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Research article

Existence and controllability of Hilfer fractional neutral differential equations with time delay via sequence method

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Abstract: This paper deals with the existence and approximate controllability outcomes for Hilfer fractional neutral evolution equations. To begin, we explore existence outcomes using fractional computations and Banach contraction fixed point theorem. In addition, we illustrate that a neutral system with a time delay exists. Further, we prove the considered fractional time-delay system is approximately controllable using the sequence approach. Finally, an illustration of our main findings is offered.

Keywords: hilfer fractional derivative; approximate controllability; neutral system; time delay; mild solutions; sequence approach; nonlocal conditions
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1. Introduction

Fraction systems have been demonstrated to be important tools for providing many complex miracles in numerous sectors of science and engineering, and this pairing has received a lot of traction recently. Fractional differential equations (FDEs), calculations have become increasingly important in mathematics, see [3, 8, 9, 15, 16, 23, 24, 32, 33]. Hilfer [17] launched a separate sort of derivative, alongside Riemann-Liouville and Caputo fractional derivatives, that is, Hilfer fractional derivative (HFD). For additional information, see [2, 4, 9, 17, 20, 22, 29, 30, 43].

Thermal science, chemical engineering, and mechanics all use the time-fractional advection-reaction-diffusion equation. An analytic solution to this equation is nearly impossible to find. Recently, numeral modalities are provided, including a finite differentiation optimization approach and a homotope perturbation method. The Taylor's formula, also known as the Delta function, was employed for three decades to build the replicating kernel space, which has proven to be an excellent technique for three decades, the Taylor's formula, also known as the Delta function, was used to construct the replicating kernel space and it has proven to be a useful method for resolving different forms. In [1], the authors proposed various new reproductive kernel spaces for numerical approaches to time-fractional advection-reaction-diffusion equations based on Legendre polynomials.

References [2, 9] explored the approximate controllability of semilinear inclusions with respect to HFD. Furati, et al. [7] discussed the existence and uniqueness of a problem involving HFD.

Neutral systems have gotten increasing attention in the present generation because among their widespread applicability in various domains of pragmatic mathematics. Several neutral systems, including heat flow in materials, visco-elasticity, wave propagation, and several natural developments, benefit from neutral systems with or without delay. To know more details on neutral system and its application reader can refer [4, 20, 21, 53].

The advancement of current mathematical control theory has been aided by approximate controllability. The difficulties of approximation controllability of differential systems are extensively used in theory connected to system analysis with control. The system with fractional order generated by the fractional evolution system has attracted attention in recent years, list of these distributions may be found in [21, 52]. Li et al. [26] and He et al. [12] developed a fractal differential model as well as a fractal Duffing-Van der Pol oscillator (DVdP) with two-scale fractal derivatives.

An analytic approximate solution can be obtained using two-scale transforms and the He-Laplace method. He and Ji [1] focused on two-scale mathematics and fractional calculus for thermodynamics, and found it is required to show the information lost owing to the reduced dimensional method. In general, one scale is set by utlization, in which case regular calculus is used, and the other scale is determined by the need to reveal lost information, in which case the continuity assumption is allowed and fractional or fractal calculus must be used. For numerical results of space fractional variable coefficient kdv-modified kdv equation via Fourier spectral approach, see [49, 50]. Many academics are now using the Sequence method to represent the approximate controllability outcomes using Riemann-Liouville fractional derivative, fractional evolution with damping, and an impulsive system. See articles [4–6, 18, 19, 26, 28, 29, 31, 34–42, 44–48, 54, 55, 57] for further information.

Consider

$$\begin{cases}\mathcal{D}_{0^+}^{\alpha,\beta}[x(\vartheta)-\mathrm{g}(\vartheta,x_\vartheta)]=\mathcal{A}x(\vartheta)+\mathcal{A}_1x(\vartheta-\sigma)+Bu(\vartheta)+\mathcal{G}(\vartheta,x(\vartheta-\sigma)),\ \vartheta>0,\\ I_{0^+}^{(1-\alpha)(1-\beta)}x(\vartheta)=\hbar(\vartheta),\ \vartheta\in[-\sigma,0].\end{cases}\tag{1.1}$$

The Hilfer fractional derivative is symbolized by $\mathcal{D}_{0^+}^{\alpha,\beta}$, whose order and type are $0<\beta<1$, $0\le\alpha\le1$ on Hilbert space $\mathcal{H}$, $\mathcal{A}$ refers to a C_0 semigroup $\{S(\vartheta)\}_{\vartheta\ge0}$'s infinitesimal generator.

On a Hilbert space $\mathcal{H}$, $\mathcal{A}_1$ denotes a bounded linear operator. We choose, $\mathcal{K}$ is a function space associate to W, and $\mathcal{Y}$ is the space of values $\vartheta(\cdot)$, then the control function $w(\cdot) \in W$, $B : W \to \mathcal{Y}$; Assume $L^b(\mathcal{H}, Y)$, $\|\mathcal{G}\| \in L^p(\mathcal{K}, R^+)$, for some p with $0 \le p \le \infty$, $\mathcal{G}$ mapping from $\mathcal{K}$ into $\mathcal{H}$ into $\mathcal{H}$, and $\hbar(\cdot)$ is from $\mathbb{C}([-\sigma, 0]; \mathcal{H})$.

We split this work into the sections below: The fundamentals of fractional differential systems, semigroup and control systems are addressed in Section 2. Existence outcomes for the system (1.1) is given in Section 3. The filter diagram is included in Section 4. Further we evaluated the results in Section 5 with respect to approximate controllability, 6 we establish the outcomes with time delay by utilizing the sequence method and nonlocal conditions. In 7, we provide an application to demonstrate our main arguments and some inference are established in the end.

2. Preliminary results

$C(\mathcal{K}, \mathcal{H}) : \mathcal{K} \to \mathcal{H}$ symbolizes the continuous function throughout this paper along with $\|x\|^*_{\mathbb{C}} = \sup_{\vartheta \in \mathcal{K}} e^{-\mathfrak{r}\vartheta}\|x(\vartheta)\|$, where $\mathfrak{r}$ is a fixed positive constant. Now characterize $C_{1-\mathfrak{b}}(\mathcal{K}, \mathcal{H}) = \{x : \vartheta^{1-\mathfrak{b}}x(\vartheta) \in C(\mathcal{K}, \mathcal{H})\}$, $\| \cdot \|_{\mathfrak{b}}$ represented as $\|x\|_{\mathfrak{b}} = \sup\{\vartheta^{1-\mathfrak{b}}\|x(\vartheta)\|, \vartheta \in \mathcal{K}\}$, where $(1 - \mathfrak{b}) = (1 - \alpha)(1 - \beta)$ since $\mathfrak{b} = \alpha + \beta - \alpha\beta$.

Following are the properties of $\mathcal{A}^\kappa$:

$\mathcal{A}^\kappa$ is a fractional power, $0 < \kappa \le 1$, as a closed linear operator on $D(\mathcal{A}^\kappa)$ along inverse $\mathcal{A}^{-\kappa}$.

(i) Let $D(\mathcal{A}^\kappa)$ be a Banach space along $\|u\|_\kappa = \|\mathcal{A}^\kappa u\|$ for $x \in D(\mathcal{A}^\kappa)$.
(ii) $S(\vartheta) : U \to U_\kappa$ for $\vartheta \ge 0$.
(iii) $\mathcal{A}^\kappa S(\vartheta)x = S(\vartheta)A^\kappa x$ for all $u \in D(\mathcal{A}^\kappa)$, $\vartheta \ge 0$.
(iv) For each $\kappa \in (0, 1)$, $\mathcal{A}^\kappa S(\vartheta)$ is bounded, $\mathbb{N}_\kappa > 0$, such that

$$\|\mathcal{A}^\kappa S(\vartheta)\| \le \frac{\mathbb{N}_\kappa}{\vartheta^\kappa}, \quad \vartheta \in (0, \mathfrak{b}].$$

Definition 2.1. *[33] Suppose $\vartheta : [d, +\infty) \to \mathbb{R}$, then RLI is defined as*

$$I^\beta_{d^+}\, \mathcal{G}(\vartheta) = \frac{1}{\Gamma(\beta)} \int_d^\vartheta \frac{\mathcal{G}(\mathrm{r})}{(\vartheta - \mathrm{r})^{1-\beta}} d\mathrm{r},\ \vartheta > d;\ \beta > 0.$$

Definition 2.2. *[33] Type $\beta \in [\jmath - 1, \jmath)$, $\jmath \in \mathbb{Z}$ for $\mathcal{G} : [d, +\infty) \to \mathbb{R}$, the RLD is defined as*

$${}^L D^\beta{}_{d^+}\, \mathcal{G}(\vartheta) = \frac{1}{\Gamma(\jmath - \beta)} \frac{d^\jmath}{dt^\jmath} \int_d^\vartheta \frac{\mathcal{G}(\mathrm{r})}{(\vartheta - \mathrm{r})^{\beta+1-\jmath}} d\mathrm{r}, \quad \vartheta > d,\ \jmath - 1 < \beta < \jmath.$$

Definition 2.3. *[33] Type $\beta \in [\jmath - 1, \jmath)$, $\jmath \in \mathbb{Z}$ for $\mathcal{G} : [d, +\infty) \to \mathbb{R}$, we have the RLD in the form of*

$${}^C D^\beta{}_{d^+}\, \mathcal{G}(\vartheta) = \frac{1}{\Gamma(\jmath - \beta)} \int_d^\vartheta \frac{\mathcal{G}^\jmath(\mathrm{r})}{(\vartheta - \mathrm{r})^{\beta+1-\jmath}} d\mathrm{r}, \quad \vartheta > d,\ \jmath - 1 < \beta < \jmath.$$

Definition 2.4. *[33] $0 \le \beta \le 1$, $0 < \alpha < 1$, for $\mathcal{G}(\vartheta)$, then the HFD is*

$$D^{\alpha,\beta}_{d^+}\mathcal{G}(\vartheta) = (I^{\alpha(1-\beta)}_{d^+} D(I^{(1-\alpha)(1-\beta)}_{d^+}\mathcal{G}))(\vartheta).$$

Remark 2.5. *[17] RLI and CFD's Hilfer fractional derivatives are characterized as follows:*

$$D_{0^+}^{\alpha,\beta}\mathcal{G}(\vartheta) = \begin{cases} \frac{d}{dt} I_{0^+}^{1-\beta}\mathcal{G}(\vartheta) = {}^{L}D^{\wp}{}_{0^+}\mathcal{G}(\vartheta), \beta = 0, 0 < \beta < 1, d = 0; \\ I_{0^+}^{1-\wp}\frac{d}{dt}\mathcal{G}(\vartheta) = {}^{e}D^{\wp}{}_{0^+}\ \mathcal{G}(\vartheta), \beta = 1, 0 < \beta < 1, d = 0. \end{cases}$$

Definition 2.6. *[10, 11] He's fractional derivative:*

In fractal space, fractional evolution equations are established using He's fractional derivative. The fractional evolution equation is converted into its traditional form via He's fractional real transform, and the solutions are obtained using the homotopy perturbation method.

Definition 2.7. *[14] Two-scale fractal derivative:*

The standard differential derivatives and the two-scale fractal derivative are conformable. The two-scale transform is used to convert the nonlinear Zhiber-Shabat oscillator with the fractal derivatives to the traditional model.

$$\frac{\partial\vartheta}{\partial x^{\alpha}} = \Gamma(1+\alpha)\lim_{x\to x_0}\frac{\vartheta - T_0}{(x-x_0)^{\alpha}},$$

where x_0 is the smallest scale beyond which there is no physical understanding and it is the porous size. Refer [10, 11, 14], for the variational iteration method refer [13].

