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Research article

Semi-topological properties of the Marcus-Wyse topological spaces

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Abstract: Since the Marcus-Wyse (MW-, for brevity) topological spaces play important roles in the fields of pure and applied topology (see Remark 2.2), the paper initially proves that the MW-topological space satisfies the semi-T_3-separation axiom. To do this work more efficiently, we first propose several techniques discriminating between the semi-openness or the semi-closedness of a set in the MW-topological space. Using this approach, we suggest the condition for simple MW-paths to be semi-closed, which confirms that while every MW-path P with $|P| \geq 2$ is semi-open, it may not be semi-closed. Besides, for each point $p \in \mathbb{Z}^2$ the smallest open neighborhood of the point p is proved to be a regular open set so that it is semi-closed. Note that the MW-topological space is proved to satisfy the semi-T_3-separation axiom, i.e., it is proved to be a semi-T_3-space so that we can confirm that it also satisfies an s-T_3-separation axiom. Finally, we prove that the semi-T_3-separation axiom is a semi-topological property.

Keywords: semi-open; semi-closed; semi-T_3-space; Alexandroff space; semi-regular; semi-T_1-space; regular open; Marcus-Wyse topology
Mathematics Subject Classification: 54A05, 54D10, 54F05, 54C08, 54C10, 54F65

1. Introduction

In relation to the study of semi-separation axioms, many concepts were established such as a regular open set [40], a semi-open set [29], a semi-regular set [11, 35], a semi-closed set [11, 35], an s-regular set [31, 32], and so on. Furthermore, based on these notions, various types of mappings were developed such as a semi-continuous mapping [37], an irresolute map [9], a pre-semi-open mapping [9], a semi-homeomorphism (a bijection such that the images of semi-open sets are semi-open (or pre-semi-open mapping) and inverses of semi-open sets are semi-open (or irresolute mapping) [9] and so forth. Since both separation axioms and semi-separation axioms play important roles in modern mathematics including pure and applied topology such as digital topology, computational topology, and so on, many works dealing with these axioms include the papers [3, 4, 6–12, 14, 17, 23, 26, 29–32, 35–37]. In detail, the semi-T_i-separation axioms, where $i = 0, \frac{1}{2}, 1, 2$, etc (see [6, 29, 31]), are obtained from the definitions of the usual separation axioms T_i after replacing open sets by semi-open ones. Hence the axiom T_i obviously implies the axiom semi-T_i [10] but the converse does not hold. Moreover, in case $i \leq j$, the axiom semi-T_j implies the axiom semi-T_i, and the converse does not hold [6]. As usual, a property is called a semi-topological property if the property is preserved by semi-homeomorphisms [9, 34]. Then each of the semi-T_i-axioms, $i \in \{0, \frac{1}{2}, 1, 2\}$, is proved to be the semi-topological property [31]. Besides, the typical regularity, normality, complete normality, T_3, T_4, T_5, paracompactness, metrizability are not semi-topological properties [9].

The paper [41] introduced a topology on $\mathbb{Z}^2$, called the Marcus-Wyse topology, which can be used in applied topology as well as computer sciences. After that, many works dealt with the structure and used it in applied sciences (see Remark 2.2). Since we will often use the name " Marcus-Wyse" in this paper, hereafter we will use the terminology "MW-" instead of " Marcus-Wyse", if there is no danger of ambiguity. Furthermore, we use the notation " $\subset$" (resp. $|X|$) to denote a 'proper subset or equal' (resp. the cardinality of the given set X). Besides, $\mathbb{Z}_o$ is used for indicating the set of odd integers. In addition, the notation " :=" will be used to introduce a new term.

The aim of the present paper is initially to prove that the MW-topological space satisfies the semi-T_3-separation axiom. In detail, the following topics will be treated.

• Study of various properties of the union and intersection of semi-closed and semi-open sets in the MW-topological space.
• Establishment of some techniques of discriminating between semi-openness and semi-closedness of subsets of the MW-topological space.
• Investigation of the MW-topological space with respect to the semi-T_3-separation axiom.
• Examination of the semi-topological property of the semi-T_3-separation axiom.

This paper is organized as follows: Section 2 provides some basic notions associated with the MW-topology. Section 3 studies various properties of semi-closed and semi-open sets in the MW-topological space. Section 4 studies both semi-openness and semi-closedness of some sets in the MW-topological space. Section 5 investigates some properties relating to the study of the semi-T_3-separation axiom of MW-topological spaces. Section 6 proves a semi-topological property of the semi-T_3-separation axiom. Section 7 concludes the paper with summary and a further work.

2. Preliminaries

For a nonempty binary symmetric relation set (X, π), we say that X is π-connected [25] if for any two elements x and y of X there is a finite sequence $(x_i)_{i \in [0,l]_{\mathbb{Z}}}$ of elements in X such that $x = x_0$, $y = x_l$ and $(x_j, x_{j+1}) \in \pi$ for $j \in [0, l-1]_{\mathbb{Z}}$, where for distinct integers $a, b \in \mathbb{Z}$ $[a, b]_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, a \leq x \leq b\}$. More precisely, a topological space (X, T) is called an Alexandroff space for each $x \in X$, the intersection of all open sets of X containing x (denoted by $SN_T(x)$) is T-open in X [1]. Let us recall some properties of MW-topological spaces associated with the semi-separation axioms. As an Alexandroff topological space [1, 2], the MW-topological space, denoted by $(\mathbb{Z}^2, \gamma)$, was established and many works investigated various properties of it including the papers [16, 18, 21, 23, 24, 41].

In relation to the study of digital images in $\mathbb{Z}^2$, let us recall some basic notations named by the digital k-neighborhood of a given point $p \in \mathbb{Z}^2$, as follows:

For a point $p := (x, y) \in \mathbb{Z}^2$ we follow the notation [38].

$$N_4(p) := \{(x \pm 1, y), p, (x, y \pm 1)\}. \tag{2.1}$$

called the 4-neighborhood of a given point $p := (x, y) \in \mathbb{Z}^2$.

Motivated by the 4-adjacency of $\mathbb{Z}^2$ of (2.1), the $k(m, n)$-adjacency relations of $\mathbb{Z}^n$ were established, as follows (see [15] in detail). The papers [15, 17, 21] initially developed some k-adjacency relations for high dimensional digital images $(X, k), X \subset \mathbb{Z}^n$ (see also (2.2) below). More precisely, the digital k-adjacency relations (or digital k-connectivity) for $X \subset \mathbb{Z}^n, n \in \mathbb{N}$, were initially developed in [17] (see also [15, 16, 25]), as follows:

For a natural number t, $1 \leq t \leq n$, the distinct points $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n$ are $k(t, n)$-adjacent if at most t of their coordinates differ by ± 1 and the others coincide.

