AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 12718–12741.
DOI: 10.3934/math.2022704
Received: 07 December 2021
Revised: 18 April 2022
Accepted: 20 April 2022
Published: 29 April 2022

Research article

On a nonlinear coupled system of differential equations involving Hilfer fractional derivative and Riemann-Liouville mixed operators with nonlocal integro-multi-point boundary conditions

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Abstract: We study a coupled system of multi-term Hilfer fractional differential equations of different orders involving non-integral and autonomous type Riemann-Liouville mixed integral nonlinearities supplemented with nonlocal coupled multi-point and Riemann-Liouville integral boundary conditions. The uniqueness result for the given problem is based on the contraction mapping principle, while the existence results are derived with the aid of Krasnosel'skiĭ's fixed point theorem and Leray-Schauder nonlinear alternative. Examples illustrating the main results are presented.

Keywords: Hilfer fractional differential equations; Riemann-Liouville mixed integral operators; nonlocal boundary conditions; fixed point theorems
Mathematics Subject Classification: 34A12, 34A40

1. Introduction

In recent years, the researchers and modelers have shown a keen interest in the topic of fractional differential equations. In fact, such equations appear in the mathematical models of several real-world phenomena occurring in pure, applied and technical sciences, for instance, see the books [1–3]. Unlike the classical derivative, there do exist many definitions of fractional derivatives and integrals. In [4], Hilfer proposed an important definition of fractional derivative (known as Hilfer fractional derivative), which represents both Riemann-Liouville and Caputo fractional derivatives under suitable choice of parameters. Several authors studied initial value problems involving Hilfer fractional derivatives, for example, see [5–9]. Some interesting results on boundary value problems involving Hilfer fractional differential equations can be found in the literature. For example, we refer the reader to works on nonlocal Hilfer problems [10, 11], Hilfer Langevin equations [12, 13], Hilfer Katugampola operators [14], Hilfer Erdelyi-Kober operators [15], Hilfer inclusion problems [16], Hilfer stochastic differential equations [17], ψ-Hilfer problems [18], ψ-Hilfer coupled systems [19], delay Hilfer fractional differential equations [20], Hilfer equations with variable coefficients [21], Hilfer sequential fractional differential equations [22, 23], Hilfer approximate controllability [24] and Hilfer-Hadamard boundary value problems [25]. A variety of recent results on boundary value problems and coupled systems of Hilfer fractional differential equations and inclusions can be found in the survey paper [26].

In [27], the authors introduced and developed the existence and uniqueness of solutions for a new class of coupled systems of Hilfer-type fractional differential equations with nonlocal integral boundary conditions of the form

$$\begin{cases} {}^{H}D^{\alpha,\beta}x(t) = f(t, x(t), y(t)), \quad t \in [a, b], \\ {}^{H}D^{\alpha_1,\beta_1}y(t) = g(t, x(t), y(t)), \quad t \in [a, b], \\ x(a) = 0, \quad x(b) = \displaystyle\sum_{i=1}^{m} \theta_i I^{\varphi_i} y(\xi_i), \quad y(a) = 0, \quad y(b) = \displaystyle\sum_{j=1}^{n} \zeta_j I^{\psi_j} x(z_j), \end{cases} \tag{1.1}$$

where ${}^{H}D^{\alpha,\beta}$, ${}^{H}D^{\alpha_1,\beta_1}$ are the Hilfer fractional derivatives of orders α, α_1, $1 < \alpha, \alpha_1 < 2$, and parameters β, β_1, $0 \le \beta, \beta_1 \le 1$, respectively, and I^{φ_i}, I^{ψ_j} are the Riemann-Liouville fractional integrals of order $\varphi_i > 0$ and $\psi_j > 0$, respectively, the points $\xi_i, z_j \in (a, b), a \ge 0$, $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and θ_i, $\zeta_j \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ are given real constants.

Recently, in [28], the authors studied a coupled system of ψ-Hilfer fractional order Langevin equations with nonlocal integral boundary conditions given by

$$\begin{cases} {}^{H}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\psi}\left({}^{H}\mathfrak{D}_{a^+}^{p_1,q_1;\psi} + \lambda_1\right)x(t) = f(t, x(t), y(t)), \quad t \in J := [a, b], \\ {}^{H}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\psi}\left({}^{H}\mathfrak{D}_{a^+}^{p_2,q_2;\psi} + \lambda_2\right)y(t) = g(t, x(t), y(t)), \quad t \in J := [a, b], \\ x(a) = 0, \ x(b) = \displaystyle\sum_{i=1}^{m} \eta_i \mathcal{I}_{a^+}^{\delta_i;\psi} y(\theta_i), \ y(a) = 0, \ y(b) = \displaystyle\sum_{j=1}^{n} \mu_j \mathcal{I}_{a^+}^{\kappa_j;\psi} x(\xi_j), \end{cases} \tag{1.2}$$

where ${}^{H}\mathfrak{D}_{a^+}^{u,v;\psi}$ is ψ-Hilfer fractional derivatives of order $u \in \{\alpha_1, \alpha_2, p_1, p_2\}$ with $0 < u \le 1$ and $v \in \{\beta_1, \beta_2, q_1, q_2\}$ with $0 \le v \le 1$, $\mathcal{I}_{a^+}^{w;\psi}$ is ψ-Riemann-Liouville fractional integral of order $w = \{\delta_i, \kappa_j\}$, $w > 0$, the points θ_i, $\xi_j \in (a, b)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, λ_1, $\lambda_2 \in \mathbb{R}$, f, $g \in C([a, b] \times \mathbb{R}^2, \mathbb{R})$ and $b > a \ge 0$.

The objective of the present paper is to investigate the existence and uniqueness of solutions for a new class of coupled systems of Langevin type Hilfer fractional differential equations of different orders involving non-integral and autonomous type Riemann-Liouville mixed integral nonlinearities complemented with nonlocal coupled multi-point and Riemann-Liouville integral boundary conditions.

This work is motivated by [27] and [28]. In precise terms, we consider the following problem:

$$\begin{cases} {}^HD^{\alpha_1,\beta_1}({}^HD^{\alpha_2,\beta_2}+\lambda_1)x(t) = I_{a^+}^{\zeta_1}g_1(x(t),y(t)) + f_1(t,x(t),y(t)),\ t\in[a,b],\\ {}^HD^{\alpha_3,\beta_3}({}^HD^{\alpha_4,\beta_4}+\lambda_2)y(t) = I_{b^-}^{\zeta_2}g_2(x(t),y(t)) + f_2(t,x(t),y(t)),\ t\in[a,b],\\ x(a)=0,\ x(b)=\sum\limits_{i=1}^{m}\mu_i y(\eta_i) + \sum\limits_{k=1}^{n}\nu_k I_{a^+}^{q_k}y(\xi_k),\ q_k>0,\\ y(a)=0,\ y(b)=\sum\limits_{i=1}^{m}\delta_i x(\eta_i) + \sum\limits_{k=1}^{n}\theta_k I_{a^+}^{p_k}x(\xi_k),\ p_k>0, \end{cases} \tag{1.3}$$

where ${}^HD^{\alpha_j,\beta_j}$ represents Hilfer fractional derivative operator of order $\alpha_j\in(0,1)$ with parameter $\beta_j\in[0,1]$, $j=1,2,3,4$, $\lambda_1,\lambda_2,\mu_i,\nu_k,\delta_i$ and $\theta_k, i=1,2,...,m, k=1,2,...,n$ are constants, $a<\eta_i,\xi_k<b$, where $a\geq 0$ and $m,n\in\mathbb{N}$, $I_{a^+}^{\zeta_1}, I_{a^+}^{q_k}, I_{a^+}^{p_k}$ denote the left Riemann-Liouville fractional integral operators of orders $\zeta_1>0, q_k>0, p_k>0$ respectively, while $I_{b^-}^{\zeta_2}$ denotes the right Riemann-Liouville fractional integral operator of order $\zeta_2>0$, and $f_1,f_2:[a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $g_1,g_2:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

Note that problem (1.3) is more general than problem (1.2), since it contains non-integral as well as Riemann-Liouville mixed integral nonlinearities and nonlocal coupled multi-point and Riemann-Liouville integral boundary conditions.

The rest of the paper is organized as follows. In Section 2, we present some necessary material related to our study and prove an auxiliary lemma to define the solution for the problem at hand. Section 3 contains the main results which rely on Banach contraction mapping principle, Krasnosel'skiĭ's fixed point theorem and Leray-Schauder alternative. In Section 4, we construct examples for the illustration of the results obtained in Section 3.

2. Preliminaries

We begin this section with some basic concepts used in our study.

Definition 2.1. *([3]) The left and right Riemann–Liouville fractional integrals of order $\omega>0$ for a continuous function g, existing almost everywhere on $[a,b]$, are respectively defined by*

$$I_{a+}^{\omega}g(t)=\int_a^t\frac{(t-s)^{\omega-1}}{\Gamma(\omega)}g(s)ds \quad and \quad I_{b-}^{\omega}g(t)=\int_t^b\frac{(s-t)^{\omega-1}}{\Gamma(\omega)}g(s)ds.$$

For the sake of simplicity, we write I_{a+}^{ω} and I_{b-}^{ω} as I_a^{ω} and I_b^{ω} respectively.

Definition 2.2. *([4]) For $n-1<\alpha<n,\ 0\leq\beta\leq 1$, the Hilfer fractional derivative of order α and parameter β for a continuous function g is defined by*

$${}^HD^{\alpha,\beta}g(t)=I_a^{\beta(n-\alpha)}D^nI_a^{(1-\beta)(n-\alpha)}g(t), D=\frac{d}{dt},$$

where

$$I_a^{\omega}g(t)=\frac{1}{\Gamma(\omega)}\int\limits_a^t(t-s)^{\omega-1}g(s)ds,\ a\geq 0,$$

with $\omega\in\{\beta(n-\alpha),(1-\beta)(n-\alpha)\}$.

Lemma 2.1. *([16]) Let $h \in L(a,b)$, $n-1<\gamma_1\le n$, $n\in\mathbb{N}$, $0\le\gamma_2\le 1$ and $I_a^{(n-\gamma_1)(1-\gamma_2)}h\in AC^k[a,b]$. Then*

$$I_a^{\gamma_1}(^H D^{\gamma_1,\gamma_2}h)(t)=h(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{k-(n-\gamma_1)(1-\gamma_2)}}{\Gamma(k-(n-\gamma_1)(1-\gamma_2)+1)}\lim_{t\longrightarrow a^+}\frac{d^k}{dt^k}(I_a^{(1-\gamma_2)(n-\gamma_1)}h)(t).$$

In the following lemma, we solve the linear variant of the problem (1.3).

