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Research article

Solving a class of variable order nonlinear fractional integral differential equations by using reproducing kernel function

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Abstract: In this paper, reproducing kernel interpolation collocation method is explored for nonlinear fractional integral differential equations with Caputo variable order. In order to testify the feasibility of this method, several examples are studied from the different values of parameters. In addition, the influence of the parameters of the Jacobi polynomial on the numerical results is studied. Our results reveal that the present method is effective and provide highly precise numerical solutions for solving such fractional integral differential equations.

Keywords: reproducing kernel method; variable fractional order; Jacobi polynomials; nonlinear integral differential equations
Mathematics Subject Classification: 65R20, 65L10

1. Introduction

Fractional order models [1–5] have important applications in materials sciences, information science, and so on [6–14]. Reproducing kernel functions [15, 16] in reproducing kernel Hilbert spaces [17–21] and related theory have important application in stochastic processes, signal analysis, machine learning and pattern recognition [22–32]. the reproducing kernel method [6–14] can not only obtain the exact solution in the form of series but also obtain the approximate solution with higher accuracy, the method has been widely used in linear and nonlinear problems, integral and differential equations, fractional partial differential equation and so on [22–32]. In [33], we use reproducing kernel interpolation collocation method to solve the linear integro differential equations of fractional order.

In this paper, reproducing kernel method with reproducing kernel function in the form of Jacobi polynomials is applied to solving the following variable fractional order nonlinear integral differential equations:

$$\begin{cases} D^{\alpha(x)}u_1(x) + \int_0^t k_{11}(x,t)u_1(t) + k_{12}(x,t)u_2(t)dt = f_1(x,u_1(x),u_2(x)), \\ D^{\beta(x)}u_2(x) + \int_0^t k_{21}(x,t)u_1(t) + k_{22}(x,t)u_2(t)dt = f_2(x,u_1(x),u_2(x)), \quad 0 < x,t \leqslant 1, \\ u_1(0) = 0, \quad u_2(0) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha(x) \leqslant 1$, $0 < \beta(x) \leqslant 1$, $f_n(x,u_1(x),u_2(x)), n = 1,2$ and $k_{ij}(x,t)$, $i,j = 1,2$ are given functions. $D^{\alpha(x)}u_1(x)$ indicates the $\alpha(x)$ is the Caputo fractional derivative defined of $u_1(x)$, $D^{\beta(x)}u_2(x)$ indicates the $\beta(x)$ is the Caputo fractional derivative defined of $u_2(x)$.

Definition 1.1. *The Caputo fractional derivative operator of variable order* $0 < \alpha(x) \le 1$ *is defined as*

$$D^{\alpha(x)}u(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha(x))}\int_0^t (t-\tau)^{-\alpha(x)}\frac{\partial u(\tau)}{\partial \tau}d\tau, & 0 < \alpha(x) < 1, \\ \frac{\partial u(t)}{\partial t}, & \alpha(x) \equiv 1. \end{cases} \tag{1.2}$$

2. Reproducing kernel space

The well-known Jacobi polynomials are defined on the interval $[-1,1]$ and can be generated with the aid of the following recurrence formula:

$$J_i^{\mu,\nu}(t) = \frac{(\mu+\nu+2i-1)\{\mu^2-\nu^2+t(\mu+\nu+2i)(\mu+\nu+2i-2)\}}{2i(\mu+\nu+i)(\mu+\nu+2i-2)}J_{i-1}^{\mu,\nu}(t) - \frac{(\mu+i-1)(\nu+i-1)(\mu+\nu+2i)}{i(\mu+\nu+i)(\mu+\nu+2i-2)}J_{i-2}^{\mu,\nu}(t), \quad i = 2,3,\ldots,$$

and

$$J_0^{\mu,\nu}(t) = 1, \quad J_1^{\mu,\nu}(t) = \frac{(\mu+\nu+2)}{2}t + \frac{(\mu-\nu)}{2}.$$

The weight function of Jacobi polynomials is $\omega(t) = (1-t)^{\mu}(1+t)^{\nu}, t \in [-1,1]$. If $\mu = \nu = 0$, Jacobi polynomials are Legendre polynomials, if $\mu = \nu = -\frac{1}{2}$, Jacobi polynomials are the first chebyshev polynomials, if $\mu = \nu = \frac{1}{2}$, Jacobi polynomials are the second kind of Chebyshev polynomials. Some polynomials are shown in Figures 1–3.

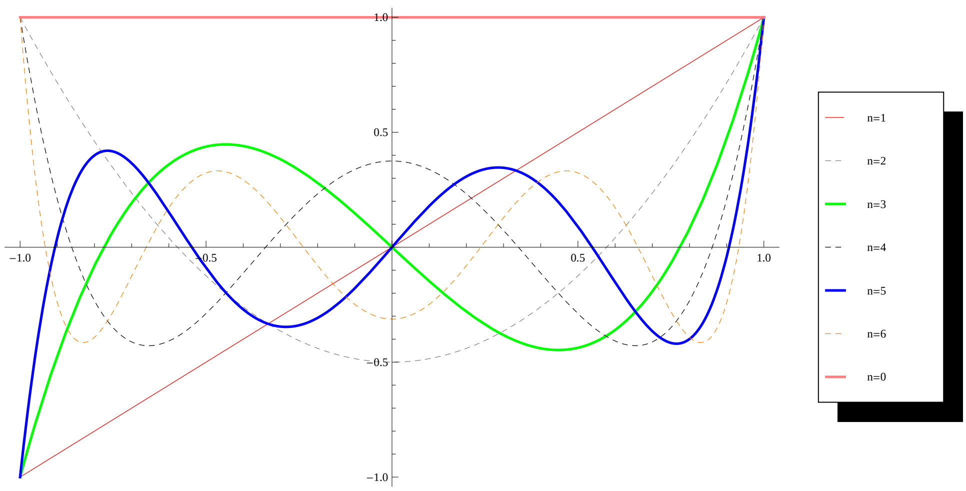


Figure 1. Legendre polynomial.