According to this statement, the $k(t, n)$-adjacency relations of $\mathbb{Z}^n, n \in \mathbb{N}$, were formulated [17] (see also [21, 28]) as follows:

$$k := k(t, n) = \sum_{i=1}^{t} 2^i C_i^n, \text{where } C_i^n := \frac{n!}{(n-i)!\ i!}. \tag{2.2}$$

For instance, the following are obtained [15, 21]:

$$(n, t, k) \in \left\{\begin{array}{l}(1, 1, 2), \\ (2, 1, 4), (2, 2, 8), \\ (3, 1, 6), (3, 2, 18), (3, 3, 26); \text{and} \\ (4, 1, 8), (4, 2, 32), (4, 3, 64), (4, 4, 80).\end{array}\right\}$$

Using the k-adjacency relations of $\mathbb{Z}^n$ in (2.2), $n \in \mathbb{N}$, we will call the pair (X, k) a digital image on $\mathbb{Z}^n$, where $X \subset \mathbb{Z}^n$. Besides, these k-adjacency relations can be essential to studying digital products with normal adjacencies [15] and pseudo-normal adjacencies [27] and calculating digital k-fundamental groups of digital products [20, 21].

Let us now recall basic concepts on the MW-topology. The MW-topology on $\mathbb{Z}^2$, denoted by $(\mathbb{Z}^2, \gamma)$, is induced by the set $\{U(p) \mid p \in \mathbb{Z}^2\}$ in (2.3) below as a base [41], where for each point $p = (x, y) \in \mathbb{Z}^2$

$$U(p) := \left\{\begin{array}{l}N_4(p) \text{ if } x + y \text{ is even, and} \\ \{p\} : \text{ else.}\end{array}\right\} \tag{2.3}$$

In relation to the further statement of a point in $\mathbb{Z}^2$, in the paper we call a point $p = (x_1, x_2)$ *double even* if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; *even* if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and *odd* if $x_1 + x_2$ is an odd number [39].

In all subspaces of $(\mathbb{Z}^2, \gamma)$ of Figures 1–5, the symbols $\Diamond$ and $\bullet$ mean a *double even or even point* and an *odd point*, respectively. In view of (2.3), we can obviously obtain the following: Under $(\mathbb{Z}^2, \gamma)$ the singleton with either a double even point or an even point is a closed set. In addition, the singleton with an odd point is an open set. Besides, for a subset $X \subset \mathbb{Z}^2$, the subspace induced by $(\mathbb{Z}^2, \gamma)$ is obtained, denoted by (X, γ_X) and called an MW-topological space. It is clear that (X, γ_X) is an Alexandroff space.

In terms of this perspective, we clearly observe that the *smallest (open) neighborhood* of the point $p := (p_1, p_2)$ of $\mathbb{Z}^2$, denoted by $SN_\gamma(p) \subset \mathbb{Z}^2$, is the following [18]:

$$SN_\gamma(p) := \begin{Bmatrix} \{p\} \text{ if } p \text{ is an odd point} \\ N_4(p) \text{ if } p \text{ is a double even or even point.} \end{Bmatrix} \tag{2.4}$$

Hereafter, in (X, γ_X), for a point $p \in X$ we use the notation $SN_\gamma(p) := SN_\gamma(p) \cap X$ [18] for short. Using the smallest open set of (2.4), the notion of MW-adjacency in $(\mathbb{Z}^2, \gamma)$ is defined, as follows: For distinct points $p, q \in (\mathbb{Z}^2, \gamma)$, we say that p is MW-adjacent to q [16] if

$$p \in SN_\gamma(p) \text{ or } q \in SN_\gamma(q)$$

In view of the properties of (2.1) and (2.4), we obviously obtain the following: Given the point $p := (p_1, p_2)$ of $\mathbb{Z}^2$, the closure of the singleton $\{p\}$ is obtained and denoted by $Cl_\gamma(\{p\}) \subset \mathbb{Z}^2$, as follows [16]:

$$Cl_\gamma(\{p\}) := \begin{Bmatrix} \{p\} \text{ if } p \text{ is a double even or even point,} \\ N_4(p) \text{ if } p \text{ is an odd point.} \end{Bmatrix} \tag{2.5}$$

Hereinafter, when studying the MW-topological space, we will use the term 'Cl' for brevity instead of Cl_γ if there is no danger of confusion.

Definition 2.1. *[18] Let $X := (X, \gamma_X)$ be an MW-topological space. Then we define the following:*
(1) We say that an MW-path from x to y in X is a sequence $(p_i)_{i\in[0,l]_\mathbb{Z}} \subset X, l \in \mathbb{N}$, in X such that $p_0 = x$, $p_l = y$ and each point p_i is MW-adjacent to p_{i+1} and $i \in [0, l-1]_\mathbb{Z}$. The number l is the length of this path. In particular, a singleton in $(\mathbb{Z}^2, \gamma)$ is assumed to be an MW-path.
(2) Distinct points $x, y \in X$ are called MW-path connected (or MW-connected) if there is a finite MW-path $(p_0, p_1, ..., p_m)$ on X with $p_0 = x$ and $p_m = y$. For arbitrary points $x, y \in X$, if there is an MW-path $(p_i)_{i\in[0,m]_\mathbb{Z}} \subset X$ such that $p_0 = x$ and $p_m = y$, then we say that X is MW-path connected (or MW-connected).
(3) A simple MW-path in X means a finite MW-path $(p_i)_{i\in[0,m]_\mathbb{Z}}$ in X such that the points p_i and p_j are MW-adjacent if and only if $|\,i - j\,| = 1$.

It is well known that each of the subspaces of $(\mathbb{Z}^2, \gamma)$ is a semi-$T_{\frac{1}{2}}$ space [5] and further, they are Alexandroff spaces with the axiom T_0.

When studying digital objects X in $\mathbb{Z}^2$, the properties of (2.4) and (2.5) enable us to get the following utilities of the MW-topological structure of X.

Remark 2.2. *(Utilities of the MW-topological structure)*
(1) When studying a self-homeomorphism of $(\mathbb{Z}^2, \gamma)$, we should consider the following map

$$\begin{Bmatrix} h : (\mathbb{Z}^2, \gamma) \to (\mathbb{Z}^2, \gamma) \text{ defined by:} \\ \text{for each point } x := (x_1, x_2) \in \mathbb{Z}^2, \\ h(x) = (x_1 + t_1, x_2 + t_2), \\ \text{where } t_i \in \mathbb{Z}_o \text{ for each } i \in [1, 2]_\mathbb{Z}, \text{ or} \\ h(x) = (x_1 + 2m_1, x_2 + 2m_2), \\ \text{for some } m_i \in \mathbb{Z}, i \in M \subset [1, 2]_\mathbb{Z}. \end{Bmatrix} \tag{2.6}$$

Since the modern electronic devices are usually operated on the finite digital planes with more than ten million pixels to support the high-level display resolution, the mapping of (2.6) can be very admissible. At the moment, note that the following map g cannot be a homeomorphism, where

$$\left\{\begin{array}{l} g : (\mathbb{Z}^2, \gamma) \to (\mathbb{Z}^2, \gamma) \text{ defined by:} \\ \text{for each point } x := (x_1, x_2) \in \mathbb{Z}^2, \\ g(x) = (x_1 + t_1, x_2 + t_2) \\ \text{such that there is at least } t_i \in \mathbb{Z}_o, i \in M \subsetneq [1, 2]_{\mathbb{Z}}. \end{array}\right\} \tag{2.7}$$

