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Research article

The finite time blow-up for Caputo-Hadamard fractional diffusion equation involving nonlinear memory

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Abstract: In this article, we focus on the blow-up problem of solution to Caputo-Hadamard fractional diffusion equation with fractional Laplacian and nonlinear memory. By virtue of the fundamental solutions of the corresponding linear and nonhomogeneous equation, we introduce a mild solution of the given equation and prove the existence and uniqueness of local solution. Next, the concept of a weak solution is presented by the test function and the mild solution is demonstrated to be a weak solution. Finally, based on the contraction mapping principle, the finite time blow-up and global solution for the considered equation are shown and the Fujita critical exponent is determined. The finite time blow-up of solution is also confirmed by the results of numerical experiment.

Keywords: Caputo-Hadamard derivative; fractional Laplacian; nonlinear memory; finite time blow-up; fixed point argument
Mathematics Subject Classification: 26A33, 35R11, 35B44

1. Introduction

The main purpose of this paper is to study the finite time blow-up to time-space fractional partial differential equation in the following form

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a,t}u(\mathrm{x},t) + (-\Delta)^s u(\mathrm{x},t) = {}_{H}\mathrm{D}^{-(1-\gamma)}_{a,t}(|u|^{p-1}u)(\mathrm{x},t), \ \mathrm{x} \in \mathbb{R}^d, \ t > a > 0, \\ u(\mathrm{x},a) = u_a(\mathrm{x}), \ \mathrm{x} \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $d \in \mathbb{N}$, $0 < \alpha < \gamma < 1$, $0 < s < 1$, $p > 1$, the operators ${}_{CH}\mathrm{D}^{\alpha}_{a,t}$, $(-\Delta)^s$, and ${}_{H}\mathrm{D}^{-(1-\gamma)}_{a,t}$ respectively denote the Caputo-Hadamard fractional derivative, fractional Laplacian, and Hadamard fractional integral, and the initial value $u_a(\mathrm{x}) \in C_0(\mathbb{R}^d)$, where $C_0(\mathbb{R}^d) = \{\upsilon \in C(\mathbb{R}^d) \mid \lim_{|\mathrm{x}|\to\infty} \upsilon(\mathrm{x}) = 0\}$.

Fractional calculus has attracted considerable attention during recent years because of its widespread applications in science and engineering fields such as physics, chemistry, biology, anomalous diffusion, control theory of dynamical systems, etc., see [1–4]. It has been found that Hadamard-type fractional calculus had many potential applications [5–11], for example, the ultraslowly diffusive process such as Sinai diffusion [5], fractal analysis [8], the Lomnitz logarithmic creep law in rheology [9], and some studies in this respect have been available [12–19]. The fractional Laplacian is a typically nonlocal pseudo-differential operator, which appears in different disciplines of mathematics and various applications, see [20–23] and the list of references therein.

We next recall some pioneering work on the blow-up problem for fractional diffusion equation, here we only mention the results related to our studies.

In the 1960s, Fujita [24] first considered the following semilinear heat equation

$$\begin{cases} u_t = \Delta u + u^{1+\alpha}, \ \mathrm{x} \in \mathbb{R}^d, \ t > 0, \\ u|_{t=0} = u_0(\mathrm{x}), \ \mathrm{x} \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where $\alpha > 0$ and $u_0(\mathrm{x}) \geq 0$. In that paper, the author shown that: If $u_0(\mathrm{x}) \not\equiv 0$ and $0 < \alpha < \frac{2}{d}$ then the solution of (1.2) blows up in finite time; If $\alpha > \frac{2}{d}$ and the initial value $u_0(\mathrm{x})$ can be bounded by sufficiently small Gaussian then the solution of (1.2) exists globally. As for the critical case $\alpha = \frac{2}{d}$, Weissler [25] proved that (1.2) has a global solution when $\|u_0\|_{L^{\frac{\alpha d}{2}}(\mathbb{R}^d)}$ is sufficiently small.

Later, Cazenave et al. [26] studied the following Cauchy problem of heat equation with nonlinear memory

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-\tau)^{-\gamma} |u(\tau)|^{p-1} u(\tau) \mathrm{d}\tau, \ \mathrm{x} \in \mathbb{R}^d, \ t > 0, \\ u|_{t=0} = u_0(\mathrm{x}), \ \mathrm{x} \in \mathbb{R}^d, \end{cases} \tag{1.3}$$

where $p > 1, 0 \leq \gamma < 1$, and $u_0 \in C_0(\mathbb{R}^d)$. Let $p_\gamma = \frac{1+2(2-\gamma)}{(d-2+2\gamma)_+}$ with $(d-2+2\gamma)_+ = \max\{0, d-2+2\gamma\}$. They proved that: If $\gamma \neq 0, p \leq \max\{\frac{1}{\gamma}, p_\gamma\}, u_0 \geq 0$, and $u_0 \not\equiv 0$, then the solution of (1.3) blows up in finite time; if $\gamma \neq 0, p > \max\{\frac{1}{\gamma}, p_\gamma\}$ and $\|u_0\|_{L^{q_{sc}}(\mathbb{R}^d)}$ is sufficiently small with $q_{sc} = \frac{d(p-1)}{4-2\gamma}$, then (1.3) has global solution. In the case with $\gamma = 0$, every nontrivial positive solution of (1.3) will blow up [27].

In [28], Fino and Kirane further investigated the equation involving fractional Laplacian with nonlinear memory

$$\begin{cases} u_t + (-\Delta)^{\frac{\beta}{2}} u = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-\tau)^{-\gamma} |u(\tau)|^{p-1} u(\tau) \mathrm{d}\tau, \ \mathrm{x} \in \mathbb{R}^d, \ t > 0, \\ u|_{t=0} = u_0(\mathrm{x}), \ \mathrm{x} \in \mathbb{R}^d, \end{cases} \tag{1.4}$$

where $0 < \beta \leq 2, 0 < \gamma < 1, p > 1$, and $u_0 \in C_0(\mathbb{R}^d)$. They derived that: If $u_0 \geq 0, u_0 \not\equiv 0$, and $p \leq \max\{1+\frac{\beta(2-\gamma)}{(d-\beta+\beta\gamma)_+}, \frac{1}{\gamma}\}$, then the solution of (1.4) will blow up in finite time; if $p > \max\{1+\frac{\beta(2-\gamma)}{(d-\beta+\beta\gamma)_+}, \frac{1}{\gamma}\}$ and $\|u_0\|_{L^{p_{sc}}(\mathbb{R}^d)}$ is very small with $p_{sc} = \frac{d(p-1)}{\beta(2-\gamma)}$, then (1.4) exists global solution.

Shortly after, Li and Zhang [29] discussed the following time fractional diffusion equation involving Caputo derivative with nonlinear memory

$$\begin{cases} {}_C\mathrm{D}_{0,t}^{\alpha} u - \Delta u = \dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-\tau)^{-\gamma} |u(\tau)|^{p-1} u(\tau) \mathrm{d}\tau, \ \mathrm{x} \in \mathbb{R}^d, \ t > 0, \\ u|_{t=0} = u_0(\mathrm{x}), \ \mathrm{x} \in \mathbb{R}^d, \end{cases} \tag{1.5}$$

where $0 < \alpha < \gamma < 1$, $p > 1$, and $u_0 \in C_0(\mathbb{R}^d)$. They proved that: If $1 < p < p^* = \max\{1 + \frac{1-\gamma}{\alpha}, 1 + \frac{2(1+\alpha-\gamma)}{\alpha d}\}$ and $u_0 \geq 0$ with $u_0 \not\equiv 0$, then the solution of (1.5) will blow up in finite time; if $d < \frac{2(1+\alpha-\gamma)}{1-\gamma}$ with $p \geq p^*$ or $d \geq \frac{2(1+\alpha-\gamma)}{1-\gamma}$ with $p > p^*$, and $\|u_0\|_{L^{q_c}(\mathbb{R}^d)}$ is small enough, where $q_c = \frac{\alpha d(p-1)}{2(1+\alpha-\gamma)}$, then (1.5) has global solution.

Recently, Li and Li [16] investigated the semilinear time-space fractional diffusion equation involving Caputo-Hadamard derivative and fractional Laplacian,

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a,t}u(\mathrm{x}, t) + (-\Delta)^s u(\mathrm{x}, t) = |u(\mathrm{x}, t)|^{p-1}u(\mathrm{x}, t), \ \mathrm{x} \in \mathbb{R}^d, \ t > a > 0, \\ u(\mathrm{x}, a) = u_a(\mathrm{x}), \ \mathrm{x} \in \mathbb{R}^d, \end{cases} \tag{1.6}$$

where $0 < \alpha < 1$, $0 < s < 1$, $p > 1$, and $u_a \in C_0(\mathbb{R}^d)$. They obtained that: If $1 < p < \frac{2s}{d}$ and $u_a \geq 0$ with $u_a \not\equiv 0$, then the solution of (1.6) will blow up in finite time; Conversely, if $p \geq 1 + \frac{2s}{d}$ and $\|u_a\|_{L^{q_*}(\mathbb{R}^d)}$ is sufficiently small, where $q_* = \frac{d(p-1)}{2s}$, then (1.6) has a global solution.

Motivated mathematically by the results and methods in [16], this paper will further study the blow-up property and global solution to time-space fractional diffusion equation (1.1) with nonlinear memory. The main result is displayed in the following theorem.

Theorem 1.1. *Let $d \in \mathbb{N}$, $0 < \alpha < \gamma < 1$, $0 < s < 1$, and $p > 1$. Assume that $u_a \in C_0(\mathbb{R}^d)$ and $u_a \geq 0$ with $u_a \not\equiv 0$.*
(1) *If $1 < p < \widetilde{p} = \max\{1 + \frac{1-\gamma}{\alpha}, 1 + \frac{2s(1+\alpha-\gamma)}{\alpha d}\}$, then the mild solution of Eq (1.1) will blow up in finite time.*
(2) *If $d < \frac{2s(1+\alpha-\gamma)}{1-\gamma}$, $p \geq \widetilde{p}$ or $d \geq \frac{2s(1+\alpha-\gamma)}{1-\gamma}$, $p > \widetilde{p}$, and $\|u_a\|_{L^{p^*}(\mathbb{R}^d)}$ is small enough with $p^* = \frac{\alpha d(p-1)}{2s(1+\alpha-\gamma)}$, then Eq (1.1) exists global solution.*

The organization of this paper is as follows. Section 2 recalls some basic definitions and presents several important lemmas. In Section 3, we define a mild solution to Eq (1.1) and then prove the local existence and uniqueness of the mild solution. Then, a weak solution of Eq (1.1) is introduced and the mild solution is actually proved to be a weak solution. Next, we show the finite time blow-up and global existence of the solution to Eq (1.1) in Section 4. Finally, an illustrative example is provided to verify the blow-up of solution in finite time in Section 5. The conclusions are given in the last section. Throughout the paper, we use the letter C to denote a generic positive constant which may take different values at different places.

2. Preliminaries

Let us recall some basic definitions and several important lemmas, which will be applied in the next sections.

Definition 2.1. [4, 30] *Let a function $f(t)$ be defined on the interval $(a, b)\,(0 \leq a < b \leq +\infty)$ and $\alpha > 0$. The left- and right- sided Hadamard fractional integrals of the function $f(t)$ with order α are given by*

$${}_{H}\mathrm{D}^{-\alpha}_{a,t}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}, \ t > a, \tag{2.1}$$

and

$${}_{H}\mathrm{D}^{-\alpha}_{t,b}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\log\frac{\tau}{t}\right)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}, \ t < b, \tag{2.2}$$

where the Gamma function $\Gamma(\xi) = \int_0^{\infty} e^{-t} t^{\xi-1} \mathrm{d}t$.