Lemma 2.2. *Let $h_1,h_2:[a,b]\to\mathbb{R}$ be continuous functions and $\Delta\neq 0$. Then the unique solution of the following coupled system:*

$$\begin{cases} ^H D^{\alpha_1,\beta_1}(^H D^{\alpha_2,\beta_2}+\lambda_1)x(t)=h_1(t), & t\in[a,b],\\ ^H D^{\alpha_3,\beta_3}(^H D^{\alpha_4,\beta_4}+\lambda_2)y(t)=h_2(t), & t\in[a,b],\\ x(a)=0,\ x(b)=\displaystyle\sum_{i=1}^{m}\mu_i y(\eta_i)+\sum_{k=1}^{n}\nu_k I_a^{q_k}y(\xi_k),\ \ q_k>0,\\ y(a)=0,\ y(b)=\displaystyle\sum_{i=1}^{m}\delta_i x(\eta_i)+\sum_{k=1}^{n}\theta_k I_a^{p_k}x(\xi_k),\ \ p_k>0,\end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} x(t) &= I_a^{\alpha_1+\alpha_2}h_1(t)-\lambda_1 I_a^{\alpha_2}x(t)+\frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)}\Big\{\Omega_2\Big(\lambda_1 I_a^{\alpha_2}x(b)-I_a^{\alpha_1+\alpha_2}h_1(b)\\ &\quad -\lambda_2\sum_{i=1}^{m}\mu_i I_a^{\alpha_4}y(\eta_i)+\sum_{i=1}^{m}\mu_i I_a^{\alpha_3+\alpha_4}h_2(\eta_i)+\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_3+\alpha_4}h_2(\xi_k)\\ &\quad -\lambda_2\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_4}y(\xi_k)\Big)+\Omega_4\Big(\lambda_2 I_a^{\alpha_4}y(b)-I_a^{\alpha_3+\alpha_4}h_2(b)+\sum_{i=1}^{m}\delta_i I_a^{\alpha_1+\alpha_2}h_1(\eta_i)\\ &\quad -\lambda_1\sum_{i=1}^{m}\delta_i I_a^{\alpha_2}x(\eta_i)+\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_1+\alpha_2}h_1(\xi_k)-\lambda_1\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_2}x(\xi_k)\Big)\Big\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} y(t) &= I_a^{\alpha_3+\alpha_4}h_2(t)-\lambda_2 I_a^{\alpha_4}y(t)+\frac{(t-a)^{\alpha_4+\epsilon_3-1}}{\Delta\Gamma(\alpha_4+\epsilon_3)}\Big\{\Omega_3\Big(\lambda_1 I_a^{\alpha_2}x(b)-I_a^{\alpha_1+\alpha_2}h_1(b)\\ &\quad -\lambda_2\sum_{i=1}^{m}\mu_i I_a^{\alpha_4}y(\eta_i)+\sum_{i=1}^{m}\mu_i I_a^{\alpha_3+\alpha_4}h_2(\eta_i)+\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_3+\alpha_4}h_2(\xi_k)\\ &\quad -\lambda_2\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_4}y(\xi_k)\Big)+\Omega_1\Big(-I_a^{\alpha_3+\alpha_4}h_2(b)+\lambda_2 I_a^{\alpha_4}y(b)+\sum_{i=1}^{m}\delta_i I_a^{\alpha_1+\alpha_2}h_1(\eta_i)\\ &\quad -\lambda_1\sum_{i=1}^{m}\delta_i I_a^{\alpha_2}x(\eta_i)+\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_1+\alpha_2}h_1(\xi_k)-\lambda_1\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_2}x(\xi_k)\Big)\Big\}, \end{aligned} \tag{2.3}$$

where Δ,Ω_i, $i=1,2,3,4$ are given by

$$\begin{aligned} \Omega_1 &= \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)},\ \ \Omega_2=\frac{(b-a)^{\alpha_4+\epsilon_3-1}}{\Gamma(\alpha_4+\epsilon_3)},\\ \Omega_3 &= \sum_{i=1}^{m}\delta_i\frac{(\eta_i-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)}+\sum_{k=1}^{n}\theta_k\frac{(\xi_k-a)^{p_k+\alpha_2+\epsilon_1-1}}{\Gamma(p_k+\alpha_2+\epsilon_1)}, \end{aligned}$$

$$\Omega_4 = \sum_{i=1}^{m}\mu_i\frac{(\eta_i-a)^{\alpha_4+\epsilon_3-1}}{\Gamma(\alpha_4+\epsilon_3)}+\sum_{k=1}^{n}\nu_k\frac{(\xi_k-a)^{q_k+\alpha_4+\epsilon_3-1}}{\Gamma(q_k+\alpha_4+\epsilon_3)},$$
$$\Delta = \Omega_1\Omega_2-\Omega_3\Omega_4, \tag{2.4}$$

and $\epsilon_i=\alpha_i+\beta_i-\alpha_i\beta_i,\ i=1,2,3,4.$

Proof. Applying the integral operators $I_a^{\alpha_1}$ and $I_a^{\alpha_3}$ on the first and second Hilfer fractional differential equations in (2.1) respectively and using Lemma 2.1, we obtain

$$({}^HD^{\alpha_2,\beta_2}+\lambda_1)x(t)-\frac{c_0(t-a)^{\epsilon_1-1}}{\Gamma(\epsilon_1)}=I_a^{\alpha_1}h_1(t), \tag{2.5}$$

$$({}^HD^{\alpha_4,\beta_4}+\lambda_2)y(t)-\frac{d_0(t-a)^{\epsilon_3-1}}{\Gamma(\epsilon_3)}=I_a^{\alpha_3}h_2(t). \tag{2.6}$$

Now operating $I_a^{\alpha_2}$ and $I_a^{\alpha_4}$ respectively to the Eqs (2.5) and (2.6), we get

$$x(t)+\lambda_1I_a^{\alpha_2}x(t)-\frac{c_1(t-a)^{\epsilon_2-1}}{\Gamma(\epsilon_2)}-\frac{c_0(t-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)}=I_a^{\alpha_1+\alpha_2}h_1(t), \tag{2.7}$$
$$y(t)+\lambda_2I_a^{\alpha_4}y(t)-\frac{d_1(t-a)^{\epsilon_4-1}}{\Gamma(\epsilon_4)}-\frac{d_0(t-a)^{\alpha_4+\epsilon_3-1}}{\Gamma(\alpha_4+\epsilon_3)}=I_a^{\alpha_3+\alpha_4}h_2(t). \tag{2.8}$$

Using the conditions $x(a)=0$ and $y(a)=0$ in (2.7) and (2.8) respectively, we find that $c_1=d_1=0$. Thus we have

$$x(t)=I_a^{\alpha_1+\alpha_2}h_1(t)-\lambda_1I_a^{\alpha_2}x(t)+\frac{c_0(t-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)}, \tag{2.9}$$
$$y(t)=I_a^{\alpha_3+\alpha_4}h_2(t)-\lambda_2I_a^{\alpha_4}y(t)+\frac{d_0(t-a)^{\alpha_4+\epsilon_3-1}}{\Gamma(\alpha_4+\epsilon_3)}. \tag{2.10}$$

Inserting (2.9) and (2.10) in the condition $x(b)=\sum_{i=1}^{m}\mu_iy(\eta_i)+\sum_{k=1}^{n}\nu_kI_a^{q_k}y(\xi_k)$, we find that

$$I_a^{\alpha_1+\alpha_2}h_1(b)-\lambda_1I_a^{\alpha_2}x(b)+\frac{c_0(b-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)} = \sum_{i=1}^{m}\mu_i\Big\{I_a^{\alpha_3+\alpha_4}h_2(\eta_i)-\lambda_2I_a^{\alpha_4}y(\eta_i)$$
$$+\frac{d_0(\eta_i-a)^{\alpha_4+\epsilon_3-1}}{\Gamma(\alpha_4+\epsilon_3)}\Big\}+\sum_{k=1}^{n}\nu_kI_a^{q_k}\Big\{I_a^{\alpha_3+\alpha_4}h_2(\xi_k)-\lambda_2I_a^{\alpha_4}y(\xi_k)+\frac{d_0(\xi_k-a)^{\alpha_4+\epsilon_3-1}}{\Gamma(\alpha_4+\epsilon_3)}\Big\},$$

which can alternatively be written as

$$c_0\frac{(b-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)}-d_0\Big\{\sum_{i=1}^{m}\mu_i\frac{(\eta_i-a)^{\alpha_4+\epsilon_3-1}}{\Gamma(\alpha_4+\epsilon_3)}+\sum_{k=1}^{n}\nu_k\frac{(\xi_k-a)^{q_k+\alpha_4+\epsilon_3-1}}{\Gamma(q_k+\alpha_4+\epsilon_3)}\Big\}$$
$$= \lambda_1I_a^{\alpha_2}x(b)-I_a^{\alpha_1+\alpha_2}h_1(b)-\lambda_2\sum_{i=1}^{m}\mu_iI_a^{\alpha_4}y(\eta_i)+\sum_{i=1}^{m}\mu_iI_a^{\alpha_3+\alpha_4}h_2(\eta_i)$$
$$+\sum_{k=1}^{n}\nu_kI_a^{q_k+\alpha_3+\alpha_4}h_2(\xi_k)-\lambda_2\sum_{k=1}^{n}\nu_kI_a^{q_k+\alpha_4}y(\xi_k). \tag{2.11}$$

In a similar manner, making use of (2.9) and (2.10) in the condition: $y(b) = \sum_{i=1}^{m} \delta_i x(\eta_i) + \sum_{k=1}^{n} \theta_k I_a^{p_k} x(\xi_k)$, leads to

$$\begin{aligned}
&-c_0\Big\{\sum_{i=1}^{m}\delta_i\frac{(\eta_i-a)^{\alpha_2+\epsilon_1-1}}{\Gamma(\alpha_2+\epsilon_1)}+\sum_{k=1}^{n}\theta_k\frac{(\xi_k-a)^{p_k+\alpha_2+\epsilon_1-1}}{\Gamma(p_k+\alpha_2+\epsilon_1)}\Big\}+d_0\frac{(b-a)^{\alpha_4+\epsilon_3-1}}{\Gamma(\alpha_4+\epsilon_3)}\\
&=\lambda_2 I_a^{\alpha_4}y(b)-I_a^{\alpha_3+\alpha_4}h_2(b)+\sum_{i=1}^{m}\delta_i I_a^{\alpha_1+\alpha_2}h_1(\eta_i)-\lambda_1\sum_{i=1}^{m}\delta_i I_a^{\alpha_2}x(\eta_i)\\
&+\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_1+\alpha_2}h_1(\xi_k)-\lambda_1\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_2}x(\xi_k).
\end{aligned}\tag{2.12}$$

Making use of the notation in (2.4), we can write (2.11) and (2.12) as

$$\begin{aligned}
\Omega_1 c_0-\Omega_4 d_0 &= \lambda_1 I_a^{\alpha_2}x(b)-I_a^{\alpha_1+\alpha_2}h_1(b)-\lambda_2\sum_{i=1}^{m}\mu_i I_a^{\alpha_4}y(\eta_i)+\sum_{i=1}^{m}\mu_i I_a^{\alpha_3+\alpha_4}h_2(\eta_i)\\
&\quad+\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_3+\alpha_4}h_2(\xi_k)-\lambda_2\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_4}y(\xi_k),\\
-\Omega_3 c_0+\Omega_2 d_0 &= \lambda_2 I_a^{\alpha_4}y(b)-I_a^{\alpha_3+\alpha_4}h_2(b)+\sum_{i=1}^{m}\delta_i I_a^{\alpha_1+\alpha_2}h_1(\eta_i)-\lambda_1\sum_{i=1}^{m}\delta_i I_a^{\alpha_2}x(\eta_i)\\
&\quad+\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_1+\alpha_2}h_1(\xi_k)-\lambda_1\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_2}x(\xi_k),
\end{aligned}$$

which, on solving for c_0 and d_0, yields

$$\begin{aligned}
c_0 &= \frac{1}{\Delta}\Big\{\Omega_2\Big(\lambda_1 I_a^{\alpha_2}x(b)-I_a^{\alpha_1+\alpha_2}h_1(b)-\lambda_2\sum_{i=1}^{m}\mu_i I_a^{\alpha_4}y(\eta_i)+\sum_{i=1}^{m}\mu_i I_a^{\alpha_3+\alpha_4}h_2(\eta_i)\\
&\quad+\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_3+\alpha_4}h_2(\xi_k)-\lambda_2\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_4}y(\xi_k)\Big)+\Omega_4\Big(\lambda_2 I_a^{\alpha_4}y(b)-I_a^{\alpha_3+\alpha_4}h_2(b)\\
&\quad+\sum_{i=1}^{m}\delta_i I_a^{\alpha_1+\alpha_2}h_1(\eta_i)-\lambda_1\sum_{i=1}^{m}\delta_i I_a^{\alpha_2}x(\eta_i)+\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_1+\alpha_2}h_1(\xi_k)\\
&\quad-\lambda_1\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_2}x(\xi_k)\Big)\Big\},\\
d_0 &= \frac{1}{\Delta}\Big\{\Omega_3\Big(\lambda_1 I_a^{\alpha_2}x(b)-I_a^{\alpha_1+\alpha_2}h_1(b)-\lambda_2\sum_{i=1}^{m}\mu_i I_a^{\alpha_4}y(\eta_i)+\sum_{i=1}^{m}\mu_i I_a^{\alpha_3+\alpha_4}h_2(\eta_i)\\
&\quad+\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_3+\alpha_4}h_2(\xi_k)-\lambda_2\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_4}y(\xi_k)\Big)+\Omega_1\Big(\lambda_2 I_a^{\alpha_4}y(b)-I_a^{\alpha_3+\alpha_4}h_2(b)\\
&\quad+\sum_{i=1}^{m}\delta_i I_a^{\alpha_1+\alpha_2}h_1(\eta_i)-\lambda_1\sum_{i=1}^{m}\delta_i I_a^{\alpha_2}x(\eta_i)+\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_1+\alpha_2}h_1(\xi_k)
\end{aligned}$$

$$-\lambda_1 \sum_{k=1}^{n} \theta_k I_a^{p_k+\alpha_2} x(\xi_k)\Big)\Big\}.$$

Substituting the values of c_0 and d_0 in (2.9) and (2.10) respectively together with (2.4), we get the solution (2.2) and (2.3). By direct computation, one can obtain the converse of this lemma. The proof is finished. □

3. Existence and uniqueness results

Let $\mathcal{X} = C([a,b],\mathbb{R})$ denote the Banach space of all continuous functions from $[a,b]$ to $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$. Then the product space $(\mathcal{X}\times\mathcal{X}, \|\cdot\|)$ is also a Banach space endowed with the norm $\|(x,y)\| = \|x\| + \|y\|$ for $(x,y) \in \mathcal{X}\times\mathcal{X}$.

