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The Cauchy problem to a gkCH equation with peakon solutions

Yunxi Guo* **and Ying Wang**

Department of Mathematics, Zunyi Normal University, Zunyi 563006, China

* **Correspondence:** Email: yunxigmaths@163.com.

Abstract: Considered in this paper is a generalized Camassa-Holm equation, which includes both the Camassa-Holm equation and the Novikov equation as two special cases. Firstly, two blow-up criteria are established for the generalized Camassa-Holm equation. Then we derive two blow-up phenomena, where a new L^{2k} estimate plays a crucial role. In addition, we also show that peakon solutions are global weak solutions.

Keywords: L^{2k} estimate; blow-up solutions; blow-up phenomena; peakon solutions; generalized Camassa-Holm equation
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1. Introduction

In this paper, we consider the initial value problem for a high-order nonlinear dispersive wave equation

$$u_t - u_{txx} + (k+2)u^k u_x - (k+1)u^{k-1}u_x u_{xx} - u^k u_{xxx} = 0, \tag{1.1}$$

$$u(0,x) = u_0(x), \tag{1.2}$$

where $k \in N_0$ (N_0 denotes the set of nonnegative integers) and u stands for the unknown function on the line $\mathbb{R}$. Equation (1.1) was known as gkCH equation [18], which admits single peakon and multi-peakon traveling wave solutions, and possesses conserved laws

$$\int_{\mathbb{R}} (u^2 + u_x^2)dx = \int_{\mathbb{R}} (u_0^2 + u_{0x}^2)dx. \tag{1.3}$$

It is shown in [18] that this equation is well-posedness in Sobolev spaces H^s with $s > 3/2$ on both the circle and the line in the sense of Hadamard by using a Galerkin-type approximation scheme. That is, the data-to-solution map is continuous. Furthermore, it is proved in [18] that this dependence is sharp by showing that the solution map is not uniformly continuous. The nonuniform dependence is proved using the method of approximate solutions and well-posedness estimates.

For $k = 1$, we obtain the integrable equation with quadratic nonlinearities

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.4}$$

which was derived by Camassa and Holm [1] and by Fokas and Fuchssteiner [12]. It was called as Camassa-Holm equation. It describes the motion of shallow water waves and possesses a Lax pair, a bi-Hamiltonian structure and infinitely many conserved integrals [1], and it can be solved by the inverse scattering method. One of the remarkable features of the CH equation is that it has the single peakon solutions

$$u(t,x) = ce^{-|x-ct|}, c \in \mathbb{R}$$

and the multi-peakon solutions

$$u(t,x) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|},$$

where $p_i(t)$, $q_i(t)$ satisfy the Hamilton system [1]

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} = \sum_{i\neq j} p_i p_j sign(q_i - q_j) e^{|q_i - q_j|},$$

$$\frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i} = \sum_{j} p_j e^{|q_i - q_j|}$$

with the Hamiltonian $H = \frac{1}{2}\sum_{i,j=1}^{N} p_i p_j e^{|q_i|}$. It is shown that those peaked solitons were orbitally stable in the energy space [6]. Another remarkable feature of the CH equation is the so-called wave breaking phenomena, that is, the wave profile remains bounded while its slope becomes unbounded in finite time [7–9]. Hence, Eq (1.4) has attracted the attention of lots of mathematicians. The dynamic properties related to the equation can be found in [3–5, 10, 11, 13–17, 22–26, 34–36] and the references therein.

For $k = 2$, we obtain the integrable equation with cubic nonlinearities

$$u_t - u_{txx} + 4u^2 u_x - 3uu_x u_{xx} - u^2 u_{xxx} = 0, \tag{1.5}$$

which was derived by Vladimir Novikov in a symmetry classification of nonlocal PDEs [29] and was known as the Novikov equation. It is shown in [29] that Eq (1.5) possesses soliton solutions, infinitely many conserved quantities, a Lax pair in matrix form and a bi-Hamiltonian structure. Equation (1.5) can be thought as a generalization of the Camassa-Holm equation. The conserved quantities

$$H_1[u(t)] = \int_{\mathbb{R}} (u^2 + u_x^2) dx$$

and

$$H_2(t) = \int_{\mathbb{R}} (u^4 + 2u^2 u_x^2 - \frac{1}{3} u_x^4) dx$$

play an important role in the study of the dynamic properties related to the Eq (1.5). More information about the Novikov equation can be found in Himonas and Holliman [19], F. Tiglay [30], Ni and Zhou [28], Wu and Yin [31, 32], Yan, Li and Zhang [33], Mi and Mu [27] and the references therein.

Inspired by the reference cited above, the objective of this paper is to investigate the dynamic properties for the Problems (1.1) and (1.2). More precisely, we firstly establish two blow up criteria and derive a lower bound of the maximal existence time. Then for $k = 2^p + 1$, p is nonnegative integer, we derive two blow-up phenomena under different initial data. Motivated by the idea from Chen etal's work [2], we apply the characteristic dynamics of $P = \sqrt{2}u - u_x$ and $Q = \sqrt{2}u + u_x$ to deduce the first blow-up phenomenon. For the Problems (1.1) and (1.2), in fact, the estimates of P and Q can be closed in the form of

$$P'(t) \leq \alpha(u)PQ + \Theta_1, \quad Q'(t) \geq -\alpha(u)PQ + \Theta_2, \tag{1.6}$$

where $\alpha(u) \geq 0$ and the nonlocal term Θ_i $(i = 1, 2)$ can be bounded by the estimates from Problems (1.1) and (1.2). From (1.6) the monotonicity of P and Q can be established, and hence the finite-time blow-up follows. In blow-up analysis, one problematic issue is that we have to deal with high order nonlinear term $u^{k-2}u_x^3$ to obtain accurate estimate. Luckily, we overcome the problem by finding a new L^{2k} estimate $\| u_x \|_{L^{2k}} \leq e^{c_0 t} \| u_{0x} \|_{L^{2k}} + c_0 t$ (see Lemma 4.1). It is shown in [18] that Eq (1.1) has peakon travelling solution $u(t, x) = c^{\frac{1}{k}} e^{-|x-ct|}$. Follow the Definition 2.1, we show that the peakon solutions are global weak solutions.

The rest of this paper is organized as follows. For the convenience, Section 2 give some preliminaries. Two blow-up criteria are established in Section 3. Section 4 give two blow-up phenomena. In Section 5, we prove that the peakon solutions are global weak solutions.