Definition 2.8. *[33] $x(\cdot; w) \in C((0, d], \vartheta)$ is a mild solution of* (1.1) *only if for all $w \in L^2(\mathcal{K}, \mathcal{H})$, the integral equation*

$$x(\vartheta) = \begin{cases} \mathcal{R}_{\alpha,\beta}(\vartheta)[\hbar(0) - \mathrm{g}(0, \hbar(0))] + \mathrm{g}(\vartheta, x_{\vartheta}) + \int_0^{\vartheta} \mathcal{Q}_{\beta}(\vartheta - \mathrm{r})[\mathcal{A}\mathrm{g}(\mathrm{r}, x_{\mathrm{r}}) \\ +\mathcal{A}_1 x(\mathrm{r}-\sigma) + B(\mathrm{r})u(\mathrm{r}) + \mathcal{G}(\mathrm{r}, x(\mathrm{r}-\sigma)), \quad \text{for } \vartheta > 0, \\ \hbar(\vartheta), \quad \vartheta \in [-\sigma, 0), \end{cases} \tag{2.1}$$

where

$$\mathcal{Q}_{\beta}(\vartheta) = \vartheta^{\beta-1}\mathcal{V}_{\beta}(\vartheta); \quad \mathcal{R}_{\alpha,\beta}(\vartheta) = I_{0^+}^{\alpha(1-\beta)}\mathcal{Q}_{\beta}(\vartheta); \quad \mathcal{V}_{\beta}(\vartheta) = \int_0^{\infty} \beta\vartheta N_{\beta}(\vartheta) S(\vartheta^{\beta}\vartheta)dt.$$

(2.1) *implies*

$$\begin{aligned} x(\vartheta) =& \mathcal{R}_{\alpha,\beta}(\vartheta)[\hbar(0) - \mathrm{g}(0, \hbar(0))] + \mathrm{g}(\vartheta, x_{\vartheta}) + \int_0^{\vartheta} (\vartheta - \mathrm{r})^{\beta-1}\mathcal{V}_{\beta}(\vartheta - \mathrm{r})[\mathcal{A}\mathrm{g}(\mathrm{r}, x_{\mathrm{r}}) \\ &+ \mathcal{A}_1 x(\mathrm{r}-\sigma) + Bu(\mathrm{r}) + \mathcal{G}(\mathrm{r}, x(\mathrm{r}-\sigma))\Big]d\mathrm{r}, \quad \text{for } \vartheta \in \mathcal{K}. \end{aligned} \tag{2.2}$$

Wright function: $(N_{\beta}(\vartheta))$:

$$N_{\beta}(\vartheta) = 1 + \sum_{k=1}^{\infty}\frac{(-\vartheta)^{k-1}}{(k-1)!\Gamma(1-dk)}, \quad 0 < d < 1, \quad \vartheta \in C,$$

where $N_{\beta}(\vartheta)$ belongs to $(0, \infty)$ satisfying

$$\int_0^{\infty}\vartheta^{v}N_{\beta}(\vartheta)dt = \frac{\Gamma(1+v)}{\Gamma(1+\beta v)}; \quad \int_0^{\infty}N_{\beta}(\vartheta)dt = 1. \quad \vartheta \geq 0.$$

Lemma 2.9. *[17, 56]*

- *The* $\mathcal{V}_\beta(\vartheta)$ *is continuous.*
- *For* $\vartheta > 0$, $\{\mathcal{R}_{\alpha,\beta}(\vartheta)\}$, $\{Q_\beta(\vartheta)\}$ *are strongly continuous.*
- *For* $\vartheta > 0$ *and for all* $x \in \mathcal{H}$*, then*

$$\|\mathcal{R}_{\alpha,\beta}(\vartheta)x\| \leq \frac{\mathbb{N}\vartheta^{\mathfrak{b}-1}}{\Gamma(\alpha(1-\beta)+\beta)}\|x\|,$$
$$\|Q_\beta(\vartheta)x\| \leq \frac{\mathbb{N}\vartheta^{\beta-1}}{\Gamma(\beta)}\|x\|, \ \ (or)\ \|\mathcal{V}_\beta(\vartheta)x\| = \frac{\mathbb{N}}{\Gamma(\beta)}\|x\|.$$

Lemma 2.10. *[29] In any case* $x \in \mathcal{B}$, $\kappa \in (0,1]$*, then*

$$\mathcal{A}\mathcal{V}_\beta(\vartheta)x = \mathcal{A}^{1-\kappa}\mathcal{V}_\beta(\vartheta)\mathcal{A}^\kappa u;$$
$$\|\mathcal{A}^\kappa\mathcal{V}_\beta(\vartheta)\| \leq \frac{\beta C_\kappa \Gamma(2-\kappa)}{\vartheta^{\beta\kappa}\Gamma(1+\beta(1-\kappa))}, \ 0 < \vartheta \leq b.$$

3. Existence results

In order to obtain the existence of mild solution for the system (1.1), the following assumptions are made.

$\mathcal{F}_1$: There exists $\mathbb{N} \geq 0$, such that the semigroup $S(\vartheta)$ is uniformly bounded on $\mathcal{H}$,

$$\sup_{\vartheta\in[0,\infty)} \|S(\vartheta)\| \leq \mathbb{N}.$$

$\mathcal{F}_2$: There exist $\vartheta_1 \in (0,\beta)$ and $\eta \in L^{\frac{1}{\vartheta_1}}(\mathcal{K}, R^+)$, $x \in \mathcal{H}$, the function $\mathcal{G}(\vartheta, x)$ is continuous at ϑ then

$$\|\mathcal{G}(\vartheta, x_1) - \mathcal{G}(\vartheta, x_2)\| \leq \eta\vartheta^{1-\mathfrak{b}}\|x_1 - x_2\|_{\mathcal{H}}, \ x_1, \ x_2 \in \mathcal{H},$$

with

$$\max_{\vartheta\in(0,d]} \|\mathcal{G}(\vartheta, 0)\| = \mathbb{N}_0.$$

$\mathcal{F}_3$:

$$\max\left\{\|\hbar\|_C^*, \frac{\mathbb{N}}{\Gamma(\alpha(1-\beta)+\beta)}\|\hbar(0)\| + \frac{d^{1-\mathfrak{b}}\mathbb{N}\mathbb{N}_b}{\Gamma(\beta)}\Delta_1\|w\|_C + \frac{\mathbb{N}\mathbb{N}_0 d^{2(1-\mathfrak{b})+\beta}}{\Gamma(\beta+1)}\right\} < \mathfrak{q}.$$

$\mathcal{F}_4$: $\mathfrak{g} : (0, \mathfrak{b}] \times \mathcal{J} \to \mathcal{B}$ is continuous and there is a $\kappa \in (0,1)$ such that $\mathfrak{g} \in D(\mathcal{A}^\kappa)$, $x, \hat{x} \in \mathbb{C}$, $\vartheta \in \mathcal{J}$, $\mathcal{A}^\kappa\mathfrak{g}(\cdot, x)$ is strongly measurable, there exist $\mathcal{L}_1, \mathcal{L}_2 > 0$ and $\mathcal{A}^\kappa\mathfrak{g}(\vartheta, \cdot)$ satisfies

$$\|\mathcal{A}^\kappa\mathfrak{g}(\vartheta, x) - \mathcal{A}^\kappa\mathfrak{g}(\vartheta, \hat{x}) \leq \vartheta^{1-\mathfrak{b}}\mathcal{L}_1\|x(\vartheta) - \hat{x}(\vartheta)\|,$$
$$\|\mathcal{A}^\kappa\mathfrak{g}(\vartheta, x)\| \leq \mathcal{L}_2(1 + \vartheta^{1-\mathfrak{b}}\|x(\vartheta)\|).$$

For convenience

$$\|\mathcal{A}_1\| = \mathbb{N}_1; \ \|B\| \leq \mathbb{N}_b; \ \Delta_1 = \frac{d^{(\mathfrak{r}+1)(1-\vartheta_1)}}{(\mathfrak{r}+1)^{(1-\vartheta_1)}}; \ \Delta_1 = \mathfrak{r} = \frac{\beta-1}{1-\vartheta_1};$$
$$\|\mathcal{A}^{-\kappa}\| = \mathbb{N}_0; \ \ \mathbb{N}_3 = \frac{C_{1-\kappa}\Gamma(1+\kappa)}{\kappa\Gamma(1+\beta\kappa)}; \ \ \Delta_2 = \frac{C_{1-\kappa}\Gamma(1+\kappa)}{\kappa\Gamma(1+\beta\kappa)}.$$

Theorem 3.1. *For every control function* $w(\cdot) \in W$ *and the assumptions* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are true then* (1.1) *has a mild solution on* $C([-\sigma, d]; \mathcal{H})$.

Proof. Γ has a fixed point in $\mathcal{H}$:

Define

$$\Gamma : B_{\mathfrak{q}} = \{x \in C([-\sigma, d], \mathcal{H}) : \|x\|_C^* \leq \mathfrak{q}\},$$

$$(\Gamma x)(\vartheta) = \begin{cases} \mathcal{R}_{\alpha,\beta}(\vartheta)[\hbar(0) - \mathfrak{g}(0, \hbar(0))] + \mathfrak{g}(\vartheta, x_\vartheta) + \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} \mathcal{V}_\beta(\vartheta - \mathfrak{r}) \mathcal{A} \mathfrak{g}(\mathfrak{r}, x_\mathfrak{r}) d\mathfrak{r} \\ + \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} \mathcal{V}_\beta(\vartheta - \mathfrak{r}) \mathcal{A}_1 x(\mathfrak{r} - \sigma) d\mathfrak{r} + \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} \mathcal{V}_\beta(\vartheta - \mathfrak{r}) B u(\mathfrak{r}) d\mathfrak{r} \\ + \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} \mathcal{V}_\beta(\vartheta - \mathfrak{r}) \mathcal{G}(\mathfrak{r}, x(\mathfrak{r} - \sigma)) d\mathfrak{r}, \vartheta \geq 0; \\ \hbar(\vartheta); \quad \text{for } -\sigma \leq \vartheta < 0; \end{cases} \tag{3.1}$$

Step 1: Fix $\mathfrak{q} > 0$ and $B_{\mathfrak{q}} = \{x \in C([-\sigma, d], \mathcal{H}) : \|x\|_C^* \leq \mathfrak{q}\}$.