Definition 2.2. [4, 31] *Let a function* $f(t)$ *be defined on the interval* $(a, b)\,(0 \leq a < b \leq +\infty)$ *and* $n - 1 < \alpha < n \in \mathbb{N}$. *The left- and right- sided Caputo-Hadamard fractional derivative of the function* $f(t)$ *with order* α *can be written as*

$$\begin{aligned} {}_{CH}\mathrm{D}_{a,t}^{\alpha} f(t) =& {}_{H}\mathrm{D}_{a,t}^{-(n-\alpha)}[\delta^n f(t)] \\ =& \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{n-\alpha-1} \delta^n f(\tau) \frac{\mathrm{d}\tau}{\tau},\ t > a, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} {}_{CH}\mathrm{D}_{t,b}^{\alpha} f(t) =& (-1)^n\, {}_{H}\mathrm{D}_{t,b}^{-(n-\alpha)}[\delta^n f(t)] \\ =& \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \left(\log \frac{\tau}{t}\right)^{n-\alpha-1} \delta^n f(\tau) \frac{\mathrm{d}\tau}{\tau},\ t < b, \end{aligned} \tag{2.4}$$

where $\delta^n f(t) = \left(t \frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t)$.

Definition 2.3. [2, 20, 30] *The fractional Laplacian* $(-\Delta)^s$ *with* $s \in (0, 1)$ *is defined by*

$$(-\Delta)^s v(\mathrm{x}) = C(d, s)\ \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{v(\mathrm{x}) - v(\mathrm{y})}{|\mathrm{x} - \mathrm{y}|^{d+2s}} \mathrm{d}\mathrm{y},\ \forall\, \mathrm{x} \in \mathbb{R}^d, \tag{2.5}$$

where P.V. denotes the Cauchy principle value and the constant

$$C(d, s) = \left(\int_{\mathbb{R}^d} \frac{1 - \cos y_1}{|\mathrm{y}|^{d+2s}} \mathrm{d}\mathrm{y}\right)^{-1}$$

for any $\mathrm{y} = (y_1, y_2, \cdots, y_d) \in \mathbb{R}^d$.

To define a mild solution of Eq (1.1), let us consider the following linear equation,

$$\begin{cases} {}_{CH}\mathrm{D}_{a,t}^{\alpha} u(\mathrm{x}, t) + (-\Delta)^s u(\mathrm{x}, t) = f(\mathrm{x}, t),\ \mathrm{x} \in \mathbb{R}^d,\ t > a > 0, \\ u(\mathrm{x}, a) = u_a(\mathrm{x}),\ \mathrm{x} \in \mathbb{R}^d, \end{cases} \tag{2.6}$$

whose solution is expressed by [14]

$$\begin{aligned} u(\mathrm{x}, t) =& G_a(\mathrm{x}, t) * u_a(\mathrm{x}) + \int_a^t G_f\left(\mathrm{x}, a\frac{t}{\tau}\right) * f(\mathrm{x}, \tau) \frac{\mathrm{d}\tau}{\tau} \\ :=& \int_{\mathbb{R}^d} G_a(\mathrm{x} - \mathrm{y}, t) u_a(\mathrm{y}) \mathrm{d}\mathrm{y} + \int_a^t \int_{\mathbb{R}^d} G_f\left(\mathrm{x} - \mathrm{y}, a\frac{t}{\tau}\right) f(\mathrm{y}, \tau) \mathrm{d}\mathrm{y} \frac{\mathrm{d}\tau}{\tau}, \end{aligned} \tag{2.7}$$

where $G_a(\mathrm{x}, t)$ and $G_f(\mathrm{x}, t)$ are the fundamental solutions given by

$$G_a(\mathrm{x}, t) = \frac{1}{|\mathrm{x}|^d \pi^{\frac{d}{2}}} H_{23}^{21}\left(\frac{|\mathrm{x}|^{2s}}{2^{2s}(\log \frac{t}{a})^{\alpha}} \middle| \begin{matrix} (1, 1); (1, \alpha) \\ (1, 1), (\frac{d}{2}, s); (1, s) \end{matrix}\right), \tag{2.8}$$

and

$$G_f(\mathrm{x}, t) = \frac{\left(\log \frac{t}{a}\right)^{\alpha-1}}{|\mathrm{x}|^d \pi^{\frac{d}{2}}} H_{23}^{21}\left(\frac{|\mathrm{x}|^{2s}}{2^{2s}(\log \frac{t}{a})^{\alpha}} \middle| \begin{matrix} (1, 1); (\alpha, \alpha) \\ (1, 1), (\frac{d}{2}, s); (1, s) \end{matrix}\right). \tag{2.9}$$

The special function $H^{21}_{23}(z)$ in the above equalities is the Fox H-function and some details regarding this function can be found in [4, 32, 33].

In the sequel, we list some properties of the functions $G_a(\mathrm{x}, t)$ and $G_f(\mathrm{x}, t)$.

Lemma 2.1. [16] *Let $d \in \mathbb{N}$, $0 < \alpha < 1$, and $0 < s < 1$. Then the functions $G_a(\mathrm{x}, t)$ and $G_f(\mathrm{x}, t)$ in Eqs* (2.8) *and* (2.9) *have the following properties.*
(1) $G_a(\mathrm{x}, t) > 0,\ G_f(\mathrm{x}, t) > 0$.
(2) $\int_{\mathbb{R}^d} G_a(\mathrm{x}, t)\mathrm{dx} = 1,\ \int_{\mathbb{R}^d} G_f(\mathrm{x}, t)\mathrm{dx} = \frac{1}{\Gamma(\alpha)}\left(\log \frac{t}{a}\right)^{\alpha-1}$. (3) ${}_H\mathrm{D}^{-(1-\alpha)}_{a,t} G_f(\mathrm{x}, t) = G_a(\mathrm{x}, t)$.

Lemma 2.2. [16] *Let $d \in \mathbb{N}$, $0 < \alpha < 1$, and $0 < s < 1$. If $u_a(\mathrm{x}) \geq 0$ and $u_a(\mathrm{x}) \not\equiv 0$, then we have $G_a(\mathrm{x}, t) * u_a(\mathrm{x}) > 0$ and $\|G_a(\mathrm{x}, t) * u_a(\mathrm{x})\|_{L^1(\mathbb{R}^d)} = \|u_a(\mathrm{x})\|_{L^1(\mathbb{R}^d)}$. Furthermore, when $1 \leq r \leq q \leq +\infty$ and $\frac{1}{r} - \frac{1}{q} < \min\{1, \frac{2s}{d}\}$, it holds that*

$$\|G_a(\mathrm{x}, t) * u_a(\mathrm{x})\|_{L^q(\mathbb{R}^d)} \leq C \left(\log \frac{t}{a}\right)^{-\frac{\alpha d}{2s}(\frac{1}{r}-\frac{1}{q})} \|u_a(\mathrm{x})\|_{L^r(\mathbb{R}^d)},\ t > a. \tag{2.10}$$

Lemma 2.3. [16] *Let $d \in \mathbb{N}$, $0 < \alpha < 1$, and $0 < s < 1$. If $u_a(\mathrm{x}) \geq 0$ and $u_a(\mathrm{x}) \not\equiv 0$, then we have $G_f(\mathrm{x}, t) * u_a(\mathrm{x}) > 0$ and $\|G_f(\mathrm{x}, t) * u_a(\mathrm{x})\|_{L^1(\mathbb{R}^d)} = \frac{1}{\Gamma(\alpha)}\left(\log \frac{t}{a}\right)^{\alpha-1} \|u_a(\mathrm{x})\|_{L^1(\mathbb{R}^d)}$. Furthermore, when $1 \leq r \leq q \leq +\infty$ and $\frac{1}{r} - \frac{1}{q} < \min\{1, \frac{4s}{d}\}$, it holds that*

$$\|G_f(\mathrm{x}, t) * u_a(\mathrm{x})\|_{L^q(\mathbb{R}^d)} \leq C \left(\log \frac{t}{a}\right)^{\alpha-1-\frac{\alpha d}{2s}(\frac{1}{r}-\frac{1}{q})} \|u_a(\mathrm{x})\|_{L^r(\mathbb{R}^d)},\ t > a. \tag{2.11}$$

Lemma 2.4. [16] *Let $d \in \mathbb{N}$, $0 < \alpha < 1$, and $0 < s < 1$. Assume $u_a(\mathrm{x}) \in C_0(\mathbb{R}^d)$. Then for $t > a > 0$, we have $G_a(\mathrm{x}, t) * u_a(\mathrm{x}) \in C_0(\mathbb{R}^d)$ and*

$${}_{CH}\mathrm{D}^{\alpha}_{a,t}\ [G_a(\mathrm{x}, t) * u_a(\mathrm{x})] = -(-\Delta)^s [G_a(\mathrm{x}, t) * u_a(\mathrm{x})],\ t > a > 0.$$

And there exists a constant $C > 0$ such that

$$\|(-\Delta)^s [G_a(\mathrm{x}, t) * u_a(\mathrm{x})]\|_{L^\infty(\mathbb{R}^d)} \leq C \left(\log \frac{t}{a}\right)^{-\alpha} \|u_a(\mathrm{x})\|_{L^\infty(\mathbb{R}^d)},\ t > a > 0.$$

For simplicity of representation, from now on, we denote $G_a(t) = G_a(\mathrm{x}, t)$, $G_f(t) = G_f(\mathrm{x}, t)$, and so on.

Lemma 2.5. [16] *Let $d \in \mathbb{N}$, $0 < \alpha < 1$, $0 < s < 1$, and $T > a > 0$. Let also $f \in L^q((a, T), C_0(\mathbb{R}^d))$ with $q > 1$ and*

$$\theta(t) = \int_a^t G_f\left(a\frac{t}{\tau}\right) * f(\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Then we have

$${}_H\mathrm{D}^{-(1-\alpha)}_{a,t}\theta(t) = \int_a^t G_a\left(a\frac{t}{\tau}\right) * f(\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Furthermore, one has $\theta(t) \in C([a, T], C_0(\mathbb{R}^d))$ provided that $q\alpha > 1$.

3. The mild solution and weak solution

In this part, we first define a mild solution of Eq (1.1) and then prove the local existence and uniqueness of the mild solution in terms of the contraction mapping principle. Next, the definition of a weak solution is introduced to Eq (1.1). We can also prove that the mild solution is just a weak solution. Let us begin by introducing the definition of a mild solution to Eq (1.1).

Definition 3.1. *Let $d \in \mathbb{N}$, $0 < \alpha < \gamma < 1$, $0 < s < 1$, $p > 1$ and $T > a > 0$. Let $u_a \in C_0(\mathbb{R}^d)$. Then a mild solution $u \in C([a, T], C_0(\mathbb{R}^d))$ of Eq (1.1) is given by*

$$u(t) = G_a(t) * u_a + \int_a^t G_f\left(a\frac{t}{\tau}\right) * [\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)(\tau)]\frac{\mathrm{d}\tau}{\tau},\ t \in [a, T]. \tag{3.1}$$

Theorem 3.1. *Let $d \in \mathbb{N}$, $0 < \alpha < \gamma < 1$, $0 < s < 1$, $p > 1$, and $T > a > 0$. Let $u_a \in C_0(\mathbb{R}^d)$. Then there is a maximal time $T_{\max} > a$ such that Eq (1.1) has a unique mild solution $u \in C([a, T_{\max}), C_0(\mathbb{R}^d))$, where, either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\|u\|_{L^\infty((a,T),L^\infty(\mathbb{R}^d)} \to \infty$ as $t \to T_{\max}^-$. Moreover, if $u_a \geq 0$ and $u_a \not\equiv 0$, then $u(t) > 0$ for any $a < t < T_{\max}$. Besides, if $u_a \in L^r(\mathbb{R}^d)$ for $1 \leq r < \infty$, then one has $u \in C([a, T_{\max}), L^r(\mathbb{R}^d))$.*

Proof. For given $T > a > 0$ and $u_a \in C_0(\mathbb{R}^d)$, let

$$E_{a,T} = \left\{u \in C([a, T], C_0(\mathbb{R}^d)) \mid \|u\|_{L^\infty((a,T),L^\infty(\mathbb{R}^d))} \leq 2\|u_a\|_{L^\infty(\mathbb{R}^d)}\right\}$$

and

$$d(u, v) = \max_{t\in[a,T]} \|u(t) - v(t)\|_{L^\infty(\mathbb{R}^d)},\ \forall\, u, v \in E_{a,T}.$$

Obviously, $(E_{a,T}, d)$ is a complete metric space. By means of the fundamental solutions $G_a(t)$ and $G_f(t)$, we define the following operator $\mathscr{F}$ on the metric space $(E_{a,T}, d)$,

$$\mathscr{F}(u)(t) = G_a(t) * u_a + \int_a^t G_f\left(a\frac{t}{\tau}\right) * [\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)(\tau)]\frac{\mathrm{d}\tau}{\tau},\ u \in E_{a,T}.$$

It follows from Lemma 2.5 that $\mathscr{F}(u) \in C([a, T], C_0(\mathbb{R}^d))$.