2. Preliminaries

We rewrite Problems (1.1) and (1.2) as follows

$$u_t + u^k u_x = -\partial_x (1 - \partial_x^2)^{-1} \Big[u^{k+1} + \frac{2k-1}{2} u^{k-1} u_x^2 \Big] - (1 - \partial_x^2)^{-1} \Big[\frac{k-1}{2} u^{k-2} u_x^3 \Big], \tag{2.1}$$

$$u(0, x) = u_0(x), \tag{2.2}$$

which is also equivalent to

$$y_t + (k+1)u^{k-1} u_x y + u^k y_x = 0, \tag{2.3}$$

$$y = u - u_{xx}, \tag{2.4}$$

$$u(0, x) = u_0(x), y_0 = u_0 - u_{0xx}. \tag{2.5}$$

Recall that

$$(1 - \partial_x^2)^{-1} f = G * f, \quad \text{where} \quad G(x) = \frac{1}{2} e^{-|x|}$$

and $*$ denotes the convolution product on $\mathbb{R}$, defined by

$$(f * G)(x) = \int_{\mathbb{R}} f(y) G(x - y) dy. \tag{2.6}$$

Lemma 2.1. *Given initial data $u_0 \in H^s$, $s > \frac{3}{2}$, the function u is said to be a weak solution to the initial-value Problem (1.1) and (1.2) if it satisfies the following identity*

$$\int_0^T \int_{\mathbb{R}} u\varphi_t - \frac{1}{k+1}u^{k+1}\varphi_x - G*(u^{k+1} + \frac{2k-1}{2}u^{k-1}u_x^2)\varphi_x$$
$$-G*(\frac{k-1}{2}u^{k-2}u_x^3)\varphi dxdt - \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0 \tag{2.7}$$

for any smooth test function $\varphi(t,x) \in C_c^\infty([0,T)\times\mathbb{R})$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

The characteristics $q(t,x)$ relating to (2.3) is governed by

$$q_t(t,x) = u^k(t,q(t,x)), \quad t \in [0,T)$$

$$q(0,x) = x, \quad x \in \mathbb{R}.$$

Applying the classical results in the theory of ordinary differential equations, one can obtain that the characteristics $q(t,x) \in C^1([0,T)\times\mathbb{R})$ with $q_x(t,x) > 0$ for all $(t,x) \in [0,T)\times\mathbb{R}$. Furthermore, it is shown in [21] that the potential $y = u - u_{xx}$ satisfies

$$y(t,q(t,x))q_x^2(t,x) = y_0(x)e^{\int_0^t (k-1)u^{k-1}u_x(\tau,q(\tau,x))d\tau}. \tag{2.8}$$

3. Blow-up criteria and a lower bound of the maximal existence time

In this section, we investigate blow-up criteria and a lower bound of the maximal existence time. Now, we firstly give the first blow-up criterion.

Theorem 3.1. *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Let $T > 0$ be the maximum existence time of the solution u to the Problem (1.1) and (1.2) with the initial data u_0. Then the corresponding solution u blows up in finite time if and only if*

$$\lim_{t\to T^-}\inf_{x\in\mathbb{R}} u^{k-1}u_x = -\infty.$$

Proof. Applying a simple density argument, it suffices to consider the case $s = 3$. Let $T > 0$ be the maximal time of existence of solution u to the Problem (1.1) and (1.2) with initial data $u_0 \in H^3(\mathbb{R})$. Due to $y = u - u_{xx}$, by direct computation, one has

$$\| y \|_{L^2}^2 = \int_{\mathbb{R}} (u - u_{xx})^2 dx = \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2)dx. \tag{3.1}$$

So,

$$\| u \|_{H^2}^2 \leq \| y \|_{L^2}^2 \leq 2 \| u \|_{H^2}^2 . \tag{3.2}$$

Multiplying Eq (2.3) by $2y$ and integrating by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} y^2 dx = 2\int_{\mathbb{R}} yy_t dx = -2(k+1)\int_{\mathbb{R}} u^{k-1}u_x y^2 dx - \int_{\mathbb{R}} 2u^k yy_x dx$$

$$= -(k+2)\int_{\mathbb{R}} u^{k-1}u_x y^2 dx. \tag{3.3}$$

If there is a $M > 0$ such that $u^{k-1}u_x > -M$, from (3.3) we deduce

$$\frac{d}{dt}\int_{\mathbb{R}} y^2 dx \leq (k+2)M\int_{\mathbb{R}} y^2 dx. \tag{3.4}$$

By virtue of Gronwall's inequality, one has

$$\| y \|_{L^2}^2 = \int_{\mathbb{R}} y^2 dx \leq e^{(k+2)Mt} \| y_0 \|_{L^2}^2 . \tag{3.5}$$

This completes the proof of Theorem 3.1. □

Now we give the second blow-up criterion.

Theorem 3.2. *Let $u_0 \in H^s(R)$ with $s > \frac{3}{2}$. Let $T > 0$ be the maximum existence time of the solution u to the Problem (1.1) with the initial data u_0. Then the corresponding solution u blows up in finite time if and only if*

$$\lim_{t\to T^-}\inf_{x\in R} | u_x | = \infty.$$

To complete the proof of Theorem 3.2, the following two lemmas is essential.

Lemma 3.1. *([11]) The following estimates hold*
(i) *For $s \geq 0$,*

$$\| fg \|_{H^s} \leq C(\| f \|_{H^s}\| g \|_{L^\infty} + \| f \|_{L^\infty}\| g \|_{H^s}). \tag{3.6}$$

(ii) *For $s > 0$,*

$$\| f\partial_x g \|_{H^s} \leq C(\| f \|_{H^{s+1}}\| g \|_{L^\infty} + \| f \|_{L^\infty}\| \partial_x g \|_{H^s}). \tag{3.7}$$

Lemma 3.2. *([20]) Let $r > 0$. If $u \in H^r \cap W^{1,\infty}$ and $v \in H^{r-1} \cup L^\infty$, then*

$$\| [\Lambda^r, u]v \|_{L^2} \leq C(\| u_x \|_{L^\infty}\| \Lambda^{r-1}v \|_{L^2} + \| \Lambda^r u \|_{L^2}\| v \|_{L^\infty}), \tag{3.8}$$

where $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$.

Proof. Applying Λ^r with $r \geq 1$ to two sides of Eq (2.1) and multiplying by $\Lambda^r u$ and integrating on $\mathbb{R}$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(\Lambda^r u)^2 = -\int_{\mathbb{R}}\Lambda^r(u^k u_x)\Lambda^r u dx - \int_{\mathbb{R}}\Lambda^r f(u)\Lambda^r u dx. \tag{3.9}$$

where $f(u) = \partial_x(1-\partial_x^2)^{-1}\Big[u^{k+1} + \frac{2k-1}{2}u^{k-1}u_x^2\Big] + (1-\partial_x^2)^{-1}\Big[\frac{k-1}{2}u^{k-2}u_x^3\Big]$.

Notice that

$$\int_{\mathbb{R}}\Lambda^r(u^k u_x)\Lambda^r u dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} [\Lambda^r, u^k] u_x \Lambda^r u dx + \int_{\mathbb{R}} u^k \Lambda^r u_x \Lambda^r u dx \\
. \quad &\leq \| [\Lambda^r, u^k] u_x \|_{L^2} \| \Lambda^r u \|_{L^2} + c \| u \|_{L^\infty}^{k-1} \| u_x \|_{L^\infty} \| u \|_{H^r}^2 \\
&\leq c \| u \|_{H^r} (\| u \|_{L^\infty}^{k-1} \| u_x \|_{L^\infty} \| u \|_{H^r} + \| u^k \|_{H^r} \| u_x \|_{L^\infty}) \\
&\quad + c \| u \|_{L^\infty}^{k-1} \| u_x \|_{L^\infty} \| u \|_{H^r}^2 \\
&\leq c \| u_x \|_{L^\infty} \| u \|_{H^r}^2,
\end{aligned} \tag{3.10}
$$

where Lemmas 3.1, 3.2 and the inequality $\| u^k \|_{H^r} \leq k \| u \|_{L^\infty}^{k-1} \| u \|_{H^r}$ were used.