$$\begin{aligned} \mathfrak{q} \leq & e^{-\mathfrak{r}\vartheta} \|(\Gamma x)(\vartheta)\|_{\mathcal{H}} \\ \leq & \sup \vartheta^{1-\mathfrak{b}} e^{-\mathfrak{r}\vartheta} \Big\| \mathcal{R}_{\alpha,\beta}(\vartheta)[\hbar(0) - \mathfrak{g}(0, \hbar(0))] + \mathfrak{g}(\vartheta, x_\vartheta) + \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} \mathcal{V}_\beta(\vartheta - \mathfrak{r}) \mathcal{A} \mathfrak{g}(\mathfrak{r}, x_\mathfrak{r}) d\mathfrak{r} \\ & + \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} \mathcal{V}_\beta(\vartheta - \mathfrak{r}) \mathcal{A}_1 x(\mathfrak{r} - \sigma) d\mathfrak{r} + \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} \mathcal{V}_\beta(\vartheta - \mathfrak{r}) B u(\mathfrak{r}) d\mathfrak{r} \\ & + \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} \mathcal{V}_\beta(\vartheta - \mathfrak{r}) \mathcal{G}(\mathfrak{r}, x(\mathfrak{r} - \sigma)) d\mathfrak{r} \Big\| \leq \sum_{\jmath=1}^{7} \mathcal{J}_\jmath, \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}_1 = & \sup \vartheta^{1-\mathfrak{b}} e^{-\mathfrak{r}\vartheta} \Big\| \mathcal{R}_{\alpha,\beta}(\vartheta) \hbar(0) \Big\| \leq \frac{\mathbb{N} e^{-\mathfrak{r}\vartheta}}{\Gamma(\alpha(1-\beta)+\beta)} \|\hbar(0)\|, \\ \mathcal{J}_2 = & \sup \vartheta^{1-\mathfrak{b}} e^{-\mathfrak{r}\vartheta} \Big\| \mathcal{R}_{\alpha,\beta}(\vartheta) \mathfrak{g}(0, \hbar(0)) \Big\| \leq \frac{\mathcal{L}_2 d^{1-\mathfrak{b}} \mathbb{N}_0 \mathbb{N} e^{-\mathfrak{r}\vartheta}}{\Gamma(\alpha(1-\beta)+\beta)}, \\ \mathcal{J}_3 = & \sup \vartheta^{1-\mathfrak{b}} e^{-\mathfrak{r}\vartheta} \Big\| \mathfrak{g}(\vartheta, x_\vartheta) \Big\| \leq d^{1-\mathfrak{b}} \mathbb{N}_0 e^{-\mathfrak{r}\vartheta} \mathcal{L}_2 (1 + \vartheta^{1-\mathfrak{b}} \|x(\vartheta)\|) \leq \mathbb{N}_0 \mathcal{L}_2 (1 + \mathfrak{q}) d^{1-\mathfrak{b}} e^{-\mathfrak{r}\vartheta}, \\ \mathcal{J}_4 = & \sup \vartheta^{1-\mathfrak{b}} e^{-\mathfrak{r}\vartheta} \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} \|\mathcal{A}^{1-\kappa} \mathcal{V}_\beta(\vartheta - \mathfrak{r}) \mathcal{A}^\kappa \mathfrak{g}(\mathfrak{r}, x_\mathfrak{r})\| d\mathfrak{r} \\ \leq & d^{1-\mathfrak{b}} e^{-\mathfrak{r}\vartheta} \frac{\beta C_{1-\kappa} \Gamma(1+\kappa))}{\Gamma(1+\beta\kappa)} \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta\kappa-1} \|\mathcal{A}^\kappa \mathfrak{g}(\mathfrak{r}, x_\mathfrak{r})\| d\mathfrak{r} \\ \leq & \frac{d^{1-\mathfrak{b}} e^{-\mathfrak{r}\vartheta} \beta C_{1-\kappa} \Gamma(1+\kappa))}{\Gamma(1+\beta\kappa)} \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta\kappa-1} \|\mathcal{A}^\kappa \mathfrak{g}(\mathfrak{r}, x_\mathfrak{r})\| d\mathfrak{r} \\ \leq & \frac{d^{1-\mathfrak{b}} e^{-\mathfrak{r}\vartheta} \beta C_{1-\kappa} \Gamma(1+\kappa))}{\Gamma(1+\beta\kappa)} \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta\kappa-1} \mathcal{L}_2 (1 + \vartheta^{1-\mathfrak{b}} \|x(\vartheta)\|) d\mathfrak{r} \\ \leq & d^{1-\mathfrak{b}+\beta\kappa} e^{-\mathfrak{r}\vartheta} \mathcal{L}_2 (1 + \mathfrak{q}) \Delta_2, \\ \mathcal{J}_5 = & \sup \vartheta^{1-\mathfrak{b}} e^{-\mathfrak{r}\vartheta} \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} \mathcal{V}_\beta(\vartheta - \mathfrak{r}) \mathcal{A}_1 x(\mathfrak{r} - \sigma) d\mathfrak{r} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\mathbb{N}d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\int_0^{\vartheta}(\vartheta-\mathfrak{r})^{\beta-1}\|\mathcal{A}_1x(\mathfrak{r}-\sigma)\|d\mathfrak{r}\\
&\leq \frac{\mathbb{N}\mathbb{N}_1d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\int_0^{\vartheta}(\vartheta-\mathfrak{r})^{\beta-1}e^{\mathfrak{r}(\mathfrak{r}-\sigma)}\|x\|_C^*d\mathfrak{r}\\
&\leq \frac{\mathbb{N}\mathbb{N}_1d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\Big(\int_0^{\vartheta}(\vartheta-\mathfrak{r})^{\frac{\beta-1}{1-\vartheta_1}}d\mathfrak{r}\Big)^{1-\vartheta_1}\Big(\int_0^{\vartheta}e^{\frac{\mathfrak{r}(\mathfrak{r}-\sigma)}{1-\vartheta_1}}d\mathfrak{r}\Big)^{1-\vartheta_1}\mathfrak{q}\\
&\leq \frac{\mathbb{N}\mathbb{N}_1d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\Delta_1e^{\frac{-\mathfrak{r}\sigma}{1-\vartheta_1}}\Big(\int_0^{\vartheta}e^{\frac{\mathfrak{r}\mathfrak{r}}{1-\vartheta_1}}d\mathfrak{r}\Big)^{1-\vartheta_1}\mathfrak{q}\leq \frac{\mathbb{N}\mathbb{N}_1d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\mathfrak{q}\Delta_1,\\
\mathcal{J}_6&=\sup\vartheta^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}\Big\|\int_0^{\vartheta}(\vartheta-\mathfrak{r})^{\beta-1}\mathcal{V}_{\beta}(\vartheta-\mathfrak{r})Bu(\mathfrak{r})d\mathfrak{r}\Big\|\\
&\leq \frac{\mathbb{N}d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\int_0^{\vartheta}(\vartheta-\mathfrak{r})^{\beta-1}\|Bu(\mathfrak{r})\|d\mathfrak{r}\leq \frac{d^{1-\mathfrak{b}}\mathbb{N}\mathbb{N}_be^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\Delta_1\|w\|_C,\\
\mathcal{J}_7&=\sup\vartheta^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}\Big\|\int_0^{\vartheta}(\vartheta-\mathfrak{r})^{\beta-1}\mathcal{V}_{\beta}(\vartheta-\mathfrak{r})\mathcal{G}(\mathfrak{r},x(\mathfrak{r}-\sigma))d\mathfrak{r}\Big\|\\
&\leq \frac{\mathbb{N}d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\int_0^{\vartheta}(\vartheta-\mathfrak{r})^{\beta-1}\|\mathcal{G}(\mathfrak{r},x(\mathfrak{r}-\sigma))-\mathcal{G}(\mathfrak{r},0)+\mathcal{G}(\mathfrak{r},0))\|d\mathfrak{r}\\
&\leq \frac{\mathbb{N}d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\Big[\int_0^{\vartheta}(\vartheta-\mathfrak{r})^{\beta-1}\eta\vartheta^{1-\mathfrak{b}}\|x(\mathfrak{r}-\sigma)\|d\mathfrak{r}+\frac{\mathbb{N}_0d^{\beta}}{\beta}\Big]\\
&\leq \frac{\mathbb{N}d^{2(1-\mathfrak{b})}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\Big[\eta\Delta_1\mathfrak{q}+\frac{\mathbb{N}_0d^{\beta}}{\beta}\Big].
\end{aligned}
$$

Combining $\mathcal{J}_1$ to $\mathcal{J}_7$, we get

$$
\begin{aligned}
e^{-\mathfrak{r}\vartheta}\|\Gamma x(\vartheta)\|\leq& \frac{\mathbb{N}e^{-\mathfrak{r}\vartheta}}{\Gamma(\alpha(1-\beta)+\beta)}\|\hbar(0)\|+\frac{\mathcal{L}_2d^{1-\mathfrak{b}}\mathbb{N}_0\mathbb{N}e^{-\mathfrak{r}\vartheta}}{\Gamma(\alpha(1-\beta)+\beta)}+\mathbb{N}_0\mathcal{L}_2(1+\mathfrak{q})d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}\\
&+d^{1-\mathfrak{b}+\beta\kappa}e^{-\mathfrak{r}\vartheta}\mathcal{L}_2(1+\mathfrak{q})\Delta_2+\frac{\mathbb{N}\mathbb{N}_1d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\mathfrak{q}\Delta_1+\frac{d^{1-\mathfrak{b}}\mathbb{N}\mathbb{N}_be^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\Delta_1\|w\|_C\\
&+\frac{\mathbb{N}d^{2(1-\mathfrak{b})}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\Big[\eta\Delta_1\mathfrak{q}+\frac{\mathbb{N}_0d^{\beta}}{\beta}\Big]\\
\leq& \frac{\mathbb{N}e^{-\mathfrak{r}\vartheta}}{\Gamma(\alpha(1-\beta)+\beta)}\Big[\|\hbar(0)\|+\mathcal{L}_2d^{1-\mathfrak{b}}\Big]+\mathcal{L}_2(1+\mathfrak{q})d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}\Big[\mathbb{N}_0+d^{\beta\kappa}\Delta_2\Big]\\
&+\frac{\mathbb{N}d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\Big([\mathbb{N}_1\mathfrak{q}+\mathbb{N}_b\|w\|_C]\Delta_1+d^{1-\mathfrak{b}}[\eta\Delta_1\mathfrak{q}+\frac{\mathbb{N}_0d^{\beta}}{\beta}]\Big)\\
\leq& \mathbb{P}^*+\mathcal{L}_2(1+\mathfrak{q})d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}\Big[\mathbb{N}_0+d^{\beta\kappa}\Delta_2\Big]+\frac{\mathbb{N}d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\Delta_1(\mathbb{N}_1+d^{1-\mathfrak{b}}\eta)\mathfrak{q}.
\end{aligned}
$$

Where

$$
\mathbb{P}^*=\frac{\mathbb{N}}{\Gamma(\alpha(1-\beta)+\beta)}\Big[\|\hbar(0)\|+\mathcal{L}_2d^{1-\mathfrak{b}}\Big]+\frac{d^{1-\mathfrak{b}}\mathbb{N}\mathbb{N}_b}{\Gamma(\beta)}\Delta_1\|w\|_C+\frac{\mathbb{N}\mathbb{N}_0d^{2(1-\mathfrak{b})+\beta}}{\Gamma(\beta+1)}.
$$

A positive constant $\mathfrak{q}$ appearing from the norm $\|\cdot\|_C^*$,

$$
\mathfrak{q}\geq \frac{\mathbb{N}d^{1-\mathfrak{b}}}{\Gamma(\beta)}\Delta_1(\mathbb{N}_1+d^{1-\mathfrak{b}}\eta)+d^{1-\mathfrak{b}}\Big[\mathbb{N}_0+d^{\beta\kappa}\Delta_2\Big]>0, \tag{3.2}
$$

and the radius of the sphere

$$\mathfrak{q} \geq \max\Big\{\|\hbar\|_C^*, \frac{\mathbb{N}}{\Gamma(\alpha(1-\beta)+\beta)}\Big[\|\hbar(0)\| + \mathcal{L}_2 d^{1-\mathfrak{b}}\Big] + \frac{d^{1-\mathfrak{b}}\mathbb{N}\mathbb{N}_b}{\Gamma(\beta)}\Delta_1\|w\|_C + \frac{\mathbb{N}\mathbb{N}_0 d^{2(1-\mathfrak{b})+\beta}}{\Gamma(\beta+1)}\Big\}. \tag{3.3}$$

From (3.2) and (3.3) we are getting a contradiction to $\mathcal{F}_3$. Therefore $\|\Gamma x\|^* \leq \mathfrak{q}$.