For instance, $g : (\mathbb{Z}^2, \gamma) \to (\mathbb{Z}^2, \gamma)$ *defined by* $g(x_1, x_2) = (x_1 + 2m_1 + 1, x_2 + 2m_2), m_1, m_2 \in \mathbb{Z}$ *cannot be a homeomorphism. Meanwhile, it is clear that the map* $h : (\mathbb{Z}^2, \gamma) \to (\mathbb{Z}^2, \gamma)$ *defined by* $h(x_1, x_2) = (x_1 + 2m_1 + 1, x_2 + 2m_2 + 1)$ *or* $(x_1 + 2m_1, x_2 + 2m_2)$, $m_1, m_2 \in \mathbb{Z}$*, is a homeomorphism.*
(2) Since the MW-topological structure is one of the fundamental frames, motivated by this structure, some more generalized topological structures on $\mathbb{Z}^n$ *can be established [19].*
(3) Based on the MW-topological structure of $\mathbb{Z}^2$*, we can obtain the digital* 4*-connectivity induced by the given topological structure [18]. In detail, for distinct elements* $x, y \in (\mathbb{Z}^2, \gamma)$*, they are MW-adjacent if* $x \in SN_\gamma(y)$ *or* $y \in SN_\gamma(x)$ *[18]. Namely, the MW-adjacency is equivalent to the* 4*-adjacency of* $\mathbb{Z}^2$ *as in (2.2).*
(4) When digitizing a set X in the 2*-dimensional real space with respect to the MW-topological structure, we can use some local rule in [21] to obtain its digitized set* $D_\gamma(X) \subset \mathbb{Z}^2$ *and finally use it in the fields of mathematical morphology, rough set theory, digital geometry [21], and so on.*

3. Techniques of discriminating between a semi-open and a semi-closed set

This section first recalls the notions of a semi-open and a semi-closed set. Namely, a subset A of a topological space (X, T) is said to be *semi-open* if there is an open set O in (X, T) such that $O \subset A \subset Cl(O)$. Besides, we say that a subset B of a topological space (X, T) is *semi-closed* if the complement of B in X (or B^c) is semi-open in (X, T). Then it turns out that a subset A of (X, T) is semi-open if and only if $A \subset Cl(Int(A))$ [29] and a subset B of (X, T) is semi-closed if and only if $Int(Cl(B)) \subset B$) [8]. Hence it is clear that an empty set both semi-open and semi-closed. Besides, "open" (resp. "closed") is stronger than "semi-open" (resp. "semi-closed"). The notions of semi-openness and semi-closedness enable us to get the following [11, 29, 35]:
($\star$1) Given two semi-open sets, the intersection of them need not be semi-open.
($\star$2) Given two semi-closed sets, the union of them need not be semi-closed.
($\star$3) Given two semi-open sets, the union of them is semi-open.
($\star$4) Given two semi-closed sets, the intersection of them is semi-closed.

Remark 3.1. *In* $(\mathbb{Z}^2, \gamma)$*, we obtain the following:*
(1) The singleton $\{p\}$ *is both semi-closed and semi-open, where p is an odd point. Namely,* $\mathbb{Z}^2 \setminus \{p\}$ *is both semi-closed and semi-open, where p is an odd point.*
(2) The singleton $\{q\}$ *is not semi-open but semi-closed, where q is a double even or even point. Namely,* $\mathbb{Z}^2 \setminus \{q\}$ *is semi-open, where q is a double even or even point.*
(3) For distinct elements $a, b \in \mathbb{Z}$*, the Cartesian product* $[a, b]_{\mathbb{Z}} \times \{c\}$*, where* $c \in \mathbb{Z}$*, e.g.,* $[a, b]_{\mathbb{Z}} \times \{0\}$*, is both semi-closed and semi-open.*

Proof. The proofs of (1) and (2) are straightforward.
(3) Regardless of the choice of a, b, and c in $\mathbb{Z}$, any set $X := [a, b]_{\mathbb{Z}} \times \{c\}$ can be assumed as a finite sequence such as $X := (x_1, x_2, \cdots, x_n)$ in $(\mathbb{Z}^2, \gamma)$ such that $\{x_i\}$ is an open set and $\{x_{i+1}\}$ is a closed set (or $\{x_i\}$ is a closed set and $\{x_{i+1}\}$ is an open set) in $(\mathbb{Z}^2, \gamma)$, $i \in [1, n_1]_{\mathbb{Z}}$. Hence X is obviously semi-open. Besides, $\mathbb{Z}^2 \setminus X$ is also semi-open so that X is also semi-closed in $(\mathbb{Z}^2, \gamma)$. □

Unlike the property of ($\star$2) above, we obtain the following property of the union of two semi-closed sets in $(\mathbb{Z}^2, \gamma)$.

Theorem 3.2. *In a topological space (X, T), assume two semi-closed sets $B_i, i \in \{1, 2\}$, such that $Cl(B_1) \cap Cl(B_2) = \emptyset$. Then the union of them is semi-closed.*

Proof. It is clear that the following properties hold [33]

$$\begin{cases} Int(Cl(B_1 \cup B_2)) = Int(Cl(B_1) \cup Cl(B_2)) \text{ and} \\ Int(Cl(B_1)) \cup Int(Cl(B_2)) \subset Int(Cl(B_1) \cup Cl(B_2)), \end{cases} \tag{3.1}$$

$$Int(Cl(B_1) \cup Cl(B_2)) \text{ need not be a subset of } Int(Cl(B_1)) \cup Int(Cl(B_2)).$$

However, with the hypothesis, we now prove the identity

$$Int(Cl(B_1) \cup Cl(B_2)) = Int(Cl(B_1)) \cup Int(Cl(B_2)). \tag{3.2}$$

Namely, in case $Cl(B_1) \cap Cl(B_2) = \emptyset$, in view of (3.1), we only need to prove the following:

$$Int(Cl(B_1) \cup Cl(B_2)) \subset Int(Cl(B_1)) \cup Int(Cl(B_2)). \tag{3.3}$$

To be precise, if $x \in Int(Cl(B_1) \cup Cl(B_2))$, then there is an open set U in (X, T) such that

$$p \in U \subset Cl(B_1) \cup Cl(B_2). \tag{3.4}$$

Owing to the given hypothesis, i.e.,

$$Cl(B_1) \cap Cl(B_2) = \emptyset, \tag{3.5}$$

we have

$$U \subset Cl(B_1) \text{ or } U \subset Cl(B_2). \tag{3.6}$$

The former implies that $p \in U \subset Int(Cl(B_1))$ and the latter supports $p \in U \subset Int(Cl(B_2))$. Thus we have

$$p \in Int(Cl(B_1)) \cup Int(Cl(B_2)).$$

By (3.1) and (3.3), we have the identity of (3.2) so that we have

$$Int(Cl(B_1 \cup B_2)) = Int(Cl(B_1)) \cup Int(Cl(B_2)) \subset B_2 \cup B_2,$$

which leads to the semi-closedness of $B_1 \cup B_2$. □

Example 3.1. *(1) Let us consider the two sets B_1 and B_2 in $(\mathbb{Z}^2, \gamma)$, where*

$$\begin{cases} B_1 := \{b_0 = (-2, 0), b_1 = (-1, 0), b_2 = (0, -1)\}, \text{ and} \\ B_2 := \{d_0 = (2, 0), d_1 = (1, 0), d_2 = (0, 1)\}. \end{cases} \tag{3.7}$$

Note that each $B_i, i \in \{1, 2\}$, are not MW-paths. Even though each $B_i, i \in \{1, 2\}$, are semi-closed in $(\mathbb{Z}^2, \gamma)$, the union $B_1 \cup B_2$ is not semi-closed in $(\mathbb{Z}^2, \gamma)$ since the point $p := (0, 0)$ has $SN_\gamma(p)$ such that $SN_\gamma(p) \subset Cl(B_1) \cup Cl(B_2)$ (see Figure 1(a) and (b)) so that $Int(Cl(B_1 \cup B_2)) \not\subseteq B_1 \cup B_2$.