In similar way, from Lemmas 3.1 and 3.2, we have

$$
\left| \int_{\mathbb{R}} \Lambda^r f(u) \Lambda^r u dx \right| \leq \| u \|_{H^r} \| \Lambda^r f(u) \|_{L^2} . \tag{3.11}
$$

$$
\begin{aligned}
\| \Lambda^r f(u) \|_{L^2} &\leq \| \Lambda^r \partial_x (1 - \partial_x^2)^{-1} \left[u^{k+1} + \frac{2k-1}{2} u^{k-1} u_x^2 \right] \|_{L^2} \\
&\quad + \| \Lambda^r (1 - \partial_x^2)^{-1} \left[\frac{k-1}{2} u^{k-2} u_x^3 \right] \|_{L^2} \\
&\leq c(\| u^{k+1} \|_{H^{r-1}} + \| u^{k-1} u_x^2 \|_{H^{r-1}} + \| u^{k-2} u_x^3 \|_{H^{r-2}}).
\end{aligned} \tag{3.12}
$$

Notice that

$$
\begin{aligned}
\| u^{k-1} u_x^2 \|_{H^{r-1}} &= \| \Lambda^{r-1} u^{k-1} u_x^2 \|_{L^2} \\
&\leq \| [\Lambda^{r-1}, u^{k-1}] u_x^2 \|_{L^2} + \| u^{k-1} \Lambda^{r-1} u_x^2 \|_{L^2} \\
&\leq c(\| u_x \|_{L^\infty}^2 \| u \|_{H^r} + \| u_x \|_{L^\infty} \| u \|_{H^r})
\end{aligned} \tag{3.13}
$$

and

$$
\begin{aligned}
\| u^{k-2} u_x^3 \|_{H^{r-2}} &= \| \Lambda^{r-2} u^{k-2} u_x^3 \|_{L^2} \\
&\leq \| [\Lambda^{r-2}, u^{k-2}] u_x^3 \|_{L^2} + \| u^{k-2} \Lambda^{r-2} u_x^3 \|_{L^2} \\
&\leq \| u \|_{L^\infty}^{k-3} \| u_x \|_{L^\infty} \| u_x^3 \|_{H^{r-1}} + \| u_x \|_{L^\infty}^3 \| u^{k-2} \|_{H^{r-1}} \\
&\quad + \| u^{k-2} \|_{L^\infty} \| u_x^3 \|_{H^{r-1}} \\
&\leq c(\| u_x \|_{L^\infty}^3 \| u \|_{H^r} + c \| u_x \|_{L^\infty}^2 \| u \|_{H^r}).
\end{aligned} \tag{3.14}
$$

Thus, we obtain

$$
\left| \int_{\mathbb{R}} \Lambda^r f(u) \Lambda^r u dx \right| \leq c \| u \|_{H^r}^2 (1 + \| u_x \|_{L^\infty} + \| u_x \|_{L^\infty}^2 + \| u_x \|_{L^\infty}^3). \tag{3.15}
$$

It follows from (3.9), (3.10) and (3.15) that

$$
\frac{d}{dt} \| u \|_{H^r}^2 \leq c \| u \|_{H^r}^2 (1 + \| u_x \|_{L^\infty} + \| u_x \|_{L^\infty}^2 + \| u_x \|_{L^\infty}^3). \tag{3.16}
$$

Therefore, if there exists a positive number M such that $\| u_x \|_{L^\infty} \leq M$. The Gronwall' inequality gives rise to

$$
\| u \|_{H^r}^2 \leq c \| u_0 \|_{H^r}^2 e^{(1+M+M^2+M^3)t}. \tag{3.17}
$$

This completes the proof of Theorem 3.2. □

Theorem 3.3. *Assume that $u_0 \in H^s$ with $s > \frac{3}{2}$, $N = max\{3, \frac{8k-3}{6}\}$. Let $\| u_x(0) \|_{L^\infty} < \infty$, $T > 0$ be the maximum existence time of the solution u to (2.1) and (2.2) with the initial data u_0. Then T satisfies*

$$T \leq \frac{1}{2N \| u_0 \|_{H^1}^{k-2} (\| u(0) \|_{L^\infty} + \| u_x(0) \|_{L^\infty})^2}.$$

Proof. Notice that the Eq (2.1) is equivalent to the following equation

$$u_t + u^k u_x + \partial_x G * \left[u^{k+1} + \frac{2k-1}{2} u^{k-1} u_x^2\right] + G * \left[\frac{k-1}{2} u^{k-2} u_x^3\right] = 0, \tag{3.18}$$

where $G(x) = \frac{1}{2} e^{-|x|}$ is the Green function of $(1 - \partial_x^2)^{-1}$. Multiplying the above equation by u^{2n-1} and integrating the resultant with respect to x, in view of Hölder's inequality, we obtain

$$\int_{\mathbb{R}} u^{2n-1} u_t dx = \frac{1}{2n} \frac{d}{dt} \| u \|_{L^{2n}}^{2n} = \| u \|_{L^{2n}}^{2n-1} \frac{d}{dt} \| u \|_{L^{2n}}, \tag{3.19}$$

$$\begin{aligned} &\int_{\mathbb{R}} u^{2n-1} G_x * [u^{k+1} + \frac{2k-1}{2} u^{k-1} u_x^2] dx \\ &\leq c \| u \|_{L^{2n}}^{2n-1} (\| u \|_{L^\infty}^k \| u \|_{L^{2n}} + \frac{2k-1}{2} \| u_x \|_{L^\infty}^2 \| u \|_{L^\infty}^{k-2} \| u \|_{L^{2n}}), \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\int_{\mathbb{R}} u^{2n-1} G * [\frac{k-1}{2} u^{k-2} u_x^3] dx \\ &\leq \frac{k-1}{2} \| u \|_{L^{2n}}^{2n-1} \| u_x \|_{L^\infty}^2 \| u \|_{L^\infty}^{k-2} \| u_x \|_{L^{2n}} . \end{aligned} \tag{3.21}$$

Combining (3.19)–(3.21), integrating over $[0, t]$, it follows that

$$\begin{aligned} &\| u \|_{L^{2n}} \\ &\leq \| u(0) \|_{L^{2n}} + \int_0^t \| u \|_{L^{2n}} (\| u \|_{L^\infty}^k + \frac{2k-1}{2} \| u \|_{L^\infty}^{k-2} \| u_x \|_{L^\infty}^2) dt \\ &+ \frac{k-1}{2} \int_0^t \| u_x \|_{L^\infty}^2 \| u \|_{L^\infty}^{k-2} \| u_x \|_{L^{2n}} dt. \end{aligned} \tag{3.22}$$

Letting n tend to infinity in the above inequality, we have

$$\begin{aligned} \| u \|_{L^\infty} \leq \| u(0) \|_{L^\infty} + \int_0^t \| u \|_{L^\infty}^{k-2} (\| u \|_{L^\infty}^3 \\ + \frac{2k-1}{2} \| u \|_{L^\infty} \| u_x \|_{L^\infty}^2 + \frac{k-1}{2} \| u_x \|_{L^\infty}^3) dx. \end{aligned} \tag{3.23}$$

Differentiating (3.18) with respect to x, we obtain

$$u_{tx} + k u^{k-1} u_x^2 + u^k u_{xx} - u^{k+1} - \frac{2k-1}{2} u^{k-1} u_x^2 + G * \Big[u^{k+1}$$

$$+\frac{2k-1}{2}u^{k-1}u_x^2\Big] + G_x * \Big[\frac{k-1}{2}u^{k-2}u_x^3\Big] = 0. \tag{3.24}$$