We next show that $\mathscr{F} : E_{a,T} \to E_{a,T}$. For $u \in E_{a,T}$ and $t \in [a, T]$, by Definition 2.1 and Lemma 2.1, we get

$$\begin{aligned}
&\|\mathscr{F}(u)(t)\|_{L^\infty(\mathbb{R}^d)}\\
\leq&\|G_a(t) * u_a\|_{L^\infty(\mathbb{R}^d)} + \int_a^t \left\|G_f\left(a\frac{t}{\tau}\right) * [\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)(\tau)]\right\|_{L^\infty(\mathbb{R}^d)}\frac{\mathrm{d}\tau}{\tau}\\
\leq&\|u_a\|_{L^\infty(\mathbb{R}^d)} + \frac{2^p}{\Gamma(\alpha)\Gamma(1-\gamma)}\int_a^t\int_a^\tau \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{\tau}{w}\right)^{-\gamma}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}\tau}{\tau}\|u_a\|_{L^\infty(\mathbb{R}^d)}^p\\
=&\|u_a\|_{L^\infty(\mathbb{R}^d)} + \frac{2^p}{\Gamma(\alpha)\Gamma(2-\gamma)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{\tau}{a}\right)^{1-\gamma}\frac{\mathrm{d}\tau}{\tau}\|u_a\|_{L^\infty(\mathbb{R}^d)}^p\\
=&\|u_a\|_{L^\infty(\mathbb{R}^d)} + \frac{2^p}{\Gamma(\alpha)\Gamma(2-\gamma)}\left(\log\frac{t}{a}\right)^{\alpha-\gamma+1}\int_0^1 \tau^{\alpha-1}(1-\tau)^{1-\gamma}\mathrm{d}\tau\|u_a\|_{L^\infty(\mathbb{R}^d)}^p
\end{aligned}$$

$$
\begin{aligned}
=&\|u_a\|_{L^\infty(\mathbb{R}^d)} + \frac{2^p\Gamma(\alpha)\Gamma(2-\gamma)}{\Gamma(\alpha)\Gamma(2-\gamma)\Gamma(2+\alpha-\gamma)}\left(\log\frac{t}{a}\right)^{\alpha-\gamma+1}\|u_a\|^p_{L^\infty(\mathbb{R}^d)}\\
\le&\|u_a\|_{L^\infty(\mathbb{R}^d)} + \frac{2^p\|u_a\|^{p-1}_{L^\infty(\mathbb{R}^d)}}{\Gamma(2+\alpha-\gamma)}\left(\log\frac{T}{a}\right)^{\alpha-\gamma+1}\|u_a\|_{L^\infty(\mathbb{R}^d)}.
\end{aligned}
$$

Choosing $T > a$ sufficiently close to a such that

$$
\frac{2^p}{\Gamma(2+\alpha-\gamma)}\left(\log\frac{T}{a}\right)^{\alpha-\gamma+1}\|u_a\|^{p-1}_{L^\infty(\mathbb{R}^d)} \le 1,
$$

then we obtain $\|\mathscr{F}(u)\|_{L^\infty((a,T),L^\infty(\mathbb{R}^d))} \le 2\|u_a\|_{L^\infty(\mathbb{R}^d)}$ and $\mathscr{F}(u) \in E_{a,T}$, viz., the operator $\mathscr{F}$ maps $E_{a,T}$ into itself.

We need to show that the operator $\mathscr{F}$ is contractive on $E_{a,T}$. For $u, v \in E_{a,T}$ and $t \in [a, T]$, one can deduce that

$$
\begin{aligned}
&\|\mathscr{F}(u)(t) - \mathscr{F}(v)(t)\|_{L^\infty(\mathbb{R}^d)}\\
\le&\frac{1}{\Gamma(\alpha)\Gamma(1-\gamma)}\int_a^t\int_a^\tau\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{\tau}{w}\right)^{-\gamma}\\
&\times\|\,|u(w)|^{p-1}u(w) - |v(w)|^{p-1}v(w)\|_{L^\infty(\mathbb{R}^d)}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}\tau}{\tau}\\
\le&\frac{2^pC(p)}{\Gamma(2+\alpha-\gamma)}\left(\log\frac{T}{a}\right)^{\alpha-\gamma+1}\|u_a\|^{p-1}_{L^\infty(\mathbb{R}^d)}\|u-v\|_{L^\infty((a,T),L^\infty(\mathbb{R}^d))}.
\end{aligned}
$$

Taking $T > a$ sufficiently close to a gives rise to

$$
\frac{2^pC(p)}{\Gamma(2+\alpha-\gamma)}\left(\log\frac{T}{a}\right)^{\alpha-\gamma+1}\|u_a\|^{p-1}_{L^\infty(\mathbb{R}^d)} \le \frac{1}{2},
$$

which means $\|\mathscr{F}(u)(t) - \mathscr{F}(v)(t)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{2}\|u-v\|_{L^\infty((a,T),C_0(\mathbb{R}^d))}$. This illustrates the operator $\mathscr{F}$ is contractive on $E_{a,T}$ and thus it has a fixed point $u \in E_{a,T}$ by the contraction mapping principle. Moreover, using Gronwall inequality immediately knows the uniqueness of the mild solutions to Eq (1.1) holds.

In view of the uniqueness, there is a maximal time $T_{\max} > a$ such that the solution of Eq (1.1) exists on the interval $[a, T_{\max})$, where

$$
T_{\max} = \sup\left\{T > a \,|\, \text{there is a mild solution } u \in C([a,T], C_0(\mathbb{R}^d)) \text{ to (1.1)}\right\} \le +\infty.
$$

Next, we show $\|u\|_{L^\infty((a,T),L^\infty(\mathbb{R}^d))} \to \infty$ as $t \to T^-_{\max}$ provided that $T_{\max} < \infty$. If $T_{\max} < \infty$ and there is $M > 0$ satisfying $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le M$ for $t \in [a, T_{\max})$, then we have for $a < \xi < \eta < T_{\max}$,

$$
\begin{aligned}
&\|u(\xi) - u(\eta)\|_{L^\infty(\mathbb{R}^d)}\\
\le&\|\,[G_a(\xi) - G_a(\eta)] * u_a\|_{L^\infty(\mathbb{R}^d)} + \left\|\int_\xi^\eta G_f\left(a\frac{\eta}{\tau}\right) * [\,{}_H\mathrm{D}^{-(1-\gamma)}_{a,\tau}(|u|^{p-1}u)(\tau)]\frac{\mathrm{d}\tau}{\tau}\right\|_{L^\infty(\mathbb{R}^d)}\\
&+\left\|\int_a^\xi\left(G_f\left(a\frac{\xi}{\tau}\right) - G_f\left(a\frac{\eta}{\tau}\right)\right) * [\,{}_H\mathrm{D}^{-(1-\gamma)}_{a,\tau}(|u|^{p-1}u)(\tau)]\frac{\mathrm{d}\tau}{\tau}\right\|_{L^\infty(\mathbb{R}^d)}
\end{aligned}
$$

$$\leq \| [G_a(\xi) - G_a(\eta)] * u_a\|_{L^\infty(\mathbb{R}^d)} + \frac{M^p}{\Gamma(\alpha)\Gamma(2-\gamma)} \int_\xi^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha-1} \left(\log \frac{\tau}{a}\right)^{1-\gamma} \frac{\mathrm{d}\tau}{\tau}$$
$$+\frac{CM^p}{\Gamma(2-\gamma)} \int_a^\xi \left(\log \frac{\tau}{a}\right)^{1-\gamma} \min\left\{\left(\log \frac{\xi}{\tau}\right)^{\alpha-1}, \left(\log \frac{\xi}{\tau}\right)^{\alpha-2} \left(\log \frac{\eta}{\xi}\right)\right\} \frac{\mathrm{d}\tau}{\tau}$$
$$\leq \| [G_a(\xi) - G_a(\eta)] * u_a\|_{L^\infty(\mathbb{R}^d)} + \frac{M^p}{\Gamma(\alpha)\Gamma(2-\gamma)} \left(\log \frac{T_{\max}}{a}\right)^{1-\gamma} \int_\xi^\eta \left(\log \frac{\eta}{\tau}\right)^{\alpha-1} \frac{\mathrm{d}\tau}{\tau}$$
$$+\frac{CM^p}{\Gamma(2-\gamma)} \left(\log \frac{T_{\max}}{a}\right)^{1-\gamma} \int_a^\xi \min\left\{\left(\log \frac{\xi}{\tau}\right)^{\alpha-1}, \left(\log \frac{\xi}{\tau}\right)^{\alpha-2} \left(\log \frac{\eta}{\xi}\right)\right\} \frac{\mathrm{d}\tau}{\tau}$$
$$\leq \| [G_a(\xi) - G_a(\eta)] * u_a\|_{L^\infty(\mathbb{R}^d)} + \frac{M^p \left(\log \frac{T_{\max}}{a}\right)^{1-\gamma}}{\Gamma(\alpha+1)\Gamma(2-\gamma)} \left(\log \frac{\eta}{\xi}\right)^\alpha$$
$$+ CM^p \left(\log \frac{T_{\max}}{a}\right)^{1-\gamma} \left(\log \frac{\eta}{\xi}\right)^\alpha,$$

which implies $\lim\limits_{t\to T_{\max}^-} u(t)$ exists in $C_0(\mathbb{R}^d)$.

Now we define $\lim\limits_{t\to T_{\max}^-} u(t) = u_{T_{\max}}$. Therefore one gets $u \in C([a, T_{\max}], C_0(\mathbb{R}^d))$. Furthermore, using Lemma 2.5 yields that

$$\int_a^t G_f\left(a\frac{t}{\tau}\right) * [\,_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)(\tau)]\frac{\mathrm{d}\tau}{\tau} \in C([a, T_{\max}], C_0(\mathbb{R}^d)).$$

For $h > 0$ and $\sigma > 0$, consider a set

$$\widetilde{E}_{h,\sigma} = \left\{u \in C([T_{\max}, T_{\max}+h], C_0(\mathbb{R}^d)) \,|\, u(T_{\max}) = u_{T_{\max}}, d(u, u_{T_{\max}}) \leq \sigma\right\}$$

equipped with

$$d(u, v) = \max_{t\in[T_{\max}, T_{\max}+h]} \|u(t) - v(t)\|_{L^\infty(\mathbb{R}^d)},\ \forall\, u, v \in \widetilde{E}_{h,\sigma}.$$

Then the metric space $(\widetilde{E}_{h,\sigma}, d)$ is complete.