Step 2: Contraction: For every $\vartheta \in (0, d]$ using ($\mathcal{F}_2$) and there exists constants $x_1, x_2 \in \mathbb{C}([-\sigma, d]; \mathcal{H})$, we obtain

$$\begin{aligned}
\|(\Gamma x_2)(\vartheta) - \Gamma x_1)(\vartheta)\| =& \Big\|[\mathfrak{g}(\vartheta, x_{2_t}) - \mathfrak{g}(\vartheta, x_{1_t})] + \int_0^\vartheta (\vartheta-\mathfrak{r})^{\beta-1}\mathcal{V}_\beta(\vartheta-\mathfrak{r})[\mathcal{A}\mathfrak{g}(\mathfrak{r}, x_{2_\mathfrak{r}}) \\
& - \mathcal{A}\mathfrak{g}(\mathfrak{r}, x_{1_\mathfrak{r}})] + [\mathcal{A}_1 x_2(\mathfrak{r}-\sigma) - \mathcal{A}_1 x_1(\mathfrak{r}-\sigma)] \\
& + [\mathcal{G}(\mathfrak{r}, x_2(\mathfrak{r}-\sigma)) - \mathcal{G}(\mathfrak{r}, x_1(\mathfrak{r}-\sigma))]d\mathfrak{r}\Big\| \\
\leq& \mathbb{N}_0 d^{1-\mathfrak{b}}\mathcal{L}_1\|x_1 - x_2\|_\mathfrak{b} + d^{1-\mathfrak{b}+\beta\kappa}\mathcal{L}_1\|x_1 - x_2\|_\mathfrak{b}\Delta_2 \\
& + \frac{\mathbb{N}}{\Gamma\beta}\Big[\mathbb{N}_1\int_0^\vartheta (\vartheta-\mathfrak{r})^{\beta-1}\Big\|x_2(\vartheta-\sigma) - x_1(\vartheta-\sigma)\Big\|d\mathfrak{r} \\
& + \int_0^\vartheta (\vartheta-\mathfrak{r})^{\beta-1}\Big\|\mathcal{G}(\mathfrak{r}, x_2(\mathfrak{r}-\sigma)) - \mathcal{G}(\mathfrak{r}, x_1(\mathfrak{r}-\sigma))\Big\|d\mathfrak{r}\Big] \\
\leq& (\mathbb{N}_0 + d^{\beta\kappa}\Delta_2)d^{1-\mathfrak{b}}\mathcal{L}_1\|x_1 - x_2\|_\mathfrak{b} \\
& + \frac{\mathbb{N}(\mathbb{N}_1 + d^{1-\mathfrak{b}}\eta)}{\Gamma\beta}\int_0^\vartheta (\vartheta-\mathfrak{r})^{\beta-1}\Big\|x_2(\mathfrak{r}-\sigma) - x_1(\mathfrak{r}-\sigma)\Big\|d\mathfrak{r} \\
\leq& e^{-\mathfrak{r}\vartheta}\frac{\mathbb{N}(\mathbb{N}_1 + d^{1-\mathfrak{b}}\eta)}{\Gamma(\beta)}\Big(\int_0^\vartheta (\vartheta-\mathfrak{r})^{\beta-1}e^{ss}d\mathfrak{r}\Big)\|x_2 - x_1\|_C^*, \\
e^{\mathfrak{r}\vartheta}\|(\Gamma x_2)(\vartheta) - \Gamma x_1)(\vartheta)\| \leq& \Big(\frac{\mathbb{N}(\mathbb{N}_1 + d^{1-\mathfrak{b}}\eta)}{\Gamma(\beta)}\Delta_1 + (\mathbb{N}_0 + d^{\beta\kappa}\Delta_2)d^{1-\mathfrak{b}}\mathcal{L}_1\Big)\|x_2 - x_1\|_C^*.
\end{aligned}$$

From the definition of $\mathfrak{r}$ from (3.2), we obtain

$$\|\Gamma x_2 - \Gamma x_1\|_\mathbb{C}^* \leq \mathfrak{r}^*\|x_2 - x_1\|_\mathbb{C}^*, \ \mathfrak{r}^* < 1.$$

Therefore Γ is contraction on $\mathbb{C}([-\sigma, d]; \mathcal{H})$. Hence x has a fixed point of Γ, i.e., it is a mild solution of (1.1). □

4. Filter system

By referring the articles [44,58], we have given a filter design for our system (1.1) shown in Figure 1 and it shows a rough diagram format, it contributes to the structure's practicality by reducing the number of input sources.

(a) Product modulators 1 and 2 accept the $\mathcal{A}$ and $\mathfrak{g}(\mathfrak{r}, x_\mathfrak{r})$, $u(\mathfrak{r})$ and B gives the outputs as $\mathcal{A}\mathfrak{g}(\mathfrak{r}, x_\mathfrak{r})$ and $Bu(\mathfrak{r})$.
(b) Product modulator 3 accepts the input $[\hbar(0) - \mathfrak{g}(0, \hbar(0))]$ and $\mathcal{R}_{\alpha,\beta}(\vartheta)$ at time $\vartheta = 0$, gives the output as $\mathcal{R}_{\alpha,\beta}(\vartheta)[\hbar(0) - \mathfrak{g}(0, \hbar(0))]$.

(c) $\mathcal{A}_1$ and $x(\mathfrak{r}-\sigma)$, produced $\mathcal{A}_1 x(\mathfrak{r}-\sigma)$.
(d) $Q_\beta(\vartheta-\mathfrak{r})$, $\mathcal{G}(\mathfrak{r}, x(\mathfrak{r}-\sigma))$ are the inputs. Over ϑ, the inputs are joined and multiplied with an integrator output.
(e) $Q_\beta(\vartheta-\mathfrak{r})$, $\mathcal{A}_1 x(\mathfrak{r}-\sigma)$ are the inputs. Over ϑ, the inputs are joined and multiplied with an integrator output.
(g) $Q\beta(\vartheta-\mathfrak{r})$, $\mathcal{A}\mathfrak{g}(\mathfrak{r}, x_\mathfrak{r})$ are the inputs. Over ϑ, the inputs are joined and multiplied with an integrator output.
(h) $Q_\beta(\vartheta-\mathfrak{r})$, $Bu(\mathfrak{r})$ are the inputs. Over ϑ, the inputs are joined and multiplied with an integrator output.
(f) The following integrators sum up with the above mentioned modulators over the period ϑ,

$$
\begin{aligned}
I_1 &= \int_0^\vartheta (\vartheta-\mathfrak{r})^{\beta-1} Q_\beta(\vartheta-\mathfrak{r})\mathcal{A}\mathfrak{g}(\mathfrak{r}, x_\mathfrak{r})d\mathfrak{r},\\
I_2 &= \int_0^\vartheta (\vartheta-\mathfrak{r})^{\beta-1} Q_\beta(\vartheta-\mathfrak{r})\mathcal{A}_1 x(\mathfrak{r}-\sigma)d\mathfrak{r},\\
I_3 &= \int_0^\vartheta (\vartheta-\mathfrak{r})^{\beta-1} Q_\beta(\vartheta-\mathfrak{r})Bu(\mathfrak{r})d\mathfrak{r},\\
I_4 &= \int_0^\vartheta (\vartheta-\mathfrak{r})^{\beta-1} Q_\beta(\vartheta-\mathfrak{r})\mathcal{G}(\mathfrak{r}, x(\mathfrak{r}-\sigma))d\mathfrak{r},
\end{aligned}
$$

where $Q_\beta(\vartheta-\mathfrak{r}) = (\vartheta-\mathfrak{r})^{\beta-1}\mathcal{V}_\beta(\vartheta-\mathfrak{r})$.

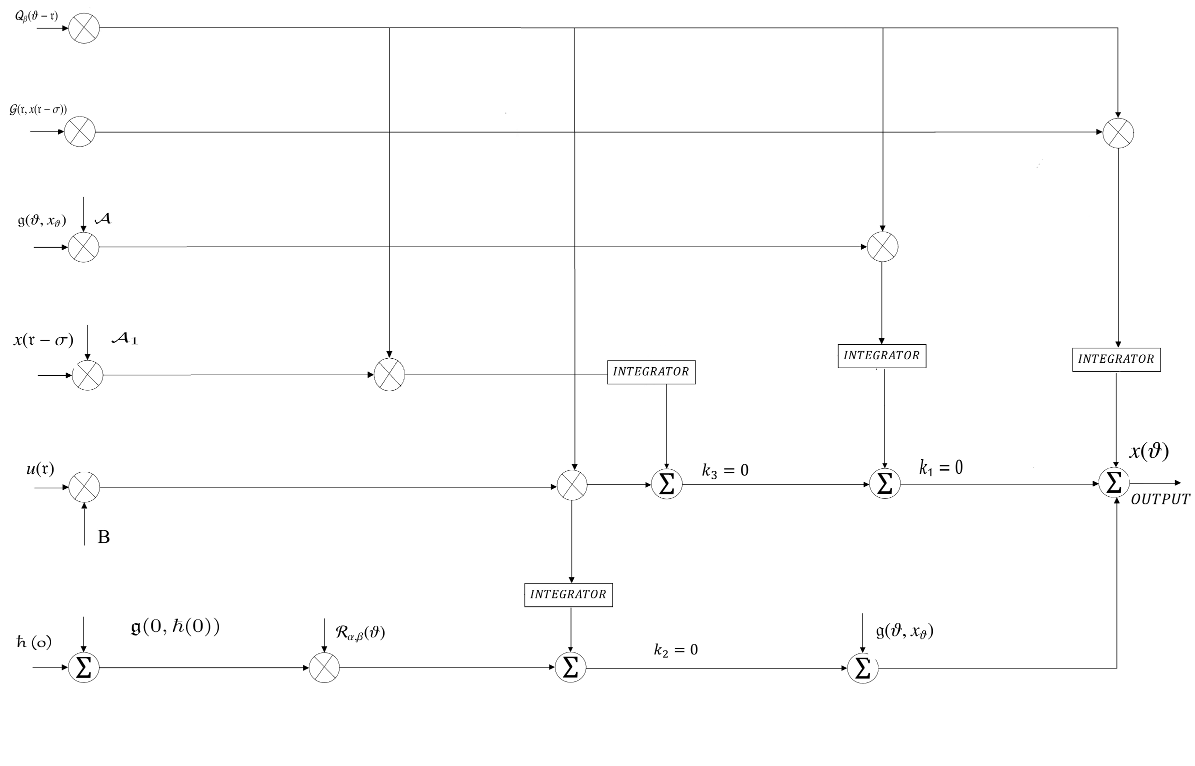

Figure 1. Filter system.

Finally, we move all integrator outputs to the network. As a result, we have our output result $x(\vartheta)$.

5. Approximate controllability results

Nonlinear control systems with approximate controllability are operated by fractional-order with time delay.

Definition 5.1. *Let* $E(\mathcal{G}) = \{x(d; w) : u(\cdot) \in U\}$ *be the reachable set of* (1.1) *at time* d*. Suppose* $\mathcal{G}$ *is identically zero then* (1.1) *is said to be corresponding linear system and* $E(0)$ *is defined as the reachable set of* (1.1)*.*

Definition 5.2. *Suppose* $\overline{E}(\mathcal{G}) = \mathcal{K}$*, then* (1.1) *is approximately controllable at time* d $(d > \sigma)$*, where* $\overline{E}(\mathcal{G})$ *signifies the closure of* $E(\mathcal{G})$*. If* $\overline{E}(0) = \mathcal{K}$ *then* (1.1) *is also approximately controllable.*

Following hypotheses are used to prove the main outcome.

$\mathcal{F}_5$: For every $\mu > 0$ and $\ell(\cdot)$ from $\mathcal{Y}$, then there exists $u(\cdot) \in U$ such that

$$\|\mathcal{E}\ell - \mathcal{E}Bu\|_{\mathcal{H}} < \mu.$$

$\mathcal{F}_6$: For $\nu > 0$ independent of $\ell(\cdot) \in \mathcal{Y}$ such that

$$\|Bu(\cdot)\|_{L^2((0,d];\mathcal{K})} < \nu\|\ell(\cdot)\|_{L^2((0,d];\mathcal{K})}.$$

Lemma 5.3. *Assumptions* $(\mathcal{F}_1), (\mathcal{F}_2)$ *are true then the mild solutions of* (1.1) *satisfies*

$$\|x\|^*_{\mathbb{C}} \le \mathbb{P}^* E_\beta(M(\mathbb{N}_1 + d^{1-\mathrm{b}}\eta)d^\beta), \quad \textit{for all } u(\cdot) \in W,$$
$$\|x_1(\cdot) - x_2(\cdot)\|^*_{\mathbb{C}} \le \theta E_\beta(M(\mathbb{N}_1 + d^{1-\mathrm{b}}\eta)d^\beta)\mathbb{N}_b\|w_1 - w_2\|_{\mathcal{Y}}, \textit{ for all } w_1,\ w_2(\cdot) \in \mathcal{X},$$

where

$$\mathbb{P}^* = \frac{\mathbb{N}}{\Gamma(\alpha(1-\beta)+\beta)}\|\hbar(0)\| + \frac{d^{1-\mathrm{b}}\mathbb{N}\mathbb{N}_b}{\Gamma(\beta)}\Delta_1\|w\|_C + \frac{\mathbb{N}\mathbb{N}_0 d^{2(1-\mathrm{b}+\beta)}}{\Gamma(\beta+1)},$$
$$\theta = \frac{\mathbb{N}\mathbb{N}_b d^{1-\mathrm{b}}}{\Gamma(\beta)}\Delta_1.$$