Multiplying the above equation by u_x^{2n-1} and integrating the resultant with respect to x over $\mathbb{R}$, still in view of Hölder's inequality, we get following estimates

$$\int_{\mathbb{R}} u_x^{2n-1}u_{tx}dx = \frac{1}{2n}\frac{d}{dt} \parallel u_x \parallel_{L^{2n}}^{2n} = \parallel u_x \parallel_{L^{2n}}^{2n-1} \frac{d}{dt} \parallel u_x \parallel_{L^{2n}}, \tag{3.25}$$

$$\Big| \int_{\mathbb{R}} ku^{k-1}u_x^{2n+1}dx \Big| \leq k \parallel u_x \parallel_{L^\infty}^2 \parallel u_x \parallel_{L^{2n}}^{2n-1} \parallel u \parallel_{L^\infty}^{k-2} \parallel u \parallel_{L^{2n}}, \tag{3.26}$$

$$\Big| \int_{\mathbb{R}} u^k u_x^{2n-1}u_{xx}dx \Big| \leq \frac{k}{2n} \parallel u_x \parallel_{L^{2n}}^{2n-1} \parallel u_x \parallel_{L^\infty}^2 \parallel u \parallel_{L^\infty}^{k-2} \parallel u \parallel_{L^{2n}}, \tag{3.27}$$

$$\Big| \int_{\mathbb{R}} u^{k+1}u_x^{2n-1}dx \Big| \leq \parallel u_x \parallel_{L^{2n}}^{2n-1} \parallel u \parallel_{L^\infty}^{k} \parallel u \parallel_{L^{2n}}, \tag{3.28}$$

$$\begin{aligned} &\Big| \int_{\mathbb{R}} \frac{2k-1}{2}u^{k-1}u_x^{2n+1}dx \Big| \\ &\leq \frac{2k-1}{2} \parallel u_x \parallel_{L^{2n}}^{2n-1} \parallel u_x \parallel_{L^\infty}^2 \parallel u \parallel_{L^\infty}^{k-2} \parallel u \parallel_{L^{2n}}, \end{aligned} \tag{3.29}$$

$$\begin{aligned} &\Big| \int_{\mathbb{R}} u_x^{2n-1} G * [u^{k+1} + \frac{2k-1}{2}u^{k-1}u_x^2]dx \Big| \\ &\leq \parallel u_x \parallel_{L^{2n}}^{2n-1} (\parallel u \parallel_{L^\infty}^{k} \parallel u \parallel_{L^{2n}} + \frac{2k-1}{2} \parallel u_x \parallel_{L^\infty}^2 \parallel u \parallel_{L^\infty}^{k-2} \parallel u \parallel_{L^{2n}}), \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} &\Big| \int_{\mathbb{R}} u_x^{2n-1} G_x * [\frac{k-1}{2}u^{k-2}u_x^3]dx \Big| \\ &\leq \frac{k-1}{2} \parallel u_x \parallel_{L^{2n}}^{2n-1} \parallel u_x \parallel_{L^\infty}^2 \parallel u \parallel_{L^\infty}^{k-2} \parallel u_x \parallel_{L^{2n}} . \end{aligned} \tag{3.31}$$

Combining (3.25)–(3.31), it gives rise to

$$\begin{aligned} \frac{d}{dt} \parallel u_x \parallel_{L^{2n}} \leq \parallel u \parallel_{L^\infty}^{k-2} ((&\frac{k}{2n} + 3k - 1) \parallel u_x \parallel_{L^\infty}^2 \parallel u \parallel_{L^{2n}} \\ &+ \parallel u \parallel_{L^\infty}^2 \parallel u \parallel_{L^{2n}} + \frac{k-1}{2} \parallel u_x \parallel_{L^\infty}^2 \parallel u_x \parallel_{L^{2n}}). \end{aligned} \tag{3.32}$$

Integrating (3.32) over $[0, t]$ and letting n tend to infinity, it follows that

$$\begin{aligned} \parallel u_x \parallel_{L^\infty} \leq \parallel u_x(0) \parallel_{L^\infty} + \int_0^t &\parallel u \parallel_{L^\infty}^{k-2} (2 \parallel u \parallel_{L^\infty}^3 \\ &+(3k-1) \parallel u \parallel_{L^\infty} \parallel u_x \parallel_{L^\infty}^2 + \frac{k-1}{2} \parallel u_x \parallel_{L^\infty}^3) d\tau. \end{aligned} \tag{3.33}$$

Combining (3.23) and (3.33), we deduce that

$$\begin{aligned}&\| u \|_{L^\infty} + \| u_x \|_{L^\infty}\\ &\leq \| u(0) \|_{L^\infty} + \| u_x(0) \|_{L^\infty} + \int_0^t \| u \|_{L^\infty}^{k-2} (3 \| u \|_{L^\infty}^3\\ &+\frac{(8k-3)}{2} \| u \|_{L^\infty} \| u_x \|_{L^\infty}^2 + (k-1) \| u_x \|_{L^\infty}^3) d\tau. \end{aligned} \tag{3.34}$$

Define $h(t) = \| u \|_{L^\infty} + \| u_x \|_{L^\infty}$ and $N = max\{3, \frac{8k-3}{6}\}$. Then $h(0) = \| u(0) \|_{L^\infty} + \| u_x(0) \|_{L^\infty}$. One can obtain

$$\| u_x \|_{L^\infty} \leq h(t) \leq h(0) + \int_0^t N \| u_0 \|_{H^1}^{k-2} h^3(t) d\tau. \tag{3.35}$$

Solving (3.35), we get

$$\| u_x \|_{L^\infty} \leq h(t) \leq \frac{h(0)}{\sqrt{1 - 2h^2(0) N \| u_0 \|_{H^1}^{k-2} t}}. \tag{3.36}$$

Therefore, let $T = \frac{1}{2N\|u_0\|_{H^1}^{k-2}(\|u(0)\|_{L^\infty} + \|u_x(0)\|_{L^\infty})^2}$, for all $t \leq T$, $\| u_x \|_{L^\infty} \leq h(t)$ holds. □

4. Blow-up phenomena

For the convenience, we firstly give several Lemmas.

Lemma 4.1. *(L^{2k} estimate)Let $u_0 \in H^s$, $s \geq 3/2$ and $\| u_{0x} \|_{L^{2k}} < \infty$, $k = 2^p + 1$, p is nonnegative integer. Let T be the lifespan of the solution to problem (1.1). The estimate*

$$\| u_x \|_{L^{2k}} \leq e^{c_0 t} (\| u_{0x} \|_{L^{2k}} + c_0 t)$$

holds for $t \in [0, T)$.

Proof. Differentiating the first equation of Problem (2.1) respect with to x, we have

$$\begin{aligned} u_{tx} + \frac{1}{2} u^{k-1} u_x^2 + u^k u_{xx} = u^{k+1} - (1 - \partial_x^2)^{-1} (u^{k+1} + \frac{2k-1}{2} u^{k-1} u_x^2)\\ -\partial_x (1 - \partial_x^2)^{-1} (\frac{k-1}{2} u^{k-2} u_x^3). \end{aligned} \tag{4.1}$$

Multiplying the above equation by $2ku_x^{2k-1}$ and integrating the resultant over $\mathbb{R}$, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} u_x^{2k} dx = \frac{d}{dt} \| u_x \|_{L^{2k}}^{2k} = 2k \| u_x \|_{L^{2k}}^{2k-1} \frac{d}{dt} \| u_x \|_{L^{2k}}\\ \leq c_0 \| u_x \|_{L^{2k}}^{2k-1} + c_0 \| u_x^{2k-1} \|_{L^1} \| u_x^3 \|_{L^1}. \end{aligned} \tag{4.2}$$

In view of Hölder's inequality, we derive the following estimates

$$\| u_x^{2k-1} \|_{L^1} \leq (\int_{\mathbb{R}} u_x^{2k} dx)^{\frac{1}{2}} (\int_{\mathbb{R}} u_x^{2k-2} dx)^{\frac{1}{2}}, \tag{4.3}$$

$$\| u_x^{2k-1} \|_{L^1} \leq \Big(\int_{\mathbb{R}} u_x^{2k} dx\Big)^{\frac{1}{2}+\frac{1}{4}} \Big(\int_{\mathbb{R}} u_x^{2k-4} dx\Big)^{\frac{1}{4}}, \tag{4.4}$$

$$\| u_x^{2k-1} \|_{L^1} \leq \Big(\int_{\mathbb{R}} u_x^{2k} dx\Big)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} \Big(\int_{\mathbb{R}} u_x^{2k-8} dx\Big)^{\frac{1}{8}}, \tag{4.5}$$

and

$$\| u_x^{2k-1} \|_{L^1} \leq \Big(\int_{\mathbb{R}} u_x^{2k} dx\Big)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}} \Big(\int_{\mathbb{R}} u_x^{2k-16} dx\Big)^{\frac{1}{16}}, \tag{4.6}$$

......