In view of Lemma 2.2, we introduce an operator $\mathcal{T} : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ as

$$\mathcal{T}(x,y)(t) = \begin{pmatrix} \mathcal{T}_1(x,y)(t) \\ \mathcal{T}_2(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned}
\mathcal{T}_1(x,y)(t) &= I_a^{\alpha_1+\alpha_2+\zeta_1} g_1(x(t),y(t)) + I_a^{\alpha_1+\alpha_2} f_1(t,x(t),y(t)) - \lambda_1 I_a^{\alpha_2} x(t) \\
&\quad + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)} \times \Big\{\Omega_2\Big(\lambda_1 I_a^{\alpha_2} x(b) - I_a^{\alpha_1+\alpha_2+\zeta_1} g_1(x(b),y(b)) \\
&\quad - I_a^{\alpha_1+\alpha_2} f_1(b,x(b),y(b)) - \lambda_2 \sum_{i=1}^{m} \mu_i I_a^{\alpha_4} y(\eta_i) \\
&\quad + \sum_{i=1}^{m} \mu_i I_a^{\alpha_3+\alpha_4} (I_b^{\zeta_2} g_2(x(\eta_i),y(\eta_i))) + \sum_{i=1}^{m} \mu_i I_a^{\alpha_3+\alpha_4} f_2(\eta_i,x(\eta_i),y(\eta_i)) \\
&\quad + \sum_{k=1}^{n} \nu_k I_a^{q_k+\alpha_3+\alpha_4} (I_b^{\zeta_2} g_2(x(\xi_k),y(\xi_k))) + \sum_{k=1}^{n} \nu_k I_a^{q_k+\alpha_3+\alpha_4} f_2(\xi_k,x(\xi_k),y(\xi_k)) \\
&\quad - \lambda_2 \sum_{k=1}^{n} \nu_k I_a^{q_k+\alpha_4} y(\xi_k)\Big) + \Omega_4\Big(\lambda_2 I_a^{\alpha_4} y(b) - I_a^{\alpha_3+\alpha_4} f_2(b,x(b),y(b)) \\
&\quad + \sum_{i=1}^{m} \delta_i I_a^{\alpha_1+\alpha_2+\zeta_1} g_1(x(\eta_i),y(\eta_i)) + \sum_{i=1}^{m} \delta_i I_a^{\alpha_1+\alpha_2} f_1(\eta_i,x(\eta_i),y(\eta_i)) \\
&\quad - \lambda_1 \sum_{i=1}^{m} \delta_i I_a^{\alpha_2} x(\eta_i) + \sum_{k=1}^{n} \theta_k I_a^{p_k+\alpha_1+\alpha_2+\zeta_1} g_1(x(\xi_k),y(\xi_k)) \\
&\quad + \sum_{k=1}^{n} \theta_k I_a^{p_k+\alpha_1+\alpha_2} f_1(\xi_k,x(\xi_k),y(\xi_k)) - \lambda_1 \sum_{k=1}^{n} \theta_k I_a^{p_k+\alpha_2} x(\xi_k)\Big)\Big\},
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\mathcal{T}_2(x,y)(t) &= I_a^{\alpha_3+\alpha_4} (I_b^{\zeta_2} g_2(x(t),y(t))) + I_a^{\alpha_3+\alpha_4} f_2(t,x(t),y(t)) - \lambda_2 I_a^{\alpha_4} y(t) \\
&\quad + \frac{(t-a)^{\alpha_4+\epsilon_3-1}}{\Delta\Gamma(\alpha_4+\epsilon_3)} \times \Big\{\Omega_3\Big(\lambda_1 I_a^{\alpha_2} x(b) - I_a^{\alpha_1+\alpha_2+\zeta_1} g_1(x(b),y(b))
\end{aligned}$$

$$
\begin{aligned}
&-I_a^{\alpha_1+\alpha_2} f_1(b, x(b), y(b)) - \lambda_2 \sum_{i=1}^{m} \mu_i I_a^{\alpha_4} y(\eta_i) \\
&+ \sum_{i=1}^{m} \mu_i I_a^{\alpha_3+\alpha_4} (I_b^{\zeta_2} g_2(x(\eta_i), y(\eta_i))) + \sum_{i=1}^{m} \mu_i I_a^{\alpha_3+\alpha_4} f_2(\eta_i, x(\eta_i), y(\eta_i)) \\
&+ \sum_{k=1}^{n} \nu_k I_a^{q_k+\alpha_3+\alpha_4} (I_b^{\zeta_2} g_2(x(\xi_k), y(\xi_k))) + \sum_{k=1}^{n} \nu_k I_a^{q_k+\alpha_3+\alpha_4} f_2(\xi_k, x(\xi_k), y(\xi_k)) \\
&-\lambda_2 \sum_{k=1}^{n} \nu_k I_a^{q_k+\alpha_4} y(\xi_k)\Big) + \Omega_1\Big(\lambda_2 I_a^{\alpha_4} y(b) - I_a^{\alpha_3+\alpha_4} f_2(b, x(b), y(b)) \\
&+ \sum_{i=1}^{m} \delta_i I_a^{\alpha_1+\alpha_2+\zeta_1} g_1(x(\eta_i), y(\eta_i)) + \sum_{i=1}^{m} \delta_i I_a^{\alpha_1+\alpha_2} f_1(\eta_i, x(\eta_i), y(\eta_i)) \\
&-\lambda_1 \sum_{i=1}^{m} \delta_i I_a^{\alpha_2} x(\eta_i) + \sum_{k=1}^{n} \theta_k I_a^{p_k+\alpha_1+\alpha_2+\zeta_1} g_1(x(\xi_k), y(\xi_k)) \\
&+ \sum_{k=1}^{n} \theta_k I_a^{p_k+\alpha_1+\alpha_2} f_1(\xi_k, x(\xi_k), y(\xi_k)) - \lambda_1 \sum_{k=1}^{n} \theta_k I_a^{p_k+\alpha_2} x(\xi_k)\Big)\Big\}. \qquad (3.2)
\end{aligned}
$$

For computational facilitation, we set

$$
\begin{aligned}
\sigma_1 &= \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{\Omega_1}{|\Delta|}\Big\{\frac{\Omega_2(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + |\Omega_4|\Big(\sum_{i=1}^{m} \frac{|\delta_i|(\eta_i-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} \\
&\quad + \sum_{k=1}^{n} \frac{|\theta_k|(\xi_k-a)^{p_k+\alpha_1+\alpha_2}}{\Gamma(p_k+\alpha_1+\alpha_2+1)}\Big)\Big\}, \qquad (3.3) \\
\sigma_2 &= \frac{\Omega_1}{|\Delta|}\Big\{\Omega_2\Big(\sum_{i=1}^{m} |\frac{\mu_i|(\eta_i-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)} + \sum_{k=1}^{n} \frac{|\nu_k|(\xi_k-a)^{q_k+\alpha_3+\alpha_4}}{\Gamma(q_k+\alpha_3+\alpha_4+1)}\Big) \\
&\quad + \frac{|\Omega_4|(b-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}\Big\}, \qquad (3.4) \\
\sigma_3 &= \frac{\Omega_2}{|\Delta|}\Big\{\frac{|\Omega_3|(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \Omega_1\Big(\sum_{i=1}^{m} \frac{|\delta_i|(\eta_i-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} \\
&\quad + \sum_{k=1}^{n} \frac{|\theta_k|(\xi_k-a)^{p_k+\alpha_1+\alpha_2}}{\Gamma(p_k+\alpha_1+\alpha_2+1)}\Big)\Big\}, \qquad (3.5) \\
\sigma_4 &= \frac{(b-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)} + \frac{\Omega_2}{|\Delta|}\Big\{|\Omega_3|\Big(\sum_{i=1}^{m} \frac{|\mu_i|(\eta_i-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)} \\
&\quad + \sum_{k=1}^{n} \frac{|\nu_k|(\xi_k-a)^{q_k+\alpha_3+\alpha_4}}{\Gamma(q_k+\alpha_3+\alpha_4+1)}\Big) + \frac{\Omega_1(b-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}\Big\}, \qquad (3.6) \\
\sigma_5 &= \frac{1}{|\Delta|}\Big\{\frac{|\lambda_1|(|\Delta|+\Omega_2\Omega_1)(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{\Omega_1|\lambda_2\Omega_4|(b-a)^{\alpha_4}}{\Gamma(\alpha_4+1)} \\
&\quad + \Omega_1|\lambda_1\Omega_4|\Big(\sum_{i=1}^{m} \frac{|\delta_i|(\eta_i-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{k=1}^{n} \frac{|\theta_k|(\xi_k-a)^{p_k+\alpha_2}}{\Gamma(p_k+\alpha_2+1)}\Big)
\end{aligned}
$$

$$+|\lambda_2|\Omega_2\Omega_1\Big(\sum_{k=1}^{n}\frac{|\nu_k|(\xi_k-a)^{q_k+\alpha_4}}{\Gamma(q_k+\alpha_4+1)}+\sum_{i=1}^{m}\frac{|\mu_i|(\eta_i-a)^{\alpha_4}}{\Gamma(\alpha_4+1)}\Big)\Big\}, \tag{3.7}$$

$$\begin{aligned}
\sigma_6 &= \frac{1}{|\Delta|}\Big\{\frac{\Omega_2|\lambda_1\Omega_3|(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{|\lambda_2|(|\Delta|+\Omega_1\Omega_2)(b-a)^{\alpha_4}}{\Gamma(\alpha_4+1)}\\
&\quad+|\lambda_1|\Omega_1\Omega_2\Big(\sum_{i=1}^{m}\frac{|\delta_i|(\eta_i-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\sum_{k=1}^{n}\frac{|\theta_k|(\xi_k-a)^{p_k+\alpha_2}}{\Gamma(p_k+\alpha_2+1)}\Big)\\
&\quad+\Omega_2|\lambda_2\Omega_3|\Big(\sum_{i=1}^{m}\frac{|\mu_i|(\eta_i-a)^{\alpha_4}}{\Gamma(\alpha_4+1)}+\sum_{k=1}^{n}\frac{|\nu_k|(\xi_k-a)^{q_k+\alpha_4}}{\Gamma(q_k+\alpha_4+1)}\Big)\Big\},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\sigma_7 &= \frac{(b-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}+\frac{\Omega_1}{|\Delta|}\Big\{\frac{\Omega_2(b-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}\\
&\quad+|\Omega_4|\Big(\sum_{i=1}^{m}\frac{|\delta_i|(\eta_i-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}+\sum_{k=1}^{n}\frac{|\theta_k|(\xi_k-a)^{p_k+\alpha_1+\alpha_2+\zeta_1}}{\Gamma(p_k+\alpha_1+\alpha_2+\zeta_1+1)}\Big)\Big\},
\end{aligned} \tag{3.9}$$

$$\sigma_8 = \frac{\Omega_1\Omega_2(b-a)^{\zeta_2}}{|\Delta|\Gamma(\zeta_2+1)}\Big(\sum_{i=1}^{m}\frac{|\mu_i|(\eta_i-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}+\sum_{k=1}^{n}\frac{|\nu_k|(\xi_k-a)^{q_k+\alpha_3+\alpha_4}}{\Gamma(q_k+\alpha_3+\alpha_4+1)}\Big), \tag{3.10}$$

$$\begin{aligned}
\sigma_9 &= \frac{\Omega_2}{|\Delta|}\Big\{\frac{|\Omega_3|(b-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}+\Omega_1\Big(\sum_{i=1}^{m}\frac{|\delta_i|(\eta_i-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}\\
&\quad+\sum_{k=1}^{n}\frac{|\theta_k|(\xi_k-a)^{p_k+\alpha_1+\alpha_2+\zeta_1}}{\Gamma(p_k+\alpha_1+\alpha_2+\zeta_1+1)}\Big)\Big\},
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\sigma_{10} &= \frac{(b-a)^{\zeta_2}}{\Gamma(\zeta_2+1)}\Big(\frac{(b-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}+\frac{\Omega_2|\Omega_3|}{|\Delta|}\Big(\sum_{i=1}^{m}\frac{|\mu_i|(\eta_i-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}\\
&\quad+\sum_{k=1}^{n}\frac{|\nu_k|(\xi_k-a)^{q_k+\alpha_3+\alpha_4}}{\Gamma(q_k+\alpha_3+\alpha_4+1)}\Big)\Big).
\end{aligned} \tag{3.12}$$