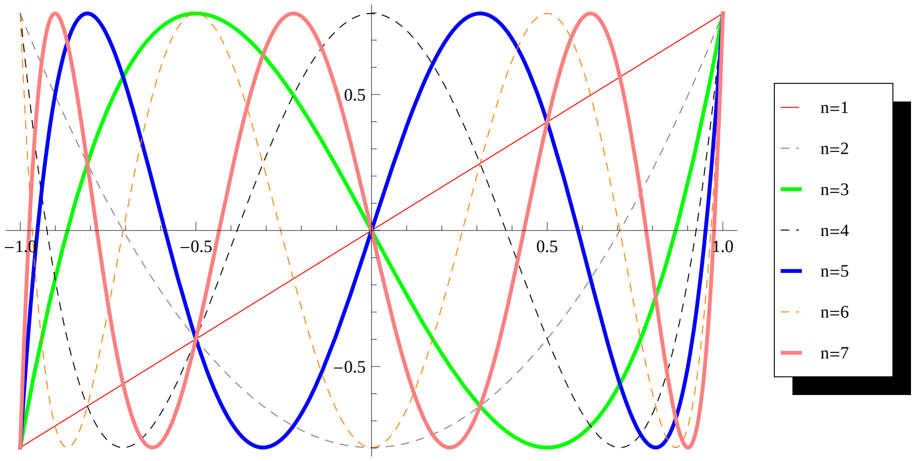


Figure 2. Chebyshev polynomial of first kind.

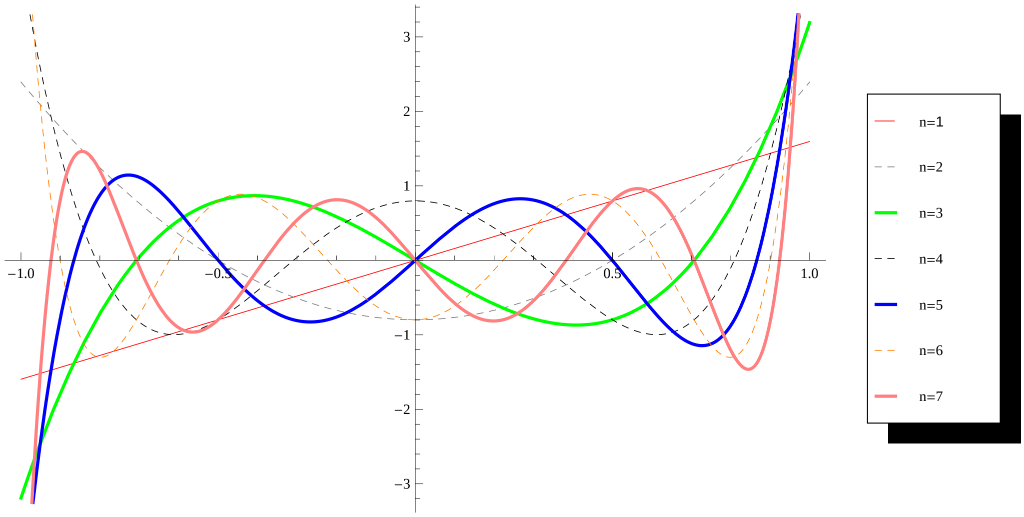


Figure 3. Chebyshev polynomial of second kind.

In order to use these polynomials on the interval $x \in [0, 1]$. Let $t = 2x - 1, t \in [-1, 1]$, the shifted Jacobi polynomials are denoted by $J_i^{\mu,\nu}(x)$. Then $J_i^{\mu,\nu}(x)$ can be generated from:

$$\begin{aligned} J_i^{\mu,\nu}(x) = {} & \frac{(\mu+\nu+2i-1)\{\mu^2-\nu^2+(2x-1)(\mu+\nu+2i)(\mu+\nu+2i-2)\}}{2i(\mu+\nu+i)(\mu+\nu+2i-2)} J_{i-1}^{\mu,\nu}(x) \\ & - \frac{(\mu+i-1)(\nu+i-1)(\mu+\nu+2i)}{i(\mu+\nu+i)(\mu+\nu+2i-2)} J_{1,i-2}^{\mu,\nu}(x), \qquad i = 2, 3, \ldots, \end{aligned} \tag{2.1}$$

and

$$J_0^{\mu,\nu}(x) = 1, \qquad J_1^{\mu,\nu}(x) = \frac{(\mu+\nu+2)}{2}(2x-1) + \frac{(\mu-\nu)}{2}. \tag{2.2}$$

The analytic form of the shifted Jacobi polynomials $J_i^{\mu,\nu}(x)$ of degree i is given by

$$J_i^{\mu,\nu}(x) = \sum_{k=0}^{i}(-1)^{(i-k)}\frac{\Gamma(i+\nu+1)\Gamma(i+k+1+\mu+\nu)}{\Gamma(k+1+\nu)\Gamma(i+\mu+\nu+1)(i-k)!k!}x^k, \tag{2.3}$$

and

$$J_i^{\mu,\nu}(0) = (-1)^i\frac{\Gamma(i+\nu+1)}{\Gamma(1+\nu)i!}, \quad J_i^{\mu,\nu}(1) = \frac{\Gamma(i+\mu+1)}{\Gamma(1+\mu)i!}. \tag{2.4}$$

Definition 2.1. *Let*

$$H_n[0,1] = Span\{J_0^{\mu,\nu}(x), J_1^{\mu,\nu}(x), ..., J_n^{\mu,\nu}(x)\}. \tag{2.5}$$

The inner product and norm are defined as:

$$\langle u(x), v(x)\rangle = \int_0^1 \omega(x)u(x)v(x)dx, \quad ||u(x)||_{H_n} = \sqrt{\langle u(x), u(x)\rangle_{H_n}}, \tag{2.6}$$

where $\omega(x) = x^\nu(1-x)^\mu$ is a weight function. So, the shifted Jacobi polynomials have the following properties:

$$\langle P_i^{\mu,\nu}(x), P_j^{\mu,\nu}(x)\rangle = \int_0^1 \omega(x)P_i^{\mu,\nu}(x)P_j^{\mu,\nu}(x)dx = h_i, \tag{2.7}$$

where

$$h_i = \begin{cases} \frac{\Gamma(i+\alpha+1)\Gamma(i+\beta+1)}{(2i+\alpha+\beta+1)k!\Gamma(i+\alpha+\beta+1)}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.8}$$

$$||P_i^{\mu,\nu}(x)||_{H_n}^2 = \langle P_i^{\mu,\nu}(x), P_i^{\mu,\nu}(x)\rangle_{H_n} = \frac{\Gamma(i+\mu+1)\Gamma(i+\nu+1)}{(2i+\mu+\nu+1)i!\Gamma(i+\mu+\nu+1)}. \tag{2.9}$$