On the other hand,

$$\| u_x^{3} \|_{L^1} \leq \Big(\int_{\mathbb{R}} u_x^{2} dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} u_x^{4} dx\Big)^{\frac{1}{2}}, \tag{4.7}$$

$$\| u_x^{3} \|_{L^1} \leq \Big(\int_{\mathbb{R}} u_x^{2} dx\Big)^{\frac{1}{2}+\frac{1}{4}} \Big(\int_{\mathbb{R}} u_x^{6} dx\Big)^{\frac{1}{4}}, \tag{4.8}$$

$$\| u_x^{3} \|_{L^1} \leq \Big(\int_{\mathbb{R}} u_x^{2} dx\Big)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} \Big(\int_{\mathbb{R}} u_x^{10} dx\Big)^{\frac{1}{8}}, \tag{4.9}$$

and

$$\| u_x^{3} \|_{L^1} \leq \Big(\int_{\mathbb{R}} u_x^{2} dx\Big)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}} \Big(\int_{\mathbb{R}} u_x^{18} dx\Big)^{\frac{1}{16}}, \tag{4.10}$$

......

Therefore, if $2k-2=2$ ($k=2$), combining (4.3) and (4.7), we have

$$\| u_x^{2k-1} \|_{L^1} \| u_x^{3} \|_{L^1} \leq \| u_0 \|_{H^1(\mathbb{R})} \int_{\mathbb{R}} u_x^{2k} dx. \tag{4.11}$$

If $2k-4=2$ ($k=3$), combining (4.4) and (4.8), we have

$$\| u_x^{2k-1} \|_{L^1} \| u_x^{3} \|_{L^1} \leq \| u_0 \|_{H^1(\mathbb{R})} \int_{\mathbb{R}} u_x^{2k} dx. \tag{4.12}$$

If $2k-8=2$ ($k=5$), combining (4.5) and (4.9), we have

$$\| u_x^{2k-1} \|_{L^1} \| u_x^{3} \|_{L^1} \leq \| u_0 \|_{H^1(\mathbb{R})} \int_{\mathbb{R}} u_x^{2k} dx. \tag{4.13}$$

and if $2k-16=2$ ($k=9$), combining (4.6) and (4.10), we have

$$\| u_x^{2k-1} \|_{L^1} \| u_x^{3} \|_{L^1} \leq \| u_0 \|_{H^1(\mathbb{R})} \int_{\mathbb{R}} u_x^{2k} dx. \tag{4.14}$$

......

Therefore, if and only if $k = 2^p + 1$, p is nonnegative integer, the following inequality follows

$$\begin{aligned} 2k \parallel u_x \parallel_{L^{2k}}^{2k-1} \frac{d}{dt} \parallel u_x \parallel_{L^{2k}} &\leq c_0 \parallel u_x \parallel_{L^{2k}}^{2k-1} + c_0 \parallel u_x^{2k-1} \parallel_{L^1} \parallel u_x^3 \parallel_{L^1} \\ &\leq c_0 \parallel u_x \parallel_{L^{2k}}^{2k-1} + c_0 \parallel u_x \parallel_{L^{2k}}^{2k}, \end{aligned} \tag{4.15}$$

where $c_0 = c_0(\parallel u_0 \parallel_{H^1(\mathbb{R})})$.

$$\frac{d}{dt} \parallel u_x \parallel_{L^{2k}} \leq c_0 + c_0 \parallel u_x \parallel_{L^{2k}} . \tag{4.16}$$

In view of Gronwall's inequality, we have for $t \in [0, T)$

$$\parallel u_x \parallel_{L^{2k}} \leq e^{c_0 t}(\parallel u_{0x} \parallel_{L^{2k}} + c_0 t). \tag{4.17}$$

This completes the proof of Lemma 4.1. □

Remark. In case of $p = 0$, then $k = 2$, and we obtain

$$\parallel u_x \parallel_{L^4} \leq e^{c_0 t}(\parallel u_{0x} \parallel_{L^4} + c_0 t), \tag{4.18}$$

which is exactly the same as the Lemma 3.2 in [23].

Lemma 4.2. *Given that $u_0 \in H^s$, $s \geq 3/2$. Let $\parallel u_{0x} \parallel_{L^{2k}} < \infty$, $k = 2^p + 1$, p is nonnegative integer and T be the lifespan of the solution to Problem (1.1) and (1.2). The estimate*

$$\int_{\mathbb{R}} |u^{k-2} u_x^3| dx \leq \alpha := (e^{c_0 T}(\parallel u_{0x} \parallel_{L^{2k}} + c_0 T))^3 \parallel u_0 \parallel_{H^1}^{k-2}$$

holds.

Proof. Applying Hölder'inequality, Lemma 4.1 and (1.3), we have

$$\begin{aligned} \int_{\mathbb{R}} |u^{k-2} u_x^3| dx &\leq (\int_{\mathbb{R}} (u_x^3)^{\frac{2k}{3}} dx)^{\frac{3}{2k}} (\int_{\mathbb{R}} (u^{k-2})^{\frac{2k}{2k-3}} dx)^{\frac{2k-3}{2k}} \\ &\leq (\int_{\mathbb{R}} (u_x^{2k}) dx)^{\frac{3}{2k}} (\int_{\mathbb{R}} u^{\frac{2k(k-2)}{2k-3}} dx)^{\frac{2k-3}{2k}} \\ &\leq (e^{c_0 t} \parallel u_{0x} \parallel_{L^{2k}} + c_0 t)^3 \parallel u \parallel_{L^\infty}^{(\frac{2k(k-2)}{2k-3} - 2)\frac{2k-3}{2k}} (\int_{\mathbb{R}} u^2 dx)^{\frac{2k-3}{2k}} \\ &\leq (e^{c_0 t} \parallel u_{0x} \parallel_{L^{2k}} + c_0 t)^3 \parallel u_0 \parallel_{H^1}^{k-2} . \end{aligned}$$

□

Lemma 4.3. *Given that $u_0 \in H^s$, $s \geq 3$. For $k = 2^p + 1$ and p is nonnegative integer. Then*

$$u(t, q(t, x_1)) > 0, for \quad 0 < t < T_1 := \frac{u_0(x_1)}{c_0 + c\alpha}.$$