In the sequel, we suppose that $f_1, f_2 : [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g_1, g_2 : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying the following assumptions:

(H_1) $\forall(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$, there exist positive real constants K_i, i=1,2, such that

$$\begin{aligned}
|f_1(t,x_1,y_1)-f_1(t,x_2,y_2)| &\le K_1(|x_1-x_2|+|y_1-y_2|),\\
|f_2(t,x_1,y_1)-f_2(t,x_2,y_2)| &\le K_2(|x_1-x_2|+|y_1-y_2|);
\end{aligned}$$

(H_2) $\forall(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$, there exist positive real constants L_i, i=1,2, such that

$$\begin{aligned}
|g_1(x_1,y_1)-g_1(x_2,y_2)| &\le L_1(|x_1-x_2|+|y_1-y_2|),\\
|g_2(x_1,y_1)-g_2(x_2,y_2)| &\le L_2(|x_1-x_2|+|y_1-y_2|);
\end{aligned}$$

(H_3) We can find real constants $u_k, v_k, \omega_k, \tau_k \geqslant 0, k=0,1,2$ with $u_0, v_0, \omega_0, \tau_0 \neq 0$ such that

$$\begin{aligned}
|f_1(t,x,y)| &\le u_0+u_1|x|+u_2|y|, |f_2(t,x,y)|\le v_0+v_1|x|+v_2|y|,\\
|g_1(x,y)| &\le \omega_0+\omega_1|x|+\omega_2|y|, |g_2(x,y)|\le \tau_0+\tau_1|x|+\tau_2|y|;
\end{aligned}$$

(H_4) There exist nonnegative functions $\phi_1, \phi_2 \in C([a,b], \mathbb{R}^+)$, and positive constants Λ_1, Λ_2 such that $|f_1(t,x,y)| \leq \phi_1(t)$, $|f_2(t,x,y)| \leq \phi_2(t)$, $|g_1(x,y)| \leqslant \Lambda_1$, $|g_2(x,y)| \leqslant \Lambda_2$ for all $(t,x,y) \in [a,b]\times\mathbb{R}\times\mathbb{R}$.

Now we present our first main result dealing with the uniqueness of solutions for the system (1.3), which relies on Banach contraction mapping principle [29].

Theorem 3.1. *Assume that conditions (H_1) and (H_2) hold. Then the system (1.3) has a unique solution on $[a,b]$ provided that*

$$(\sigma_1+\sigma_3)K_1 + (\sigma_4+\sigma_2)K_2 + (\sigma_9+\sigma_7)L_1 + (\sigma_{10}+\sigma_8)L_2 + \sigma_5 + \sigma_6 < 1, \tag{3.13}$$

where $\sigma_1, \ldots, \sigma_{10}$ are given in (3.3)–(3.12).

Proof. Let us fix $\sup_{t\in[a,b]} |f_i(t,0,0)| = M_i < \infty, |g_i(0,0)| = 0,\ i = 1,2$. In order to satisfy the hypotheses of Banach contraction mapping principle, we first show that $\mathcal{T} B_\rho \subset B_\rho$, where B_ρ is a closed bounded ball $B_\rho \subset \mathcal{X}\times\mathcal{X}$ defined by

$$B_\rho = \{(x,y) \in \mathcal{X}\times\mathcal{X} : \|(x,y)\| \leq \rho\},$$

with

$$\rho \geq \frac{M_1(\sigma_1+\sigma_3) + M_2(\sigma_2+\sigma_4)}{1 - [K_1(\sigma_1+\sigma_3) + K_2(\sigma_2+\sigma_4) + L_1(\sigma_7+\sigma_9) + L_2(\sigma_8+\sigma_{10}) + \sigma_5 + \sigma_6]}. \tag{3.14}$$

For an arbitrary element $(x,y) \in B_\rho$ and for each $t \in [a,b]$, we have

$$\begin{aligned}
|\mathcal{T}_1(x,y)(t)| \leq\ & I_a^{\alpha_1+\alpha_2+\zeta_1}(|g_1(x(t),y(t)) - g_1(0,0)| + |g_1(0,0)|) \\
&+ I_a^{\alpha_1+\alpha_2}(|f_1(t,x(t),y(t)) - f_1(t,0,0)| + |f_1(t,0,0)|) \\
&+ |\lambda_1| I_a^{\alpha_2}|x(t)| + \frac{(b-a)^{\alpha_2+\epsilon_1-1}}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big\{\Omega_2\Big(|\lambda_1| I_a^{\alpha_2}|x(b)| \\
&+ I_a^{\alpha_1+\alpha_2+\zeta_1}(|g_1(x(b),y(b)) - g_1(0,0)| + |g_1(0,0)|) \\
&+ I_a^{\alpha_1+\alpha_2}(|f_1(b,x(b),y(b)) - f_1(b,0,0)| + |f_1(b,0,0)|) + |\lambda_2| \sum_{i=1}^{m} |\mu_i| I_a^{\alpha_4}|y(\eta_i)| \\
&+ \sum_{i=1}^{m} |\mu_i| I_a^{\alpha_3+\alpha_4} I_b^{\zeta_2}(|g_2(x(\eta_i),y(\eta_i)) - g_2(0,0)| + |g_2(0,0)|) \\
&+ \sum_{i=1}^{m} |\mu_i| I_a^{\alpha_3+\alpha_4}(|f_2(\eta_i,x(\eta_i),y(\eta_i)) - f_2(\eta_i,0,0)| + |f_2(\eta_i,0,0)|) \\
&+ \sum_{k=1}^{n} |\nu_k| I_a^{q_k+\alpha_3+\alpha_4} I_b^{\zeta_2}(|g_2(x(\xi_k),y(\xi_k)) - g_2(0,0)| + |g_2(0,0)|) \\
&+ \sum_{k=1}^{n} |\nu_k| I_a^{q_k+\alpha_3+\alpha_4}(|f_2(\xi_k,x(\xi_k),y(\xi_k)) - f_2(\xi_k,0,0)| + |f_2(\xi_k,0,0)|) \\
&+ |\lambda_2| \sum_{k=1}^{n} |\nu_k| I_a^{q_k+\alpha_4}|y(\xi_k)|\Big) + \Omega_4\Big(|\lambda_2| I_a^{\alpha_4}|y(b)| \\
&+ I_a^{\alpha_3+\alpha_4}(|f_2(b,x(b),y(b)) - f_2(b,0,0)| + |f_2(b,0,0)|)
\end{aligned}$$

$$
\begin{aligned}
& +\sum_{i=1}^{m}|\delta_i|I_a^{\alpha_1+\alpha_2+\zeta_1}(|g_1(x(\eta_i),y(\eta_i))-g_1(0,0)|+|g_1(0,0)|)\\
& +\sum_{i=1}^{m}|\delta_i|I_a^{\alpha_1+\alpha_2}(|f_1(\eta_i,x(\eta_i),y(\eta_i))-f_1(\eta_i,0,0)|+|f_1(\eta_i,0,0)|)\\
& +\sum_{k=1}^{n}|\theta_k|I_a^{p_k+\alpha_1+\alpha_2+\zeta_1}(|g_1(x(\xi_k),y(\eta_i))-g_1((0,0)|+|g_1(0,0)|)\\
& +\sum_{k=1}^{n}|\theta_k|I_a^{p_k+\alpha_1+\alpha_2}(|f_1(\xi_k,x(\xi_k),y(\xi_k))-f_1(\xi_k,0,0)|+|f_1(\xi_k,0,0)|)\\
& +|\lambda_1|\sum_{i=1}^{m}|\delta_i|I_a^{\alpha_2}|x(\eta_i)|+|\lambda_1|\sum_{k=1}^{n}|\theta_k|I_a^{p_k+\alpha_2}|x(\xi_k)|\Big)\Big\}\\
\leq\ & \sigma_1[K_1(\|x\|+\|y\|)+M_1]+\sigma_2[K_2(\|x\|+\|y\|)+M_2]+\sigma_7L_1(\|x\|+\|y\|)\\
& +\sigma_8L_2(\|x\|+\|y\|)+\sigma_5(\|x\|+\|y\|).
\end{aligned}
$$

In a similar manner, one can find that

$$
\begin{aligned}
\|\mathcal{T}_2(x,y)\| \leq\ & \sigma_3[K_1(\|x\|+\|y\|)+M_1]+\sigma_4[K_2(\|x\|+\|y\|)+M_2]\\
& +\sigma_9L_1(\|x\|+\|y\|)+\sigma_{10}L_2(\|x\|+\|y\|)+\sigma_6(\|x\|+\|y\|).
\end{aligned}
$$

Adding the last two inequalities and using (3.14), we obtain

$$
\begin{aligned}
\|\mathcal{T}(x,y)\| \leq\ & (\sigma_1+\sigma_3)[K_1(\|x\|+\|y\|)+M_1]+(\sigma_2+\sigma_4)[K_2(\|x\|+\|y\|)+M_2]\\
& +(\sigma_7+\sigma_9)L_1(\|x\|+\|y\|)+(\sigma_8+\sigma_{10})L_2(\|x\|+\|y\|)\\
& +(\sigma_5+\sigma_6)(\|x\|+\|y\|)\\
\leq\ & \rho,
\end{aligned}
$$

which shows that $\mathcal{T}B_\rho\subset B_\rho$.