Definition 2.2. *Let*

$$\bar{H}_n[0,1] = \{u | u \in H_n[0,1], u(0) = 0\}.$$

Its the norm as same as the norm of $H_n[0,1]$. From [33], we can prove that $H_n[0,1]$ and $\bar{H}_n[0,1]$ are two reproducing kernel Hilbert spaces. The reproducing kernel of $H_n[0,1]$ is

$$R_n(x,y) = \sum_{k=0}^{n}\frac{(2k+\mu+\nu+1)k!\Gamma(k+\mu+\nu+1)}{\Gamma(k+\mu+1)\Gamma(k+\nu+1)}J_k^{\mu,\nu}(x)J_k^{\mu,\nu}(y), \tag{2.10}$$

The reproducing kernel of $\bar{H}_n[0,1]$ is

$$K(x,y) = K_x(y) = R(x,y) - \frac{R(0,x)R(y,0)}{||\ R(0,0)\ ||^2}. \tag{2.11}$$

Definition 2.3. *The inner product space is defined as:*

$$\bar{H}_n[0,1] \bigoplus \bar{H}_n[0,1] = \{U(x) = [u_1(x), u_2(x)]^T | u_1(x), u_2(x) \in \bar{H}_n[0,1]\},$$

its inner product and norm are defined by

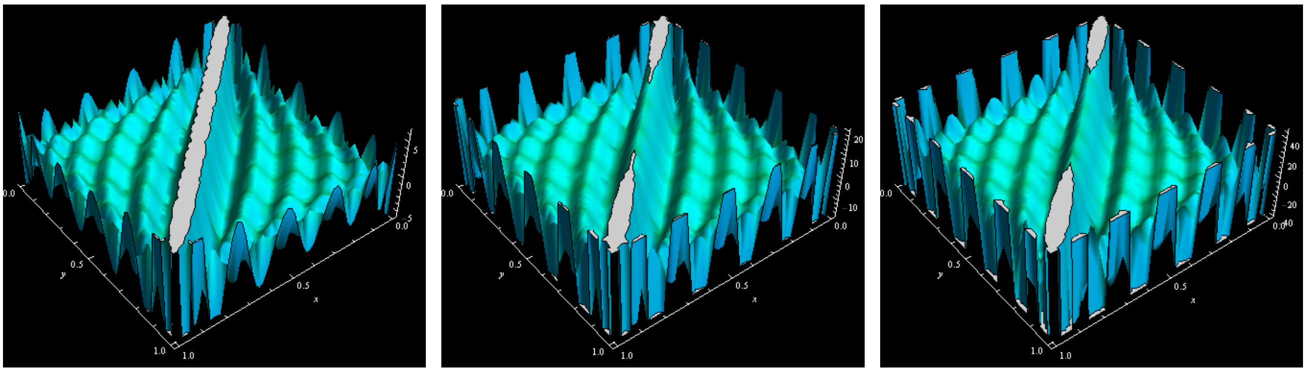
$$\langle U(x), V(x)\rangle = \sum_{i=1}^{2}\langle u_i(x), v_i(x)\rangle_{\bar{H}_n[0,1]\bigoplus\bar{H}_n[0,1]}, \quad ||U(x)||^2 = \sum_{i=1}^{2}||u_i(x)||_{\bar{H}_n[0,1]\bigoplus\bar{H}_n[0,1]}. \tag{2.12}$$

It is easy to verify that $\bar{H}_n[0,1]\bigoplus\bar{H}_n[0,1]$ is a Hilbert space with the definition of inner product (2.12).

Some reproducing kernels are shown in Table 1 and Figures 4–7.

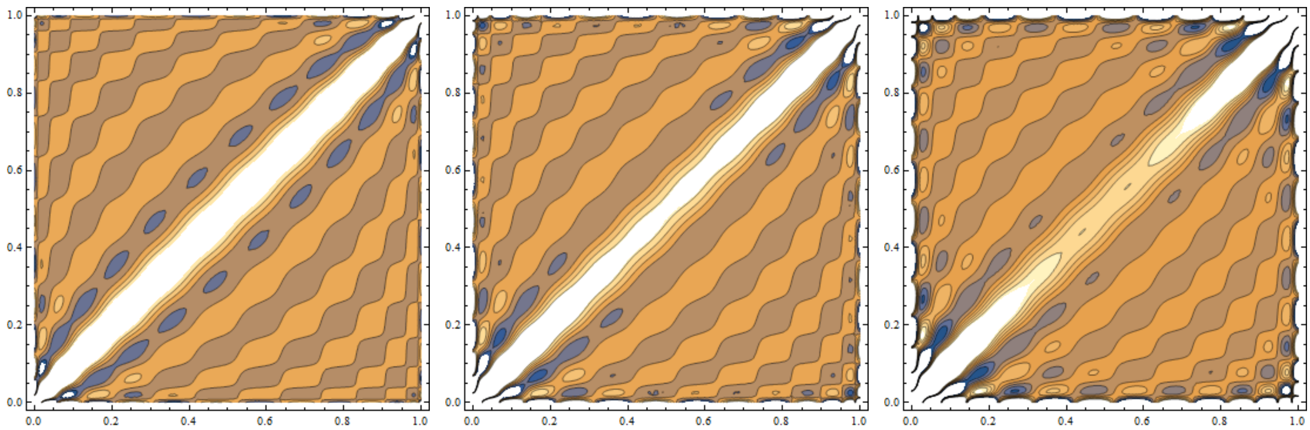
Table 1. Some reproducing kernel functions.