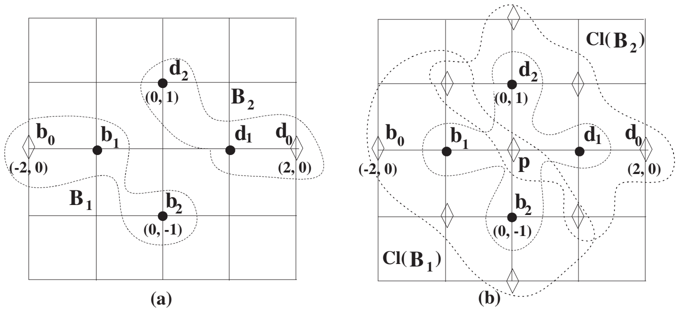

Figure 1. Configuration of the non-semi-closedness of the set $B_1 \cup B_2$ in (a) because the point $p := (0, 0)(\notin B_1 \cup B_2)$ has the set $SN_\gamma(p) \subset Cl(B_1 \cup B_2)$ (see (b)). This means $Int(Cl(B_1 \cup B_2)) \nsubseteq B_1 \cup B_2$, i.e., $B_1 \cup B_2$ is not semi-closed in $(\mathbb{Z}^2, \gamma)$.

Theorem 3.3. *In $(\mathbb{Z}^2, \gamma)$, assume a non-empty set $B \subset \mathbb{Z}^2$. B is semi-closed in $(\mathbb{Z}^2, \gamma)$ if and only if there are at least $q \in N_4(p) \setminus \{p\}$ such that $q \notin B$ whenever $p \notin B$, where p is a double even or even point.*

Proof. ($\Rightarrow$) Let $B(\subset \mathbb{Z}^2)$ be semi-closed in $(\mathbb{Z}^2, \gamma)$ and $p \notin B$, where p is a double even or even point. By contrary, suppose that each $q \in N_4(p) \setminus \{p\}$ belong to B. Then there is $SN_\gamma(p)$ in $(\mathbb{Z}^2, \gamma)$ such that

$$SN_\gamma(p) = N_4(p) \subset Int(Cl(B)),$$

because $SN_\gamma(p) \subset Cl(B)$. Hence we have $Int(Cl(B)) \nsubseteq B$, since $p \notin B$ (see Figure 2(a), (b)). Thus B is not semi-closed, which invokes a contradiction to the hypothesis of the semi-closedness of B.
($\Leftarrow$) In case $Int(Cl(B)) = \emptyset$, the proof is straightforward. Thus we may assume $Int(Cl(B)) \neq \emptyset$. To prove $Int(Cl(B)) \subset B$, take an arbitrary element $x \in Int(Cl(B))$. Since there is $SN_\gamma(x)$ such that $SN_\gamma(x) \subset Cl(B)$, we need to consider the following two cases:
(Case 1) Assume that x is an odd point. Owing to the inclusion $SN_\gamma(x) \subset Cl(B)$, since we obtain

$$x \in Cl(B) \text{ and } SN_\gamma(x) = \{x\} \text{ so that } SN_\gamma(x) \cap B = \{x\} \cap B \neq \emptyset.$$

Hence it is clear that $x \in B$.
(Case 2) Assume that x is a double even or even point. Owing to the MW-topological structure and the given hypothesis, based on the property

$$x \in Int(Cl(B)) \Rightarrow SN_\gamma(x) \subset Cl(B) \text{ (see above)},$$

we now prove

$$SN_\gamma(x) \subset Cl(B) \Rightarrow SN_\gamma(x) \subset B. \tag{3.8}$$

Indeed, without the given condition, i.e.,

$$\begin{Bmatrix} \text{there are at least } q \in N_4(p) \setminus \{p\} \text{ such that } q \notin B, \\ \text{whenever } p \notin B, \text{where } p \text{ is a double even or even point,} \end{Bmatrix} \tag{3.9}$$

note that the property of (3.8) does not hold. For instance, for a double even point $p := (0,0)$ in $(\mathbb{Z}^2, \gamma)$, assume $B := \{y \mid y \in N_4(p) \setminus \{p\}\}$. Then it is obvious that while $SN_\gamma(p) \subset Cl(B)$, $SN_\gamma(p) \not\subset B$ because $p \notin B$ (see Figure 2(a), (b)).

However, using both the condition $SN_\gamma(p) \subset Cl(B)$ and the given condition of (3.9), let us prove the property of (3.8) with two cases according to the point $x \in SN_\gamma(p)$. More precisely, depending on the situation of the point $x \in SN_\gamma(p)$, i.e., x is an odd point or double even or even point, we prove the property of (3.8). For our purposes, we will equivalently represent the statement of (3.9) by using the contraposition of it, as follows (see Figure 2(a), (b)):

$$\begin{Bmatrix} \text{for all } q \in N_4(p) \setminus \{p\} \text{ we have } q \in B, \text{ then } p \in B, \\ \text{where } p \text{ is a double even or even point.} \end{Bmatrix} \tag{3.10}$$

This statement of (3.10) will be essentially used in the proof of (Case 2-2) below.
(Case 2-1) Consider the case in which $p \in SN_\gamma(x) \subset Cl(B)$, where p is an odd point, i.e., $p \neq x$. Since x is a double even or even point, we get

$$\{p\} = SN_\gamma(p) \subset SN_\gamma(x) \subset Cl(B). \tag{3.11}$$

Using a method similar to the proof of the (Case 1) above, we clearly obtain $p \in B$.
(Case 2-2) Consider the case in which $p \in SN_\gamma(x) \subset Cl(B)$, where p is a double even or even point, i.e., $p = x$. Owing to the statement of (3.9) (or in particular (3.10)) and the proof of (Case 2-1) above, since all of the odd points $q \in SN_\gamma(p) \setminus \{p\} = N_4(p) \setminus \{p\}$ belong to B, by (3.10), we have $p \in B$. Based on the proofs of (Case 2-1) and (Case 2-2), we complete the proof of (3.8). □

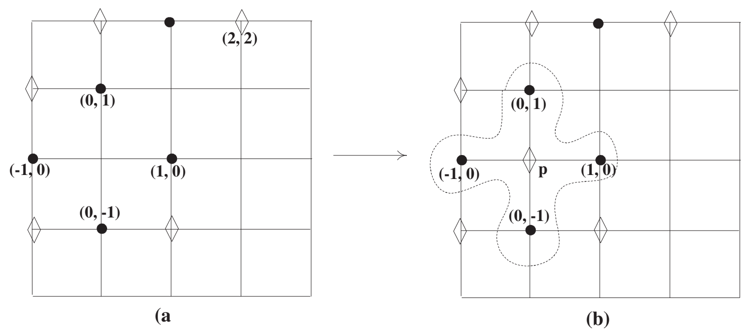

Figure 2. Given the set $B = \{(-1,0),(0,-1),(1,0),(0,1)\}$ as in (Case 2) of the proof of Theorem 3.3 (see (a)), we obtain $SN_\gamma(p) \subset Int(Cl(B))$ (see (b)) which leads $Int(Cl(B)) \not\subset B$ without the condition of (3.9). However, as in (Case 2-2), with the given condition of (3.9), we obtain $Int(Cl(B)) \subset B$.