Proof. Define $\mathcal{E} : \mathcal{Y} \to C((0,d],\mathcal{K})$ by

$$\mathcal{E}\sigma = \int_0^d (d-\mathrm{r})^{\beta-1}\mathcal{V}_\beta(d-\mathrm{r})\sigma(\mathrm{r})d\mathrm{r}, \text{ for } \sigma(\cdot) \in \mathcal{Y},$$

choosing a desired final state function Ψ and $\mu > 0$ then we have

$$\|\Psi - \mathcal{R}_{\alpha,\beta}(d)[\hbar(0) - \mathrm{g}(0,\hbar(0))] - \mathrm{g}\lambda_\mu - \mathcal{E}\mathcal{A}\lambda_\mu - \mathcal{E}\mathcal{A}_1\lambda_\mu - \mathcal{E}\mathcal{G}\lambda_\mu - \mathcal{E}Bw_\mu\| < \mu.$$

For any $\Psi \in D(\mathcal{A})$ and $x_0 \in \mathcal{H}$, there exists $\ell > 0$ such that

$$\mathcal{E}\ell = \Psi - \mathcal{R}_{\alpha,\beta}(d)\hbar(0).$$

In the above

$$(\mathrm{g}\lambda)(\vartheta) = (\mathrm{g}(\vartheta, x_\vartheta),$$
$$(\mathcal{G}\lambda)(\vartheta) = \mathcal{G}(\vartheta, x(\vartheta - \sigma)),$$
$$\mathcal{A}_1\lambda = \mathcal{A}_1 x(\vartheta - \sigma),$$

and

$$x_\mu(\vartheta) = x(\vartheta; w_\mu),$$

is a mild solution (1.1) according to $w_\mu(\cdot)$ belongs to $\mathcal{K}$. Suppose $x(\cdot, w) = x(\cdot)$ is a mild solution of (1.1) with respect to $u(\cdot) \in W$ then

$$\begin{aligned}
\|x(\vartheta)\|_{\mathcal{H}} =& \vartheta^{1-\mathfrak{b}}\Big\|\mathcal{R}_{\alpha,\beta}(\vartheta)[\hbar(0) - \mathrm{g}(0, \hbar(0))] + \mathrm{g}(\vartheta, x_\vartheta) + \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}\mathcal{V}_\beta(\vartheta - \mathfrak{r})\mathcal{A}\mathrm{g}(\mathfrak{r}, x_\mathfrak{r})d\mathfrak{r} \\
&+ \int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}\mathcal{V}_\beta(\vartheta - \mathfrak{r})\Big[\mathcal{A}_1 x(\mathfrak{r} - \sigma) + Bu(\mathfrak{r}) + \mathcal{G}(\mathfrak{r}, x(\vartheta - \sigma))\Big]d\mathfrak{r}\Big\| \\
\le& \frac{\mathbb{N}}{\Gamma(\alpha(1-\beta)+\beta)}\|\hbar(0)\| + \frac{\mathcal{L}_2 d^{1-\mathfrak{b}}\mathbb{N}_0\mathbb{N}}{\Gamma(\alpha(1-\beta)+\beta)} + d^{1-\mathfrak{b}}\mathbb{N}_0\mathcal{L}_2(1 + \vartheta^{1-\mathfrak{b}}\|x(\vartheta)\|) \\
&+ \frac{\mathbb{N}d^{1-\mathfrak{b}}}{\Gamma(\beta)}\int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}\|\mathcal{A}\mathrm{g}(\mathfrak{r}, x_\mathfrak{r})\|d\mathfrak{r} \\
&+ \frac{\mathbb{N}d^{1-\mathfrak{b}}}{\Gamma(\beta)}\int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}\|\mathcal{A}_1 x(\mathfrak{r} - \sigma)\|d\mathfrak{r} \\
&+ \frac{\mathbb{N}\mathbb{N}_b d^{1-\mathfrak{b}}}{\Gamma(\beta)}\int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}\|u(\mathfrak{r})\|d\mathfrak{r} + \frac{\mathbb{N}d^{1-\mathfrak{b}}}{\Gamma(\beta)}\int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}\|\mathcal{G}(\mathfrak{r}, x(\mathfrak{r} - \sigma))\|d\mathfrak{r} \\
\le& \frac{\mathbb{N}\|\hbar(0)\|}{\Gamma(\alpha(1-\beta)+\beta)} + \frac{\mathcal{L}_2 d^{1-\mathfrak{b}}\mathbb{N}_0\mathbb{N}}{\Gamma(\alpha(1-\beta)+\beta)} + d^{1-\mathfrak{b}}\mathbb{N}_0\mathcal{L}_2(1 + \vartheta^{1-\mathfrak{b}}\|x(\vartheta)\|) \\
&+ d^{1-\mathfrak{b}+\beta\kappa}\mathcal{L}_2(1 + \mathfrak{q})\Delta_2 + \frac{\mathbb{N}\mathbb{N}_1 d^{1-\mathfrak{b}}}{\Gamma(\beta)}\int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}\|x(\mathfrak{r} - \sigma)\|d\mathfrak{r} \\
&+ \frac{\mathbb{N}d^{1-\mathfrak{b}}}{\Gamma(\beta)}\int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}\|\mathcal{G}(\mathfrak{r}, x(\mathfrak{r} - \sigma)) - \mathcal{G}(\mathfrak{r}, 0) + \mathcal{G}(\mathfrak{r}, 0)\|d\mathfrak{r} \\
&+ \frac{\mathbb{N}\mathbb{N}_b d^{1-\mathfrak{b}}}{\Gamma(\beta)}\int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}\|u(\mathfrak{r})\|d\mathfrak{r} \\
\le& \frac{\mathbb{N}\|\hbar(0)\|}{\Gamma(\alpha(1-\beta)+\beta)} + \frac{\mathbb{N}d^{1-\mathfrak{b}+\beta}}{\Gamma(\beta+1)}\Big[\mathbb{N}_0 + \mathbb{N}_b\|w\|_{\mathcal{Y}}\Big] \\
&+ d^{1-\mathfrak{b}}\mathbb{N}_0\mathcal{L}_2(1 + \vartheta^{1-\mathfrak{b}}\|x(\vartheta)\|) + d^{1-\mathfrak{b}+\beta\kappa}\mathcal{L}_2(1 + \mathfrak{q})\Delta_2 \\
&+ \frac{\mathbb{N}(\mathbb{N}_1 + 1)d^{1-\mathfrak{b}}}{\Gamma(\beta)}\int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}[x(\mathfrak{r} - \sigma) + \mathcal{G}(\mathfrak{r}, 0))]d\mathfrak{r} \\
&+ \frac{\mathbb{N}\eta d^{1-\mathfrak{b}}e^{-\mathfrak{r}\vartheta}}{\Gamma(\beta)}\int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1}e^{\mathfrak{r}(\mathfrak{r}-\sigma)}\|x\|^* d\mathfrak{r}.
\end{aligned} \tag{5.1}$$

By using Gronwall's inequality, Mittag-Leffler function

$$E_\beta(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(k\beta + 1)}, \quad \text{and} \quad \|x\|_C^* = \sup_{\vartheta\in(0,b]} e^{-\mathfrak{r}\vartheta}\|x(\vartheta)\|,$$

(5.1) implies

$$\|x\|_C^* \leq \mathbb{P}^* E_\beta(\mathbb{N}_1 + d^{1-\mathfrak{b}}\eta)d^\beta,$$

where

$$\mathbb{P}^* = \frac{\mathbb{N}}{\Gamma(\alpha(1-\beta)+\beta)}\Big[\|\hbar(0)\| + \mathcal{L}_2 d^{1-\mathfrak{b}}\Big] + \frac{d^{1-\mathfrak{b}}\mathbb{N}\mathbb{N}_b}{\Gamma(\beta)}\Delta_1\|w\|_C + \frac{\mathbb{N}\mathbb{N}_0 d^{2(1-\mathfrak{b})+\beta}}{\Gamma(\beta+1)}.$$

Now

$$\begin{aligned}\|x_1(\vartheta) - x_2(\vartheta)\| \leq& e^{-\mathfrak{r}\vartheta}\frac{\mathbb{N}\mathbb{N}_b d^{1-\mathfrak{b}}}{\Gamma(\beta)}\Big(\int_0^\vartheta (\vartheta - \mathfrak{r})^{\beta-1} d\mathfrak{r}\Big)\|w_2 - w_1\|_C^*,\\ \|x_1 - x_2\|_{\mathbb{C}}^* \leq& \theta E_\beta(\mathbb{N}(\mathbb{N}_1 + d^{1-\mathfrak{b}}\eta)d^\beta)\|w_2 - w_1\|_C^*.\end{aligned}$$

This completes the proof. □

Theorem 5.4. *If* ($\mathcal{F}_1$)–($\mathcal{F}_3$) *are true then* (1.1) *is approximately controllable.*

Proof. To verify $\overline{D(\mathcal{A})} \subset \mathcal{H}$ for all $\Psi \in D(\mathcal{A})$, there is a control $w_\mu(\cdot) \in W$, such that

$$\|\Psi - \mathcal{R}_{\alpha,\beta}(d)[\hbar(0) - \mathfrak{g}(0,\hbar(0))] - \mathfrak{g}\lambda_\mu - \mathcal{E}\mathcal{A}_1 x_\mu - \mathcal{E}\mathcal{G}x_\mu - \mathcal{E}Bw_\mu\| < \mu, \quad \mu > 0. \tag{5.2}$$

For any x_0 belongs to $\mathcal{H}$ and there exists a function $\ell(\cdot)$ belongs to $\mathcal{Y}$, then

$$\mathcal{E}\ell = \Psi - \mathcal{R}_{\alpha,\beta}(d)[\hbar(0) - \mathfrak{g}(0,\hbar(0))] - \mathfrak{g}\lambda_\mu.$$

Suppose $w_1(\cdot)$ belongs to W and $\mu > 0$ then from ($\mathcal{F}_3$) we choose an arbitrary value $w_2(\cdot)$ belongs to W such that

$$\|\Psi - \mathcal{R}_{\alpha,\beta}(d)[\hbar(0) - \mathfrak{g}(0,\hbar(0))] - \mathfrak{g}\lambda_\mu - \mathcal{E}\mathcal{A}_1 x_1 - \mathcal{E}\mathcal{G}x_1 - \mathcal{E}Bw_2\| < \frac{\mu}{2^2}. \tag{5.3}$$

From the above we note $x_1(\vartheta)$ takes $x(\vartheta; w_1)$ and $x_2(\vartheta)$ takes $x(\vartheta; w_2)$ for $0 \leq \vartheta \leq d$.

Again from ($\mathcal{F}_3$) there exists $\varpi_2(\cdot) \in W$,such that

$$\Big\|\mathcal{E}[\mathcal{A}_1 x_2 - \mathcal{A}_1 x_1 + \mathcal{G}x_2 - \mathcal{G}x_1] - \mathcal{E}B\varpi_2\Big\| < \frac{\mu}{2^3},$$

we now consider

$$\begin{aligned}\Big\|B\varpi_2\Big\|_{\mathcal{Y}} &\leq \nu\Big\|\mathcal{A}_1 x_2 - \mathcal{A}_1 x_1 + \mathcal{G}x_2 - \mathcal{G}x_1\Big\|\\ &\leq \nu(\mathbb{N}_1 + d^{1-\beta}\eta)\|x_2(\cdot) - x_1(\cdot)\|\\ &\leq \theta\nu(\mathbb{N}_1 + d^{1-\beta}\eta)E_\beta(\mathbb{N}(\mathbb{N}_1 + d^{1-\mathfrak{b}}\eta)d^\beta)\|w_2 - w_1\|_C^*.\end{aligned}$$

Define $w_3(\vartheta) = w_2(\vartheta) - \nu_2(\vartheta)$, $w_3(\cdot) \in W$, then

$$\begin{aligned}&\|\xi - \mathcal{R}_{\alpha,\beta}(\vartheta))[\hbar(0) - \mathfrak{g}(0,\hbar(0))] - \mathfrak{g}\lambda_\mu - \mathcal{E}\mathcal{A}_1 x_2 - \mathcal{E}\mathcal{G}x_2 - \mathcal{E}Bw_3\|\\ \leq&\|\xi - \mathcal{R}_{\alpha,\beta}(\vartheta))[\hbar(0) - \mathfrak{g}(0,\hbar(0))] - \mathfrak{g}\lambda_\mu - \mathcal{E}\mathcal{A}_1 x_1 - \mathcal{E}\mathcal{G}x_1 - \mathcal{E}Bw_2\|\end{aligned}$$

$$+\|\mathcal{E}Bv_2-\mathcal{E}[\mathcal{A}_1x_2-\mathcal{A}_1x_1+\mathcal{G}x_2-\mathcal{G}x_1]\|\leq(\frac{1}{2^2}+\frac{1}{2^3})\mu.$$