On the space $(\widetilde{E}_{h,\sigma}, d)$, define an operator $\mathscr{Q}$ as follows,

$$\mathscr{Q}(v)(t) = G_a(t) * u_a + \int_a^{T_{\max}} G_f\left(a\frac{t}{\tau}\right) * [\,_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)(\tau)]\frac{\mathrm{d}\tau}{\tau}$$
$$+ \int_{T_{\max}}^t G_f\left(a\frac{t}{\tau}\right) * [\,_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|v|^{p-1}v)(\tau)]\frac{\mathrm{d}\tau}{\tau},\ v \in E_{h,\sigma}.$$

It is easy to see that $\mathscr{Q}(v) \in C([T_{\max}, T_{\max}+h], C_0(\mathbb{R}^d))$ and $\mathscr{Q}(v)(T_{\max}) = u_{T_{\max}}$.

We first prove $\mathscr{Q}(v) \in \widetilde{E}_{h,\sigma}$ for $v \in \widetilde{E}_{h,\sigma}$. As a matter of fact, if $t \in [T_{\max}, T_{\max}+h]$, then

$$\|\mathscr{Q}(v)(t) - u_{T_{\max}}\|_{L^\infty(\mathbb{R}^d)}$$
$$\leq \|G_a(t) * u_a - G_a(T_{\max}) * u_a\|_{L^\infty(\mathbb{R}^d)}$$
$$+ \left\|\int_{T_{\max}}^t G_f\left(a\frac{t}{\tau}\right) * [\,_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|v|^{p-1}v)(\tau)]\frac{\mathrm{d}\tau}{\tau}\right\|_{L^\infty(\mathbb{R}^d)}$$

$$
\begin{aligned}
&+\left\|\int_a^{T_{\max}}\left[G_f\left(a\frac{t}{\tau}\right)-G_f\left(a\frac{T_{\max}}{\tau}\right)\right]*[\,_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)(\tau)]\frac{\mathrm{d}\tau}{\tau}\right\|_{L^\infty(\mathbb{R}^d)}\\
=&\|J_1\|_{L^\infty(\mathbb{R}^d)}+\|J_2\|_{L^\infty(\mathbb{R}^d)}+\|J_3\|_{L^\infty(\mathbb{R}^d)}.
\end{aligned}
$$

By taking sufficiently small h, we arrive at

$$
\|J_1\|_{L^\infty(\mathbb{R}^d)}\le\frac{\sigma}{3},\quad \|J_3\|_{L^\infty(\mathbb{R}^d)}\le\frac{\sigma}{3}.
$$

In regard to $\|J_2\|_{L^\infty(\mathbb{R}^d)}$, one has

$$
\begin{aligned}
\|J_2\|_{L^\infty(\mathbb{R}^d)}\le&\left\|\int_{T_{\max}}^t G_f\left(a\frac{t}{\tau}\right)*\left[\,_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|v(\tau)|^{p-1}v(\tau)-|u_{T_{\max}}|^{p-1}u_{T_{\max}})\right]\frac{\mathrm{d}\tau}{\tau}\right\|_{L^\infty(\mathbb{R}^d)}\\
&+\left\|\int_{T_{\max}}^t G_f\left(a\frac{t}{\tau}\right)*[\,_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u_{T_{\max}}|^{p-1}u_{T_{\max}})]\frac{\mathrm{d}\tau}{\tau}\right\|_{L^\infty(\mathbb{R}^d)}\\
\le&\left(\frac{C\sigma}{\alpha}+\frac{\left(\log\frac{t}{a}\right)^{1-\gamma}}{\Gamma(\alpha+1)\Gamma(2-\gamma)}\|u_{T_{\max}}\|_{L^\infty(\mathbb{R}^d)}^p\right)\left(\log\frac{t}{T_{\max}}\right)^\alpha\le\frac{\sigma}{3},
\end{aligned}
$$

for $t\in[T_{\max},T_{\max}+h]$ and h small enough. Therefore, there holds $\|\mathscr{Q}(v)(t)-u_{T_{\max}}\|_{L^\infty(\mathbb{R}^d)}\le\sigma$, i.e., $d(\mathscr{Q}(v),u_{T_{\max}})\le\sigma$ for $t\in[T_{\max},T_{\max}+h]$.

We next show that the operator $\mathscr{Q}$ is contractive on $\widetilde{E}_{h,\sigma}$. Assume that $v,w\in\widetilde{E}_{h,\sigma}$ and $t\in[T_{\max},T_{\max}+h]$, it follows that

$$
\begin{aligned}
&\|\mathscr{Q}(v)(t)-\mathscr{Q}(w)(t)\|_{L^\infty(\mathbb{R}^d)}\\
\le&\frac{C(p)}{\Gamma(\alpha)\Gamma(2-\gamma)}\left(\|v\|^{p-1}_{L^\infty((T_{\max},T_{\max}+h),L^\infty(\mathbb{R}^d))}+\|w\|^{p-1}_{L^\infty((T_{\max},T_{\max}+h),L^\infty(\mathbb{R}^d))}\right)\\
&\times\int_{T_{\max}}^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{\tau}{a}\right)^{1-\gamma}\frac{\mathrm{d}\tau}{\tau}\|v-w\|_{L^\infty((T_{\max},T_{\max}+h),L^\infty(\mathbb{R}^d))}\\
\le&\frac{2C(p)\left(\log\frac{t}{a}\right)^{1-\gamma}}{\Gamma(\alpha+1)\Gamma(2-\gamma)}\left(\sigma+\|u_{T_{\max}}\|_{L^\infty(\mathbb{R}^d)}\right)^{p-1}\left(\log\frac{t}{T_{\max}}\right)^\alpha d(v,w).
\end{aligned}
$$

In this case, for $t\in[T_{\max},T_{\max}+h]$, one may take very small h such that

$$
\frac{2C(p)\left(\log\frac{t}{a}\right)^{1-\gamma}}{\Gamma(\alpha+1)\Gamma(2-\gamma)}\left(\sigma+\|u_{T_{\max}}\|_{L^\infty(\mathbb{R}^d)}\right)^{p-1}\left(\log\frac{t}{T_{\max}}\right)^\alpha\le\frac{1}{2},
$$

which suggests the operator $\mathscr{Q}$ is contractive on $\widetilde{E}_{h,\sigma}$ and thus it has a fixed point $v\in\widetilde{E}_{h,\sigma}$. In view of $v(T_{\max})=\mathscr{Q}(v)(T_{\max})=u(T_{\max})$, we set

$$
\overline{u}(t)=\begin{cases}u(t), & t\in[a,T_{\max}),\\ v(t), & t\in[T_{\max},T_{\max}+h],\end{cases}
$$

such that $\overline{u}(t)\in C([a,T_{\max}+h],C_0(\mathbb{R}^d))$ and

$$
\overline{u}(t)=G_a(t)*u_a+\int_a^t G_f\left(a\frac{t}{\tau}\right)*[\,_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|\overline{u}(\tau)|^{p-1}\overline{u})(\tau)]\frac{\mathrm{d}\tau}{\tau},
$$

which means $\overline{u}(t)$ is indeed a mild solution of Eq (1.1). Recalling the definition of $T_{\max}$, this yields a contradiction.

The proof of the remainder of this theorem follows that of Theorem 3.2 in [16] and so is omitted. The proof is thus complete. □

In the following, we present the definition of a weak solution to Eq (1.1) and show that the mild solution given by Definition 3.1 is a weak solution.

Definition 3.2. *Let $d \in \mathbb{N}$, $0 < \alpha < \gamma < 1$, $0 < s < 1$, $p > 1$, and $T > a > 0$. For given $u_a \in L^\infty_{Loc}(\mathbb{R}^d)$, a function u is said to be a weak solution of Eq (1.1) if $u \in L^p((a,T), L^\infty_{Loc}(\mathbb{R}^d))$ and*

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_a^T \left({}_H\mathrm{D}^{-(1-\gamma)}_{a,\tau}(|u|^{p-1}u)\,\varphi + u_a\, {}_{CH}\mathrm{D}^{\alpha}_{t,T}\,\varphi\right)\frac{\mathrm{d}t}{t}\mathrm{dx}\\ &= \int_{\mathbb{R}^d}\int_a^T \left(u(-\Delta)^s\varphi + u\, {}_{CH}\mathrm{D}^{\alpha}_{t,T}\,\varphi\right)\frac{\mathrm{d}t}{t}\mathrm{dx},\end{aligned} \tag{3.2}$$

for any test function $\varphi \in C^{2,1}_{\mathrm{x},t}(\mathbb{R}^d \times [a,T])$ satisfying $\operatorname{supp}_{\mathrm{x}}\varphi \subset\subset \mathbb{R}^d$ and $\varphi(\cdot,T) = 0$.

Theorem 3.2. *Let $d \in \mathbb{N}$, $0 < \alpha < \gamma < 1$, $0 < s < 1$, $p > 1$, and $T > a > 0$. If the initial value $u_a \in C_0(\mathbb{R}^d)$, then the mild solution $u \in C([a,T], C_0(\mathbb{R}^d))$ of Eq (1.1) is also its weak solution.*

Proof. Assume that $u \in C([a,T], C_0(\mathbb{R}^d))$ is a mild solution to Eq (1.1). Then Definition 3.1 gives

$$u - u_a = G_a(t) * u_a - u_a + \int_a^t G_f\left(a\frac{t}{\tau}\right) * [\, {}_H\mathrm{D}^{-(1-\gamma)}_{a,\tau}(|u|^{p-1}u)]\frac{\mathrm{d}\tau}{\tau}.$$

Use Lemma 2.5 to get

$${}_H\mathrm{D}^{-(1-\alpha)}_{a,t}(u - u_a) = {}_H\mathrm{D}^{-(1-\alpha)}_{a,t}(G_a(t) * u_a - u_a) + \int_a^t G_a\left(a\frac{t}{\tau}\right) * [\, {}_H\mathrm{D}^{-(1-\gamma)}_{a,\tau}(|u|^{p-1}u)]\frac{\mathrm{d}\tau}{\tau}.$$

Therefore, for every $\varphi \in C^{2,1}_{\mathrm{x},t}(\mathbb{R}^d \times [a,T])$ satisfying $\operatorname{supp}_{\mathrm{x}}\varphi \subset\subset \mathbb{R}^d$ and $\varphi(\cdot,T) = 0$, there holds

$$\begin{aligned}\int_{\mathbb{R}^d} {}_H\mathrm{D}^{-(1-\alpha)}_{a,t}(u - u_a)\varphi\mathrm{dx} &= \int_{\mathbb{R}^d} {}_H\mathrm{D}^{-(1-\alpha)}_{a,t}(G_a(t) * u_a - u_a)\varphi\mathrm{dx}\\ &\quad + \int_{\mathbb{R}^d}\int_a^t G_a\left(a\frac{t}{\tau}\right) * [\, {}_H\mathrm{D}^{-(1-\gamma)}_{a,\tau}(|u|^{p-1}u)]\frac{\mathrm{d}\tau}{\tau}\varphi\mathrm{dx} = I_1 + I_2.\end{aligned}$$

For I_1, an application of Lemma 2.4 leads to

$$\delta I_1 = -\int_{\mathbb{R}^d}(-\Delta)^s(G_a(t) * u_a)\varphi\mathrm{dx} + \int_{\mathbb{R}^d} {}_H\mathrm{D}^{-(1-\alpha)}_{a,t}(G_a(t) * u_a - u_a)\delta\varphi\mathrm{dx}. \tag{3.3}$$