Let us further establish some techniques to examine if a given set in $(\mathbb{Z}^2, \gamma)$ is semi-open or semi-closed. In $(\mathbb{Z}^2, \gamma)$, for a set $B \subset \mathbb{Z}^2$, we will take the following notation.

$$B_{op} := \{x \mid x \text{ is an odd point in } B\}. \tag{3.12}$$

Besides, owing to the topological structure of $(\mathbb{Z}^2, \gamma)$, we obviously have the following:

Remark 3.4. *In $(\mathbb{Z}^2, \gamma)$, we have the following:*
(1) For $x, y \in \mathbb{Z}^2$, $x \in SN_\gamma(y)$ if and only if $y \in Cl(x)$, i.e., $y \in Cl_\gamma(x)$(see the properties of (2.4) and (2.5) in the present paper).
(2) If B is an open set in $(\mathbb{Z}^2, \gamma)$, then there is an odd point $x \in B$ (see the property of (2.3)).
(3) The set B_{op} of (3.12) is an open set in $(\mathbb{Z}^2, \gamma)$.

Proof. (1) The proof is straightforward.
(2) Due to the topology $(\mathbb{Z}^2, \gamma)$ (see (2.4)), the proof is completed.
(3) Based on the topological structure of $(\mathbb{Z}^2, \gamma)$, the singleton $\{x\}$ consisting of the odd point $x \in \mathbb{Z}^2$ is equal to $SN_\gamma(x)$. Hence the set $B_{op} = \bigcup_{x \in B_{op}} \{x\}$ is an open set in $(\mathbb{Z}^2, \gamma)$. □

Given a set X in $(\mathbb{Z}^2, \gamma)$, to further examine if the set X is semi-open or semi-closed in $(\mathbb{Z}^2, \gamma)$, we now introduce the following two theorems that will be strongly used in discriminating between semi-openness and semi-closedness of subsets of the MW-topological space.

Theorem 3.5. *In $(\mathbb{Z}^2, \gamma)$, a non-empty set $B(\subset \mathbb{Z}^2)$ is semi-open if and only if each $x \in B$, $SN_\gamma(x) \cap B_{op} \neq \emptyset$.*

Before proving the assertion, if $B = \emptyset$, then the proof is straightforward.
Proof. ($\Rightarrow$) According to the choice of a point $x \in B$, we can consider the following two cases.
(Case 1) Assume that $x(\in B)$ is an odd point. From the hypothesis, we have $x \in B \subset Cl(Int(B))$ so that we obtain

$$SN_\gamma(x) \cap Int(B) \neq \emptyset. \tag{3.13}$$

Since $SN_\gamma(x) = \{x\}$, we obtain $x \in Int(B)$ and further, $x \in B_{op}$. Hence, owing to (3.13), we have $SN_\gamma(x) \cap B_{op} \neq \emptyset$.
(Case 2) Assume that $x \in B$ is a double even or even point. Owing to the hypothesis, we obtain $x \in Cl(Int(B))$ that leads to the following property as mentioned in (3.13).

$$SN_\gamma(x) \cap Int(B) \neq \emptyset.$$

Since $SN_\gamma(x) \cap Int(B)$ is a non-empty open set in $(\mathbb{Z}^2, \gamma)$, by Remark 3.4(2), we now take an odd point z in $(\mathbb{Z}^2, \gamma)$ such that

$$z \in SN_\gamma(x) \cap Int(B). \tag{3.14}$$

By the properties of (3.14), since $z \in Int(B) \subset B$, we have $z \in B_{op}$ (see Remark 3.4(2)) so that $z \in SN_\gamma(x) \cap B_{op} \neq \emptyset$. In addition, it is clear that the point z is indeed MW-adjacent to x.

($\Leftarrow$) According to the choice of a point $x \in B$, we can consider the following two cases.
(Case 1) For an arbitrary point $x \in B$, assume that x is an odd point in $(\mathbb{Z}^2, \gamma)$. Since $\{x\} = SN_\gamma(x)$, owing to the hypothesis of $SN_\gamma(x) \cap B_{op} \neq \emptyset$, we have $x \in B_{op}$, i.e., $\{x\} \cap B_{op} \neq \emptyset$. Furthermore, owing to the identity $SN_\gamma(x) = \{x\}$, by Remark 3.4(3), it is clear that

$$x \in B_{op} \Rightarrow \{x\} \subset Int(B) \Rightarrow x \in Cl(Int(B)). \tag{3.15}$$

(Case 2) For an arbitrary point $x \in B$, assume that x is a double even or even point in $(\mathbb{Z}^2, \gamma)$. Owing to the hypothesis, since $SN_\gamma(x) \cap B_{op} \neq \emptyset$, by Remark 3.4(2) and (3), there is an odd point z in $(\mathbb{Z}^2, \gamma)$

such that $z \in SN_\gamma(x) \cap B_{op}$ because $SN_\gamma(x) \cap B_{op}$ is an open set in $(\mathbb{Z}^2, \gamma)$. Hence we get $z \in SN_\gamma(x)$, by Remark 3.4(1), we have

$$x \in Cl(\{z\}) \subset Cl(Int(B)) \Rightarrow x \in Cl(Int(B)). \tag{3.16}$$

Owing to both (3.15) and (3.16), we obtain $B \subset Cl(Int(B))$ which completes the proof. □

Owing to the notion of semi-closedness, using Theorem 3.5, we obtain the following:

Theorem 3.6. *In $(\mathbb{Z}^2, \gamma)$, $B(\subset \mathbb{Z}^2)$ is semi-closed if and only if each $x \in \mathbb{Z}^2 \setminus B$, $SN_\gamma(x) \cap (\mathbb{Z}^2 \setminus B)_{op} \neq \emptyset$.*

As examples for Theorems 3.5 and 3.6, see the cases referred to in Remark 3.1(1)–(3).

As special cases of a semi-open and a semi-closed set, we have the following concepts [11].