μ,ν	n	$K(x,y)$
$\mu=\nu=0$	3	$15xy(20-60y+42y^2+7x^2(6-20y+15y^2)-4x(15-48y+35y^2))$
$\mu=\nu=\frac{1}{2}$	3	$\frac{128xy}{15\pi}(297-924y+672y^2+32x^2(21-72y+56y^2)-12x(77-254y+192y^2))$
$\mu=\nu=1$	3	$84xy(25-80y+60y^2-10x(8-27y+21y^2)+6x^2(10-35y+28y^2))$
$\mu=0,\nu=1$	3	$6xy(28x^2(10-30y+21y^2)-35x(12-35y+24y^2)+10(15-42y+28y^2))$
$\mu=0,\nu=\frac{1}{2}$	3	$\frac{33}{128}xy(65x^2(63-198y+143y^2)-78x(77-234y+165y^2)+21(99-286y+195y^2))$
$\mu=\nu=2$	3	$630xy(21-70y+55y^2+11x^2(5-18y+15y^2)-2x(35-122y+99y^2))$



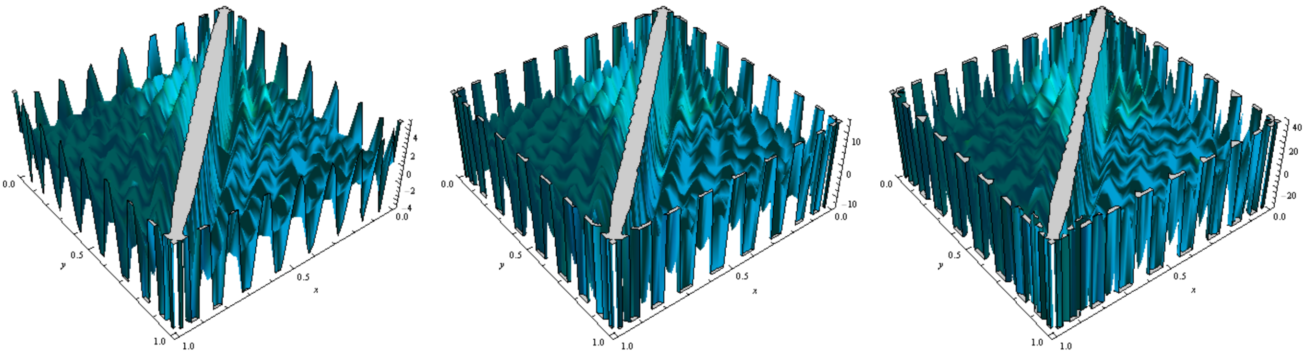
$\mu=\nu=0$ $\mu=\nu=\frac{1}{2}$ $\mu=\nu=1$

Figure 4. Reproducing kernel $R_{13}(x,y)$ at different $\mu=\nu$.



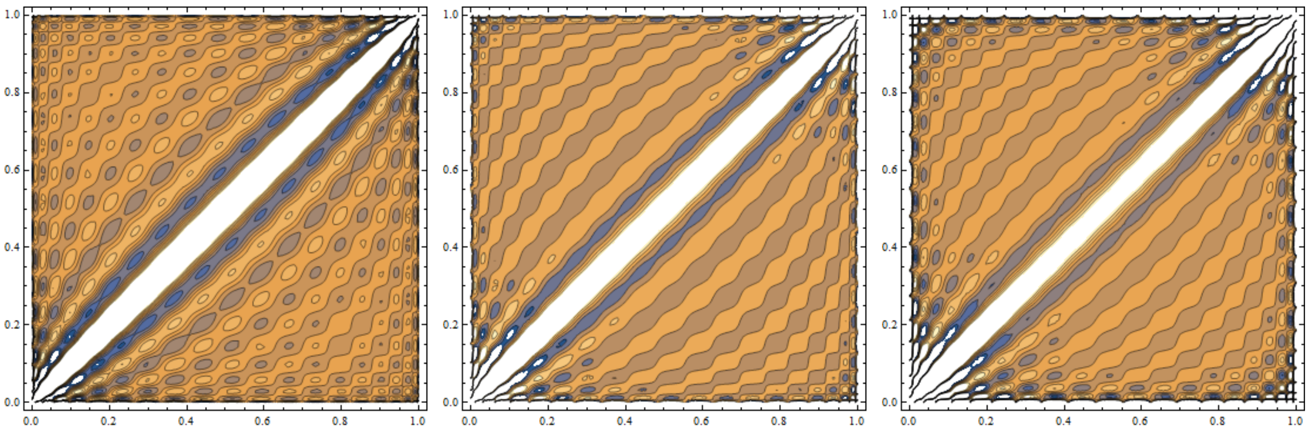
$\mu=\nu=0$ $\mu=\nu=\frac{1}{2}$ $\mu=\nu=1$

Figure 5. Contour plot of $R_{13}(x,y)$ at different $\mu=\nu$.



$\mu = \nu = 0.$ $\mu = \nu = \frac{1}{2}$ $\mu = \nu = 1$

Figure 6. Reproducing kernel $R_{20}(x, y)$ at different $\mu = \nu$.



$\mu = \nu = 0$ $\mu = \nu = \frac{1}{2}$ $\mu = \nu = 1$

Figure 7. Contour plot of $R_{20}(x, y)$ at different $\mu = \nu$.

3. The reproducing kernel method

To solve Eq (1), let

$$\begin{cases} l_{11}u_1 = D^{\alpha(x)}u_1(x) + \int_0^t k_{11}(x,t)u_1(t)dt, \\ l_{12}u_2 = \int_0^t k_{12}(x,t)u_2(x,t)dt, \\ l_{21}u_1 = \int_0^t k_{21}(x,t)u_1(t)dt, \\ l_{22}u_2 = D^{\beta(x)}u_2(x) + \int_0^t k_{22}(x,t)u_2(t)dt. \end{cases} \tag{3.1}$$

So, Eq (1) can be turn into Eq (3.2).

$$LU(x) = F(x, u_1(x), u_2(x)), \tag{3.2}$$

where

$$L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}, \quad U(X) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}, \quad F(x, u_1(x), u_2(x)) = \begin{bmatrix} f_1(x, u_1(x), u_2(x)) \\ f_2(x, u_1(x), u_2(x)) \end{bmatrix}. \tag{3.3}$$

The operator L: $\bar{H}_n[0,1] \bigoplus \bar{H}_n[0,1] \to H_1[0,1] \bigoplus H_1[0,1]$ is a bounded linear operator.

Assuming that $\{x_i\}_{i=1}^{\infty}$ is dense on the interval $[0,1]$, put $\phi_{ijk} = l_{ij}^{*}k_{x_k}(x)$, where l_{ij}^{*} is the adjoint operator of l_{ij}. From [33], we have

$$\phi_{ijk}(x) = l_{ij}K_x(x_k).\ i,j = 1,2,\ k = 1,2,\cdots.$$

Putting

$$\Psi_{i1}(x) = (\phi_{11i}(x), \phi_{12i}(x))^T, \Psi_{i2}(x) = (\phi_{21i}(x), \phi_{22i}(x))^T,\ i, = 1,2,\cdots.$$

Theorem 3.1. *For each fixed n,* $\{\Psi_{ij}\}_{(1,1)}^{(n,2)}$ *is linearly independent in* $\bar{H}_n[0,1] \bigoplus \bar{H}_n[0,1]$.