To estimate I_2, we set $h > 0$, $t \in [a,T)$ and $t + h \le T$, then

$$\frac{I_2(t+h) - I_2(t)}{\log(t+h) - \log(t)}$$

$$
\begin{aligned}
=&\frac{1}{\log(t+h)-\log(t)}\int_{\mathbb{R}^d}\int_t^{t+h}G_a\left(a\frac{t+h}{\tau}\right)*[\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)]\frac{\mathrm{d}\tau}{\tau}\varphi(t+h,\mathrm{x})\mathrm{dx}\\
+&\frac{1}{\log(t+h)-\log(t)}\int_{\mathbb{R}^d}\int_a^{t}G_a\left(a\frac{t+h}{\tau}\right)*[\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)]\frac{\mathrm{d}\tau}{\tau}[\varphi(t+h,\mathrm{x})-\varphi(t,\mathrm{x})]\mathrm{dx}\\
+&\frac{1}{\log(t+h)-\log(t)}\int_{\mathbb{R}^d}\int_a^{t}\left[G_a\left(a\frac{t+h}{\tau}\right)-G_a\left(a\frac{t}{\tau}\right)\right]*[\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)]\frac{\mathrm{d}\tau}{\tau}\varphi(t,\mathrm{x})\mathrm{dx}\\
=&I_{21}+I_{22}+I_{23}.
\end{aligned}
$$

Applying the mean value theorem yields that

$$
\lim_{h\to 0}I_{21}=\int_{\mathbb{R}^d}{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)\,\varphi\mathrm{dx},
$$

$$
\lim_{h\to 0}I_{22}=\int_{\mathbb{R}^d}\int_a^t G_a\left(a\frac{t}{\tau}\right)*[\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)]\frac{\mathrm{d}\tau}{\tau}\delta\varphi\mathrm{dx},
$$

and

$$
\lim_{h\to 0}I_{23}=-\int_{\mathbb{R}^d}\int_a^t G_f\left(a\frac{t}{\tau}\right)*[\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)]\frac{\mathrm{d}\tau}{\tau}(-\Delta)^s\varphi\mathrm{dx}.
$$

Consequently,

$$
\begin{aligned}
\delta I_2=&\int_{\mathbb{R}^d}{}_H\mathrm{D}_{a,t}^{-(1-\gamma)}(|u|^{p-1}u)\,\varphi\mathrm{dx}\\
&+\int_{\mathbb{R}^d}{}_H\mathrm{D}_{a,t}^{-(1-\alpha)}\int_a^t G_f\left(a\frac{t}{\tau}\right)*[\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)]\frac{\mathrm{d}\tau}{\tau}\delta\varphi\mathrm{dx}\\
&-\int_{\mathbb{R}^d}\int_a^t G_f\left(a\frac{t}{\tau}\right)*[\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)]\frac{\mathrm{d}\tau}{\tau}(-\Delta)^s\varphi\mathrm{dx}.
\end{aligned}\tag{3.4}
$$

Combining (3.3) and (3.4), we obtain

$$
\begin{aligned}
0=&\int_a^T\delta\int_{\mathbb{R}^d}{}_H\mathrm{D}_{a,t}^{-(1-\alpha)}(u-u_a)\varphi\mathrm{dx}\frac{\mathrm{d}t}{t}=\int_a^T(\delta I_1+\delta I_2)\frac{\mathrm{d}t}{t}\\
=&-\int_a^T\int_{\mathbb{R}^d}u(-\Delta)^s\varphi\mathrm{dx}\frac{\mathrm{d}t}{t}+\int_a^T\int_{\mathbb{R}^d}{}_H\mathrm{D}_{a,t}^{-(1-\gamma)}(|u|^{p-1}u)\,\varphi\mathrm{dx}\frac{\mathrm{d}t}{t}\\
&-\int_a^T\int_{\mathbb{R}^d}(u-u_a)\,{}_{CH}\mathrm{D}_{t,T}^{\alpha}\,\varphi\mathrm{dx}\frac{\mathrm{d}t}{t},
\end{aligned}
$$

which is the desired result and the proof is now ended. □

4. Proof of main result

Proof of Theorem 1.1.
(1) We consider two cases: (i) $1<p<\widetilde{p}=1+\frac{1-\gamma}{\alpha}$. (ii) $1<p<\widetilde{p}=1+\frac{2s(1+\alpha-\gamma)}{\alpha d}$.

(i) Assume that $1 < p < \widetilde{p} = 1 + \frac{1-\gamma}{\alpha}$. Let

$$\omega(\mathrm{x}) = \left(\int_{\mathbb{R}^d} e^{-\sqrt{d^2+|\mathrm{x}|^2}}\mathrm{dx}\right)^{-1} e^{-\sqrt{d^2+|\mathrm{x}|^2}}, \mathrm{x} \in \mathbb{R}^d,$$

and the function Φ satisfy

$$\Phi \in C_0^\infty(\mathbb{R}), \quad \Phi(\xi) = \begin{cases} 1, & |\xi| \le 1, \\ 0, & |\xi| \ge 2, \end{cases} \quad 0 \le \Phi \le 1.$$

Thanks to Theorem 3.2, we may take $\varphi_1(\mathrm{x}) = \omega(\mathrm{x})\Phi_n(\mathrm{x})$ with $\Phi_n(\mathrm{x}) = \Phi(|\mathrm{x}|/n), n = 1, 2, \ldots$, and $\varphi_2(t) = \left(1 - \frac{\log(t/a)}{\log(T/a)}\right)^m$ for $t \in [a, T]$, where $m \ge \max\{2, \frac{p(1+\alpha-\gamma)}{p-1}\}$. Now we set $\varphi(\mathrm{x}, t) = {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}(\varphi_1(\mathrm{x})\varphi_2(t))$. From Definition 3.2 of the weak solution, one has

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_a^T \left({}_H\mathrm{D}_{a,t}^{-(1-\gamma)}u^p\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}(\varphi_1\varphi_2) + u_a\, {}_{CH}\mathrm{D}_{t,T}^{\alpha}\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}(\varphi_1\varphi_2)\right)\frac{\mathrm{d}t}{t}\mathrm{dx} \\ &= \int_{\mathbb{R}^d}\int_a^T \left(u(-\Delta)^s\varphi_1\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}\varphi_2 + u\, {}_{CH}\mathrm{D}_{t,T}^{\alpha}\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}(\varphi_1\varphi_2)\right)\frac{\mathrm{d}t}{t}\mathrm{dx}. \end{aligned} \tag{4.1}$$

Furthermore, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_a^T \left(u^p\varphi_1\varphi_2 + u_a\varphi_1\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2\right)\frac{\mathrm{d}t}{t}\mathrm{dx} \\ &= \int_{\mathbb{R}^d}\int_a^T \left(u(-\Delta)^s\varphi_1\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}\varphi_2 + u\varphi_1\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2\right)\frac{\mathrm{d}t}{t}\mathrm{dx}. \end{aligned} \tag{4.2}$$

According to the inequality $(-\Delta)^s\omega(\mathrm{x}) \le \omega(\mathrm{x})$ in [16] and the Lebesgue dominated convergence theorem, we have with $n \to \infty$ in (4.2),

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_a^T u^p\omega\varphi_2\frac{\mathrm{d}t}{t}\mathrm{dx} + \int_{\mathbb{R}^d}\int_a^T u_a\omega\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2\frac{\mathrm{d}t}{t}\mathrm{dx} \\ &\le \int_{\mathbb{R}^d}\int_a^T \left(u\omega\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}\varphi_2 + u\omega\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2\right)\frac{\mathrm{d}t}{t}\mathrm{dx}. \end{aligned} \tag{4.3}$$

Using Jensen's inequality in (4.3) gives

$$\begin{aligned}&\int_a^T \left(\int_{\mathbb{R}^d} u\omega\mathrm{dx}\right)^p \varphi_2\frac{\mathrm{d}t}{t} + \int_{\mathbb{R}^d}\int_a^T u_a\omega\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2\frac{\mathrm{d}t}{t}\mathrm{dx} \\ &\le \int_{\mathbb{R}^d}\int_a^T \left(u\omega\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}\varphi_2 + u\omega\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2\right)\frac{\mathrm{d}t}{t}\mathrm{dx}. \end{aligned} \tag{4.4}$$

Denoting $f(t) = \int_{\mathbb{R}^d} u\,\omega\,\mathrm{dx}$, it is easy to see that $f(t) \ge 0$ and $f(a) > 0$. In view of inequality (4.4), Hölder inequality and Young's inequality, we obtain

$$\int_a^T f^p(t)\varphi_2(t)\frac{\mathrm{d}t}{t} + \int_a^T f(a)\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2(t)\frac{\mathrm{d}t}{t}$$

$$
\begin{aligned}
&\leq \int_a^T f(t)\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}\varphi_2(t)\frac{\mathrm{d}t}{t} + \int_a^T f(t)\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2(t)\frac{\mathrm{d}t}{t} \\
&= \int_a^T f(t)\varphi_2^{\frac{1}{p}}(t)\varphi_2^{-\frac{1}{p}}(t)\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}\varphi_2(t)\frac{\mathrm{d}t}{t} + \int_a^T f(t)\varphi_2^{\frac{1}{p}}(t)\varphi_2^{-\frac{1}{p}}(t)\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2(t)\frac{\mathrm{d}t}{t} \\
&\leq \frac{1}{2}\int_a^T f^p(t)\varphi_2(t)\frac{\mathrm{d}t}{t} + C\int_a^T \varphi_2^{-\frac{1}{p-1}}(t)\left({}_{CH}\mathrm{D}_{t,T}^{1-\gamma}\varphi_2(t)\right)^{\frac{p}{p-1}}\frac{\mathrm{d}t}{t} \\
&\quad + C\int_a^T \varphi_2^{-\frac{1}{p-1}}(t)\left({}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2(t)\right)^{\frac{p}{p-1}}\frac{\mathrm{d}t}{t}.
\end{aligned}
$$

Hence there holds

$$
\frac{1}{2}\int_a^T f^p(t)\varphi_2(t)\frac{\mathrm{d}t}{t} + Cf(a)\left(\log\frac{T}{a}\right)^{\gamma-\alpha} \leq C\left(\log\frac{T}{a}\right)^{\frac{p\gamma-1}{p-1}} + C\left(\log\frac{T}{a}\right)^{\frac{p\gamma-p\alpha-1}{p-1}}.
$$

Then we get

$$
f(a) < C\left(\log\frac{T}{a}\right)^{\frac{p\gamma-1}{p-1}+\alpha-\gamma} + C\left(\log\frac{T}{a}\right)^{\frac{p\gamma-p\alpha-1}{p-1}+\alpha-\gamma}. \tag{4.5}
$$

If Eq (1.1) has a global solution, we know that $f(a) = 0$ as $T \to \infty$ in (4.5) by $0 < \alpha < \gamma < 1$ and $p < 1 + \frac{1-\gamma}{\alpha}$, which is inconsistent with $f(a) > 0$. Hence, the mild solution of Eq (1.1) blows up in finite time.

(ii) Suppose that $1 < p < \widetilde{p} = 1 + \frac{2s(1+\alpha-\gamma)}{\alpha d}$. For $t \in [a, T]$ with $T > a > 0$, we take

$$
\varphi_1(\mathrm{x}) = \left(\Phi\left(\left(\log\frac{T}{a}\right)^{-\frac{\alpha}{2s}}|\mathrm{x}|\right)\right)^{\frac{2p}{p-1}}, \quad \varphi_2(t) = \left(1 - \frac{\log(t/a)}{\log(T/a)}\right)^m
$$

with $m \geq \max\{2, \frac{p(1+\alpha-\gamma)}{p-1}\}$, and $\varphi(\mathrm{x}, t) = {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}(\varphi_1(\mathrm{x})\varphi_2(t))$.

