3(|x - y|)$ *for all* $t \in [0, T]$ *where* $K_3 > 0$ *is constant.*

Remark 1. *Assumptions* 1 *and* 3 *are the linear growth condition and Assumptions* 2 *and* 4 *are the Lipschitz condition. From Mao [13], it can be confirmed that the stochastic differential equation (2.1) has a unique solution.*

Let $\Omega = C([0, T]; \mathbb{R}^2)$ be the space of continuous functions from $[0, T]$ into $\mathbb{R}^2$. Consider the canonical process $(X, W^*) = (X_t, W_t^*, t \in [0, T])$ on Ω where $(X_t, W_t^*)(x, y) = (x_t, y_t)$ for any $(x, y) \in \Omega$. The probability $\widetilde{\mathbb{P}}$ denotes the unique probability measure on Ω such that defining the variable ξ by $\xi = W_0^*$ and $\widetilde{W} = (\widetilde{W}_t), t \in [0, T]$ by $\widetilde{W}_t = W_t^* - W_0^*, t \in [0, T]$, the pair (X, ξ) is independent of $\widetilde{W}$ and the process $\widetilde{W}$ is fractional Brownian motion with the Hurst parameter H. The canonical filtration on Ω is $(F_t, t \in [0, T])$ where $F_t = \sigma\{(X_s, W_s^*), 0 \le s \le t\} \vee \mathcal{N}$ with $\mathcal{N}$ denoting the set of null sets of $(\Omega, \widetilde{\mathbb{P}})$.

Define the function $a(\theta, x)$ on $[0, T]$ for all continuous functions $x = (x_t, t \in [0, T])$ by

$$a(\theta, x)(t) = \frac{\theta f(x_t)}{g(x_t)}, t \in [0, T]). \tag{2.2}$$

Let $k_{a(\theta,x)}^t = (k_{a(\theta,x)}^t)$.

Define the processes $N = (N_t, t \in [0, T])$ and $< N >= (< N >_t, t \in [0, T])$ as follows:

$$N_t := N_t^{a(\theta,X)}, \quad < N >_t := < N^{a(\theta,X)} >_t . \tag{2.3}$$

Notice that N_t and $< N >_t$ depend only on the values of $X^{(t)} = (X_s, 0 \le s \le t)$.

Let

$$< N, N^* >_t := < N^{a(\theta,X)}, N^* >_t = \int_0^t k_*^t(s) a(\theta, X)(s) ds, \quad t \in [0, T], \tag{2.4}$$

and

$$b_t(\theta, X) := b_t^{a(\theta,X)} = \frac{d < N, N^* >_t}{d < N^* >_t}, \quad t \in [0, T], \tag{2.5}$$

where $\widetilde{b}_t(X) := \frac{b_t(\theta,X)}{\theta}$.

Define the processes

$$\widetilde{N}_t(\theta, x) = \int_0^t k^t_{a(\theta,x)} d\widetilde{W}^H_s, \quad < \widetilde{N} >_t (\theta, x) := \int_0^t a(\theta, x)(s) k^t_h(s) ds, \quad t \in [0, T], \tag{2.6}$$

where $\widetilde{N}_t(\theta, x)$ is a Gaussian martingale under $\widetilde{\mathbb{P}}$.

Let

$$\Lambda_t(\theta, x) = \exp\{\widetilde{N}_t(\theta, x) - \frac{1}{2} < \widetilde{N} >_t (\theta, x)\}, \quad t \in [0, T], \tag{2.7}$$

and

$$\Lambda_t(\theta) = \Lambda_t(\theta, X). \tag{2.8}$$

Let $\mathbb{P} = \Lambda_T(\theta)\widetilde{\mathbb{P}}$, $\mathcal{Y}_t = \sigma(\{Y_s, 0 \le s \le t\}), t \in [0, T]$, the optimal filter $\pi_t(\phi) = \mathbb{E}[\phi(X_t)|\mathcal{Y}_t]$ and the unnormalized filter $\sigma_t(\phi) = \widetilde{\mathbb{E}}[\phi(X_t)\Lambda_t|\mathcal{Y}_t]$, $t \in [0, T]$.