Proof. From the Young's inequality, Lemma 4.2 and (1.3), it follows that

$$\begin{aligned}
| u'(t) | =& | -\partial_x(1-\partial_x^2)^{-1}(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2) \\
&+(1-\partial_x^2)^{-1}(\frac{k-1}{2}u^{k-2}u_x^3) | \\
\leq& | -\partial_x(1-\partial_x^2)^{-1}(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2) | \\
&+ | (1-\partial_x^2)^{-1}(\frac{k-1}{2}u^{k-2}u_x^3) | \\
\leq& c_0 + c\alpha. \qquad (4.19)
\end{aligned}$$

Therefore,

$$-c_0 - c\alpha \leq u'(t) \leq c_0 + c\alpha.$$

Integrating over the time interval $[0, t]$ yields

$$\begin{aligned}
u_0(x_1) - [c_0 + c\alpha]t \leq& u(t, q(t, x_1)) \\
\leq& u_0(x_1) + [c_0 + c\alpha]t. \qquad (4.20)
\end{aligned}$$

So,

$$u(t, q(t, x_1)) > 0, for \quad 0 < t < T_1 := \frac{u_0(x_1)}{c_0 + c\alpha}.$$

□

Theorem 4.1. *Let* $\| u_{0x} \|_{L^{2k}} < \infty$, $k = 2^p + 1$, *p is nonnegative integer and* $u_0 \in H^s(\mathbb{R})$ *for* $s > \frac{3}{2}$. *Suppose that there exist some* $0 < \lambda < 1$ *and* $x_1 \in \mathbb{R}$ *such that* $\frac{1}{2}u_0^{k-1}(2u_0^2 - u_{0x}^2) + C^2 \leq 0$, $\sqrt{2}u_0 < -u_{0x}$, $u_0(x_1) > 0$ *and*

$$\ln(\frac{(\lambda u_0(x_1))^{\frac{k-1}{2}}(2u_0^2 - u_{0x}^2) + \sqrt{2}C}{(\lambda u_0(x_1))^{\frac{k-1}{2}}(2u_0^2 - u_{0x}^2) - \sqrt{2}C}) \leq \frac{(1-\lambda)u_0(x_1)}{\sqrt{C}}\sqrt{2}(\lambda u_0(x_1))^{\frac{k-1}{2}}.$$

Then the corresponding solution $u(t, x)$ blows up in finite time T^* with

$$T^* \leq T_0 = \frac{1}{\sqrt{2}(\lambda u_0(x_1))^{\frac{k-1}{2}}C}\ln(\frac{(\lambda u_0(x_1))^{\frac{k-1}{2}}(2u_0^2 - u_{0x}^2) + \sqrt{2}C}{(\lambda u_0(x_1))^{\frac{k-1}{2}}(2u_0^2 - u_{0x}^2) - \sqrt{2}C}),$$

where $C = \sqrt{c\alpha + c_0}$.

Proof. We track the dynamics of $P(t) = (\sqrt{2}u - u_x)(t, q(t, x_1))$ and $Q(t) = (\sqrt{2}u + u_x)(t, q(t, x_1))$ along the characteristics

$$\begin{aligned}
P'(t) =& \sqrt{2}(u_t + u_x q_t) - (u_{tx} + u_{xx}q_t) \\
=& -\frac{1}{2}u^{k-1}(2u^2 - u_x^2) - \sqrt{2}\partial_x(1-\partial_x^2)^{-1}(u^{k+1} + \frac{2k-1}{2}u^{k-1}u_x^2)
\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{2}(1-\partial_x^2)^{-1}(\frac{k-1}{2}u^{k-2}u_x^3)+(1-\partial_x^2)^{-1}(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2)\\
&-\partial_x(1-\partial_x^2)^{-1}(\frac{k-1}{2}u^{k-2}u_x^3)\\
&\geq -\frac{1}{2}u^{k-1}PQ-C^2 \qquad (4.21)
\end{aligned}
$$

and

$$
\begin{aligned}
Q'(t)=&\ \sqrt{2}(u_t+u_xq_t)+(u_{tx}+u_{xx}q_t)\\
&=\frac{1}{2}u^{k-1}(2u^2-u_x^2)-\ \sqrt{2}\partial_x(1-\partial_x^2)^{-1}(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2)\\
&+\sqrt{2}(1-\partial_x^2)^{-1}(\frac{k-1}{2}u^{k-2}u_x^3)-(1-\partial_x^2)^{-1}(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2)\\
&+\partial_x(1-\partial_x^2)^{-1}(\frac{k-1}{2}u^{k-2}u_x^3)\\
&\leq \frac{1}{2}u^{k-1}PQ+C^2. \qquad (4.22)
\end{aligned}
$$

Then we obtain

$$
P'(t)\geq -\frac{1}{2}u^{k-1}PQ-C^2,\quad Q'(t)\leq \frac{1}{2}u^{k-1}PQ+C^2. \qquad (4.23)
$$

The expected monotonicity conditions on P and Q indicate that we would like to have

$$
\frac{1}{2}u^{k-1}PQ+C^2<0. \qquad (4.24)
$$

It is shown from assumptions of Theorem 4.1 that the initial data satisfies

$$
\frac{1}{2}u_0^{k-1}(2u_0^2-u_{0x}^2)+C^2<0,\quad \sqrt{2}u_0<-u_{0x}. \qquad (4.25)
$$

Therefore, along the characteristics emanating from x_1, the following inequalities hold

$$
\frac{1}{2}u_0^{k-1}P(0)Q(0)+C^2<0,\quad P(0)>0,\quad Q(0)<0 \qquad (4.26)
$$

and

$$
P'(0)>0,\quad Q'(0)<0. \qquad (4.27)
$$

Therefore over the time of existence the following inequalities always hold

$$
P'(t)>0,\quad Q'(t)<0. \qquad (4.28)
$$

Letting $h(t)=\ \sqrt{-PQ(t)}$ and using the estimate $\frac{Q-P}{2}\geq h(t)$, we have

$$
h'(t)=-\frac{P'Q+PQ'}{2\sqrt{-PQ}}\geq \frac{(\frac{1}{2}u^{k-1}PQ+C^2)Q-P(\frac{1}{2}u^{k-1}PQ+C^2)}{2\sqrt{-PQ}}
$$

$$
\begin{aligned}
&= \frac{-(\frac{1}{2}u^{k-1}PQ + C^2)(P-Q)}{2\sqrt{-PQ}} \\
&\geq \frac{1}{2}u^{k-1}h^2 - C^2,
\end{aligned} \tag{4.29}
$$

We focus on the time interval $0 \leq t \leq T_2 := \frac{(1-\lambda)u_0(x_1)}{c_0+c\alpha}$, it implies that

$$
0 < \lambda u_0(x_1) \leq u(t, q(t, x_1)) \leq (2-\lambda)u_0(x_1). \tag{4.30}
$$

Solving for $0 \leq t \leq T_2$

$$
h'(t) \geq \frac{1}{2}\lambda^{k-1}u_0(x_1)^{k-1}h^2 - C^2, \tag{4.31}
$$

we obtain

$$
\begin{aligned}
&\ln \frac{(\lambda u_0(x_1))^{\frac{k-1}{2}}h - \sqrt{2}C}{(\lambda u_0(x_1))^{\frac{k-1}{2}}h + \sqrt{2}C} \\
&\geq \ln(\frac{(\lambda u_0(x_1))^{\frac{k-1}{2}}h_0 - \sqrt{2}C}{(\lambda u_0(x_1))^{\frac{k-1}{2}}h_0 + \sqrt{2}C}) + \sqrt{2}(\lambda u_0(x_1))^{\frac{k-1}{2}}Ct,
\end{aligned} \tag{4.32}
$$