Theorem 3.2. $\{\Psi_{ij}\}_{(1,1)}^{(\infty,2)}$*is complete in space in* $\bar{H}_n[0,1] \bigoplus \bar{H}_n[0,1]$.

Proof. See of Theorems 3.1 and 3.2 in [33].

Using Theorems 3.1 and 3.2, the exact solution of Eq (1) can be expressed as

$$U(X) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} c_{ij}\Psi_{ij}(x), \tag{3.4}$$

and truncate the infinite series of the analytic solution, we obtain the approximate solution of Eq (1).

$$U_m(X) = \sum_{i=1}^{m}\sum_{j=1}^{2} c_{ij}\Psi_{ij}(x). \tag{3.5}$$

If we can obtain the coefficients of each $\Psi_{ij}(x)$, the approximate solution $U_m(x)$ can be obtained also. Use $\Psi_{ij}(x)$ to do the inner products with both sides of Eq (3.5) and let $u_{1,0}(x) = u_{2,0}(x) = 0$, we have

$$\begin{cases} \sum\limits_{i=1}^{m} c_{i1}\langle\Psi_{i1},\Psi_{n1}\rangle + \sum\limits_{j=1}^{m} c_{j2}\langle\Psi_{j2},\Psi_{n1}\rangle = f_1(x_k, u_{1,n-1}(x_k), u_{2,n-1}(x_k)), \\ \sum\limits_{i=1}^{m} c_{i1}\langle\Psi_{i1},\Psi_{n2}\rangle + \sum\limits_{j=1}^{m} c_{j2}\langle\Psi_{j2},\Psi_{n2}\rangle = f_2(x_k, u_{1,n-1}(x_k), u_{2,n-1}(x_k)), \end{cases} \quad k = 1,2,\cdots,m. \tag{3.6}$$

Letting

$$L_{2m} = \begin{bmatrix} \langle\Psi_{i1},\Psi_{m1}\rangle & \cdots & \langle\Psi_{j2},\Psi_{m1}\rangle \\ \cdots & \cdots & \cdots \\ \langle\Psi_{i1},\Psi_{m2}\rangle & \cdots & \langle\Psi_{j2},\Psi_{m2}\rangle \end{bmatrix}_{i,j,n=1,2,\cdots,\ m,}$$

$$F = (f_1(x_1),\cdots,f_1(x_m), f_2(x_1),\cdots,f_1(x_m))^T.$$

It is obvious that the inverse of A_{2m} exists by Theorem 3.1. So, we have

$$(c_{11}, c_{12},\cdots,c_{1m}, c_{21}, c_{22},\cdots,c_{2m})^T = L_{2m}^{-1}F.$$

Theorem 3.3. *Let* $U \in \bar{H}_n[0,1] \bigoplus \bar{H}_n[0,1]$ *be the exact solution of Eq (1),* $U_{m,n}$ *be the solution of* (3.6). *If*

$$\|f_i(x,y,z) - f_i(x,\bar{y},z)\| < c|y-\bar{y}|$$

and

$$\|f_i(x,y,z) - f_i(x,y\bar{z})\| < c|z-\bar{z}|,$$

$0 < c < 1$, *then* $U_{m,n}$ *converges uniformly to U [17].*

4. Numerical experiment

Example 1. We consider the following nonlinear integro-differential equations of fractional order [34].

$$\begin{cases} D^{\alpha(x)}u_1(x) - \int_0^x (x+t)u_2(t)dt = f_1(x, u_1(x), u_2(x)), \\ D^{\beta(x)}u_2(x) - \int_0^x (x-t)u_1(t)dt = f_2(x, u_1(x), u_2(x)), \quad 0 < x, t \leqslant 1, \\ u_1(0) = 0, \quad u_2(0) = 0. \end{cases} \tag{4.1}$$

(1) Where

$$\alpha(x) = \frac{1}{3}, \;\; \beta(x) = \frac{2}{3},$$
$$f_1(x, u_1(x), u_2(x)) = \frac{x^{2/3}}{\Gamma(5/3)} + \frac{5x^3}{6},$$
$$f_2(x, u_1(x), u_2(x)) = \frac{x^{1/3}}{\Gamma(4/3)} - \frac{x}{2} + \frac{1}{3}.$$

The exact solution $u_1(x) = x$, $u_2(x) = -x$, The numerical results which of Example 1 for $m = 10, \mu = \nu = 0, n = 2$ are given in Table 2, and the absolute errors of this example for $m = 10, n = 2$ are given in Figures 8 and 9.

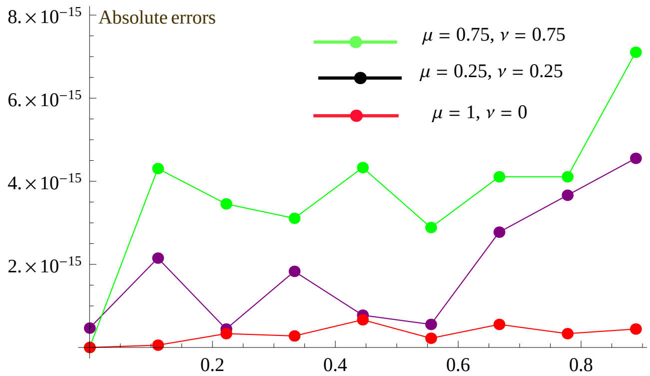


Figure 8. Absolute errors of u_1 for Example 1.

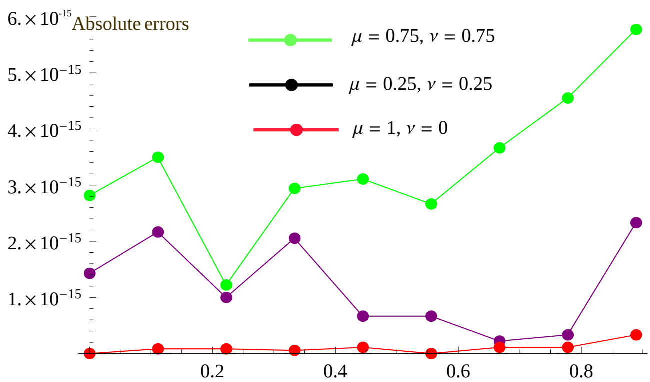


Figure 9. Absolute errors of u_2 for Example 1.