Then, for all $t \in [0, T]$, it can be checked that

$$\pi_t(\phi) = \frac{\sigma_t(\phi)}{\sigma_t(1)}. \tag{2.9}$$

Define

$$Z_t = \int_0^t k^t_{a(\theta,X)}(s) g^{-1}(X_s) dY_s, \quad t \in [0, T], \tag{2.10}$$

and

$$Z^*_t = \int_0^t k^t_*(s) g^{-1}(X_s) dY_s, \quad t \in [0, T]. \tag{2.11}$$

Thus, it can be checked that the processes Z and Z^* are semimartingales with the following decomposition:

$$Z_t =< N >_t + N_t, \quad t \in [0, T], \tag{2.12}$$

and

$$Z^*_t =< N, N^* >_t + N^*_t, \quad t \in [0, T]. \tag{2.13}$$

Then, we have

$$Z_t = \int_0^t b^2_s(\theta, X) d < N^* >_s + \int_0^t b_s(\theta, X) dN^*_s, \quad t \in [0, T], \tag{2.14}$$

and

$$Z^*_t = \int_0^t b_s(\theta, X) d < N^* >_s + N^*_t, \quad t \in [0, T]. \tag{2.15}$$

Thus, we obtain

$$Z_t = \int_0^t b_s(\theta, X) dZ^*_t, \quad t \in [0, T]. \tag{2.16}$$

Let

$$\nu_t = Z^*_t - \int_0^t \pi_s(b) d < N^* >_s, \quad t \in [0, T], \tag{2.17}$$

which plays the role of the innovation process in the usual situation where the noise is Brownian motions.

Recall the notation $\pi_s(b) = \mathbb{E}[b_s(\theta, X)|\mathcal{Y}_s], \quad 0 \le s \le t$.
The particular case of an unnormalized filter is

$$\widetilde{\Lambda}_t(\theta) = \sigma_t(1) = \widetilde{\mathbb{E}}[\Lambda_t|\mathcal{Y}_t], \quad t \in [0, T]. \tag{2.18}$$

Then, we have

$$\widetilde{\Lambda}_T(\theta, \mathcal{Y}_t) = \exp\{\theta \int_0^T \pi_s(\widetilde{b}) dZ_s^* - \frac{\theta^2}{2} \int_0^T \pi_s^2(\widetilde{b}) d < N^* >_s\}. \tag{2.19}$$

Therefore, the maximum likelihood estimator of θ is

$$\widehat{\theta}_T = \frac{\int_0^T \pi_s(\widetilde{b}) dZ_s^*}{\int_0^T \pi_s^2(\widetilde{b}) d < N^* >_s}. \tag{2.20}$$

In the next section, we shall prove the strong consistency and derive the asymptotic normality of the estimator.

3. Main results and proofs

In the following theorem, we prove the strong consistency of the maximum likelihood estimator.

Theorem 1. *Under the Assumptions 1–4, when $T \to \infty$, $\widehat{\theta}_T$ is a strongly consistent estimator of θ, namely*

$$\widehat{\theta}_T \overset{a.s.}{\to} \theta.$$

Proof. Note that

$$dZ_t^* = \pi_t(b) d < N^* >_t + d\nu_t. \tag{3.1}$$

It is known that ν is a continuous Gaussian martingale on $(\mathcal{Y}_t, \mathbb{P})$ such that $< \nu > = < N^* >$.
Then, we have

$$\widehat{\theta}_T = \frac{\int_0^T \pi_s(\widetilde{b}) dZ_s^*}{\int_0^T \pi_s^2(\widetilde{b}) d < N^* >_s} = \theta + \frac{\int_0^T \pi_s(\widetilde{b}) d\nu_s}{\int_0^T \pi_s^2(\widetilde{b}) d < N^* >_s}, \tag{3.2}$$

which means that

$$\widehat{\theta}_T - \theta = \frac{\int_0^T \pi_s(\widetilde{b}) d\nu_s}{\int_0^T \pi_s^2(\widetilde{b}) d < N^* >_s}. \tag{3.3}$$

By the strong law of large numbers for continuous martingales, we have

$$\frac{\int_0^T \pi_s(\widetilde{b}) d\nu_s}{\int_0^T \pi_s^2(\widetilde{b}) d < N^* >_s} \overset{a.s.}{\to} 0. \tag{3.4}$$

Therefore, we obtain

$$\widehat{\theta}_T \overset{a.s.}{\to} \theta.$$

The proof is complete. □

Remark 2. *According to Assumptions 1–4, we could also obtain that*

$$\limsup_{T} \frac{A_T^{\frac{1}{2}}|\widehat{\theta}_T - \theta|}{(2\log\log A_T)^{\frac{1}{2}}} = 1, a.s.$$

where $A_T = \int_0^T \pi_s^2(\widetilde{b})d < N^* >_s$.

In the following theorem, the asymptotic normality of the estimator is proved.