Suppose there is a sequence $\{x_k(\cdot)\}\subset\mathcal{X}$, then

$$\|\xi-\mathcal{R}_{\alpha,\beta}(\vartheta)[\hbar(0)-\mathrm{g}(0,\hbar(0))]-\mathrm{g}\lambda_\mu-\mathcal{E}\mathcal{A}_1x_k-\mathcal{E}\mathcal{G}x_k-\mathcal{E}Bw_{k+1}\|<\Big(\frac{1}{2^2}+\cdots+\frac{1}{2^k}\Big)\mu.$$

In the above $x_k(\cdot)=x(\cdot;w_k)$ for $0\leq\vartheta\leq\mu$, and

$$\|Bw_{k+1}-Bw_k\|_{\mathcal{H}}\leq\theta\mathbb{N}(\mathbb{N}_1+d^{1-\beta}\eta)\nu E_\eta(\mathbb{N}(\mathbb{N}_1+d^{1-\beta}\eta)d^\beta)\|w_k(\cdot)-w_{k-1}(\cdot)\|_{\mathcal{H}}.$$

By referring (5.3) and there exists a $\chi(\cdot)\in W$, such that

$$\lim_{k\to\infty}Bw_k(\cdot)=\chi(\cdot)\quad\in\ \mathcal{Y}.$$

As a result, for every $\mu>0$, there exists a positive integer number N, such that

$$\|\mathcal{E}Bw_{N+1}-\mathcal{E}Bw_N\|<\frac{\mu}{2}.$$

Hence, we get

$$\begin{aligned}&\|\Psi-\mathcal{R}_{\alpha,\beta}(\vartheta)[\hbar(0)-\mathrm{g}(0,\hbar(0))]-\mathrm{g}\lambda_\mu-\mathcal{E}\mathcal{A}_1x_N-\mathcal{E}\mathcal{G}x_N-\mathcal{E}Bw_N\|\\ \leq&\|\Psi-\mathcal{R}_{\alpha,\beta}(\vartheta)[\hbar(0)-\mathrm{g}(0,\hbar(0))]-\mathrm{g}\lambda_\mu-\mathcal{E}\mathcal{A}_1x_N-\mathcal{E}\mathcal{G}x_N-\mathcal{E}Bw_{N+1}\|\\ &+\|\mathcal{E}Bw_{N+1}-\mathcal{E}Bw_N\|\\ \leq&\Big(\frac{1}{2^2}+\cdots+\frac{1}{2^k}\Big)\mu+\frac{\mu}{2}<\mu.\end{aligned}$$

Therefore (1.1) is approximate controllability. Thus this ends the proof . □

6. Nonlocal condition

Byszewski [15, 16] investigated the idea of "nonlocal conditions", proving the existence and uniqueness of mild, strong, and classical nonlocal Cauchy problem solutions for semilinear evolution equations. In [51] the authors considered the controllability with nonlocal conditions by utilizing fixed point methods and fractional calculus. A valuable conversation about the nonlocal conditions are given in [25, 27, 51].

Apparently, the controllability of neutral differential problems in particular of time delay with nonlocal conditions with respect to Hilfer fractional differential equations has not been explored at this point. Motivated by the articles [25, 53, 56], consider

$$\begin{cases}\mathcal{D}^{\alpha,\beta}_{0^+}[x(\vartheta)-\mathrm{g}(\vartheta,x_\vartheta)]=\mathcal{A}x(\vartheta)+\mathcal{A}_1x(\vartheta-\sigma)+Bu(\vartheta)+\mathcal{G}(\vartheta,x(\vartheta-\sigma)),\ \text{for all }\vartheta>0,\\ I^{(1-\alpha)(1-\beta)}_{0^+}x(\vartheta)+\mathfrak{p}(\vartheta_{t_1},\vartheta_{t_2},\vartheta_{t_3},\cdots,\vartheta_{t_n})=\hbar(\vartheta),\ \vartheta\in[-\sigma,0].\end{cases}\tag{6.1}$$

Where $\mathcal{K}$ is a positive real, $0<t_1<t_2<t_3<\cdots<t_n\leq d$, $\mathfrak{p}:C([0,\mathcal{K}],\mathcal{H})\to\mathcal{H}$ and satisfying the following assumption:

$\mathcal{F}_7$: $\mathfrak{p} : C((0,\mathcal{K}],\mathcal{H}) \to \mathbb{K}$ is continuous, there exists $\mathcal{N}_i(\mathfrak{p}) > 0$ such that

$$\|\mathfrak{p}(\lambda_1,\lambda_2,\cdots,\lambda_n) - \mathfrak{p}(\gamma_1,\gamma_2,\cdots,\gamma_n)\| \le \sum_{k=1}^{n} \mathcal{N}_k(\mathfrak{p})\|\lambda_k - \gamma_k\|_{\mathcal{H}},$$

For every $\lambda_k, \gamma_k \in \mathbb{K}$ and consider $N_h = \sup\{\|\mathfrak{p}(\lambda_1,\lambda_2,\cdots,\lambda_n)\| : \lambda_k \in \mathbb{K}\}$.

Definition 6.1. *If* $x(\cdot;w) \in C((0,\mathcal{K}],\mathcal{H})$*is a mild solution of* (6.1) *then* $w \in L^2(\mathcal{K},\mathcal{H})$ *the integral equation*

$$x(\vartheta) = \begin{cases} \mathcal{R}_{\alpha,\beta}(\vartheta)[\mathfrak{p}(\vartheta_{t_1},\vartheta_{t_2},\vartheta_{t_3},\cdots,\vartheta_{t_n})(0) + \hbar(0) - \mathfrak{g}(0,\hbar(0))] + \mathfrak{g}(\vartheta,x_\vartheta) \\ + \int_0^\vartheta (\vartheta-\mathfrak{r})^{\beta-1}\mathcal{V}_\beta(\vartheta-\mathfrak{r})\mathcal{A}\mathfrak{g}(\mathfrak{r},x_\mathfrak{r})d\mathfrak{r}] + \int_0^\vartheta \mathcal{Q}_\beta(\vartheta-\mathfrak{r})\mathcal{A}_1 x(\vartheta-\sigma)d\mathfrak{r} \\ + \int_0^\vartheta \mathcal{Q}_\beta(\vartheta-\mathfrak{r})Bu(\mathfrak{r})d\mathfrak{r} + \int_0^\vartheta \mathcal{Q}_\beta(\vartheta-\mathfrak{r})\mathcal{G}(\mathfrak{r},x(\mathfrak{r}))d\mathfrak{r}, \quad for\ \vartheta > 0, \\ \hbar(\vartheta), \quad \vartheta \in [-\sigma,0]. \end{cases} \tag{6.2}$$

Theorem 6.2. *If the assumptions* $(\mathcal{F}_1)$*–*$(\mathcal{F}_5)$ *are true then* (6.2) *is approximately controllable.*

Proof. To verify $\overline{D(\mathcal{A})} \subset \mathcal{H}$ for every $\Psi \in D(\mathcal{A})$, suppose there is a control $u_\mu(\cdot) \in U$, then

$$\begin{aligned} &\|\Psi - \mathcal{R}_{\alpha,\beta}(d)[\mathfrak{p}(\vartheta_{t_1},\vartheta_{t_2},\vartheta_{t_3},\cdots,\vartheta_{t_n})(0) + [\hbar(0) - \mathfrak{g}(0,\hbar(0))] \\ &- \mathfrak{g}\lambda_\mu - \mathcal{E}\mathcal{A}_1 x_\mu - \mathcal{E}\mathcal{G}x_\mu - \mathcal{E}Bu_\mu\| < \mu, \ \ \mu > 0. \end{aligned} \tag{6.3}$$

For any $x_0 \in \mathcal{H}$ and there exists a function $\ell(\cdot) \in \mathcal{Y}$, such that

$$\mathcal{E}\ell = \Psi - \mathcal{R}_{\alpha,\beta}(d)[\mathfrak{p}(\vartheta_{t_1},\vartheta_{t_2},\vartheta_{t_3},\cdots,\vartheta_{t_n})(0) + big[\hbar(0) - \mathfrak{g}(0,\hbar(0))] - \mathfrak{g}\lambda_\mu.$$

Suppose $u_1(\cdot) \in U$ and $\mu > 0$ then from $(\mathcal{F}_3)$ we choose an arbitrary value $w_2(\cdot) \in W$ such that

$$\begin{aligned} &\|\Psi - \mathcal{R}_{\alpha,\beta}(d)[\mathfrak{p}(\vartheta_{t_1},\vartheta_{t_2},\vartheta_{t_3},\cdots,\vartheta_{t_n})(0) + [\hbar(0) - \mathfrak{g}(0,\hbar(0))] \\ &- \mathfrak{g}\lambda_\mu - \mathcal{E}\mathcal{A}_1 x_1 - \mathcal{E}\mathcal{G}x_1 - \mathcal{E}Bu_2\| < \frac{\mu}{2^2}. \end{aligned} \tag{6.4}$$

From the above we note $x_1(\vartheta)$ takes $x(\vartheta;u_1)$ and $x_2(\vartheta)$ takes $x(\vartheta;u_2)$ for $0 \le \vartheta \le d$.

Again from $(\mathcal{F}_3)$ there exists $\varpi_2(\cdot) \in U$, such that

$$\left\|\mathcal{E}[\mathcal{A}_1 x_2 - \mathcal{A}_1 x_1 + \mathcal{G}x_2 - \mathcal{G}x_1] - \mathcal{E}B\varpi_2\right\| < \frac{\mu}{2^3},$$

we now consider

$$\begin{aligned} \left\|B\varpi_2\right\|_{\mathcal{Y}} &\le \nu\left\|\mathcal{A}_1 x_2 - \mathcal{A}_1 x_1 + \mathcal{G}x_2 - \mathcal{G}x_1\right\| \\ &\le \nu(\mathbb{N}_1 + d^{1-\beta}\eta)\|x_2(\cdot) - x_1(\cdot)\| \\ &\le \theta\nu(\mathbb{N}_1 + d^{1-\beta}\eta)E_\beta(\mathbb{N}(\mathbb{N}_1 + d^{1-\mathfrak{b}}\eta)d^\beta)\|u_2 - u_1\|_C^*. \end{aligned}$$

Define $u_3(\vartheta) = u_2(\vartheta) - v_2(\vartheta)$, $u_3(\cdot) \in U$, then

$$
\begin{aligned}
&\|\xi-\mathcal{R}_{\alpha,\beta}(\vartheta)[\mathfrak{p}(\vartheta_{t_1},\vartheta_{t_2},\vartheta_{t_3},\cdots,\vartheta_{t_n})(0)+\hbar(0)-\mathfrak{g}(0,\hbar(0))]\\
&-\mathfrak{g}\lambda_\mu-\mathcal{E}\mathcal{A}_1x_2-\mathcal{E}\mathcal{G}x_2-\mathcal{E}Bu_3\|\\
\le&\|\xi-\mathcal{R}_{\alpha,\beta}(\vartheta)[\mathfrak{p}(\vartheta_{t_1},\vartheta_{t_2},\vartheta_{t_3},\cdots,\vartheta_{t_n})(0)+\hbar(0)]-\mathcal{E}\mathcal{A}_1x_1-\mathcal{E}\mathcal{G}x_1-\mathcal{E}Bu_2\|\\
&+\|\mathcal{E}Bv_2-\mathcal{E}[\mathcal{A}_1x_2-\mathcal{A}_1x_1+\mathcal{G}x_2-\mathcal{G}x_1]\|\le(\frac{1}{2^2}+\frac{1}{2^3})\mu.
\end{aligned}
$$