Next, we show that $\mathcal{T}$ is a contraction on $\mathcal{X}\times\mathcal{X}$. For that, let $(x,y),(x_1,y_1)\in\mathcal{X}\times\mathcal{X}$. Then we have

$$
\begin{aligned}
& |\mathcal{T}_1(x,y)(t)-\mathcal{T}_1(x_1,y_1)(t)|\\
\leq\ & I_a^{\alpha_1+\alpha_2+\zeta_1}|g_1(x(t),y(t))-g_1(x_1(t),y_1(t))|+I_a^{\alpha_1+\alpha_2}|f_1(t,x(t),y(t))-f_1(t,x_1(t),y_1(t))|\\
& +|\lambda_1|I_a^{\alpha_2}|x(t)-x_1(t)|+\frac{\Omega_1}{|\Delta|}\Big\{\Omega_2\Big(I_a^{\alpha_1+\alpha_2}|f_1(b,x(b),y(b))-f_1(b,x_1(b),y_1(b))|\\
& +|\lambda_1|I_a^{\alpha_2}|x(b)-x_1(b)|+I_a^{\alpha_1+\alpha_2+\zeta_1}|g_1(x(b),y(b))-g_1(x_1(b),y_1(b))|\\
& +|\lambda_2|\sum_{i=1}^{m}|\mu_i|I_a^{\alpha_4}|y(\eta_i)-y_1(\eta_i)|\\
& +\sum_{i=1}^{m}|\mu_i|I_a^{\alpha_3+\alpha_4}I_b^{\zeta_2}(|g_2(x(\eta_i),y(\eta_i))-g_2(x_1(\eta_i),y_1(\eta_i))|)\\
& +\sum_{i=1}^{m}|\mu_i|I_a^{\alpha_3+\alpha_4}|f_2(\eta_i,x(\eta_i),y(\eta_i))-f_2(\eta_i,x_1(\eta_i),y_1(\eta_i))|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}|\nu_k|I_a^{q_k+\alpha_3+\alpha_4}|f_2(\xi_k,x(\xi_k),y(\xi_k))-f_2(\xi_k,x_1(\xi_k),y_1(\xi_k))| \\
&+\sum_{k=1}^{n}|\nu_k|I_a^{q_k+\alpha_3+\alpha_4}I_b^{\zeta_2}(|g_2(x(\xi_k),y(\xi_k))-g_2(x_1(\xi_k),y_1(\xi_k))|) \\
&+|\lambda_2|\sum_{k=1}^{n}|\nu_k|I_a^{q_k+\alpha_4}|y(\xi_k)-y_1(\xi_k)|\Big) \\
&+|\Omega_4|\Big(I_a^{\alpha_3+\alpha_4}|f_2(b,x(b),y(b))-f_2(b,x_1(b),y_1(b))|+|\lambda_2|I_a^{\alpha_4}|y(b)-y_1(b)| \\
&+\sum_{i=1}^{m}|\delta_i|I_a^{\alpha_1+\alpha_2}|f_1(\eta_i,x(\eta_i),y(\eta_i))-f_1(\eta_i,x_1(\eta_i),y_1(\eta_i))| \\
&+\sum_{i=1}^{m}|\delta_i|I_a^{\alpha_1+\alpha_2+\zeta_1}|g_1(x(\eta_i),y(\eta_i))-g_1(x_1(\eta_i),y_1(\eta_i))| \\
&+\sum_{k=1}^{n}|\theta_k|I_a^{p_k+\alpha_1+\alpha_2+\zeta_1}|g_1(x(\xi_k),y(\xi_k))-g_1(x_1(\xi_k),y_1(\xi_k))| \\
&+|\lambda_1|\sum_{k=1}^{n}|\theta_k|I_a^{p_k+\alpha_2}|x(\xi_k)-x_1(\xi_k)| \\
&+\sum_{k=1}^{n}|\theta_k|I_a^{p_k+\alpha_1+\alpha_2}|f_1(\xi_k,x(\xi_k),y(\xi_k))-f_1(\xi_k,x_1(\xi_k),y_1(\xi_k))| \\
&+|\lambda_1|\sum_{i=1}^{m}|\delta_i|I_a^{\alpha_2}|x(\eta_i)-x_1(\eta_i)|\Big)\Big\},
\end{aligned}
$$

which leads to

$$\|\mathcal{T}_1(x,y)-\mathcal{T}_1(x_1,y_1)\| \le [\sigma_1K_1+\sigma_2K_2+\sigma_7L_1+\sigma_8L_2+\sigma_5](\|x-x_1\|+\|y-y_1\|).$$

Similarly one can obtain

$$\|\mathcal{T}_2(x,y)-\mathcal{T}_2(x_1,y_1)\| \le [\sigma_3K_1+\sigma_4K_2+\sigma_9L_1+\sigma_{10}L_2+\sigma_6](\|x-x_1\|+\|y-y_1\|).$$

It follows from the last two inequalities that

$$
\begin{aligned}
\|\mathcal{T}(x,y)-\mathcal{T}(x_1,y_1)\| \le\ & [(\sigma_1+\sigma_3)K_1+(\sigma_4+\sigma_2)K_2+(\sigma_9+\sigma_7)L_1+(\sigma_{10}+\sigma_8)L_2 \\
& +\sigma_5+\sigma_6][\|x-x_1\|+\|y-y_1\|].
\end{aligned}
$$

Since $(\sigma_1+\sigma_3)K_1+(\sigma_4+\sigma_2)K_2+(\sigma_9+\sigma_7)L_1+(\sigma_{10}+\sigma_8)L_2+\sigma_5+\sigma_6<1$ by (3.13), therefore $\mathcal{T}$ is a contraction and hence by Banach's contraction mapping principle, the operator $\mathcal{T}$ has a unique fixed point. In consequence, the problem (1.3) has a unique solution on $[a,b]$. The proof is completed. □

The next existence result is based on Krasnosel'skiĭ's fixed point theorem.

Lemma 3.1. *(Krasnosel'skiĭ's fixed point theorem). [30] Let B be a closed, convex, bounded and nonempty subset of a Banach space X. Let E_1 and E_2 be the operators such that (i) $E_1x+E_2y\in B$*

whenever $x, y \in B$; (ii) E_1 is compact and continuous; (iii) E_2 is a contraction mapping. Then there exists $z \in B$ such that $z = E_1 z + E_2 z$.

Theorem 3.2. *Assume that (H_1), (H_2) and (H_4) hold and*

$$\sigma_5 + \sigma_6 < 1, \tag{3.15}$$

where σ_5 and σ_6 are given by (3.7) and (3.8) respectively. Then the problem (1.3) has at least one solution on $[a, b]$.

Proof. Let us split the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ defined by (3.1) and (3.2) respectively into four operators as follows

$$\mathcal{T}_1(x,y)(t) = \mathcal{T}_{1,1}(x,y)(t) + \mathcal{T}_{1,2}(x,y)(t),\ \ \mathcal{T}_2(x,y)(t) = \mathcal{T}_{2,1}(x,y)(t) + \mathcal{T}_{2,2}(x,y)(t),$$

where

$$\begin{aligned}
\mathcal{T}_{1,1}(x,y)(t) &= I_a^{\alpha_1+\alpha_2+\zeta_1} g_1(x(t),y(t)) + I_a^{\alpha_1+\alpha_2} f_1(t,x(t),y(t)) \\
&\quad + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)} \times \Big\{ \Omega_2\Big(- I_a^{\alpha_1+\alpha_2+\zeta_1} g_1(x(b),y(b)) - I_a^{\alpha_1+\alpha_2} f_1(b,x(b),y(b)) \\
&\quad + \sum_{i=1}^{m} \mu_i I_a^{\alpha_3+\alpha_4}(I_b^{\zeta_2} g_2(x(\eta_i),y(\eta_i))) + \sum_{i=1}^{m} \mu_i I_a^{\alpha_3+\alpha_4} f_2(\eta_i,x(\eta_i),y(\eta_i)) \\
&\quad + \sum_{k=1}^{n} \nu_k I_a^{q_k+\alpha_3+\alpha_4}(I_b^{\zeta_2} g_2(x(\xi_k),y(\xi_k))) + \sum_{k=1}^{n} \nu_k I_a^{q_k+\alpha_3+\alpha_4} f_2(\xi_k,x(\xi_k),y(\xi_k))\Big) \\
&\quad + \Omega_4\Big(- I_a^{\alpha_3+\alpha_4} f_2(b,x(b),y(b)) + \sum_{i=1}^{m} \delta_i I_a^{\alpha_1+\alpha_2+\zeta_1} g_1(x(\eta_i),y(\eta_i)) \\
&\quad + \sum_{i=1}^{m} \delta_i I_a^{\alpha_1+\alpha_2} f_1(\eta_i,x(\eta_i),y(\eta_i)) + \sum_{k=1}^{n} \theta_k I_a^{p_k+\alpha_1+\alpha_2+\zeta_1} g_1(x(\xi_k),y(\xi_k)) \\
&\quad + \sum_{k=1}^{n} \theta_k I_a^{p_k+\alpha_1+\alpha_2} f_1(\xi_k,x(\xi_k),y(\xi_k))\Big)\Big\}, \\
\mathcal{T}_{1,2}(x,y)(t) &= -\lambda_1 I_a^{\alpha_2} x(t) + \frac{(t-a)^{\alpha_2+\epsilon_1-1}}{\Delta\Gamma(\alpha_2+\epsilon_1)} \Big\{ \Omega_2\Big(-\lambda_1 I_a^{\alpha_2} x(b) - \lambda_2 \sum_{i=1}^{m} \mu_i I_a^{\alpha_4} y(\eta_i) \\
&\quad - \lambda_2 \sum_{k=1}^{n} \nu_k I_a^{q_k+\alpha_4} y(\xi_k)\Big) + \Omega_4\Big(\lambda_2 I_a^{\alpha_4} y(b) - \lambda_1 \sum_{i=1}^{m} \delta_i I_a^{\alpha_2} x(\eta_i) \\
&\quad - \lambda_1 \sum_{k=1}^{n} \theta_k I_a^{p_k+\alpha_2} x(\xi_k)\Big)\Big\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_{2,1}(x,y)(t) &= I_a^{\alpha_3+\alpha_4}(I_b^{\zeta_2} g_2(x(t),y(t))) + I_a^{\alpha_3+\alpha_4} f_2(t,x(t),y(t)) \\
&\quad + \frac{(t-a)^{\alpha_4+\epsilon_3-1}}{\Delta\Gamma(\alpha_4+\epsilon_3)} \times \Big\{ \Omega_3\Big(- I_a^{\alpha_1+\alpha_2+\zeta_1} g_1(x(b),y(b)) - I_a^{\alpha_1+\alpha_2} f_1(b,x(b),y(b))
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\mu_i I_a^{\alpha_3+\alpha_4}(I_b^{\zeta_2}g_2(x(\eta_i),y(\eta_i)))+\sum_{i=1}^{m}\mu_i I_a^{\alpha_3+\alpha_4}f_2(\eta_i,x(\eta_i),y(\eta_i))\\
&+\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_3+\alpha_4}(I_b^{\zeta_2}g_2(x(\xi_k),y(\xi_k)))+\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_3+\alpha_4}f_2(\xi_k,x(\xi_k),y(\xi_k))\Big)\\
&+\Omega_1\Big(-I_a^{\alpha_3+\alpha_4}f_2(b,x(b),y(b))+\sum_{i=1}^{m}\delta_i I_a^{\alpha_1+\alpha_2+\zeta_1}g_1(x(\eta_i),y(\eta_i))\\
&+\sum_{i=1}^{m}\delta_i I_a^{\alpha_1+\alpha_2}f_1(\eta_i,x(\eta_i),y(\eta_i))+\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_1+\alpha_2+\zeta_1}g_1(x(\xi_k),y(\xi_k))\\
&+\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_1+\alpha_2}f_1(\xi_k,x(\xi_k),y(\xi_k))\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_{2,2}(x,y)(t) \;=\;& -\lambda_2 I_a^{\alpha_4}y(t)+\frac{(t-a)^{\alpha_4+\epsilon_3-1}}{\Delta\Gamma(\alpha_4+\epsilon_3)}\times\Big\{\Omega_3\Big(\lambda_1 I_a^{\alpha_2}x(b)-\lambda_2\sum_{i=1}^{m}\mu_i I_a^{\alpha_4}y(\eta_i)\\
&-\lambda_2\sum_{k=1}^{n}\nu_k I_a^{q_k+\alpha_4}y(\xi_k)\Big)+\Omega_1\Big(\lambda_2 I_a^{\alpha_4}y(b)-\lambda_1\sum_{i=1}^{m}\delta_i I_a^{\alpha_2}x(\eta_i)\\
&-\lambda_1\sum_{k=1}^{n}\theta_k I_a^{p_k+\alpha_2}x(\xi_k)\Big)\Big\}.
\end{aligned}
$$

Now we verify the hypotheses of Krasnosel'skiĭ's fixed point theorem (Lemma 3.1) in three steps.