Let u be a mild solution of Eq (1.1), then Theorem 3.2 implies

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_a^T \left(u^p\varphi_1\varphi_2 + u_a\varphi_1\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2\right)\frac{\mathrm{d}t}{t}\mathrm{dx} \\
&= \int_{\mathbb{R}^d}\int_a^T \left(u((-\Delta)^s\varphi_1)\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}\varphi_2 + u\varphi_1\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2\right)\frac{\mathrm{d}t}{t}\mathrm{dx}.
\end{aligned} \tag{4.6}
$$

Note that the fact

$$
(-\Delta)^s\varphi_1\, {}_{CH}\mathrm{D}_{t,T}^{1-\gamma}\varphi_2 \leq C_1\left(\log\frac{T}{a}\right)^{-(1+\alpha-\gamma)}\varphi_1^{\frac{1}{p}}\varphi_2^{\frac{1}{p}}, \tag{4.7}
$$

and

$$
\varphi_1\, {}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2 \leq C_2\left(\log\frac{T}{a}\right)^{-(1+\alpha-\gamma)}\varphi_1^{\frac{1}{p}}\varphi_2^{\frac{1}{p}}, \tag{4.8}
$$

where the positive constants C_1 and C_2 are independent of T.

According to (4.6)–(4.8), together with Young's inequality and Hölder inequality, it holds that

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_a^T\left(u^p\varphi_1\varphi_2+u_a\varphi_1\,{}_{CH}\mathrm{D}_{t,T}^{1+\alpha-\gamma}\varphi_2\right)\frac{\mathrm{d}t}{t}\mathrm{dx}\\ \le& C\left(\log\frac{T}{a}\right)^{-(1+\alpha-\gamma)+(1+\frac{\alpha d}{2s})\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}\int_a^T u^p\varphi_1\varphi_2\frac{\mathrm{d}t}{t}\mathrm{dx}\right)^{\frac{1}{p}}\\ \le& C(p)\left(\log\frac{T}{a}\right)^{1+\frac{\alpha d}{2s}-\frac{p(1+\alpha-\gamma)}{p-1}}+\int_{\mathbb{R}^d}\int_a^T u^p\varphi_1\varphi_2\frac{\mathrm{d}t}{t}\mathrm{dx}.\end{aligned}\tag{4.9}$$

As a result,

$$C(\alpha,\gamma)\left(\log\frac{T}{a}\right)^{\gamma-\alpha}\int_{\mathbb{R}^d}u_a\varphi_1\mathrm{dx}\le C(p)\left(\log\frac{T}{a}\right)^{1+\frac{\alpha d}{2s}-\frac{p(1+\alpha-\gamma)}{p-1}},\tag{4.10}$$

i.e.,

$$\int_{\mathbb{R}^d}u_a\,\varphi\mathrm{dx}\le C(\alpha,\gamma,p)\left(\log\frac{T}{a}\right)^{1+\alpha-\gamma+\frac{\alpha d}{2s}-\frac{p(1+\alpha-\gamma)}{p-1}}.\tag{4.11}$$

The condition $1<p<1+\frac{2s(1+\alpha-\gamma)}{\alpha d}$ indicates $1+\alpha-\gamma+\frac{\alpha d}{2s}-\frac{p(1+\alpha-\gamma)}{p-1}<0$. If Eq (1.1) has a global solution, then $\int_{\mathbb{R}^d}u_a\varphi\mathrm{dx}=0$ as $T\to\infty$, that is $u_a\equiv 0$, which makes a contradiction with the assumption $u_a\not\equiv 0$. Therefore, blowup of the mild solution u of Eq (1.1) occurs in finite time.

(2) Based on the fixed point principle, we demonstrate the required result by constructing the global solution of Eq (1.1). Firstly, the condition $p\ge 1+\frac{2s(1+\alpha-\gamma)}{\alpha d}$ implies that

$$\frac{\alpha d(p-1)}{2s(p\alpha-p\gamma+1)_+}>1,\tag{4.12}$$

where $(p\alpha-p\gamma+1)_+=\max\{0,p\alpha-p\gamma+1\}$. If $d<\frac{2s(1+\alpha-\gamma)}{1-\gamma}$, one has $p\ge\widetilde{p}=1+\frac{2s(1+\alpha-\gamma)}{\alpha d}$, and if $d\ge\frac{2s(1+\alpha-\gamma)}{1-\gamma}$, one gets $p>\widetilde{p}=1+\frac{1-\gamma}{\alpha}$. In either case, we obtain

$$\frac{\alpha d(p-1)}{2sp(1+\alpha-\gamma-(p-1)\alpha)_+}>1.\tag{4.13}$$

In addition, by $p>1+\frac{1-\gamma}{\alpha}>\frac{1}{\gamma}$, it follows that

$$\frac{d(p-1)}{2sp}<\frac{\alpha d(p-1)}{2s(p(\alpha-\gamma)+1)_+},\tag{4.14}$$

and

$$\frac{d(p-1)}{2sp}<\frac{\alpha d(p-1)}{2sp(2\alpha+1-\gamma-\alpha p)_+}.\tag{4.15}$$

Hence, taking (4.12)–(4.15) into account, we can choose $q>p$ such that

$$\frac{1+\alpha-\gamma}{p-1}-\frac{1}{p}<\frac{\alpha d}{2sq}<\frac{\alpha}{p-1},\tag{4.16}$$

and

$$\frac{1+\alpha-\gamma}{p-1}-\alpha<\frac{\alpha d}{2sq}<\frac{\alpha}{p-1}.\tag{4.17}$$

Let

$$\beta = \frac{\alpha d}{2s}\left(\frac{1}{p^*} - \frac{1}{q}\right) = \frac{1+\alpha-\gamma}{p-1} - \frac{\alpha d}{2sq}. \tag{4.18}$$

Then (4.16) gives

$$0 < p\beta < 1. \tag{4.19}$$

If the initial value u_a satisfies

$$\sup_{t>a}\left(\log\frac{t}{a}\right)^{\beta} \|G_a(t) * u_a\|_{L^q(\mathbb{R}^d)} = \vartheta, \tag{4.20}$$

then $\vartheta < +\infty$ by (4.18) and (2.10) provided that $u_a \in L^{p^*}(\mathbb{R}^d)$ with $p^* = \frac{\alpha d(p-1)}{2s(1+\alpha-\gamma)}$.

Next we use the contractive mapping principle to obtain result. To this end, we denote

$$E = \left\{ u \in L^{\infty}((a,\infty), L^q(\mathbb{R}^d)) \,\middle|\, \|u\|_E = \sup_{t>a}\left(\log\frac{t}{a}\right)^{\beta}\|u\|_{L^q(\mathbb{R}^d)} < +\infty \right\}$$

and

$$\Psi(u)(t) = G_a(t) * u_a + \int_a^t G_f\left(a\frac{t}{\tau}\right) * [\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(|u|^{p-1}u)(\tau)]\frac{\mathrm{d}\tau}{\tau}, \ \forall\, u \in E.$$

Define

$$E_K = \{u \in E \,|\, \|u\|_E \le K, K > 0\}.$$

By $q > p$ and $q > \frac{d(p-1)}{4s}$, one gets $\frac{p}{q} - \frac{1}{q} < \min\{1, \frac{4s}{d}\}$. Thus, for any $u, v \in E_K$ and $t > a$, it follows that

$$\begin{aligned}
&\left(\log\frac{t}{a}\right)^{\beta}\|\Psi(u)(t) - \Psi(v)(t)\|_{L^q(\mathbb{R}^d)} \\
\le&\left(\log\frac{t}{a}\right)^{\beta}\int_a^t \left\|G_f\left(a\frac{t}{\tau}\right) * [\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}(u^p(\tau) - v^p(\tau))]\right\|_{L^q(\mathbb{R}^d)}\frac{\mathrm{d}\tau}{\tau} \\
\le&\frac{C}{\Gamma(1-\gamma)}\left(\log\frac{t}{a}\right)^{\beta}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1-\frac{\alpha d(p-1)}{2sq}}\int_a^{\tau}\left(\log\frac{\tau}{w}\right)^{-\gamma}\|u^p - v^p\|_{L^{\frac{q}{p}}(\mathbb{R}^d)}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}\tau}{\tau} \\
\le&\frac{C}{\Gamma(1-\gamma)}\left(\log\frac{t}{a}\right)^{\beta}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1-\frac{\alpha d(p-1)}{2sq}} \\
&\times\int_a^{\tau}\left(\log\frac{\tau}{w}\right)^{-\gamma}(\|u\|_{L^q(\mathbb{R}^d)}^{p-1} + \|v\|_{L^q(\mathbb{R}^d)}^{p-1})\|u-v\|_{L^q(\mathbb{R}^d)}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}\tau}{\tau} \\
\le&\frac{CK^{p-1}}{\Gamma(1-\gamma)}\left(\log\frac{t}{a}\right)^{\beta}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1-\frac{\alpha d(p-1)}{2sq}}\int_a^{\tau}\left(\log\frac{\tau}{w}\right)^{-\gamma}\left(\log\frac{w}{a}\right)^{-p\beta}\frac{\mathrm{d}w}{w}\frac{\mathrm{d}\tau}{\tau}\|u-v\|_E \\
=&\frac{CK^{p-1}}{\Gamma(1-\gamma)}\int_0^1(1-\tau)^{\alpha-1-\frac{\alpha d(p-1)}{2sq}}\tau^{1-\gamma-p\beta}\mathrm{d}\tau\int_0^1(1-w)^{-\gamma}w^{-p\beta}\mathrm{d}w\|u-v\|_E \\
=&\frac{CK^{p-1}}{\Gamma(1-\gamma)}\frac{\Gamma(\alpha-\frac{\alpha d(p-1)}{2sq})\Gamma(2-\gamma-p\beta)}{\Gamma(2+\alpha-p\beta-\gamma-\frac{\alpha d(p-1)}{2sq})}\frac{\Gamma(1-p\beta)\Gamma(1-\gamma)}{\Gamma(2-\gamma-p\beta)}\|u-v\|_E \\
=&CK^{p-1}\frac{\Gamma(\alpha-\frac{\alpha d(p-1)}{2sq})\Gamma(1-p\beta)}{\Gamma(2+\alpha-p\beta-\gamma-\frac{\alpha d(p-1)}{2sq})}\|u-v\|_E.
\end{aligned} \tag{4.21}$$

By taking K small enough such that

$$CK^{p-1}\frac{\Gamma(\alpha-\frac{\alpha d(p-1)}{2sq})\Gamma(1-p\beta)}{\Gamma(2+\alpha-p\beta-\gamma-\frac{\alpha d(p-1)}{2sq})}<\frac{1}{2},$$

which yields $\|\Psi(u)-\Psi(v)\|_E\le\frac{1}{2}\|u-v\|_E$ by Eq (4.21).

A similar calculation as (4.21) results in

$$\left(\log\frac{t}{a}\right)^{\beta}\|\Psi(u)(t)\|_{L^q(\mathbb{R}^d)}\le\vartheta+CK^p\frac{\Gamma(\alpha-\frac{\alpha d(p-1)}{2sq})\Gamma(1-p\beta)}{\Gamma(2+\alpha-p\beta-\gamma-\frac{\alpha d(p-1)}{2sq})}. \tag{4.22}$$

Choose sufficiently small ϑ and K such that

$$\vartheta+CK^p\frac{\Gamma(\alpha-\frac{\alpha d(p-1)}{2sq})\Gamma(1-p\beta)}{\Gamma(2+\alpha-p\beta-\gamma-\frac{\alpha d(p-1)}{2sq})}\le K.$$

This implies $\Psi(u)\in E_K$ and thus Ψ has a fixed point $u\in E_K$ by the contractive mapping principle.