Definition 3.7. *[40] In a topological space (X, T), a subset A is said to be regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$.*

Theorem 3.8. *In $(\mathbb{Z}^2, \gamma)$, for any point $p \in \mathbb{Z}^2$, $SN_\gamma(p)$ is not a regular closed set but a regular open set, i.e., it is semi-closed.*

Proof. We prove that each point $p(\in \mathbb{Z}^2)$ has the following property $Int(Cl(SN_\gamma(p))) = SN_\gamma(p)$.
(Case 1) Assume that the point p is a double even or even point. Then we obtain that $Int(Cl(SN_\gamma(p))) = SN_\gamma(p)$ (see Figure 3(a) and (b)).
(Case 2) Assume that the point p is an odd point. For an arbitrary element $x \in Int(Cl(SN_\gamma(p)))$, we have $SN_\gamma(x)$ in $(\mathbb{Z}^2, \gamma)$ such that $SN_\gamma(x) \subset Cl(SN_\gamma(p))$, which implies that $x \in SN_\gamma(p)$. Indeed, $x = p$ because $SN_\gamma(p) = \{p\}$. □

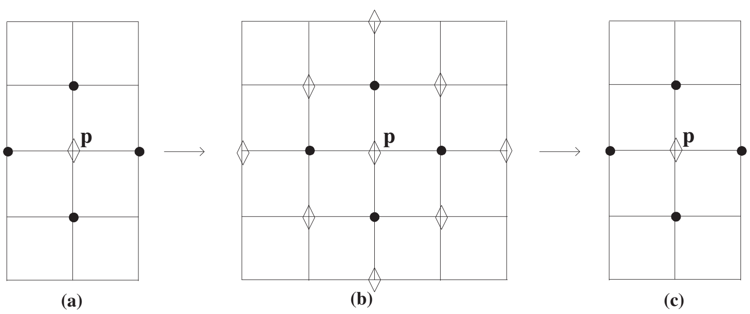

Figure 3. The process of explaining the regular openness of $SN_\gamma(p)$ referred to in Theorem 3.7, where p is a double even or even point. From the given $SN_\gamma(p)$ (see (a) and (Case 1) of the proof of Theorem 3.7), we obtain $Cl(SN_\gamma(p))$(see (b)) so that we can confirm $Int(Cl(SN_\gamma(p))) = SN_\gamma(p)$ (see (c)).

4. Some properties of semi-open and semi-closed sets of MW-paths

This section studies various properties of the semi-topological features of simple MW-paths, which will play an important role in studying the semi-T_3-separation axiom of the MW-topological space in Section 6. Hereinafter, an MW-path is assumed to be a non-empty set.

Lemma 4.1. *Assume a simple MW-path P in $(\mathbb{Z}^2, \gamma)$.*
(1) P is semi-closed whenever $|P| \leq 6$.
(2) P may not be semi-closed whenever $|P| \geq 7$.

Proof. (1) According to the length of a simple MW-path, using Theorem 3.6, the proof is completed. More precisely, assume a simple MW-path P with $|P| \leq 6$. Then, for each $x \in \mathbb{Z}^2 \setminus P$, we have the property $SN_\gamma(x) \cap (\mathbb{Z}^2 \setminus P)_{op} \neq \emptyset$, which implies the semi-closedness of P.
(2) In case of $|P| = 7$, based on Theorems 3.3 and 3.6, we can consider the following case (see Figure 4(a)). Let us consider the simple MW-path P as in Figure 4(a), i.e., $P := \{c_i \,|\, i \in [0,6]_\mathbb{Z}\}$ with $c_0 = (0,1), c_1 = (-1,1), c_2 = (-1,0), c_3 = (-1,-1), c_4 = (0,-1), c_5 = (1,-1), c_6 = (1,0)$. Then, by Theorem 3.6, we see that this P cannot be semi-closed because the point point $c := (0,0) \in \mathbb{Z}^2 \setminus P$ does not satisfy the condition of Theorem 3.6. Indeed, we obtain $Int(Cl(P)) \not\subseteq P$ owing to the point $c = (0,0) \in \mathbb{Z}^2 \setminus P$ having $SN_\gamma(c)$ such that $SN_\gamma(c) \subset Cl(P)$ so that $Int(Cl(P)) \not\subseteq P$ (see the set $SN_\gamma(c)$ in Figure 4(b)).

Meanwhile, consider another simple MW-path P with $|P| = 7$ as in Figure 4(c). Then this is semi-closed (see Theorem 3.6).
(3) Assume a simple MW-path P with $|P| \geq 7$ such that $P := (d_0, d_1, \cdots, d_{l-1})$ in $(\mathbb{Z}^2, \gamma)$ and P has the subsequence $X_1 := (d'_1, d'_2, d'_3, d'_4)$ of P whose each elements is odd point (e.g., in Figure 4(a) we may consider $X_1 = (c_0, c_2, c_4, c_6)$) and $X_1 \subset N_4(c), c \in \mathbb{Z}^2 \setminus P$. For our purpose, as an example for the subsequence X_1 of P, consider the set (c_0, c_2, c_4, c_6) in Figure 4(a). Then this P cannot be semi-closed. Indeed, the point c is a double even or even point (see the point c in Figure 4(b)). □

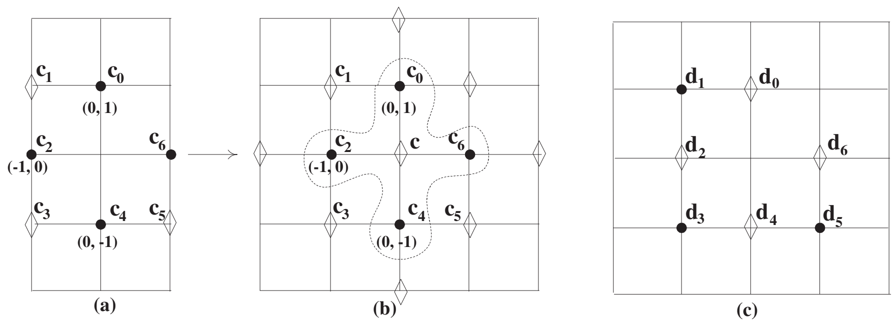

Figure 4. The objects of (a)–(c) are related to the proof of the semi-closedness of a simple MW-path in $(\mathbb{Z}^2, \gamma)$ stated in Lemma 4.1.

Motivated by Lemma 4.1(2), let us investigate some conditions for a simple MW-path to be semi-closed, as follows:

Theorem 4.2. *Assume a simple MW-path $P = (c_0, c_1, \cdots, c_{l-1})$ in $(\mathbb{Z}^2, \gamma)$ with $|P| \geq 7$ such that P does not have the subsequence $Y_1 := (c'_1, c'_2, c'_3, c'_4)$ whose each element is an odd point and $Y_1 \subset N_4(c), c \in \mathbb{Z}^2 \setminus P$. Then P is semi-closed.*

Before proving the assertion, we strongly need to recall the given hypothesis. Without the hypothesis, as mentioned in Lemma 4.1, the given path P with $|P| \geq 7$ may not be semi-closed. For instance, in Figure 4(a), the subsequence $Y_1 = (c_0, c_2, c_4, c_6)$ makes the given MW-path $P := (c_i)_{i \in [0,6]_\mathbb{Z}}$ non-semi-closed.