It is observed from assumption of Theorem 4.1 that $T_0 < T_2$, (4.32) implies that $h \to +\infty$ as $t \to T^*$

$$
T^* \leq T_0 = \frac{1}{\sqrt{2}(\lambda u_0(x_1))^{\frac{k-1}{2}}C}\ln(\frac{(\lambda u_0(x_1))^{\frac{k-1}{2}}h_0 + \sqrt{2}C}{(\lambda u_0(x_1))^{\frac{k-1}{2}}h_0 - \sqrt{2}C}). \tag{4.33}
$$

□

Theorem 4.2. *Let* $\| u_{0x} \|_{L^{2k}} < \infty$, $k = 2^p + 1$, *p is nonnegative integer and* $u_0 \in H^s(\mathbb{R})$ *for* $s > \frac{3}{2}$. *Suppose that there exist some* $0 < \lambda < 1$ *and* $x_2 \in \mathbb{R}$ *such that* $u_0(x_2) > 0$, $u_{0x}(x_2) < -\sqrt{\frac{b}{a}}$ *and*

$$
\ln\left(\frac{-u_{0x}(x_2) + \sqrt{\frac{b}{a}}}{-u_{0x}(x_2) - \sqrt{\frac{b}{a}}}\right) \leq 2(1-\lambda)u_0(x_2)\sqrt{\frac{b}{a}}, \tag{4.34}
$$

where $a = \frac{k}{2}\lambda^{k-1}u_0^{k-1}(x_2)$ *and* $b = c_0 + c\alpha$. *Then the corresponding solution* $u(t,x)$ *blows up in finite time* T^{**} *with*

$$
T^{**} \leq T_4 = \frac{1}{2\sqrt{ab}}\ln\left(\frac{-u_{0x}(x_2) + \sqrt{\frac{b}{a}}}{-u_{0x}(x_2) - \sqrt{\frac{b}{a}}}\right), \tag{4.35}
$$

Proof. Now, we prove the blow-up phenomenon along the characteristics $q(t, x_2)$. From (2.1), it follows that

$$
u'(t) = -\partial_x(1-\partial_x^2)^{-1}\left(u^{k+1} + \frac{2k-1}{2}u^{k-1}u_x^2\right)
$$

$$-(1-\partial_x^2)^{-1}\Big(\frac{k-1}{2}u^{k-2}u_x^3\Big), \tag{4.36}$$

and

$$u_x'(t)=\frac{-1}{2}u^{k-1}u_x^2+u^{k+1}-(1-\partial_x^2)^{-1}\Big(u^{k+1}+\frac{2k-1}{2}u^{k-1}u_x^2\Big)$$
$$-\partial_x(1-\partial_x^2)^{-1}\Big(\frac{k-1}{2}u^{k-2}u_x^3\Big). \tag{4.37}$$

Setting $M(t)=u(t,q(t,x_1))$ and using the Young's inequality, Lemmas 4.2 and 4.3 and (2.8) again, from (4.37), we get

$$M_x'(t)\le -\frac{1}{2}M^{k-1}M_x^2+b. \tag{4.38}$$

where $b=c_0+c\alpha$. Now, we focus on the time interval $0\le t\le T_3:=\frac{(1-\lambda)u_0(x_2)}{c_0+c\alpha}$, it implying that

$$0<\lambda u_0(x_2)\le u(t,q(t,x_2))\le(2-\lambda)u_0(x_2). \tag{4.39}$$

Therefore, for $0\le t\le T_3$, we deduce from (4.38) that

$$M_x'(t)\le -aM_x^2+b, \tag{4.40}$$

where $a=\frac{1}{2}\lambda^{k-1}u_0^{k-1}(x_2)$.

It is observed from assumption of Theorem 4.2 that $u_{0x}(x_2)<-\sqrt{\frac{b}{a}}$ and $T_4<T_3$. Solving (4.40) results in

$$M_x\to-\infty \quad as \quad t\to T^{**}, \tag{4.41}$$

where $T^{**}\le T_4=\frac{1}{2\sqrt{ab}}\ln\Bigg(\frac{u_{0x}(x_2)-\sqrt{\frac{b}{a}}}{u_{0x}(x_2)+\sqrt{\frac{b}{a}}}\Bigg)$. □

5. Peakon solutions

In this section, we will turn our attention to peakon solution for the Problem (1.1).

Theorem 5.1. *The peakon function of the form*

$$u(t,x)=c^{\frac{1}{k}}e^{-|x-ct|}, \quad c\ne0 \quad is \quad arbitary \quad constant, \tag{5.1}$$

is a global weak solution to (1.1) and (1.2) in the sense of Definition 2.1.

Proof. Let $u=ae^{-|x-ct|}$ be peakon solution for the Problems (1.1) and (1.2), where $a\ne0$ is an undetermined constant. We firstly claim that

$$u_t=asign(x-ct)u, \quad u_x=-sign(x-ct)u. \tag{5.2}$$

Hence, using (2.6), (5.2) and integration by parts, we derive that

$$\int_0^T\int_{\mathbb{R}}u\varphi_t+\frac{1}{k+1}u^{k+1}\varphi_x dxdt+\int_{\mathbb{R}}u_0(x)\varphi(0,x)dx$$

$$
\begin{aligned}
&= -\int_0^T \int_{\mathbb{R}} \varphi(u_t + u^k u_x) dxdt \\
&= -\int_0^T \int_{\mathbb{R}} \varphi sign(x-ct)(cu - u^{k+1}) dxdt. \qquad (5.3)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} G * (u^{k+1} + \frac{2k-1}{2} u^{k-1} u_x^2)\varphi_x - G * [\frac{k-1}{2} u^{k-2} u_x^3]\varphi dxdt \\
&= \int_0^T \int_{\mathbb{R}} -\varphi G_x * [\frac{2k-1}{2} u^{k-1} u_x^2] \\
&\qquad -\varphi p * [\frac{k-1}{2} u^{k-2} u_x^3 - (k+1) u^k u_x] dxdt. \qquad (5.4)
\end{aligned}
$$

Directly calculate

$$
\begin{aligned}
&\frac{k-1}{2} u^{k-2} u_x^3 + (k+1) u^k u_x \\
&= (k+1) u^k (-sign(x-ct)u) + \frac{k-1}{2} u^{k-2} (-sign^3(x-ct) u^3) \\
&= -[(k+1) + \frac{k-1}{2}] sign(x-ct) u^{k+1} \\
&= [\frac{k-1}{2(k+1)} + 1] \partial_x (u^{k+1}). \qquad (5.5)
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} G * (u^{k+1} + \frac{2k-1}{2} u^{k-1} u_x^2)\varphi_x - G * [\frac{k-1}{2} u^{k-2} u_x^3]\varphi dxdt \\
&= \int_0^T \int_{\mathbb{R}} \varphi G_x * [-\frac{2k-1}{2} u^{k-1} u_x^2 - (\frac{k-1}{2(k+1)} + 1) u^{k+1}] dxdt \\
&= -\int_0^T \int_{\mathbb{R}} \varphi G_x * (\frac{k(k+2)}{k+1} u^{k+1}) dxdt. \qquad (5.6)
\end{aligned}
$$

Note that $G_x = -\frac{1}{2} sign(x) e^{-|x|}$. For $x > ct$,

$$
\begin{aligned}
&G_x * [(\frac{k(k+2)}{k+1} u^{k+1}] \\
&= -\frac{1}{2} \int_{\mathbb{R}} sign(x-y) e^{-|x-y|} (\frac{k(k+2)}{k+1} a^{k+1} e^{-(k+1)|y-ct|} dy \\
&= -\frac{1}{2} (\int_{-\infty}^{ct} + \int_{ct}^{x} + \int_x^{\infty}) sign(x-y) e^{-|x-y|} \frac{k(k+2)}{k+1} a^{k+1} e^{-(k+1)|y-ct|} dy \\
&= I_1 + I_2 + I_3. \qquad (5.7)
\end{aligned}
$$