Table 2. Comparison of the absolute errors for Example 1.

x	Absolute errors of u_1 Ref. [34]	Absolute errors of u_2 Ref. [34]	Absolute errors of u_1 Present method	Absolute errors of u_2 Present method
0	$5.8392E-5$	$6.9382E-5$	0	0
$\frac{1}{9}$	$6.9402E-5$	$6.7372E-5$	$1.6653E-16$	$9.7145E-16$
$\frac{2}{9}$	$6.3829E-5$	$7.9300E-5$	$1.1102E-16$	$2.7756E-16$
$\frac{3}{9}$	$7.3428E-5$	$8.0320E-5$	$4.9960E-16$	$2.7756E-16$
$\frac{4}{9}$	$7.8230E-5$	$8.9324E-5$	$4.4409E-16$	$6.1062E-16$
$\frac{5}{9}$	$8.8492E-5$	$9.2803E-5$	$4.4409E-16$	$4.4409E-16$
$\frac{6}{9}$	$8.3723E-5$	$9.7832E-5$	$7.7716E-16$	$3.3307E-16$
$\frac{7}{9}$	$9.8402E-5$	$2.8943E-4$	$3.3307E-16$	$2.2205E-16$
$\frac{8}{9}$	$3.4829E-4$	$3.7231E-4$	$1.1102E-16$	$3.3307E-16$

(2) Where

$$\alpha(x) = \frac{2x}{3},\ \beta(x) = \frac{x}{3},$$

$$f_1(x, u_1(x), u_2(x)) = \frac{3x^{1-\frac{2x}{3}}}{(3-2x)\Gamma(1-\frac{2x}{3})} + \frac{5x^3}{6},$$

$$f_2(x, u_1(x), u_2(x)) = \frac{3x^{1-\frac{x}{3}}}{(x-3)\Gamma(1-\frac{x}{3})} - \frac{x^3}{6}.$$

The exact solution $u_1(x) = x$, $u_2(x) = -x$. The absolute errors of Example 1 for $m = 10, \mu = \nu = 0, n = 2$ are given in Figures 10 and 11.

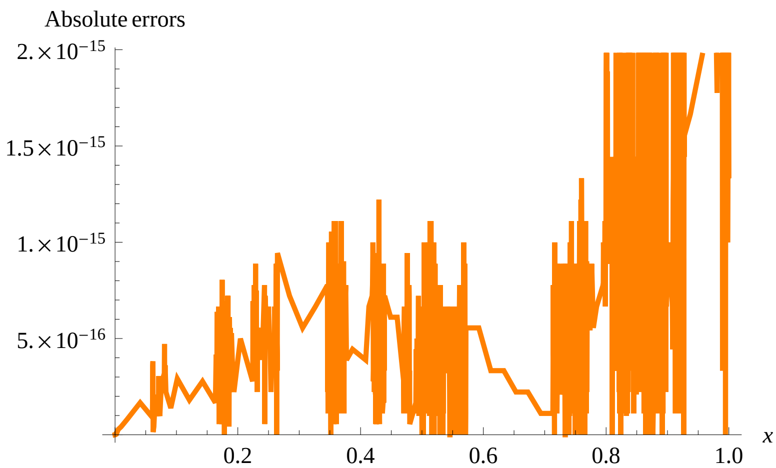


Figure 10. Absolute errors of u_1 with $\mu = \nu = 0$, $\alpha(x) = \frac{2x}{3}, \beta(x) = \frac{x}{3}$ for Example 1.

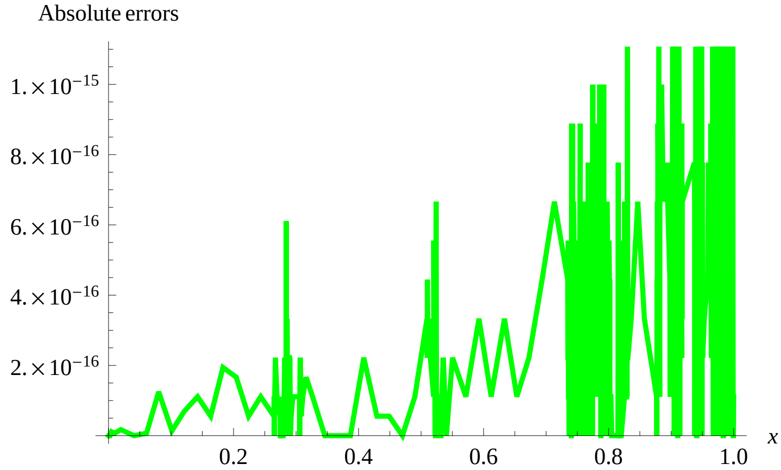


Figure 11. Absolute errors of u_2 with $\mu = \nu = 0$, $\alpha(x) = \dfrac{2x}{3}, \beta(x) = \dfrac{x}{3}$ for Example 1.

(3) Where

$$\alpha(x) = \cos x, \quad \beta(x) = \sin x,$$

$$f_1(x, u_1(x), u_2(x)) = \frac{x^{1-\cos x}}{(1-\cos x)\Gamma(1-\cos x)} + \frac{5x^3}{6},$$

$$f_2(x, u_1(x), u_2(x)) = -\frac{x^{1-\sin(x)}}{(1-\sin x)\Gamma(1-\sin x)} - \frac{x^3}{6}.$$

The exact solution $u_1(x) = x$, $u_2(x) = -x$. The absolute errors of Example 1 for $m = 10, \mu = \nu = 0, n = 2$ are given in Figures 12 and 13.