Theorem 2. *Under the Assumptions 1–4, when* $T \to \infty$,

$$\sqrt{\int_0^T \pi_s^2(\widetilde{b})d < N^* >_s}(\widehat{\theta}_T - \theta) \xrightarrow{d} N(0,1).$$

Proof.

$$\sqrt{\int_0^T \pi_s^2(\widetilde{b})d < N^* >_s}(\widehat{\theta}_T - \theta) = \frac{\int_0^T \pi_s(\widetilde{b})d\nu_s}{\sqrt{\int_0^T \pi_s^2(\widetilde{b})d < N^* >_s}}.$$

By the central limit theorem for stochastic integrals with respect to Gaussian martingales, it can be checked that

$$\frac{\int_0^T \pi_s(\widetilde{b})d\nu_s}{\sqrt{\int_0^T \pi_s^2(\widetilde{b})d < N^* >_s}} \xrightarrow{d} N(0,1).$$

Therefore,

$$\sqrt{\int_0^T \pi_s^2(\widetilde{b})d < N^* >_s}(\widehat{\theta}_T - \theta) \xrightarrow{d} N(0,1). \tag{3.5}$$

The proof is complete. □

4. Example

Consider the following stochastic system with fractional Brownian motion observation noise

$$\begin{cases} dY_t = \theta X_t dt + dW_t^H \\ dX_t = -X_t dt + dV_t, \quad t \in [0,T], \\ Y_0 = 0, X_0 = 0, \end{cases} \tag{4.1}$$

where $W^H, H \in (\frac{1}{2}, 1)$ represents the fractional Brownian motion, V denotes standard Brownian motion independent of W^H and $\theta < 0$ is an unknown parameter to be estimated on the observation $\{Y_t, 0 \le t \le T\}$.

It is easy to check that the system satisfies the conditions for Assumptions 1–4 mentioned in Section 2.

Let $\widehat{X}_t = \mathbb{E}(X_t|\mathcal{Y}_t)$ and $\lambda_t = \mathbb{E}([X_t - \widehat{X}_t]^2|\mathcal{Y}_t)$.

Then, we obtain

$$\begin{cases} d\widehat{X}_t = -\widehat{X}_t dt + \theta\lambda_t d\nu_t, t \in [0, T] \\ \widehat{X}_0 = 0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} d\lambda_t = dt - 2\lambda_t dt - \theta^2 \lambda_t^2 d < N^* >_t \\ \lambda_0 = 0. \end{cases} \tag{4.3}$$

Thus, when $t \to \infty$, we have

$$\lambda_t \to \frac{-1 + \sqrt{1 + \theta^2}}{\theta^2}. \tag{4.4}$$

Let $\lambda_\theta = \frac{-1+\sqrt{1+\theta^2}}{\theta^2}$.

When the system has reached the steady state, it follows that

$$\begin{cases} d\widehat{X}_t = -\widehat{X}_t dt + \theta\lambda_\theta d\nu_t, t \in [0, T] \\ \widehat{X}_0 = 0, \end{cases} \tag{4.5}$$

Then, we obtain

$$\widehat{X}_t = \theta\lambda_\theta \int_0^t \exp\{\sqrt{1 + \theta^2}(t - s)\} dY_s. \tag{4.6}$$

It is easy to check that the maximum likelihood estimator satisfies the asymptotic properties mentioned in Theorems 1 and 2.

Now we will describe the numerical simulations of the estimator derived in this study. The fractional Brownian motion was simulated by using the Paxson's method [15]. Let $H = 0.75$. In Table 1, T is increasing from 10000 to 50000.

Table 1. Least squares estimator simulation results for θ.

True		Average	Absolute Error
θ	Size T	$\widehat{\theta_T}$	$\lvert\theta - \widehat{\theta_T}\rvert$
1	10000	1.0531	0.0531
	30000	1.0082	0.0082
	50000	1.0007	0.0007
2	10000	2.0439	0.0439
	30000	2.0063	0.0063
	50000	2.0002	0.0002

5. Discussion

There exist several stochastic processes that are self-similar and exhibiting long-range dependence but fractional Brownian motion seems to be one of the simplest. Moreover, when controlling a system, due to some reasons, the state of the system cannot be directly obtained. Therefore, it is of great importance to consider the parameter estimation problem for partially observed stochastic differential equations driven by fractional Brownian motion. Here, we studied the parameter and state estimation problem in the meantime and discussed the strong consistency and asymptotic normality of the maximum likelihood estimator.

6. Conclusions

The aim of this study was to investigate the parameter estimation problem for partially observed stochastic differential equations driven by fractional Brownian motion. The state estimation equation has been given and the parameter estimator has been obtained. The strong consistency and asymptotic normality of the maximum likelihood estimator have been derived by applying the strong law of large numbers for continuous martingales and the central limit theorem for stochastic integrals with respect to Gaussian martingales. Further research will include investigating the parameter estimation problem for stochastic differential equations driven by Lévy noises.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grant 61403248, Soft Science Research Project of Henan Province under Grant 222400410052, Anyang Science and Technology Plan Project under Grant 2021C02ZF016 and the Key Research Projects of Universities under Grant 22A110001.

Conflict of interest

The author declares that there are no conflicts of interest.

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