We directly compute I_1 as follows

$$
I_1 = -\frac{1}{2} \int_{-\infty}^{ct} sign(x-y) e^{-|x-y|} \frac{k(k+2)}{k+1} a^{k+1} e^{-(k+1)|y-ct|} dy
$$

$$
\begin{aligned}
&= -\frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}\int_{-\infty}^{ct} e^{-x-(k+1)ct}e^{(k+2)y}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}e^{-x-(k+1)ct}\int_{-\infty}^{ct} e^{(k+2)y}dy \\
&= -\frac{1}{2(k+2)}\frac{k(k+2)}{k+1}a^{k+1}e^{-x+ct}. \qquad (5.8)
\end{aligned}
$$

In a similar procedure, we obtain

$$
\begin{aligned}
I_2 &= -\frac{1}{2}\int_{ct}^{x} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}a^{k+1}e^{-(k+1)|y-ct|}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}\int_{ct}^{x} e^{-x+(k+1)ct}e^{-ky}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}e^{-x+(k+1)ct}\int_{ct}^{x} e^{-ky}dy \\
&= \frac{1}{2k}\frac{k(k+2)}{k+1}a^{k+1}(e^{-(k+1)(x-ct)}-e^{-x+ct}). \qquad (5.9)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= -\frac{1}{2}\int_{x}^{\infty} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}a^{k+1}e^{-(k+1)|y-ct|}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}\int_{x}^{\infty} e^{x+(k+1)ct}e^{-(k+2)y}dy \\
&= -\frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}\int_{x}^{\infty} e^{-(k+2)y}dy \\
&= \frac{1}{2(k+2)}\frac{k(k+2)}{k+1}a^{k+1}e^{-(k+1)(x-ct)}. \qquad (5.10)
\end{aligned}
$$

Substituting (5.8)–(5.10) into (5.7), we deduce that for $x > ct$

$$
\begin{aligned}
G_x * [\frac{k(k+2)}{k+1}u^{k+1}] &= \frac{2(k+1)}{k(k+2)}\Omega e^{-x+ct} - \frac{2(k+1)}{k(k+2)}\Omega e^{-(k+1)(x-ct)} \\
&= -a^{k+1}e^{-x+ct} + a^{k+1}e^{-(k+1)(x-ct)}, \qquad (5.11)
\end{aligned}
$$

where $\Omega = -\frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}$.

For $x < ct$,

$$
\begin{aligned}
&G_x * [(\frac{k(k+2)}{k+1}u^{k+1}] \\
&= -\frac{1}{2}\int_{\mathbb{R}} sign(x-y)e^{-|x-y|}(\frac{k(k+2)}{k+1}a^{k+1}e^{-(k+1)|y-ct|}dy \\
&= -\frac{1}{2}(\int_{-\infty}^{x} + \int_{x}^{ct} + \int_{ct}^{\infty})sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}a^{k+1}e^{-(k+1)|y-ct|}dy \\
&= \Delta_1 + \Delta_2 + \Delta_3. \qquad (5.12)
\end{aligned}
$$

We directly compute Δ_1 as follows

$$\begin{aligned}\Delta_1 &= -\frac{1}{2}\int_{-\infty}^{x} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}a^{k+1}e^{-(k+1)|y-ct|}dy\\ &= -\frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}\int_{-\infty}^{x} e^{-x-(k+1)ct}e^{(k+2)y}dy\\ &= -\frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}e^{-x-(k+1)ct}\int_{-\infty}^{x} e^{(k+2)y}dy\\ &= -\frac{1}{2(k+2)}\frac{k(k+2)}{k+1}a^{k+1}e^{(k+1)(x-ct)}.\end{aligned}\tag{5.13}$$

In a similar procedure, one has

$$\begin{aligned}\Delta_2 &= -\frac{1}{2}\int_{x}^{ct} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}a^{k+1}e^{-(k+1)|y-ct|}dy\\ &= \frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}\int_{x}^{ct} e^{x-(k+1)ct}e^{ky}dy\\ &= \frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}e^{x-(k+1)ct}\int_{x}^{ct} e^{ky}dy\\ &= \frac{1}{2k}\frac{k(k+2)}{k+1}a^{k+1}(-e^{(k+1)(x-ct)}+e^{x-ct}).\end{aligned}\tag{5.14}$$

and

$$\begin{aligned}\Delta_3 &= -\frac{1}{2}\int_{ct}^{\infty} sign(x-y)e^{-|x-y|}\frac{k(k+2)}{k+1}a^{k+1}e^{-(k+1)|y-ct|}dy\\ &= \frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}\int_{ct}^{\infty} e^{x+(k+1)ct}e^{-(k+2)y}dy\\ &= \frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}e^{x+(k+1)ct}\int_{ct}^{\infty} e^{-(k+2)y}dy\\ &= \frac{1}{2(k+2)}\frac{k(k+2)}{k+1}a^{k+1}e^{x-ct}.\end{aligned}\tag{5.15}$$

Therefore, from (5.13)–(5.15), we deduce that for $x < ct$

$$\begin{aligned}&G_x * [\frac{k(k+2)}{k+1}u^{k+1}]\\ &= -\frac{2(k+1)}{k(k+2)}\Omega e^{x-ct} + \frac{2(k+1)}{k(k+2)}\Omega e^{(k+1)(x-ct)},\end{aligned}\tag{5.16}$$

where $\Omega = \frac{1}{2}\frac{k(k+2)}{k+1}a^{k+1}$.

Due to $u = ae^{-|x-ct|}$,

$$\begin{aligned}&sign(x-ct)(cu-u^{k+1})\\ &= \begin{cases} -ace^{-x+ct} + a^{k+1}e^{-(k+1)(x-ct)}, & for \quad x > ct,\\ ace^{x-ct} - a^{k+1}e^{(k+1)(x-ct)}, & for \quad x \le ct.\end{cases}\end{aligned}$$

To ensure that $u = ae^{-|x-ct|}$ is a global weak solution of (2.1) in the sense of Definition 2.1, we let

$$a^{k+1} = ac, \tag{5.17}$$

Solving (5.17), we get

$$a = c^{\frac{1}{k}}, \quad c > 0. \tag{5.18}$$

From (5.18), we derive that

$$u = c^{\frac{1}{k}} e^{-|x-ct|}, \quad c > 0, \tag{5.19}$$

which along with (5.11), (5.16) and (5.17) gives rise to

$$asign(x-ct)(cu - u^{k+1})(t,x) - G_x * (\frac{k(k+2)}{k+1} u^{k+1}(t,x) = 0. \tag{5.20}$$

Therefore, we conclude that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} u\varphi_t + \frac{1}{k+1} u^{k+1}\varphi_x + G * (u^{k+1} + \frac{2k-1}{2} u^{k-1} u_x^2)\varphi_x \\ + G * (\frac{k-1}{2} u^{k-2} u_x^3)\varphi dx dt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0, \end{aligned} \tag{5.21}$$

for every test function $\varphi(t,x) \in C_c^\infty([0,+\infty) \times \mathbb{R})$, which completes the proof of Theorem 5.1. □

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Conflict of interest

There are no conflict of interest.

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