Proof. Assume a simple MW-path with l elements, $l \geq 7$, say $P = (c_0, c_1, \cdots, c_{l-1})$. Then, note that P consists of only double even or even points, or odd points. Owing to Lemma 4.1, using induction, we will prove the assertion.
(Case 1) Assume $|P| = 7$ (see the object in Figure 4(c) as an example). Then, by Theorem 3.6 and the hypothesis, it is clear that P is semi-closed in $(\mathbb{Z}^2, \gamma)$. Here, we note that the MW-path in Figure 4(a) does not satisfy the hypothesis.
(Case 2) Assume $|P| \geq 7$. For any l, $6 \leq l \in \mathbb{N}$, assume $P = (c_0, c_1, \cdots, c_{l-2})$ is semi-closed in $(\mathbb{Z}^2, \gamma)$. Then we now prove that $P = (c_0, c_1, \cdots, c_{l-2}, c_{l-1})$ is semi-closed. Owing to the properties of (2.3) and (2.4), we first examine the semi-closedness of the subset of P consisting of the consecutive two elements c_{l-2} and c_{l-1} in P according to the topological properties of the points c_{l-2} and c_{l-1}. Let us now investigate only two cases. Namely, take the set $\{c_{l-2}, c_{l-1}\} \subset P$ according to the two cases depending on the situation of P, as follows:
(Case 2-1) Assume the case that c_{l-2} is an odd point and c_{l-1} is a double even or even point. Then we obtain

$$Int(Cl(\{c_{l-2}, c_{l-1}\})) = \{c_{l-2}\} \subset \{c_{l-2}, c_{l-1}\},$$

because $Cl(\{c_{l-2}, c_{l-1}\}) = N_4(c_{l-2})$ in $(\mathbb{Z}^2, \gamma)$. Thus we see that the set $\{c_{l-2}, c_{l-1}\}$ is semi-closed. Based on this approach, denote the set $(c_0, c_1, \cdots, c_{l-2})$ by B. Then, for $P = B \cup \{c_{l-1}\}$, we have

$$\left\{\begin{array}{l} Int(Cl(P)) = Int(Cl(B \cup \{c_{l-1}\})) \\ = Int(Cl(B) \cup Cl(\{c_{l-1}\})) = Int(Cl(B)) \subset B \subset P, \end{array}\right\}$$

which implies the property $Int(Cl(P)) \subset P$.
(Case 2-2) Assume the case that c_{l-2} is a double even or even point and c_{l-1} is an odd point (this case is associated with the given hypothesis. Then we obtain

$$Int(Cl(\{c_{l-2}, c_{l-1}\})) = \{c_{l-1}\} \subset \{c_{l-2}, c_{l-1}\},$$

because $Cl(\{c_{l-2}, c_{l-1}\}) = N_4(c_{l-1})$ in $(\mathbb{Z}^2, \gamma)$. Thus we see that the set $\{c_{l-2}, c_{l-1}\}$ is semi-closed. Based on this approach, denote the set $(c_0, c_1, \cdots, c_{l-2})$ by B. Then, for $P = B \cup \{c_{l-1}\}$, owing to the given hypothesis, we have

$$\left\{\begin{array}{l} Int(Cl(B \cup \{c_{l-1}\})) \\ = Int(Cl(B) \cup Cl(\{c_{l-1}\})) = Int(Cl(P)) \subset P, \end{array}\right\} \tag{4.1}$$

which implies the property $Int(Cl(P)) \subset P$.
Note that without the hypothesis, the property of (4.1) does not hold. Based on these two cases above, with the hypothesis, the simple MW-path P is proved to be semi-closed in $(\mathbb{Z}^2, \gamma)$. □

Unlike Theorem 4.2, we have the following:

Remark 4.3. *A simple MW-path need not be semi-open because the path $P := \{c_0\}$ is not semi-open (see Theorem 3.5), where c_0 is a double even or even point.*

In view of Theorem 3.5 and Remark 4.3, we obtain the following:

Proposition 4.4. *Any simple MW-path P with $|P| \geq 2$ is semi-open.*

Proof. For any $x \in P$, we have $SN_\gamma(x) \cap P_{op} \neq \emptyset$. Thus, by Theorem 3.5, the proof is completed. □

5. The semi-T_3-separation axiom of the MW-topological space

This section studies various properties relating to the semi-T_3-separation axiom of $(\mathbb{Z}^2, \gamma)$. Based on the above properties, let us now investigate some topological properties of $(\mathbb{Z}^2, \gamma)$ with respect to the s-T_3-separation axiom and the semi-T_3-separation axiom (see Definition 5.3 below), and so on. The papers [31, 32] defined the notion of s-regular as follows: The paper [32] said that that a topological space (X, T) is s-regular if for each closed subset F of X and point $x \in F^c$, there are $U, V \in SO(X, T)$ such that $F \subset U$ and $x \in V$ and $U \cap V = \emptyset$, where $SO(X, T) := \{U \subset X \,|\, U$ is semi-open in $(X, T)\}$. The paper [36] proved that this s-regularity has the finite product property. Unlike the above definition of s-regularity, the present paper will take the following approach.

Definition 5.1. *[11] A topological space (X, T) is semi-regular if for each semi-closed set C and each $x \notin C$ there exist disjoint semi-open sets U and V in (X, T) such that $x \in U$, $C \subset V$ and $U \cap V = \emptyset$.*

Definition 5.2. *[31] A topological space (X, T) is said to be a semi-T_1-space if any distinct points $p, q \in X$ have their own semi-open sets $SO(p)$ and $SO(q)$ in (X, T) such that $q \notin SO(p)$ and $p \notin SO(q)$, where $SO(x)$ means a semi-open set containing the given point x.*

Besides, it turns out that a topological space (X, T) is a semi-T_1-space if and only if every singleton is semi-closed [31].

After comparing between s-regularity and semi-regularity, while the s-regularity implies the semi-regularity, the converse does not hold. Based on the notions of s-regularity or semi-regularity above, we now define the following:

Definition 5.3. *(1) We say that a topological space (X, T) is an s-T_3-space if it is both a semi-T_1-space and an s-regular space.*
(2) We call a topological space (X, T) is a semi-T_3-space if it is both a semi-T_1-space and a semi-regular space.

Lemma 5.4. *In $(\mathbb{Z}^2, \gamma)$, if A is semi-closed, then $Int(Cl(A)) = Int(A)$.*

Proof. Since $Int(A) \subset Int(Cl(A))$, with the hypothesis we need to prove that $Int(Cl(A)) \subset Int(A)$ with the following two cases.
(Case 1) Assume $Int(Cl(A)) = \emptyset$. Then the proof is straightforward.
(Case 2) Assume $Int(Cl(A)) \neq \emptyset$. Take an arbitrary element $x \in Int(Cl(A))$. Then we obtain $SN_\gamma(x) \subset Cl(A)$. Using methods similar to the proofs of (Case 1) and (Case 2) of Theorem 3.3 (see also the property of (3.10)), since $SN_\gamma(x)$ is semi-closed, we have

$$SN_\gamma(x) \subset Cl(A) \Rightarrow SN_\gamma(x) \subset A \Rightarrow x \in Int(A). \ \square$$

Theorem 5.5. *The MW-topological space, $(\mathbb{Z}^2, \gamma)$, is a semi-T_3-space.*