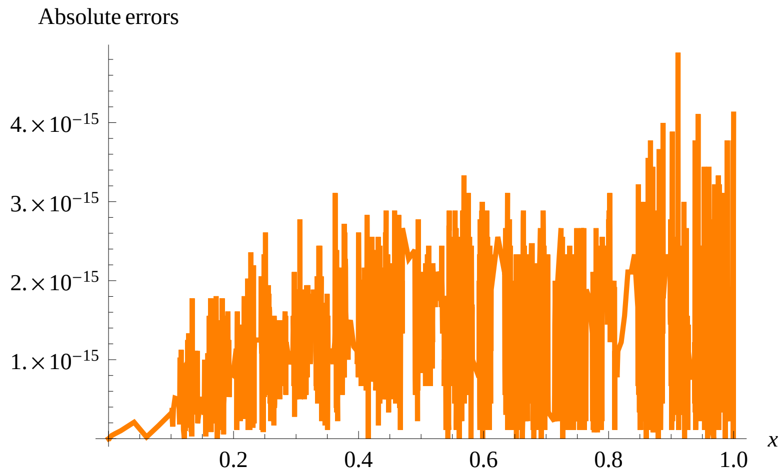


Figure 12. Absolute errors of u_1 with $\mu = \nu = 0, \alpha(x) = \cos x, \beta(x) = \sin x$ for Example 1.

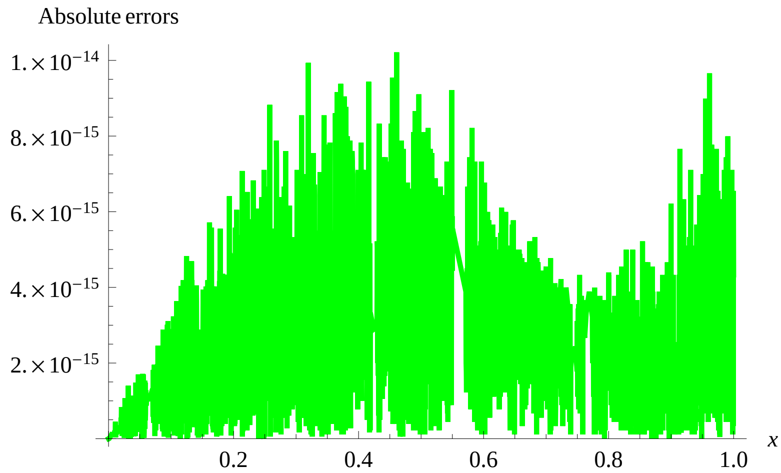


Figure 13. Absolute errors of u_2 with $\mu = \nu = 0, \alpha(x) = \cos x, \beta(x) = \sin x$ for Example 1.

Example 2. We consider the following linear integro-differential equations of fractional order [34].

$$\begin{cases} D^{\alpha}u_1(x) - \int_0^1 xt^2u_1(t)dt - \int_0^x (x^2+t)u_2(t)dt = f_1(x), \\ D^{\beta}u_2(x) - \int_0^1 (x+t^2)u_1(t)dt - \int_0^x x^2tu_2(t)dt = f_2(x), \quad 0 < x, t \leqslant 1, \\ u_1(0) = 0, \quad u_2(0) = 0. \end{cases} \tag{4.2}$$

Where $\alpha = \beta = \frac{1}{2}$,

$$f_1(x) = \frac{8x^{1.5}}{3\sqrt{\pi}} - \frac{x}{5} + \frac{x^4(4x+3)}{12},$$

$$f_2(x) = -\frac{8x^{1.5}}{3\sqrt{\pi}} + \frac{x^6}{4} - \frac{x}{3} - \frac{1}{5}.$$

The exact solution $u_1(x) = x^2$, $u_2(x) = -x^2$. The absolute errors of Example 1 with $m = 10, n = 5$ are given in Tables 3 and 4 and Figures 14–19.

Table 3. Comparison of the numerical result of $u_1(x)$ in Example 2 with $\mu = \nu = 0$.

	Exact solution	Ref. [34] Numerical solution	Present method Numerical solution	Present method Absolute errors
0.1	0.0100	0.0099	0.0100	$2.1511E-16$
0.3	0.0900	0.0897	0.0900	$8.3267E-16$
0.5	0.2500	0.2496	0.2500	$1.1102E-15$
0.7	0.4900	0.4894	0.4900	$5.9397E-15$
0.9	0.8100	0.8089	0.8100	$1.1102E-14$

Table 4. Comparison of the numerical result of $u_2(x)$ in Example 2 with $\mu = \nu = 0$.

	Exact solution	Ref. [34] Numerical solution	Present method Numerical solution	Present method Absolute errors
0.1	−0.0100	−0.0103	−0.0100	$1.6168E-15$
0.3	−0.0900	−0.0905	−0.0900	$7.7716E-16$
0.5	−0.2500	−0.2508	−0.2500	$5.5511E-17$
0.7	−0.4900	−0.4913	−0.4900	$5.7732E-15$
0.9	−0.8100	−0.8117	−0.8100	$1.9984E-14$

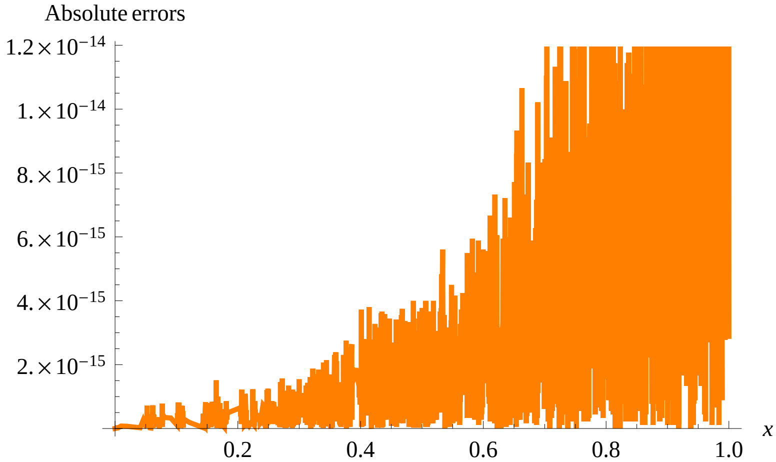


Figure 14. Absolute errors of u_1 for Example 2 with $\mu = \nu = 0$.