Suppose there is a sequence $\{x_k(\cdot)\}\subset\mathcal{X}$, then

$$\|\xi-\mathcal{R}_{\alpha,\beta}(\vartheta)[\mathfrak{p}(\vartheta_{t_1},\vartheta_{t_2},\vartheta_{t_3},\cdots,\vartheta_{t_n})(0)+\hbar(0)]-\mathcal{E}\mathcal{A}_1x_k-\mathcal{E}\mathcal{G}x_k-\mathcal{E}Bu_{k+1}\|<\left(\frac{1}{2^2}+\cdots+\frac{1}{2^k}\right)\mu.$$

In the above $x_k(\cdot)=x(\cdot;u_k)$ for $0\le\vartheta\le\mu$, and

$$\|Bu_{k+1}-Bu_k\|_{\mathcal{H}}\le\theta\mathbb{N}(\mathbb{N}_1+d^{1-\beta}\eta)\nu E_\eta(\mathbb{N}(\mathbb{N}_1+d^{1-\beta}\eta)d^\beta)\|u_k(\cdot)-u_{k-1}(\cdot)\|_{\mathcal{H}}.$$

By referring (6.4) and there exists a $\chi(\cdot)\in U$ such that

$$\lim_{k\to\infty}Bu_k(\cdot)=\chi(\cdot)\quad\in\ \mathcal{Y}.$$

As a result, any $\mu>0$, there is a positive integer number N, then

$$\|\mathcal{E}Bu_{N+1}-\mathcal{E}Bu_N\|<\frac{\mu}{2}.$$

Hence, we get

$$
\begin{aligned}
&\|\Psi-\mathcal{R}_{\alpha,\beta}(\vartheta)[\mathfrak{p}(\vartheta_{t_1},\vartheta_{t_2},\vartheta_{t_3},\cdots,\vartheta_{t_n})(0)+\hbar(0)]-\mathcal{E}\mathcal{A}_1x_N-\mathcal{E}\mathcal{G}x_N-\mathcal{E}Bu_N\|\\
\le&\|\Psi-\mathcal{R}_{\alpha,\beta}(\vartheta)[\mathfrak{p}(\vartheta_{t_1},\vartheta_{t_2},\vartheta_{t_3},\cdots,\vartheta_{t_n})(0)+\hbar(0)]-\mathcal{E}\mathcal{A}_1x_N-\mathcal{E}\mathcal{G}x_N-\mathcal{E}Bw_{N+1}\|\\
&+\|\mathcal{E}Bu_{N+1}-\mathcal{E}Bu_N\|\le\left(\frac{1}{2^2}+\cdots+\frac{1}{2^k}\right)\mu+\frac{\mu}{2}<\mu.
\end{aligned}
$$

As a consequence, the system (6.1) is approximately controllable. This ends the proof. □

7. Application

Consider

$$
\begin{cases}
D_{0+}^{\alpha,\frac{2}{3}}[x(\vartheta,\beta)-\int_0^\pi c(\beta,\mathfrak{u})x(\vartheta,\beta)d\beta]=\frac{\partial^2}{\partial\beta^2}x(\vartheta,\beta)+x(\vartheta-\sigma,\beta)\\
\qquad\qquad+\mathcal{G}(\vartheta,x(\vartheta-\sigma,\beta))+Bu(\vartheta,\beta),\ \vartheta\in(0,1],\\
x(\vartheta,0)=x(\vartheta,\pi)=0,\vartheta\ge0,\\
I_{0^+}^{\frac{1}{3}(1-\alpha)}x(0,\beta)=\hbar(\vartheta),\ \vartheta\in[-\sigma,0],\ \ \beta\in[0,\pi].
\end{cases}
\tag{7.1}
$$

The Hilfer fractional derivative is symbolized by $D_{0+}^{\alpha,\frac{2}{3}}$, whose order and type are $\frac{3}{2}$, $0\le\alpha\le1$ and $I_{0^+}^{\frac{1}{3}(1-\alpha)}$ is the Reimann-Liouoville integral of order $\frac{1}{3}(1-\alpha)$. The function $\mathcal{G}(\cdot,\cdot)\in L^2([0,\pi]\times[0,\pi],\mathbb{R}^+)$, for $\mathfrak{m}>0$.

Abstract form: Considering $\mathcal{A} : \mathcal{H} \to \mathcal{H},\ \mathcal{H} = L^2([0,\pi],\mathbb{R})$ which is defined as $\mathcal{A}v = v''$, $v \in D(\mathcal{A})$, where

$$D(\mathcal{A}) = v \in \mathcal{H} : v, v' \text{ are absolutely continuous},$$

and

$$D(\mathcal{A}) = v'' \in \mathcal{H},\ v(0) = v(\pi) = 0.$$

Also, $\mathcal{A}$ satisfies C_1, C_2 Then, we have

$$\mathcal{A}x = -\sum_{h=1}^{\infty} h^2 \langle x, \varsigma_h \rangle \varsigma_h,\ \vartheta \in D(\mathcal{A}), \tag{7.2}$$

where

$$\varsigma_h(\beta) = \sqrt{\frac{2}{\pi}} \sin(h\beta),\ h = 1, 2, \cdots .$$

For all $x \in \mathcal{H}$,

$$T(\vartheta)x = \sum_{h=1}^{\infty} e^{-h^2\vartheta} \langle x, \varsigma_h \rangle \varsigma_h,\ \ \|T(\cdot)\| \leq 1.$$

The function $\mathfrak{g} : [0, d] \times \to \mathcal{H}$ is defined by $\mathfrak{g}(\vartheta, x_\vartheta) = \int_0^\pi c(\beta, \mathfrak{u}) x(\vartheta, \beta) d\beta$.

Let $\frac{\partial}{\partial \beta} c(\beta, \mathfrak{u})$ be measurable, $c(0, u) = c(\pi, u) = 0$, and

$$\mathcal{L} = \sqrt{\int_0^\pi \int_0^\pi \Big[\frac{\partial}{\partial \beta} c(\beta, \mathfrak{u})\Big]^2 d\beta dz} < \infty,\ \ \int_0^\pi \int_0^\pi c^2(\beta, \mathfrak{u}) d\beta dz < \infty,$$

c is measurable.

Hence, $X_1(\vartheta) \in D(\mathcal{A}^{\frac{1}{2}})$ and $\|\mathcal{A}^{\frac{1}{2}}\|^2 \leq \mathcal{L}$.

Therefore,

$$\langle X_1(\vartheta), \varsigma_h \rangle = \int_0^\pi \varsigma_h(x) \int_0^\pi c(\beta, \mathfrak{u}) \mathfrak{u}(\beta) d\beta dx = \frac{1}{h} \sqrt{\frac{2}{\pi}} \Big\langle x(\vartheta), \frac{1}{\sec(h\vartheta)} \Big\rangle.$$

Let's define the B as

$$Bu(\vartheta) = \sum_{h=1}^{\infty} \hat{u}_h(\vartheta) \varsigma_h,\ \ u(\vartheta) = \sum_{h=1}^{\infty} u_h(\vartheta) \varsigma_h,$$

where $u_h(\vartheta) = \langle u(\vartheta, \varsigma_h) \rangle, h = 0, 1, 2, \cdots$.

$$\hat{u}_h(\vartheta) = \begin{cases} 0, 0 \leq \vartheta < 1 - \frac{1}{h^2}, \\ u_h(\vartheta), 1 - \frac{1}{h^2} \leq \vartheta \leq 1; \end{cases} \tag{7.3}$$

the reader can understand by $\|Bu(\cdot)\| \leq \|u(\cdot)\|$, B is bounded linear operator. The linear system of (7.1) is

$$\begin{cases} D_{0+}^{\alpha, \frac{2}{3}} [x_h(\vartheta) - \mathfrak{g}(\vartheta)] = u_h(\vartheta),\ \vartheta \in (0, 1],\ \beta \in \mathcal{K}, \\ I_{0^+}^{\frac{1}{3}(1-\alpha)} x_h(0) = \hbar(\vartheta),\ \vartheta \in [-\sigma, 0],\ \ \beta \in [0, \pi]. \end{cases} \tag{7.4}$$

Hence, for any given $f(\cdot)$

$$\rho = \int_0^1 (1-\mathfrak{r})^{-\frac{1}{3}} \mathcal{G}_{\frac{2}{3}}(1-\mathfrak{r}) f(\mathfrak{r}) d\mathfrak{r} = \sum_{h=1}^{\infty} \rho_h \varsigma_h, \ \rho_h = \langle \rho, \varsigma_h \rangle .$$

For any $f(\cdot)$ assume $\hat{u_h}$ as

$$\hat{u_h} = \frac{2h^2}{(1-e^{-2})} \rho_h e^{-h^2(1-\vartheta)}, \ -\frac{1}{h^2} \leq \vartheta \leq 1,$$

where

$$\rho_h = \int_{1-\frac{1}{h^2}}^1 \int_0^{\infty} (1-\mathfrak{r})^{\frac{-1}{3}} \vartheta N_{\frac{2}{3}}(\vartheta) e^{-h^2\vartheta(1-\vartheta)^{\frac{2}{3}}},$$

$$\int_0^1 (1-\mathfrak{r})^{-\frac{1}{3}} \mathcal{G}_{\frac{2}{3}}(1-\mathfrak{r}) Bu(\mathfrak{r}) d\mathfrak{r} = \int_0^1 (1-\mathfrak{r})^{-\frac{1}{3}} \mathcal{G}_{\frac{2}{3}}(1-\mathfrak{r}) f(\mathfrak{r}) d\mathfrak{r},$$

as a consequence, $\mathcal{F}_{\mathbf{6}}$ is fulfilled. Furthermore, we have

$$\begin{aligned} \|Bu(\cdot)\|^2 &= \sum_{h=1}^{\infty} \int_{1-\frac{1}{h^2}}^1 |\hat{u}_h(\vartheta)|^2 d\vartheta = (1-e^{-2})^{-1} \sum_{h=1}^{\infty} 2h^2 \rho_h^2 \\ &= \frac{3}{2}(1-e^{-2})^{-1} \sum_{h=1}^{\infty} (1-e^{-2h^2}) \int_0^1 |f_h(\vartheta)|^2 d\vartheta \leq \frac{3}{2}(1-e^{-2})^{-1} |f(\cdot)|^2. \end{aligned}$$

As a result, it can be observed that if the conditions $\mathcal{F}_{\mathbf{6}}$, is satisfied then (7.1) is approximately controllable on $\mathcal{H}$.

8. Conclusions

Our study investigates the existence and approximate controllability of HF neutral evolution equations with time delay. The Sequence method was used to derive the approximate controllability outcomes for HFD equations with time delay. An illustration is offered to support the analytical findings at the end and also given a filter diagram to represent the mild solution of the system with neutral term. Next, new research may use the sequence method with infinite delay to extend the Hilfer fractional stochastic differential evolution equations to approximate control results.

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Conflict of interest

Conflicts of interest are not present in this study.