(i) In this step, it will be shown that $\mathcal{T}_1(x,y)+\mathcal{T}_2(\widehat{x,y})\in B_r$ for all $(x,y),(\widehat{x,y})\in B_r$, where $B_r\subset\mathcal{X}\times\mathcal{X}$ is a bounded closed ball with radius

$$
r\geqslant\frac{(\sigma_1+\sigma_3)\|\phi_1\|+(\sigma_2+\sigma_4)\|\phi_2\|+(\sigma_7+\sigma_9)\Lambda_1+(\sigma_8+\sigma_{10})\Lambda_2}{1-\sigma_5-\sigma_6}.
$$

As in the proof of Theorem 3.1, we can find that

$$
|\mathcal{T}_{1,1}(x,y)(t)+\mathcal{T}_{1,2}(x,y)(t)|\le\sigma_1\|\phi_1\|+\sigma_2\|\phi_2\|+\sigma_7\Lambda_1+\sigma_8\Lambda_2+\sigma_5 r,
$$

and

$$
|\mathcal{T}_{2,1}(\widehat{x,y})(t)+\mathcal{T}_{2,2}(\widehat{x,y})(t)|\le\sigma_3\|\phi_1\|+\sigma_4\|\phi_2\|+\sigma_9\Lambda_1+\sigma_{10}\Lambda_2+\sigma_6 r,
$$

which lead to the inequality

$$
\begin{aligned}
\|\mathcal{T}_1(x,y)+\mathcal{T}_2(\widehat{x,y})\| \;\le\;& (\sigma_1+\sigma_3)\|\phi_1\|+(\sigma_2+\sigma_4)\|\phi_2\|+(\sigma_7+\sigma_9)\Lambda_1\\
&+(\sigma_8+\sigma_{10})\Lambda_2+(\sigma_5+\sigma_6)r\le r.
\end{aligned}
$$

Thus $\mathcal{T}_1(x,y)+\mathcal{T}_2(\widehat{x,y})\in B_r$.

(ii) Here we establish that $(\mathcal{T}_{1,2},\mathcal{T}_{2,2})$ is a contraction mapping. Let $(x,y),(\widehat{x,y})\in B_r$. Then it is easy to find that

$$
|\mathcal{T}_{1,2}(x,y)(t)-\mathcal{T}_{1,2}(\widehat{x,y})(t)|\le\sigma_5[\|x-\widehat{x}\|+\|y-\widehat{y}\|],
$$

$$
|\mathcal{T}_{2,2}(x,y)(t)-\mathcal{T}_{2,2}(\widehat{x,y})|\le\sigma_6[\|x-\widehat{x}\|+\|y-\widehat{y}\|].
$$

Consequently, we get

$$\|(\mathcal{T}_{1,2},\mathcal{T}_{2,2})(x,y)-(\mathcal{T}_{1,2},\mathcal{T}_{2,2})(\widehat{x,y})\| \le (\sigma_5+\sigma_6)[\|x-\widehat{x}\|+\|y-\widehat{y}\|],$$

which, by (3.15), implies that $(\mathcal{T}_{1,2},\mathcal{T}_{2,2})$ is a contraction.

(iii) We show that $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is compact and continuous.

Continuity of $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is obvious. For $(x,y)\in B_r$, we have

$$\begin{aligned}
|\mathcal{T}_{1,1}(x,y)(t)| \le\ & I_a^{\alpha_1+\alpha_2+\zeta_1}(|g_1(x(t),y(t))|)+I_a^{\alpha_1+\alpha_2}(|f_1(t,x(t),y(t))|)\\
&+\frac{(b-a)^{\alpha_2+\epsilon_1-1}}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big\{\Omega_2\Big(I_a^{\alpha_1+\alpha_2+\zeta_1}(|g_1(x(b),y(b))|)+I_a^{\alpha_1+\alpha_2}(|f_1(b,x(b),y(b))|)\\
&+\sum_{i=1}^m|\mu_i|I_a^{\alpha_3+\alpha_4}(I_b^{\zeta_2}(|g_2(x(\eta_i),y(\eta_i))|)+\sum_{i=1}^m|\mu_i|I_a^{\alpha_3+\alpha_4}(|f_2(\eta_i,x(\eta_i),y(\eta_i))|)\\
&+\sum_{k=1}^n|\nu_k|I_a^{q_k+\alpha_3+\alpha_4}(I_b^{\zeta_2}(|g_2(x(\xi_k),y(\xi_k))|)+\sum_{k=1}^n|\nu_k|I_a^{q_k+\alpha_3+\alpha_4}(|f_2(\xi_k,x(\xi_k),y(\xi_k))|)\Big)\\
&+\Omega_4\Big(I_a^{\alpha_3+\alpha_4}(|f_2(b,x(b),y(b))|)+\sum_{i=1}^m\delta_iI_a^{\alpha_1+\alpha_2+\zeta_1}(|g_1(x(\eta_i),y(\eta_i))|)\\
&+\sum_{i=1}^m|\delta_i|I_a^{\alpha_1+\alpha_2}(|f_1(\eta_i,x(\eta_i),y(\eta_i))|)+\sum_{k=1}^n\theta_kI_a^{p_k+\alpha_1+\alpha_2+\zeta_1}(|g_1(x(\xi_k),y(\eta_i))|)\\
&+\sum_{k=1}^n|\theta_k|I_a^{p_k+\alpha_1+\alpha_2}(|f_1(\xi_k,x(\xi_k),y(\xi_k))|)\Big)\Big\}\\
\le\ & \sigma_1\|\phi_1\|+\sigma_2\|\phi_2\|+\sigma_7\Lambda_1+\sigma_8\Lambda_2.
\end{aligned}$$

In a similar manner, we can get $|\mathcal{T}_{2,1}(x,y)(t)|\le\sigma_3\|\phi_1\|+\sigma_4\|\phi_2\|+\sigma_9\Lambda_1+\sigma_{10}\Lambda_2$. Thus

$$\|(\mathcal{T}_{1,1},\mathcal{T}_{2,1})(x,y)\|\le(\sigma_1+\sigma_3)\|\phi_1\|+(\sigma_2+\sigma_4)\|\phi_2\|+(\sigma_7+\sigma_9)\Lambda_1+(\sigma_8+\sigma_{10})\Lambda_2,$$

which means that $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is uniformly bounded on B_r.

In order to show the equicontinuity of $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$, we take $t_1,t_2\in[a,b]$ with $t_1<t_2$. Then, for arbitrary $(x,y)\in B_r$, we obtain

$$\begin{aligned}
&|\mathcal{T}_{1,1}(x,y)(t_2)-\mathcal{T}_{1,1}(x,y)(t_1)|\\
\le\ & \Big|\int_a^{t_2}\frac{(t_2-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f_1(s,x(s),y(s))ds-\int_a^{t_1}\frac{(t_1-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f_1(s,x(s),y(s))ds\Big|\\
&+\Big|\int_a^{t_2}\frac{(t_2-s)^{\alpha_1+\alpha_2+\zeta_1-1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1)}g_1(x(s),y(s))ds-\int_a^{t_1}\frac{(t_1-s)^{\alpha_1+\alpha_2+\zeta_1-1}}{\Gamma(\alpha_1+\alpha_2)+\zeta_1}g_1(x(s),y(s))ds\Big|\\
&+\frac{|(t_2-a)^{\alpha_2+\epsilon_1-1}-(t_1-a)^{\alpha_2+\epsilon_1-1}|}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big\{\Omega_2\Big(I_a^{\alpha_1+\alpha_2+\zeta_1}(|g_1(x(b),y(b))|)\\
&+I_a^{\alpha_1+\alpha_2}(|f_1(b,x(b),y(b))|)+\sum_{i=1}^m|\mu_i|I_a^{\alpha_3+\alpha_4}(I_b^{\zeta_2}(|g_2(x(\eta_i),y(\eta_i))|)\\
&+\sum_{i=1}^m|\mu_i|I_a^{\alpha_3+\alpha_4}(|f_2(\eta_i,x(\eta_i),y(\eta_i))|)+\sum_{k=1}^n|\nu_k|I_a^{q_k+\alpha_3+\alpha_4}(I_b^{\zeta_2}(|g_2(x(\xi_k),y(\xi_k))|)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}|\nu_k|I_a^{q_k+\alpha_3+\alpha_4}(|f_2(\xi_k,x(\xi_k),y(\xi_k))|)\Big)+\Omega_4\Big(I_a^{\alpha_3+\alpha_4}(|f_2(b,x(b),y(b))|)\\
&+\sum_{i=1}^{m}|\delta_i|I_a^{\alpha_1+\alpha_2+\zeta_1}(|g_1(x(\eta_i),y(\eta_i))|)+\sum_{i=1}^{m}|\delta_i|I_a^{\alpha_1+\alpha_2}(|f_1(\eta_i,x(\eta_i),y(\eta_i))|)\\
&+\sum_{k=1}^{n}|\theta_k|I_a^{p_k+\alpha_1+\alpha_2+\zeta_1}(|g_1(x(\xi_k),y(\eta_i))|)+\sum_{k=1}^{n}|\theta_k|I_a^{p_k+\alpha_1+\alpha_2}(|f_1(\xi_k,x(\xi_k),y(\xi_k))|)\Big)\Big\}\\
\leqslant\ &\frac{\|\phi_1\|}{\Gamma(\alpha_1+\alpha_2+1)}\Big\{2(t_2-t_1)^{\alpha_1+\alpha_2}+|(t_2-a)^{\alpha_1+\alpha_2}-(t_1-a)^{\alpha_1+\alpha_2}|\Big\}\\
&+\frac{\Lambda_1}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}\Big\{2(t_2-t_1)^{\alpha_1+\alpha_2+\zeta_1}+|(t_2-a)^{\alpha_1+\alpha_2+\zeta_1}-(t_1-a)^{\alpha_1+\alpha_2+\zeta_1}|\Big\}\\
&+\frac{|(t_2-a)^{\alpha_2+\epsilon_1-1}-(t_1-a)^{\alpha_2+\epsilon_1-1}|}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big\{\Omega_2\Big(\frac{\|\phi_1\|(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\\
&+\frac{\Lambda_1(b-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}+\sum_{i=1}^{m}\frac{|\mu_i|\|\phi_2\|(\eta_i-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}\\
&+\sum_{i=1}^{m}\frac{|\mu_i|\Lambda_2(b-a)^{\zeta_2}(\eta_i-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)\Gamma(\zeta_2+1)}+\sum_{k=1}^{n}\frac{|\nu_k|\|\phi_2\|(\xi_k-a)^{q_k+\alpha_3+\alpha_4}}{\Gamma(q_k+\alpha_3+\alpha_4+1)}\Big)\\
&+\sum_{k=1}^{n}\frac{|\nu_k|\Lambda_2(b-a)^{\zeta_2}(\xi_k-a)^{q_k+\alpha_3+\alpha_4}}{\Gamma(\zeta_2+1)\Gamma(q_k+\alpha_3+\alpha_4+1)}\Big)\\
&+|\Omega_4|\Big(\frac{\|\phi_2\|(b-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}+\sum_{i=1}^{m}\frac{|\delta_i|\|\phi_1\|(\eta_i-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\\
&+\sum_{i=1}^{m}\frac{|\delta_i|\Lambda_1(\eta_i-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}+\sum_{k=1}^{n}\frac{|\theta_k|\|\phi_1\|(\xi_k-a)^{p_k+\alpha_1+\alpha_2}}{\Gamma(p_k+\alpha_1+\alpha_2+1)}\\
&+\sum_{k=1}^{n}\frac{|\theta_k|\Lambda_1(\xi_k-a)^{p_k+\alpha_1+\alpha_2+\zeta_1}}{\Gamma(p_k+\alpha_1+\alpha_2+\zeta_1+1)}\Big)\Big\}\to 0,
\end{aligned}
$$

as $t_2 \to t_1$ independently of $(x, y) \in B_r$. Also

$$
\begin{aligned}
&|\mathcal{T}_{2,1}(x,y)(t_2)-\mathcal{T}_{2,1}(x,y)(t_1)|\\
\leq\ &\frac{\|\phi_2\|}{\Gamma(\alpha_3+\alpha_4+1)}\Big\{2(t_2-t_1)^{\alpha_3+\alpha_4}+|(t_2-a)^{\alpha_3+\alpha_4}-(t_1-a)^{\alpha_3+\alpha_4}|\Big\}\\
&+\frac{\Lambda_2(b-a)^{\zeta_2}}{\Gamma(\zeta_2+1)\Gamma(\alpha_3+\alpha_4+1)}\Big\{2(t_2-t_1)^{\alpha_3+\alpha_4}+|(t_2-a)^{\alpha_3+\alpha_4}-(t_1-a)^{\alpha_3+\alpha_4}|\Big\}\\
&+\frac{|(t_2-a)^{\alpha_4+\epsilon_3-1}-(t_1-a)^{\alpha_4+\epsilon_3-1}|}{|\Delta|\Gamma(\alpha_4+\epsilon_3)}\Big\{|\Omega_3|\Big(\frac{\|\phi_1\|(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\\
&+\frac{\Lambda_1(b-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}+\sum_{i=1}^{m}\frac{|\mu_i|\|\phi_2\|(\eta_i-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}\\
&+\sum_{i=1}^{m}\frac{|\mu_i|\Lambda_2(b-a)^{\zeta_2}(\eta_i-a)^{\alpha_3+\alpha_4}}{\Gamma(\zeta_2+1)\Gamma(\alpha_3+\alpha_4+1)}+\sum_{k=1}^{n}\frac{|\nu_k|\|\phi_2\|(\xi_k-a)^{q_k+\alpha_3+\alpha_4}}{\Gamma(q_k+\alpha_3+\alpha_4+1)}
\end{aligned}
$$