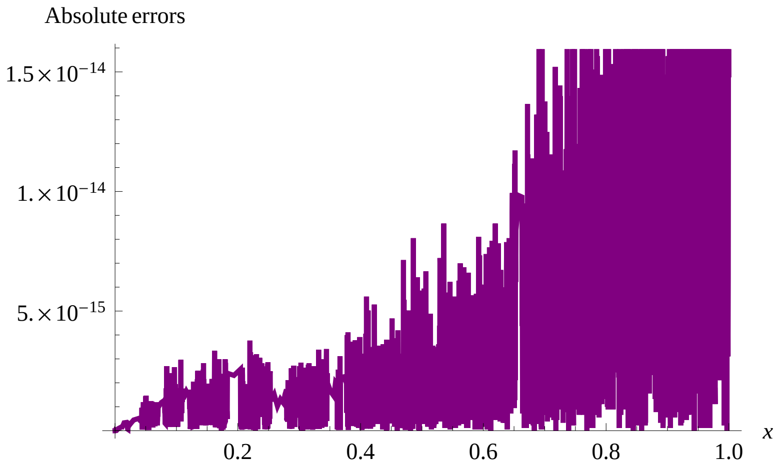


Figure 15. Absolute errors of u_2 for Example 2 with $\mu = \nu = 0$.

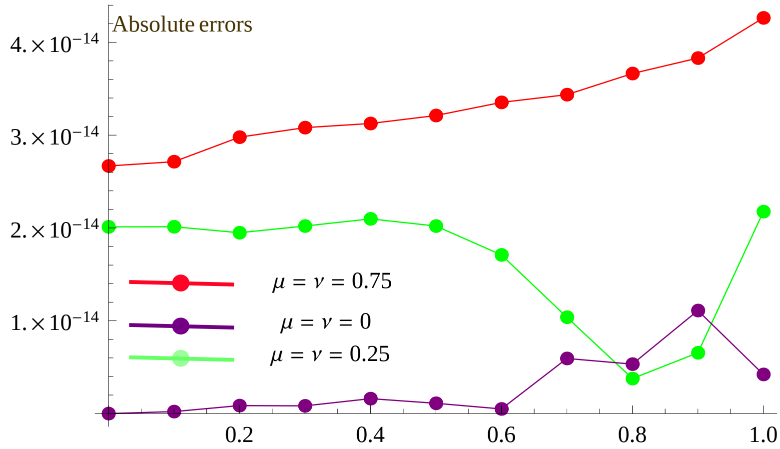


Figure 16. Absolute errors of u_1 for Example 2.

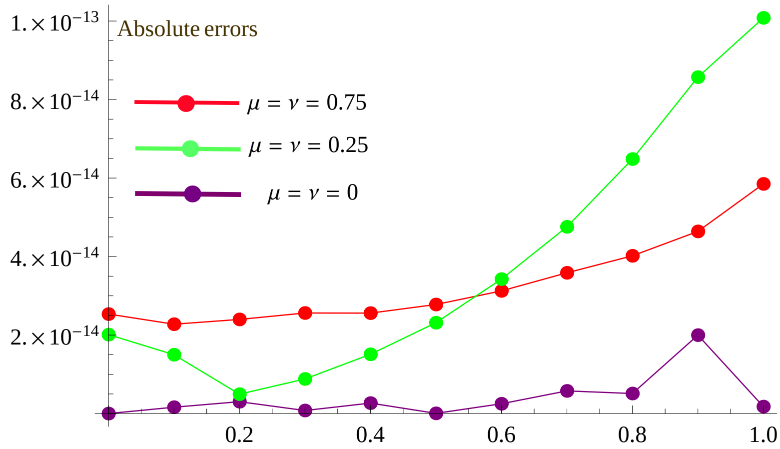


Figure 17. Absolute errors of u_2 for Example 2.

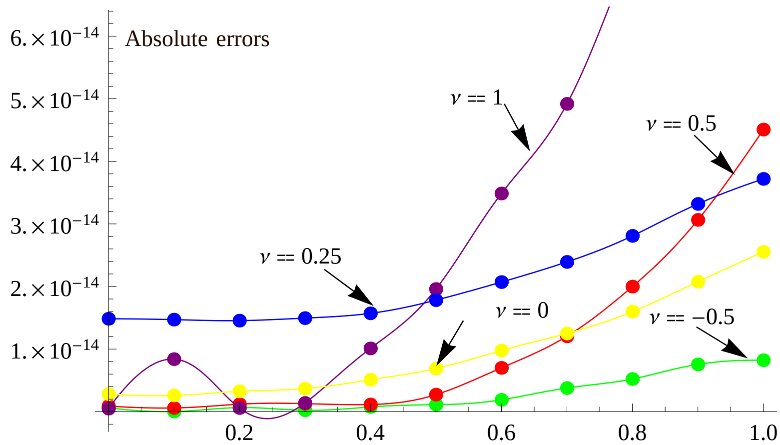


Figure 18. Absolute errors of u_1 for Example 2 with $\mu = 0.5$.

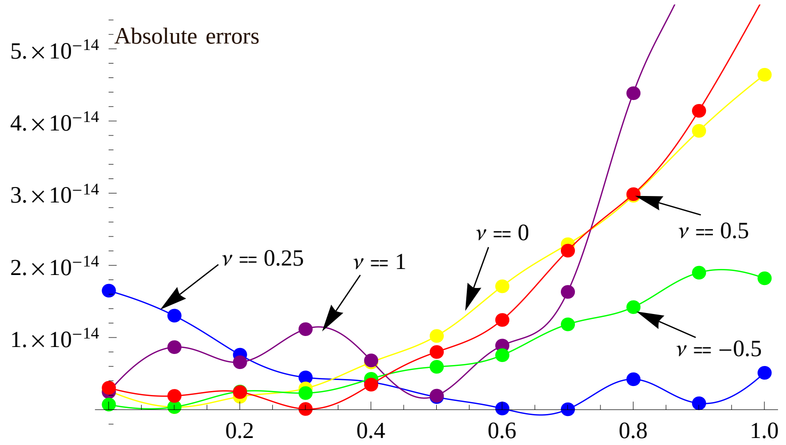


Figure 19. Absolute errors of u_2 for Example 2 with $\mu = 0.5$.

5. Conclusions and remarks

In this paper, fractional nonlinear integro-differential equations of variable order have been solved by reproducing kernel interpolation collocation method with reproducing kernel function in the form of Jacobi polynomials. By comparing the obtained numerical solutions to the exact solutions of the fractional nonlinear integro-differential equations of variable order, it is indicated that our approach is powerful for variable order time fractional nonlinear integro-differential equations. All computations are performed by the Mathematica 7.0.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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