Proof. Since $(\mathbb{Z}^2, \gamma)$ is a semi-T_1-space [20], it suffice to prove the semi-regularity of $(\mathbb{Z}^2, \gamma)$. In $(\mathbb{Z}^2, \gamma)$, let $C(\neq \emptyset)$ be semi-closed and $x \notin C$. According to the choice of the point x, we can consider the following two cases.
(Case 1) Assume x is an odd point. Since $SN_\gamma(x) = \{x\}$, we have the two sets $U := \mathbb{Z}^2 \setminus \{x\}$ and $V := \{x\}$, then we find out that $U, V \in SO(\mathbb{Z}^2, \gamma)$ (see Remark 3.1(1)) and $C \subset U, x \in V, U \cap V = \emptyset$

because U is a closed set in $(\mathbb{Z}^2, \gamma)$. Hence $(\mathbb{Z}^2, \gamma)$ is semi-regular.
(Case 2) Assume x is a double even or even point. Since $x \notin C$ and C^c is semi-open, by Theorem 3.5, for each $x \in C^c$, we obtain the following identity

$$SN_\gamma(x) \cap (C^c)_{op} \neq \emptyset. \tag{5.1}$$

Namely, we may take $p \in SN_\gamma(x) \cap (C^c)_{op}$ so that there is a simple MW-path $\{x, p\}$ because $p \in SN_\gamma(x)$. Besides, by Theorems 3.5 and 3.6, the set $\{x, p\}$ is both semi-open and semi-closed (see also Lemma 4.1). Then consider the set $\mathbb{Z}^2 \setminus \{x, p\}$ which is semi-open because $\{x, p\}$ is semi-closed and $C \subset \mathbb{Z}^2 \setminus \{x, p\}$ (see Figure 5). Hence, after putting $U := \mathbb{Z}^2 \setminus \{x, p\}, V := \{x, p\}$, we have $U, V \in SO(\mathbb{Z}^2, \gamma)$ such that $C \subset U$, $x \in V$ and $U \cap V = \emptyset$.

In view of the above two cases, we now complete that $(\mathbb{Z}^2, \gamma)$ is a semi-regular space, which completes the proof. □

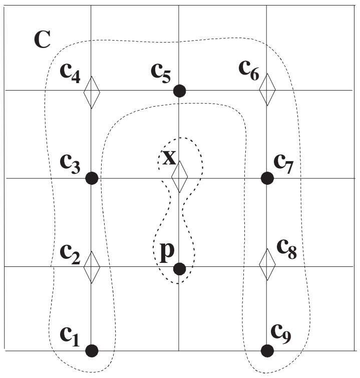

Figure 5. Configuration of an existence of $SO(C)$ and $SO(x) = \{x, p\}$ such that $SO(C) \cap SO(x) = \emptyset$ related to the (Case 2) of the proof of Theorem 5.5.

6. Semi-homeomorphic property of the semi-T_3-separation axiom

This section first refers to two kinds of semi-homeomorphisms from the literature. Next, we prove the semi-homeomorphic property of the semi-T_3-separation axiom.

Definition 6.1. *Given two topological spaces (X, T_1) and (Y, T_2), we now recall two types of semi-homeomorphisms.*
(1) [9] A bijection $h : (X, T_1) \to (Y, T_2)$ is said to be a semi-homeomorphism if $h(U) \in SO(Y, T_2)$ for each $U \in SO(X, T_1)$ (or pre-semi-open) and $h^{-1}(V) \in SO(X, T_1)$ for each $V \in SO(Y, T_2)$ (or irresolute or semi-open). This semi-homeomorphism is often called a semi-homeomorphism in the sense of "Crosseley and Hildebrand" (or semi-homeomorphism of C.H, for brevity).
(2) [3] A bijection $h : (X, T_1) \to (Y, T_2)$ is said to be a semi-homeomorphism if h is continuous and $h^{-1}(V) \in SO(X, T_1)$ for each $V \in SO(Y, T_2)$ (or irresolute or semi-open). This semi-homeomorphism is often called a semi-homeomorphism in the sense of "Biswas" (or semi-homeomorphism of B, for short).

Hereinafter, a property of topological spaces preserved by semi-homeomorphisms is called a semi-topological property [9] (see Definition 6.1(1)).

Definition 6.2. *Given a topological space* (X, T)*, let* $SC(X, T) := \{C \subset X \,|\, C$ *is semi-closed in* $(X, T)\}$*. We say that a map* $f : (X, T_1) \to (Y, T_2)$ *is a semi-closed map if* $f(C) \in SC(Y, T_2)$ *for each* $C \in SC(X, T_1)$*.*

The paper [34] has shown that there are semi-homeomorphisms of Definition 6.1(1) that are not semi-homeomorphisms of Definition 6.1(2) in general topology. Let us now investigate some properties of a semi-closed map with respect to a semi-homoeomorphism, as follows:

In view of Definition 6.1 and the notions of semi-openness and semi-closedness, after replacing the term "semi-open" by "semi-closed", we can define the semi-homeomorphism. Hereinafter, we will follow the semi-homeomorphism of C.H (see Definition 6.1(1)).

Theorem 6.3. *The semi-*T_3*-separation axiom is a semi-homeomorphic property.*

Proof. (Step 1) It is clear that the semi-T_1-separation axiom is a semi-homeomorphic property.
(Step 2) Let us prove that the semi-regularity is a semi-homeomorphic property. Assume two semi-homeomorphic spaces (X, T_1) and (Y, T_2), i.e., consider a semi-homeomorphism of C.H. (see Definition 6.1(1)) $h : (X, T_1) \to (Y, T_2)$. Further assume that (X, T_1) satisfies the semi-regularity. Then take any semi-closed subset C_2 in (Y, T_2) and the point $q \in (C_2)^c = Y \setminus C_2$ that is semi-open. Then, owing to the semi-homeomorphism of h, we have $h^{-1}(Y \setminus C_2) = X \setminus h^{-1}(C_2)$ is semi-open in (X, T_1) so that it turns out that $C_1 := h^{-1}(C_2)$ is semi-closed and further, it is clear that $p := h^{-1}(q) \notin h^{-1}(C_2) = C_1$.

Owing to the assumption that (X, T_1) satisfies the semi-regularity, there are

$$\begin{Bmatrix} SO(C_1), SO(p) \text{ in } (X, T_1) \text{ such that} \\ SO(C_1) \cap SO(p) = \emptyset. \end{Bmatrix}$$

Then, owing to the pre-semi-open mapping of h, we have

$$\begin{Bmatrix} SO(C_2) := h(SO(C_1)), SO(q) := h(SO(p)) \text{ and} \\ SO(C_2) \cap SO(q) = \emptyset, \end{Bmatrix}$$

which implies that (Y, T_2) has the semi-regularity.
In view of Steps 1 and 2, the proof is completed. □

7. Conclusions

We have proposed several techniques making a distinction between semi-open and semi-closed sets in $(\mathbb{Z}^2, \gamma)$. Owing to this approach, we have investigated more efficiently semi-topological features of some subsets of the MW-topological space. Finally, we have proved that $(\mathbb{Z}^2, \gamma)$ is a semi-T_3-space and further, the semi-T_3-separation axiom is a semi-topological property. This finding can facilitate many studies in the fields of digital topology and digital geometry. As a further work, we need to intensively study some semi-topological features of the infinite MW-sphere.

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Conflict of interest

The author declares no conflict of interest.

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