$$+\sum_{k=1}^{n}\frac{|\nu_k|\Lambda_2(b-a)^{\zeta_2}(\xi_k-a)^{q_k+\alpha_3+\alpha_4}}{\Gamma(\zeta_2+1)\Gamma(q_k+\alpha_3+\alpha_4+1)}\Big)+\Omega_1\Big(\frac{\|\phi_2\|(b-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}$$
$$+\sum_{i=1}^{m}\frac{|\delta_i|\|\phi_1\|(\eta_i-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\sum_{i=1}^{m}\frac{|\delta_i|\Lambda_1(\eta_i-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}$$
$$+\sum_{k=1}^{n}\frac{|\theta_k|\|\phi_1\|(\xi_k-a)^{p_k+\alpha_1+\alpha_2}}{\Gamma(p_k+\alpha_1+\alpha_2+1)}+\sum_{k=1}^{n}\frac{|\theta_k|\Lambda_1(\xi_k-a)^{p_k+\alpha_1+\alpha_2+\zeta_1}}{\Gamma(p_k+\alpha_1+\alpha_2+\zeta_1+1)}\Big)\Big\}\to 0,$$

as $t_2 \to t_1$ independently of $(x, y) \in B_r$. Thus $|(\mathcal{T}_{1,1}, \mathcal{T}_{2,1})(x, y)(t_2) - (\mathcal{T}_{1,1}, \mathcal{T}_{2,1})(x, y)(t_1)|$ vanishes as $t_2 \to t_1$ independently of $(x, y) \in B_r$, which shows that $(\mathcal{T}_{1,1}, \mathcal{T}_{2,1})$ is equicontinuous. So we deduce by the Arzelá-Ascoli theorem that $(\mathcal{T}_{1,1}, \mathcal{T}_{2,1})$ is compact on B_r.

It follows from the steps $(i) - (iii)$ that the hypotheses of Krasnosel'skiĭ's fixed point theorem are satisfied, so its conclusion implies that the problem (1.3) has at least one solution on $[a, b]$. This finishes the proof. □

Remark 3.1. *The conclusion of Theorem 3.2 can also be achieved by assuming* $(H_1), (H_2), (H_4)$ *and the condition:* $(\sigma_1+\sigma_3)K_1+(\sigma_2+\sigma_4)K_2+(\sigma_7+\sigma_9)L_1+(\sigma_8+\sigma_{10})L_2 < 1$, *where* $\sigma_1, \dots, \sigma_4$ *are given in (3.3)–(3.6) and* $\sigma_7, \dots, \sigma_{10}$ *are given in (3.9)–(3.12).*

In the following result, we prove the existence of solutions for the problem (1.3) by applying the Leray-Schauder alternative [29].

Lemma 3.2. *(Leray-Schauder alternative [29]) Let E be a Banach space, M be closed, convex subset of E, U is an open subset of C and* $0 \in U$. *Suppose that* $F : \overline{U} \to C$ *is continuous, compact map (that is,* $F(U)$ *is a relatively compact subset of C). Then either (i) F has a fixed point in* $\overline{U}$, *or (ii) there are* $u \in \partial U$, *and* $\lambda \in (0, 1)$ *with* $u = \lambda F(U)$.

Theorem 3.3. *Assume that* (H_3) *holds. Then there exists at least one solution for the problem (1.3) on* $[a, b]$ *provided that*

$$(\sigma_1+\sigma_3)u_i+(\sigma_2+\sigma_4)v_i+(\sigma_7+\sigma_9)\omega_i+(\sigma_8+\sigma_{10})\tau_i+(\sigma_5+\sigma_6)<1\ ,\ i=1,2, \tag{3.16}$$

where $\sigma_1, \dots, \sigma_{10}$ *are given in (3.3)–(3.12).*

Proof. For all $(x, y) \in B_\rho \subset \mathcal{X}\times\mathcal{X}$, where B_ρ defined by (3.14), there exist positive constants $N_1, \dots, N_4$ such that $|f_1(t, x, y)| \le N_1, |f_2(t, x, y)| \le N_2, |g_1(x, y)| \le N_3, |g_2(x, y)| \le N_4$, Then we show that $\mathcal{T} : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ is completely continuous. Observe that continuity of f_1, f_2, g_1, g_2 implies that of the operator $\mathcal{T}$. For $(x, y) \in B_\rho$, as in the proof of Theorem 3.1, we have

$$\begin{aligned}\|\mathcal{T}_1(x,y)\| &\le \sigma_1N_1+\sigma_2N_2+\sigma_7N_3+\sigma_8N_4+\rho\sigma_5,\\ \|\mathcal{T}_2(x,y)\| &\le \sigma_3N_1+\sigma_4N_2+\sigma_9N_3+\sigma_{10}N_4+\rho\sigma_6.\end{aligned}$$

From the preceding inequalities, we get

$$\|\mathcal{T}(x,y)\| \le (\sigma_1+\sigma_3)N_1+(\sigma_2+\sigma_4)N_2+(\sigma_7+\sigma_9)N_3+(\sigma_8+\sigma_{10})N_4+\rho(\sigma_5+\sigma_6),$$

which implies that $\mathcal{T}B_\rho$ is uniformly bounded.

Next we show that $\mathcal{T}B_\rho$ is equicontinues. Let $t_1, t_2 \in [a,b]$ with $t_2 > t_1$. Then, for arbitrary $(x,y) \in B_\rho$, we obtain

$$
\begin{aligned}
&|\mathcal{T}_1(x,y)(t_2) - \mathcal{T}_1(x,y)(t_1)| \\
\leq\ & \Big|\int_a^{t_2} \frac{(t_2-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1(s,x(s),y(s))ds - \int_a^{t_1} \frac{(t_1-s)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f_1(s,x(s),y(s))ds\Big| \\
&+\Big|\int_a^{t_2} \frac{(t_2-s)^{\alpha_1+\alpha_2+\zeta_1-1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1)} g_1(x(s),y(s))ds - \int_a^{t_1} \frac{(t_1-s)^{\alpha_1+\alpha_2+\zeta_1-1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1)} g_1(x(s),y(s))ds\Big| \\
&+\frac{|\lambda_1|}{\Gamma(\alpha_2)}\Big|\int_a^{t_1}[(t_2-s)^{\alpha_2-1}-(t_1-s)^{\alpha_2-1}]x(s)ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha_2-1}x(s)ds\Big| \\
&+\frac{|(t_2-a)^{\alpha_2+\epsilon_1-1}-(t_1-a)^{\alpha_2+\epsilon_1-1}|}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big\{\Omega_2\Big(I_a^{\alpha_1+\alpha_2+\zeta_1}(|g_1(x(b),y(b))|) \\
&+I_a^{\alpha_1+\alpha_2}(|f_1(b,x(b),y(b))|) + |\lambda_1| I_a^{\alpha_2}|x(b)| + |\lambda_2|\sum_{i=1}^m |\mu_i| I_a^{\alpha_4}|y(\eta_i)| \\
&+\sum_{i=1}^m |\mu_i| I_a^{\alpha_3+\alpha_4}(I_b^{\zeta_2}(|g_2(x(\eta_i),y(\eta_i))|) + \sum_{i=1}^m |\mu_i| I_a^{\alpha_3+\alpha_4}(|f_2(\eta_i,x(\eta_i),y(\eta_i))|) \\
&+\sum_{k=1}^n |\nu_k| I_a^{q_k+\alpha_3+\alpha_4}(I_b^{\zeta_2}(|g_2(x(\xi_k),y(\xi_k))|) + \sum_{k=1}^n |\nu_k| I_a^{q_k+\alpha_3+\alpha_4}(|f_2(\xi_k,x(\xi_k),y(\xi_k))|) \\
&+|\lambda_2|\sum_{k=1}^n |\nu_k| I_a^{q_k+\alpha_4}|y(\xi_k)|\Big) + \Omega_4\Big(|\lambda_2| I_a^{\alpha_4}|y(b)| + I_a^{\alpha_3+\alpha_4}(|f_2(b,x(b),y(b))|) \\
&+\sum_{i=1}^m |\delta_i| I_a^{\alpha_1+\alpha_2+\zeta_1}(|g_1(x(\eta_i),y(\eta_i))|) + \sum_{i=1}^m |\delta_i| I_a^{\alpha_1+\alpha_2}(|f_1(\eta_i,x(\eta_i),y(\eta_i))|) \\
&+|\lambda_1|\sum_{i=1}^m |\delta_i| I_a^{\alpha_2}|x(\eta_i)| + \sum_{k=1}^n |\theta_k| I_a^{p_k+\alpha_1+\alpha_2+\zeta_1}(|g_1(x(\xi_k),y(\eta_i))|) \\
&+\sum_{k=1}^n |\theta_k| I_a^{p_k+\alpha_1+\alpha_2}(|f_1(\xi_k,x(\xi_k),y(\xi_k))|) + |\lambda_1|\sum_{k=1}^n |\theta_k| I_a^{p_k+\alpha_2}|x(\xi_k)|\Big)\Big\} \\
&\leqslant \frac{N_1}{\Gamma(\alpha_1+\alpha_2+1)}\Big\{2(t_2-t_1)^{\alpha_1+\alpha_2} + |(t_2-a)^{\alpha_1+\alpha_2}-(t_1-a)^{\alpha_1+\alpha_2}|\Big\} \\
&+\frac{N_3}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}\Big\{2(t_2-t_1)^{\alpha_1+\alpha_2+\zeta_1} + |(t_2-a)^{\alpha_1+\alpha_2+\zeta_1}-(t_1-a)^{\alpha_1+\alpha_2+\zeta_1}|\Big\} \\
&+\frac{|\lambda_1|\rho}{\Gamma(\alpha_2+1)}\Big\{2(t_2-t_1)^{\alpha_2} + |(t_2-a)^{\alpha_2}-(t_1-a)^{\alpha_2}|\Big\} \\
&+\frac{|(t_2-a)^{\alpha_2+\epsilon_1-1}-(t_1-a)^{\alpha_2+\epsilon_1-1}|}{|\Delta|\Gamma(\alpha_2+\epsilon_1)}\Big\{\Omega_2\Big(\frac{|\lambda_1|\rho(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^m \frac{|\lambda_2\mu_i|\rho(\eta_i-a)^{\alpha_4}}{\Gamma(\alpha_4+1)} \\
&+\sum_{k=1}^n \frac{|\lambda_2\nu_k|\rho(\xi_k-a)^{q_k+\alpha_4}}{\Gamma(q_k+\alpha_4+1)} + \frac{N_1(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{N_3(b-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)} \\
&+\sum_{i=1}^m \frac{|\mu_i|N_2(\eta_i-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)} + \sum_{i=1}^m \frac{|\mu_i|N_4(b-a)^{\zeta_2}(\eta_i-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)\Gamma(\zeta_2+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\frac{|\nu_k|N_2(\xi_k-a)^{q_k+\alpha_3+\alpha_4}}{\Gamma(q_k+\alpha_3+\alpha_4+1)}+\sum_{k=1}^{n}\frac{|\nu_k|N_4(b-a)^{\zeta_2}(\xi_k-a)^{q_k+\alpha_3+\alpha_4}}{\Gamma(\zeta_2+1)\Gamma(q_k+\alpha_3+\alpha_4+1)}\Big)\\
&+|\Omega_4|\Big(\frac{N_2(b-a)^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}+\sum_{i=1}^{m}\frac{|\delta_i|N_1(\eta_i-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}\\
&+\sum_{i=1}^{m}\frac{|\delta_i|N_3(\eta_i-a)^{\alpha_1+\alpha_2+\zeta_1}}{\Gamma(\alpha_1+\alpha_2+\zeta_1+1)}+\sum_{k=1}^{n}\frac{|\theta_k|N_1(\xi_k-a)^{p_k+\alpha_1+\alpha_2}}{\Gamma(p_k+\alpha_1+\alpha_2+1)}\\
&+\sum_{k=1}^{n}\frac{|\theta_k|N_3(\xi_k-a)^{p_k+\alpha_1+\alpha_2+\zeta_1}}{\Gamma(p_k+\alpha_1+\alpha_2+\zeta_1+1)}+\frac{|\lambda_2|\rho(b-a)^{\alpha_4}}{\Gamma(\alpha_4+1)}\\
&+\sum_{i=1}^{m}\frac{|\lambda_1\delta_i|\rho(\eta_i-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}+\sum_{k=1}^{n}\frac{|\lambda_1\theta_k|\rho(\xi_k-a)^{p_k+\alpha_2}}{\Gamma(p_k+\alpha_2+1)}\Big)\Big\},
\end{aligned}
$$