References

1. D. D. Dai, T. T. Ban, Y. L. Wang, W. Zhang, The piecewise reproducing kernel method for the time variable fractional order advection-reaction-diffusion equations, *Therm. Sci.*, **25** (2021), 1261–1268. https://doi.org/10.2298/TSCI200302021D
2. A. Debbouche, V. Antonov, Approximate controllability of semilinear Hilfer fractional differential inclusions with impulsive control inclusion conditions in Hilbert spaces, *Chaos Soliton Fract.*, **102** (2017), 140–148. https://doi.org/10.1016/j.chaos.2017.03.023
3. K. Diethelm, *The analysis of fractional differential equations*, Lecture Notes in Mathematics, Springer, 2010. https://doi.org/10.1007/978-3-642-14574-2
4. C. Dineshkumar, R. Udhayakumar, V. Vijayakumar, K. S. Nisar, A. Shukla, A note on the approximate controllability of Sobolev type fractional stochastic integro-differential delay inclusions with order $1 < r < 2$, *Math. Comput. Simulat.*, **190** (2021), 1003–1026. https://doi.org/10.1016/j.matcom.2021.06.026
5. C. Dineshkumar, K. S. Nisar, R. Udhayakumar, V. Vijayakumar, New discussion about the approximate controllability of fractional stochastic differential inclusions with order $1 < r < 2$, *Asian J. Control*, 2021. https://doi.org/10.1002/asjc.2663
6. C. Dineshkumar, R. Udhayakumar, V. Vijayakumar, A. Shukla, K. S. Nisar, A note on approximate controllability for nonlocal fractional evolution stochastic integrodifferential inclusions of order with delay, *Chaos Soliton Fract.*, **153** (2021), 111565. https://doi.org/10.1016/j.chaos.2021.111565
7. K. M. Furati, M. D. Kassim, N. E. Tatar, Existence and uniqueness for a problem involving Hilfer fractional derivative, *Comput. Math. Appl.*, **64** (2012), 1616–1626. https://doi.org/10.1016/j.camwa.2012.01.009
8. F. D. Ge, H. C. Zhou, C. H. Kou, Approximate controllability of semilinear evolution equations of fractional order with nonlocal and impulsive conditions via an approximating technique, *Appl. Math. Comput.*, **275** (2016), 107–120. https://doi.org/10.1016/j.amc.2015.11.056
9. H. Gu, J. J. Trujillo, Existence of integral solution for evolution equation with Hilfer fractional derivative, *Appl. Math. Comput.*, **257** (2015), 344–354. https://doi.org/10.1016/j.amc.2014.10.083
10. C. H. He, C. Liu, J. H. He, H. M. Sedighi, A. Shokri, K. A. Gepreel, A fractal model for the internal temperature response of a porous concrete, *Appl. Math. Comput.*, **22** (2021), 71–77.
11. Y. T. Zuo, C. Liu, H. J. He, Fractal approach to mechanical and electrical properties of graphene/sic composites, *Facta Univ. Ser.: Mech. Eng.*, **19** (2021), 271–284. https://doi.org/10.22190/FUME201212003Z
12. J. H. He, G. M. Moatimid, M. H. Zekry, Forced nonlinear oscillator in a fractal space, *Facta Univ. Ser.: Mech. Eng.*, **20** (2022), 1–20. https://doi.org/10.22190/FUME220118004H
13. J. H. He, C. Liu, A modified frequency-amplitude formulation for fractal vibration systems, *Fractals*, 2022. https://doi.org/10.1142/S0218348X22500463
14. J. H. He, F. Y. Ji, Two-scale mathematics and fractional calculus for thermodynamics, *Therm. sci.*, **23** (2019), 2131–2133. https://doi.org/10.2298/TSCI1904131H

15. S. Ji, Approximate controllability of semilinear nonlocal fractional differential systems via an approximating method, *Appl. Math. Comput.*, **236** (2014), 43–53. https://doi.org/10.1016/j.amc.2014.03.027
16. J. W. He, Y. Liang, B. Ahmad, Y. Zhou, Nonlocal fractional evolution inclusions of order $\alpha \in (1, 2)$, *Mathematics*, **2019** (2019), 209. https://doi.org/10.3390/math7020209
17. R. Hilfer, *Application of fractional calculus in physics*, Singapore: World Scientific, 2000. https://doi.org/10.1142/3779
18. A. Haq, N. Sukavanam, Existence and approximate controllability of Riemann-Liouville fractional integrodifferential systems with damping, *Chaos Soliton Fract.*, **139** (2020), 110043. https://doi.org/10.1016/j.chaos.2020.110043
19. A Haq, Partial-approximate controllability of semi-linear systems involving two Riemann-Liouville fractional derivatives, *Chaos Soliton Fract.*, **157** (2022), 111923. https://doi.org/10.1016/j.chaos.2022.111923
20. K. Kavitha, V. Vijayakumar, R. Udhayakumar, N. Sakthivel, K. S. Nisar, A note on approximate controllability of the Hilfer fractional neutral differential inclusions with infinite delay, *Math. Methods Appl. Sci.*, **44** (2021), 4428–4447. https://doi.org/10.1002/mma.7040
21. K. Kavitha, V. Vijayakumar, R. Udhayakumar, C. Ravichandran, Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness, *Asian J. Control*, 2021. https://doi.org/10.1002/asjc.2549
22. K. Kavitha, V. Vijayakumar, A. Shukla, K. S. Nisar, R. Udhayakumar, Results on approximate controllability of Sobolev-type fractional neutral differential inclusions of Clarke subdifferential type, *Chaos Soliton Fract.*, **151** (2021), 111264. https://doi.org/10.1016/j.chaos.2021.111264
23. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.
24. V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, 2009.
25. J. Liang, H. Yang, Controllability of fractional integro-differential evolution equations with nonlocal conditions, *Appl. Math. Comput.*, **254** (2015), 20–29. https://doi.org/10.1016/j.amc.2014.12.145
26. X. Li, Z. Liu, C. C. Tisdell, Approximate controllability of fractional control systems with time delay using the sequence method, *Electron. J. Differ. Equ.*, **272** (2017), 1–11.
27. N. I. Mahmudov, Approximate controllability of evolution systems with nonlocal conditions, *Nonlinear Anal.*, **68** (2008), 536–46. https://doi.org/10.1016/j.na.2006.11.018
28. M. Mohan Raja, V. Vijayakumar, R. Udhayakumar, Results on the existence and controllability of fractional integro-differential system of order $1 < r < 2$ via measure of noncompactness, *Chaos Soliton Fract.*, **139** (2020), 110299. https://doi.org/10.1016/j.chaos.2020.110299
29. K. S. Nisar, V. Vijayakumar, Results concerning to approximate controllability of non-densely defined Sobolev-type Hilfer fractional neutral delay differential system, *Math. Methods Appl. Sci.*, **44** (2021), 13615–13632. https://doi.org/ 10.1002/mma.7647
30. K. S. Nisar, K. Jothimani, K. Kaliraj, C, Ravichandran An analysis of controllability results for nonlinear Hilfer neutral fractional derivatives with non-dense domain, *Chaos Soliton Fract.,* **146** (2021), 110915. https://doi.org/10.1016/j.chaos.2021.110915

31. R. Patel, A. Shukla, S. S. Jadon, Existence and optimal control problem for semilinear fractional order $(1, 2]$ control system, *Math. Methods Appl. Sci.*, 2020. https://doi.org/10.1002/mma.6662
32. A. Pazy, *Semigroups of linear operators and applications to partial differential equations,* New York Springer, 1983. https://doi.org/10.1007/978-1-4612-5561-1
33. I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to method of their solution and some of their applications*, San Diego: Academic Press, 1999.
34. R. Sakthivel, R. Ganesh, S. M. Anthoni, Approximate controllability of fractional nonlinear differential inclusions, *Appl. Math. Comput.*, **225** (2013), 708–717. https://doi.org/10.1016/j.amc.2013.09.068
35. A. Shukla, N. Sukavanam, D. N. Pandey, Controllability of semilinear stochastic control system with finite delay, *IMA J. Math. Control Inf.*, **35** (2018), 427–449. https://doi.org/10.1093/imamci/dnw059
36. A. Shukla, N. Sukavanam, D. N. Pandey Complete controllability of semi-linear stochastic system with delay, *Rend. Circ. Mat. Palermo*, **64** (2015), 209–220. https://doi.org/10.1007/s12215-015-0191-0
37. A. Shukla, N. Sukavanam, D. N. Pandey, Approximate controllability of semilinear fractional stochastic control system, *Asian-Eur. J. Math.*, **11** (2018), 1850088. https://doi.org/10.1142/S1793557118500882
38. A. Shukla, N. Sukavanam, D. N. Pandey, Controllability of semilinear stochastic system with multiple delays in control, *IFAC Proc. Vol.*, **47** (2014), 306–312. https://doi.org/10.3182/20140313-3-IN-3024.00107
39. A. Shukla, R. Patel, Existence and optimal control results for second-order semilinear system in Hilbert spaces, *Circuits Syst. Signal Proccess.*, **40** (2021), 4246–4258. https://doi.org/10.1007/s00034-021-01680-2
40. A. Shukla, R. Patel, Controllability results for fractional semilinear delay control systems, *J. Appl. Math. Comput.*, **65** (2021), 861–875. https://doi.org/10.1007/s12190-020-01418-4
41. A. Shukla, N. Sukavanam, D. N. Pandey, Approximate controllability of semilinear fractional control systems of order $\alpha \in (1, 2)$, *2015 Proceedings of the Conference on Control and its Applications*, 2015, 175–180. https://doi.org/10.1137/1.9781611974072.25
42. V. Vijayakumar, C. Ravichandran, K. S. Nisar, K. D. Kucche, New discussion on approximate controllability results for fractional Sobolev type Volterra-Fredholm integro-differential systems of order $1 < r < 2$, *Numer. Methods Partial Differ. Equ.*, 2021. https://doi.org/10.1002/num.22772
43. R. Subashini, K. Jothimani, K. S. Nisar, C. Ravichandran, New results on nonlocal functional integro-differential equations via Hilfer fractional derivative, *Alex. Eng. J.*, **5** (2020), 2891–2899. https://doi.org/10.1016/j.aej.2020.01.055
44. V. Vijayakumar, S. K. Panda, K. S. Nisar, H. M. Baskonus, Results on approximate controllability results for second-order Sobolev-type impulsive neutral differential evolution inclusions with infinite delay, *Numer. Methods Partial Differ. Equ.*, **37** (2021), 1200–1221. https://doi.org/10.1002/num.22573

45. V. Vijayakumar, R. Udhayakumar, C. Dineshkumar, Approximate controllability of second order nonlocal neutral differential evolution inclusions, *IMA J. Math. Control Inf.*, **38** (2021), 192–210. https://doi.org/10.1093/imamci/dnaa001
46. V. Vijayakumar, R. Murugesu, Controllability for a class of second order evolution differential inclusions without compactness, *Appl. Anal.*, **98** (2019), 1367–1385. https://doi.org/10.1080/00036811.2017.1422727
47. V. Vijayakumar, R. Murugesu, M. Tamil Selvan, Controllability for a class of second order functional evolution differential equations without uniqueness, *IMA J. Math. Control Inf.*, **36** (2019), 225–246. https://doi.org/10.1093/imamci/dnx048
48. V. Vijayakumar, R. Udhayakumar, S. K. Panda, K. S. Nisar, Results on approximate controllability of Sobolev type fractional stochastic evolution hemivariational inequalities, *Numer. Methods Partial Differ. Equ.*, 2020. https://doi.org/10.1002/num.22690
49. K. L. Wang, S. W. Yao, He's fractional derivative for the evolution equation, *Therm. Sci.*, **24** (2020), 2507–2513. https://doi.org/10.2298/TSCI2004507W
50. C. Han , Y. L. Wang, Z. Y. Li, Numerical solutions of space fractional variable-coefficient kdv-modified kdv equation by fourier spectral method, *Fractals*, **29** (2021), 2150246. https://doi.org/10.1142/S0218348X21502467
51. J. R. Wang, Y. R. Zhang, Nonlocal initial value problems for differential equation with Hilfer fractional derivative, *Appl. Math. Comput.*, **266** (2015), 850–859. https://doi.org/10.1016/j.amc.2015.05.144
52. W. K. Williams, V. Vijayakumar, R. Udhayakumar, K. S. Nisar, A new study on existence and uniqueness of nonlocal fractional delay differential systems of order $1 < r < 2$ in Hilbert spaces, *Numer. Methods Partial Differ. Equ.*, **37** (2021), 949–961. https://doi.org/10.1002/num.22560
53. F. Xianlong, L. Xingbo, Controllability of non-densely defined neutral functional differential systems in abstract space, *Chin. Ann. Math. Ser. B*, **28** (2007), 243–252. https://doi.org/10.1002/num.22560
54. Y. Zhou, *Basic theory of fractional differential equations*, Singapore: World Scientific, 2014. https://doi.org/10.1142/10238
55. Y. Zhou, *Fractional evolution equations and inclusions*, Analysis and Control, Elsevier, 2015. https://doi.org/10.1016/C2015-0-00813-9
56. Y. Zhou, F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, *Nonlinear Anal.*, **11** (2010), 4465–4475. https://doi.org/10.1016/j.nonrwa.2010.05.029
57. Y. Zhou, L. Zhang, X. H. Shen, Existence of mild solutions for fractional evolution equations, *J. Integral Equ. Appl.*, **25** (2013), 557–585. https://doi.org/10.1216/JIE-2013-25-4-557
58. S. Zahoor, S. Naseem, Design and implementation of an efficient FIR digital filter, *Cogent Eng.*, **4** (2017), 1323373. https://doi.org/10.1080/23311916.2017.1323373

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