Finally, We need to prove $u\in C([a,\infty),C_0(\mathbb{R}^d))$. For a T sufficiently close to a, let

$$E_{K,T}=\left\{u\in L^\infty((a,T),L^q(\mathbb{R}^d))\;\Big|\;\sup_{a<t<T}\left(\log\frac{t}{a}\right)^{\beta}\|u(t)\|_{L^q(\mathbb{R}^d)}\le K\right\}.$$

As demonstrated before, it is known that there is a unique solution u on $E_{K,T}$. It follows from Theorem 3.1 and the initial value $u_a\in C_0(\mathbb{R}^d)\cap L^q(\mathbb{R}^d)$ that there exists a unique solution $\widetilde{u}\in C([a,T],C_0(\mathbb{R}^d))\cap C([a,T],L^q(\mathbb{R}^d))$ for T sufficiently close to a. Hence, for T sufficiently close to a, one has $\sup\limits_{a<t<T}\left(\log\frac{t}{a}\right)^{\beta}\|\widetilde{u}(t)\|_{L^q(\mathbb{R}^d)}\le K$. This means that $u=\widetilde{u}$ for $t\in[a,T]$ from the uniqueness of solution and thus $u\in C([a,T],C_0(\mathbb{R}^d))\cap C([a,T],L^q(\mathbb{R}^d))$.

Our purpose is to prove $u\in C([a,\infty),C_0(\mathbb{R}^d))$. In fact, for $t>T$, it holds that

$$\begin{aligned}u-G_a(t)*u_a&=\int_a^t G_f\left(a\frac{t}{\tau}\right)*[\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}u^p(\tau)]\frac{\mathrm{d}\tau}{\tau}\\&=\int_a^T G_f\left(a\frac{t}{\tau}\right)*[\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}u^p(\tau)]\frac{\mathrm{d}\tau}{\tau}+\int_T^t G_f\left(a\frac{t}{\tau}\right)*[\,{}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}u^p(\tau)]\frac{\mathrm{d}\tau}{\tau}\\&=I_1+I_2.\end{aligned}$$

Using the fact $u\in C([a,T],C_0(\mathbb{R}^d))$, one obtains

$$I_1\in C([T,\infty),C_0(\mathbb{R}^d))\cap C([T,\infty),L^q(\mathbb{R}^d)).$$

For any $\widetilde{T}>T$, it can be easily find that $u^p\in L^\infty((T,\widetilde{T}),L^{q/p}(\mathbb{R}^d))$ and ${}_H\mathrm{D}_{a,\tau}^{-(1-\gamma)}u^p\in L^\infty((T,\widetilde{T}),L^{q/p}(\mathbb{R}^d))$. On the other hand, the condition $q>\frac{d(p-1)}{2s}$ indicates that we may choose $r>q$ such that $\frac{d}{2s}(\frac{p}{q}-\frac{1}{r})<1$. As what we have proved in Lemma 2.5, it is obvious that $I_2\in C([T,\widetilde{T}],L^r(\mathbb{R}^d))$. By the arbitrariness of $\widetilde{T}$, we see that $I_2\in C([T,\infty),L^r(\mathbb{R}^d))$ and thus $u\in C([T,\infty),L^r(\mathbb{R}^d))$.

Let $r = q\lambda^n$ and $\lambda > 1$ satisfy

$$\frac{d}{2s}\left(\frac{p}{q\lambda^{n-1}} - \frac{1}{q\lambda^n}\right) < 1,\ n = 1, 2, \ldots,$$

then $u \in C([T,\infty), L^{q\lambda^n}(\mathbb{R}^d))$. After finite steps, one has $\frac{p}{q\lambda^n} < \frac{2s}{d}$. In other words, we show $u \in C([a,\infty), C_0(\mathbb{R}^d))$. This concludes the proof of the theorem.

Remark 4.1. *It is worth noticing that, according to Theorem 1.1, the Fujita critical exponent to Eq* (1.1) *is the number* $\widetilde{p} = \max\{1 + \frac{1-\gamma}{\alpha}, 1 + \frac{2s(1+\alpha-\gamma)}{\alpha d}\}$.

Remark 4.2. *In the Eq* (1.1)*, we consider the case* $0 < \alpha < \gamma < 1$ *and prove the main result, i.e., Theorem 1.1. If* $\gamma \geq \alpha$ *with* $0 < \alpha < 1$ *and* $0 \leq \gamma < 1$*, then it is easy to verify that Theorems 3.1 and 3.2 are still valid provided that a mild solution and a weak solution are defined as Definitions 3.1 and 3.2. However, compared with Theorem 1.1, we see that the main conclusions are very different. In fact, we can derive the following result whose proof is similar to that of Theorem 1.1 or can also refer to the proof of Theorem 1 in [34].*

Theorem 4.1. *Let* $d \in \mathbb{N}$*,* $0 < \alpha < 1$*,* $0 \leq \gamma < 1$*,* $\gamma \leq \alpha$*,* $0 < s < 1$*, and* $p > 1$*. Assume that* $u_a \in C_0(\mathbb{R}^d)$ *and* $u_a \geq 0$ *with* $u_a \not\equiv 0$.
(1) *If* $1 < p \leq \overline{p} = \max\{\frac{1}{\gamma}, 1 + \frac{2s(1+\alpha-\gamma)}{(2+\alpha d-2s(1+\alpha-\gamma))_+}\}$*, then the mild solution of Eq* (1.1) *will blow up in finite time.*
(2) *If* $p > \overline{p}$ *and* $\|u_a\|_{L^{p^*}(\mathbb{R}^d)}$ *is small enough with* $p^* = \frac{\alpha d(p-1)}{2s(1+\alpha-\gamma)}$*, then Eq* (1.1) *exists global solution.*

Remark 4.3. *From Theorem 4.1, we remark that the Fujita critical exponent is* $\overline{p} = \max\{\frac{1}{\gamma}, 1 + \frac{2s(1+\alpha-\gamma)}{(2+\alpha d-2s(1+\alpha-\gamma))_+}\}$ *when* $\gamma \leq \alpha$ *for* $0 < \alpha < 1$ *and* $0 \leq \gamma < 1$.

5. Numerical simulations

In this section, we show the finite time blow-up of the solution to Eq (1.1) by numerical simulation. For this purpose, we have to approximate the Caputo-Hadamard derivative, fractional Laplacian and Hadamard fractional integral in Eq (1.1), respectively. We shall use formulaes (3.2) and (3.3) in [35] to discretize the Caputo-Hadamard derivative of order $\alpha \in (0,1)$ and apply formula (2.9) in [36] to approximate the fractional Laplacian of order $s \in (0,1)$. For the right sided Hadamard fractional integral of order $1-\gamma\,(\gamma \in (0,1))$ in Eq (1.1), we present the following discrete scheme.

Let $a = t_0 < t_1 < \ldots < t_k < \ldots < t_N = T$ be a partition of the interval $[a,T]$ with $N \in \mathbb{N}$ and some positive number $T > a$. Then the Hadamard fractional integral with order $1-\gamma\,(\gamma \in (0,1))$ can be approximated by, for $t = t_k, 1 \leq k \leq N$,

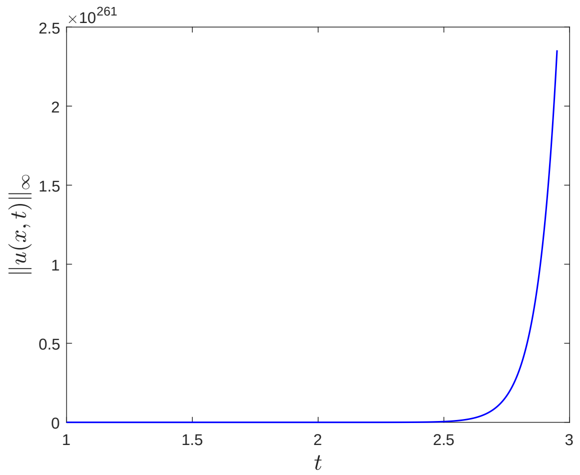
$$\begin{aligned}
{}_H\mathrm{D}_{a,t}^{-(1-\gamma)} g(t)|_{t=t_k} =& \frac{1}{\Gamma(1-\gamma)} \int_a^{t_k} \left(\log\frac{t_k}{\tau}\right)^{-\gamma} g(\tau)\frac{\mathrm{d}\tau}{\tau} \\
=& \frac{1}{\Gamma(1-\gamma)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left(\log\frac{t_k}{\tau}\right)^{-\gamma} g(\tau)\frac{\mathrm{d}\tau}{\tau} \\
\approx& \frac{1}{\Gamma(1-\gamma)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \left(\log\frac{t_k}{\tau}\right)^{-\gamma} g(t_{j-1})\frac{\mathrm{d}\tau}{\tau}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(2-\gamma)} \sum_{j=1}^{k} \left[\left(\log \frac{t_k}{t_{j-1}} \right)^{1-\gamma} - \left(\log \frac{t_k}{t_j} \right)^{1-\gamma} \right] g(t_{j-1}) \\
&= \sum_{j=1}^{k} b_{j,k} g(t_{j-1}),
\end{aligned}
$$

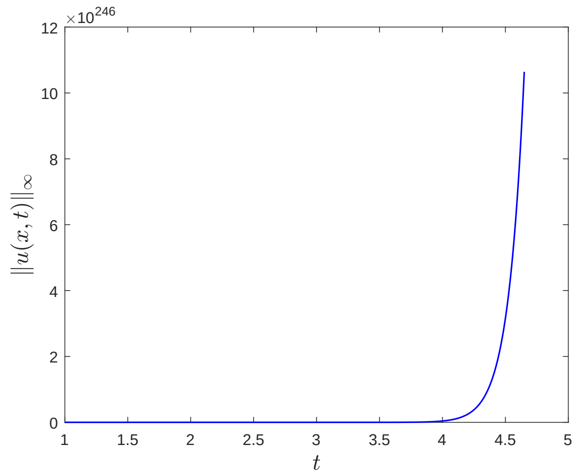
where

$$
b_{j,k} = \frac{1}{\Gamma(2-\gamma)} \left[\left(\log \frac{t_k}{t_{j-1}} \right)^{1-\gamma} - \left(\log \frac{t_k}{t_j} \right)^{1-\gamma} \right].
$$

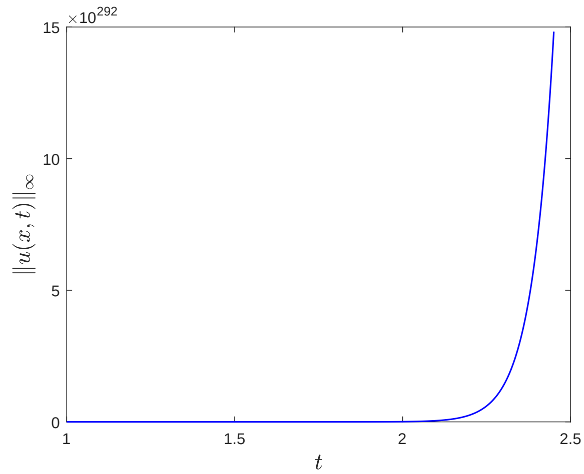
Based on these results, we obtain a numerical scheme to Eq (1.1). For simplicity, we now take $d = 1, a = 1, p = 2$ and $u_a = 10$ in Eq (1.1). Figure 1 depicts the curves of the solution to Eq (1.1) when the parameters α and s choose different values and $\gamma = 0.8$, which displays the finite time blow-up of solution of Eq (1.1) and thus shows the effectiveness of the results in Theorem 1.1. Similarly, Figure 2 presents the curves of the solution to Eq (1.1) in the case $\gamma \leq \alpha$ and illustrates the validity of the results given by Theorem 4.1.



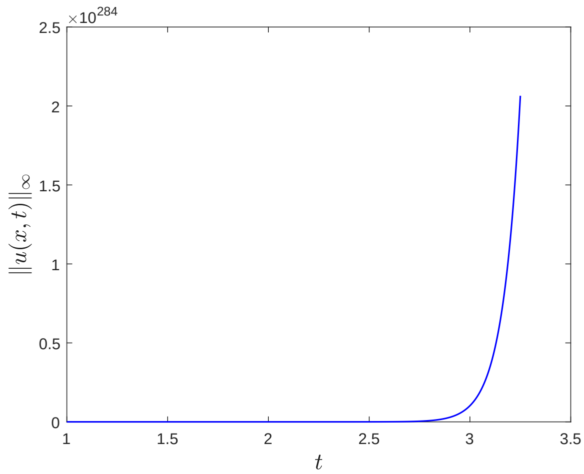
(a) $\alpha = 0.3$, $s = 0.4$



(b) $\alpha = 0.3$, $s = 0.8$

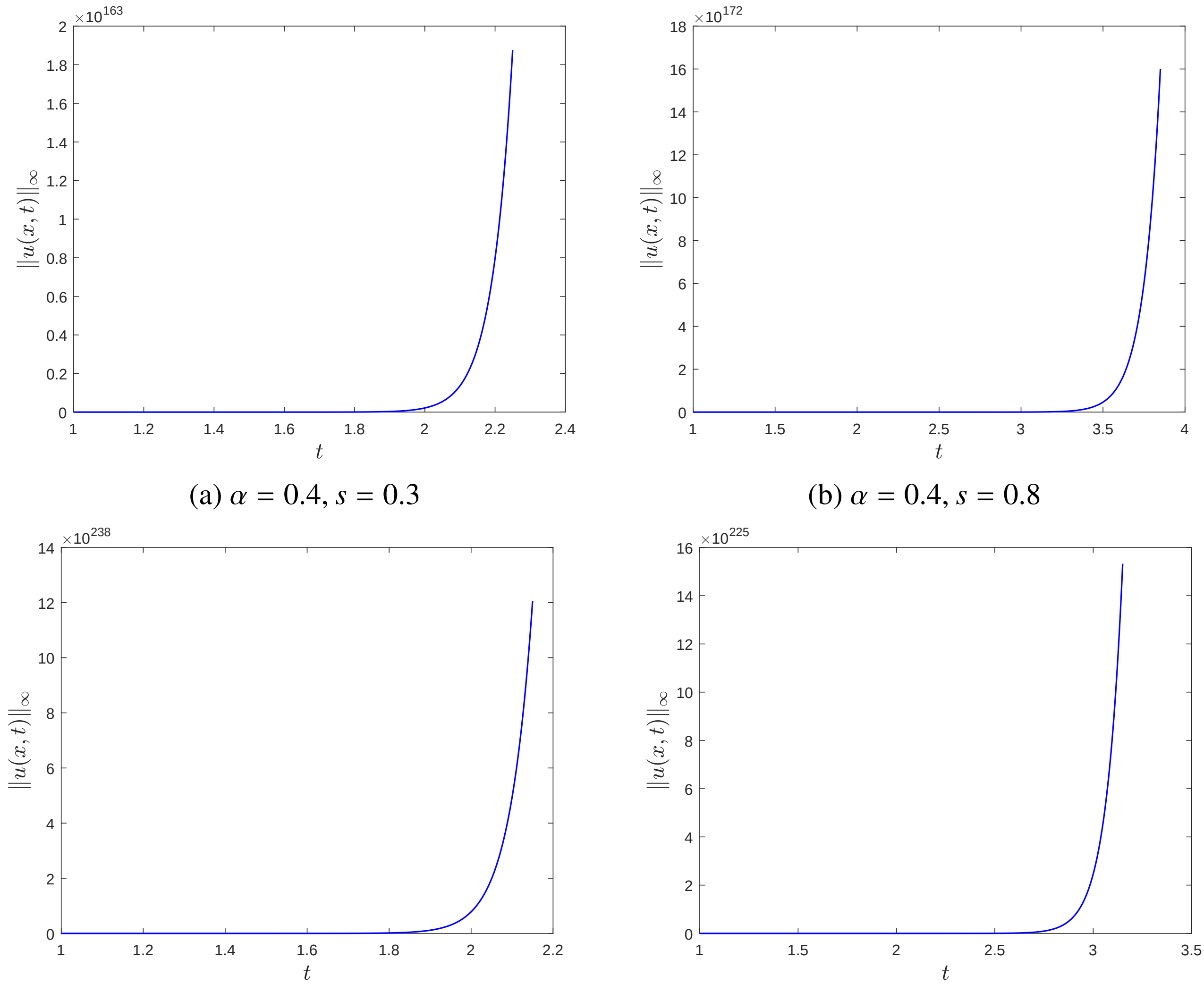


(c) $\alpha = 0.7$, $s = 0.4$



(d) $\alpha = 0.7$, $s = 0.8$

Figure 1. Solution curves for Eq (1.1) with $\gamma = 0.8$.



(a) $\alpha = 0.4, s = 0.3$ (b) $\alpha = 0.4, s = 0.8$

(c) $\alpha = 0.7, s = 0.3$ (d) $\alpha = 0.7, s = 0.8$

Figure 2. Solution curves for Eq (1.1) with $\gamma = 0.3$.

6. Conclusions

In this paper, we study the blow-up and global existence of solution of the Cauchy problem to time-space fractional partial differential Eq (1.1) with nonlinear memory. A mild solution and a weak solution are introduced to Eq (1.1) and the mild solution is actually shown to be the weak solution. We next prove the local existence and uniqueness of the mild solution of Eq (1.1) by using the fixed point argument. Finally, the finite time blow-up and global solution of Eq (1.1) are established and the Fujita critical exponent is also determined, where the blowing-up character of the solution in a finite time is verified by numerical simulations.

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Conflicts of interest

The author declare no conflict of interest.

References

1. K. B. Oldham, J. Spanier, *The fractional calculus*, New York: Academic Press, 1974.
2. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Amsterdam: Gordon and Breach Science, 1993.
3. I. Podlubny, *Fractional differential equations*, New York: Academic Press, 1999.
4. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.
5. Y. G. Sinai, The limiting behavior of a one-dimensional random walk in a random medium, *Theor. Probab. Appl.,* **27** (1983), 256–268. https://doi.org/10.1007/978-1-4419-6205-8
6. H. Schiessel, I. M. Sokolov, A. Blumen, Dynamics of a polyampholyte hooked around an obstacle, *Phys. Rev. E,* **56** (1997), R2390–R2393. https://doi.org/10.1103/PhysRevE.56.R2390
7. S. I. Denisov, H. Kantz, Continuous-time random walk theory of superslow diffusion, *Europhys Lett.,* **92** (2010), 30001. https://doi.org/10.1209/0295-5075/92/30001
8. W. T. Ang, *Hypersingular integral equations in fracture analysis*, Amsterdam: Elsevier, 2014.
9. R. Garra, F. Mainardi, G. Spada, A generalization of the Lomnitz logarithmic creep law via Hadamard fractional calculus, *Chaos Soliton. Fract.,* **102** (2017), 333–338. https://doi.org/10.1016/j.chaos.2017.03.032
10. Y. Liang, S. Wang, W. Chen, Z. Zhou, R. L. Magin, A survey of models of ultraslow diffusion in heterogeneous materials, *Appl. Mech. Rev.,* **71** (2019), 040802. https://doi.org/10.1115/1.4044055
11. A. De Gregorio, R. Garra, Hadamard-type fractional heat equations and ultra-slow diffusions, *Fractal Fract.,* **5** (2021), 48. https://doi.org/10.3390/fractalfract5020048
12. J. Hadamard, Essai sur létude des fonctions données par leur développement de Taylor, *J. Math. Pures Appl.,* **8** (1892), 101–186.
13. A. A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.,* **38** (2001), 1191–1204.
14. C. P. Li, Z. Q. Li, Asymptotic behaviors of solution to Caputo-Hadamard fractional partial differential equation with fractional Laplacian, *Int. J. Comput. Math.,* **98** (2021), 305–339.
15. C. P. Li, Z. Q. Li, Asymptotic behaviors of solution to partial differential equation with Caputo-Hadamard derivative and fractional Laplacian: Hyperbolic case, *Discrete Cont. Dyn.-S.,* **14** (2021), 3659–3683. https://doi.org/10.3934/dcdss.2021023

16. C. P. Li, Z. Q. Li, The blow-up and global existence of solution to Caputo-Hadamard fractional partial differential equation with fractional Laplacian, *J. Nonlinear Sci.,* **31** (2021), 80. https://doi.org/10.1007/s00332-021-09736-y

17. C. P. Li, Z. Q. Li, Stability and logarithmic decay of the solution to Hadamard-type fractional differential equation, *J. Nonlinear Sci.,* **31** (2021), 31. https://doi.org/10.1007/s00332-021-096918

18. L. Ma, Blow-up phenomena profile for Hadamard fractional differential systems in finite time, *Fractals,* **27** (2019), 1950093. https://doi.org/10.1142/S0218348X19500932

19. L. Ma, Comparative analysis on the blow-up occurrence of solutions to Hadamard type fractional differential systems, *Int. J. Comput. Math.,* **99** (2022), 895–908. https://doi.org/10.1080/00207160.2021.1939020

20. E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.,* **136** (2011), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004

21. C. Bucur, E. Valdinoci, *Non-local diffusion and applications*, Cham: Springer, 2016.

22. X. Ros-Oton, Nonlocal elliptic equations in bounded domains: A survey, *Publ. Mat.,* **60** (2016), 3–26. https://doi.org/10.5565/PUBLMAT_60116_01

23. M. Kwaśnicki, Ten equivalent definitions of the fractional Laplace operator, *Frac. Calc. Appl. Anal.,* **20** (2017), 7–51. https://doi.org/10.1515/fca-2017-0002

24. H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. I,* **13** (1966), 109–124.

25. F. B. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, *Israel J. Math.,* **38** (1981), 29–40.

26. T. Cazenave, F. Dickstein, F. B. Weissler, An equation whose Fujita critical exponent is not given by scaling, *Nonlinear Anal.,* **68** (2008), 862–874. https://doi.org/10.1016/j.na.2006.11.042

27. P. Souplet, Blow-up in nonlocal reaction-diffusion equations, *SIAM J. Math. Anal.,* **29** (1998), 1301–1334. https://doi.org/10.1137/S0036141097318900

28. A. Z. Fino, M. Kirane, Qualitative properties of solutions to a time-space fractional evolution equation, *Quart. Appl. Math.,* **70** (2012), 133–157. https://doi.org/10.1090/S0033-569X-2011-01246-9

29. Y. N. Li, Q. G. Zhang, Blow-up and global existence of solutions for a time fractional diffusion equation, *Frac. Calc. Appl. Anal.,* **21** (2018), 1619–1640. https://doi.org/10.1515/fca-2018-0085

30. C. P. Li, M. Cai, *Theory and numerical approximations of fractional integrals and derivatives*, Philadelphia: SIAM, 2019.

31. F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.,* **2012** (2012), 142. https://doi.org/10.1186/1687-1847-2012-142

32. A. A. Kilbas, M. Saigo, *H-transforms: Theory and applications*, Boca Raton: CRC Press, 2004.

33. H. M. Srivastava, K. C. Gupta, S. P. Goyal, *The H-functions of one and two variables with applications*, New Delhi: South Asian, 1982.

34. Q. G. Zhang, Y. N. Li, The critical exponent for a time fractional diffusion equation with nonlinear memory, *Math. Meth. Appl. Sci.,* **41** (2018), 6443–6456. https://doi.org/10.1002/mma.5169

35. M. Gohar, C. P. Li, Z. Q. Li, Finite difference methods for Caputo-Hadamard fractional differential equations, *Mediterr. J. Math.,* **17** (2020), 194. https://doi.org/10.1007/s00009-020-01605-4

36. S. W. Duo, H. W. V. Wyk, Y. Z. Zhang, A novel and accurate finite difference method for the fractional Laplacian and the fractional Poisson problem, *J. Comput. Phys.,* **355** (2018), 233–252. https://doi.org/10.1016/j.jcp.2017.11.011