which tends to zero as $t_1 \to t_2$ independent of $(x, y) \in B_\rho$. Similarly, it can be established that $|\mathcal{T}_2(x, y)(t_2) - \mathcal{T}_2(x, y)(t_1)| \to 0$ as $t_2 \to t_1$ independently of $(x, y) \in B_\rho$. Thus the operator $\mathcal{T}$ is equicontinuous. Hence, by the Arzelá-Ascoli theorem, the operator $\mathcal{T}$ is completely continuous.
Next we consider the set

$$\Omega = \{(x, y) \in \mathcal{X} \times \mathcal{X} : (x, y) = r\mathcal{T}(x, y), \ \ 0 \le r \le 1\},$$

and show that it is bounded. Let $(x, y) \in \Omega$, then $(x, y) = r\mathcal{T}(x, y)$ implies that $x(t) = r\mathcal{T}_1(x, y)(t)$, and $y(t) = r\mathcal{T}_2(x, y)(t), \forall t \in [a, b]$. By the condition (H_3), we obtain

$$
\begin{aligned}
\|x\| &\le \sigma_1(u_0+u_1|x|+u_2|y|)+\sigma_2(v_0+v_1|x|+v_2|y|)+\sigma_7(\omega_0+\omega_1|x|+\omega_2|y|)\\
&\quad+\sigma_8(\tau_0+\tau_1|x|+\tau_2|y|)+\sigma_5(\|x\|+\|y\|),\\
\|y\| &\le \sigma_3(u_0+u_1|x|+u_2|y|)+\sigma_4(v_0+v_1|x|+v_2|y|)+\sigma_9(\omega_0+\omega_1|x|+\omega_2|y|)\\
&\quad+\sigma_{10}(\tau_0+\tau_1|x|+\tau_2|y|)+\sigma_6(\|x\|+\|y\|).
\end{aligned}
$$

Adding the above inequalities, we get

$$
\begin{aligned}
\|x\|+\|y\| &\le (\sigma_1+\sigma_3)(u_0+u_1|x|+u_2|y|)+(\sigma_2+\sigma_4)(v_0+v_1|x|+v_2|y|)\\
&\quad+(\sigma_7+\sigma_9)(\omega_0+\omega_1|x|+\omega_2|y|)+(\sigma_8+\sigma_{10})(\tau_0+\tau_1|x|+\tau_2|y|)\\
&\quad+(\sigma_5+\sigma_6)(\|x\|+\|y\|),
\end{aligned}
$$

which leads to

$$\|(x, y)\| \le \frac{(\sigma_1+\sigma_3)u_0+(\sigma_2+\sigma_4)v_0+(\sigma_7+\sigma_9)\omega_0+(\sigma_8+\sigma_{10})\tau_0}{\sigma^*},$$

where

$$
\begin{aligned}
\sigma^* &= \min\{1-(\sigma_1+\sigma_3)u_1-(\sigma_2+\sigma_4)v_1-(\sigma_7+\sigma_9)\omega_1-(\sigma_8+\sigma_{10})\tau_1-(\sigma_5+\sigma_7),\\
&\quad 1-(\sigma_1+\sigma_3)u_2-(\sigma_2+\sigma_4)v_2-(\sigma_7+\sigma_9)\omega_2-(\sigma_8+\sigma_{10})\tau_2-(\sigma_5+\sigma_7)\}>0
\end{aligned}
$$

by condition (3.16). Thus the set Ω is bounded. Hence it follows by the Leray-Schauder alternative for single-valued maps [29] that the problem (1.3) has at least one solution on $[a, b]$, which completes the proof. □

4. Examples

Consider a coupled system of Hilfer fractional differential equations with boundary conditions:

$$\begin{cases} {}^HD^{1/2,3/4}({}^HD^{1/6,4/5}+1/90)x(t)=I_{0^+}^{1/2}g_1(x,y)+f_1(t,x,y),\ t\in[0,1],\\ {}^HD^{1/2,3/4}({}^HD^{1/2,1/7}+1/100)y(t)=I_{1^-}^{1/3}g_2(x,y)+f_2(t,x,y),\ t\in[0,1],\\ x(0)=y(0)=0,\\ x(1)=\frac{1}{100}y(1/10)+\frac{1}{200}y(1/5)+\frac{1}{300}y(3/10)+\frac{1}{400}y(2/5)+\frac{1}{500}y(1/2)\\ \qquad +\frac{1}{90}I_{0^+}^{1/2}y(3/5)+\frac{1}{70}I_{0^+}^{1/2}y(7/10)+\frac{1}{20}I_{0^+}^{1/2}y(4/5),\\ y(1)=\frac{1}{35}x(1/10)+\frac{1}{100}x(1/5)+\frac{1}{21}x(3/10)+\frac{1}{70}x(2/5)+\frac{1}{500}x(1/2)\\ \qquad +\frac{1}{100}I_{0^+}^{1/3}x(3/5)+\frac{1}{200}I_{0^+}^{1/3}x(7/10)+\frac{1}{300}I_{0^+}^{1/3}x(4/5). \end{cases} \tag{4.1}$$

Here $\alpha_1=1/2,\alpha_2=1/6,\alpha_3=1/2,\alpha_4=1/2,\beta_1=3/4,\beta_2=4/5,\beta_3=3/4,\beta_4=1/7,\lambda_1=1/90,\lambda_2=1/100,\epsilon_1=7/8=\epsilon_3,q_k=1/2,p_k=1/3,k=1,2,3,m=5,n=3,\eta_1=1/10,\eta_2=1/5,\eta_3=3/10,\eta_4=2/5,\eta_5=1/2,\xi_1=3/5,\xi_2=7/10,\xi_3=4/5,\mu_1=1/100,\mu_2=1/200,\mu_3=1/300,\mu_4=1/400,\mu_5=1/500,\nu_1=1/90,\nu_2=1/70,\nu_3=1/20,\delta_1=1/35,\delta_2=1/100,\delta_3=1/21,\delta_4=1/70,\delta_5=1/500,\theta_1=1/100,\theta_2=1/200,\theta_3=1/300,\zeta_1=1/2,\zeta_2=1/3$.

With the given data, it is found that $|\Delta|=1.1419,\sigma_1=2.2278,\sigma_3=0.1836,\sigma_2=0.1096,\sigma_4=2.0132,\sigma_5=0.0009,\sigma_6=0.0025,\sigma_7=1.8560,\sigma_8=0.0475,\sigma_9=0.1328,\sigma_{10}=1.1252$.

(a) *For illustrating Theorem 3.1*, we take

$$\begin{cases} f_1(t,x,y)=\dfrac{2\arctan x+\pi}{14\pi(1+t)}+\dfrac{1}{7(t+\pi)}\sin|y|,\\ f_2(t,x,y)=\dfrac{1}{7}\arctan x+\dfrac{3}{(21+t)}\dfrac{|y|}{(1+|y|)}+\dfrac{t^3}{(1+t^2)},\\ g_1(x,y)=\dfrac{1}{12}\Big(\dfrac{|x|}{(1+|x|)}+|y|\Big),\\ g_2(x,y)=\dfrac{1}{17}\Big(\sin|x|+\arctan|y|\Big). \end{cases} \tag{4.2}$$

It can easily be verified that f_1,f_2 satisfy the condition (H_1) with $K_1=1/7\pi,K_2=1/7$, respectively and g_1,g_2 satisfy the condition (H_2) with $L_1=1/12,L_2=1/17$, respectively. Furthermore

$$(\sigma_1+\sigma_3)K_1+(\sigma_4+\sigma_2)K_2+(\sigma_9+\sigma_7)L_1+(\sigma_{10}+\sigma_8)L_2+\sigma_5+\sigma_6\approx 0.65102<1.$$

Clearly the hypotheses of Theorem 3.1 are satisfied and hence it follows by its conclusion that the system (4.1) with $f_1(t,x,y),f_2(t,x,y),g_1(x,y)$ and $g_2(x,y)$ given by (4.2) has a unique solution on $[0,1]$. On the other hand, one can deduce that the system (4.1) with (4.2) has at least one solution on $[0,1]$ by the application of Remark 3.1 with $(\sigma_1+\sigma_3)K_1+(\sigma_4+\sigma_2)K_2+(\sigma_9+\sigma_7)L_1+(\sigma_{10}+\sigma_8)L_2\approx 0.6476<1$.

(b) *As an application of Theorem 3.2 , consider*

$$\begin{cases} f_1(t,x,y) = \dfrac{\arctan x}{10(t^2+1)} + \dfrac{\sin|y|}{17(1+t)}, \\ f_2(t,x,y) = \dfrac{2}{\sqrt{t^2+2}} + \dfrac{2|x|}{5\pi(8+t)(1+|x|)}, \\ g_1(x,y) = \dfrac{|x|}{2(1+|x|)} + \dfrac{1}{6}\arctan y, \\ g_2(x,y) = \dfrac{1}{3}e^{-|x|} + \dfrac{1}{7}\cos|x|. \end{cases} \tag{4.3}$$

Using the given values, we find that the assumption (H_4) is satisfied since $|f_1(t,x,y)| \leqslant \dfrac{\pi}{20(1+t^2)} + \dfrac{1}{17(t+1)} = \phi_1(t)$ and $|f_2(t,x,y)| \leqslant \dfrac{2}{\sqrt{t^2+2}} + \dfrac{2}{5\pi(8+t)} = \phi_2(t)$, $|g_1(x,y)| \leqslant (6+\pi)/12 = \Lambda_1$, $|g_2(x,y)| \leqslant 10/21 = \Lambda_2$. Also $(\sigma_5+\sigma_6) \approx 0.0034 < 1$ holds true. As all the assumptions of Theorem 3.2 are satisfied, so its conclusion implies that the system (4.1) with the nonlinearities (4.3) has at least one solution on $[0,1]$.

(c) *In order to demonstrate the application of Theorem 3.3*, let us choose

$$\begin{cases} f_1(t,x,y) = \arctan x + \dfrac{e^{-t}|x|^2}{17(1+|x|)} + \dfrac{1}{26}y\cos x, \\ f_2(t,x,y) = \dfrac{2}{\sqrt{t^2+2}} + \dfrac{2}{\pi(8+t)}x\ \arctan y + \dfrac{|x||y|}{5(1+|x|)}, \\ g_1(x,y) = \ln 7 + \dfrac{1}{21}x\sin|y| + \dfrac{1}{13}y, \\ g_2(x,y) = 3e^{-|x|} + \dfrac{1}{7}x\cos y + \dfrac{1}{11}y\arctan x. \end{cases} \tag{4.4}$$

Obviously (H_3) holds true with positive values of $u_0, v_0, \omega_0, \tau_0$ and $u_1 = 1/17, u_2 = 1/26, v_1 = 1/8, v_2 = 1/5, \omega_1 = 1/21, \omega_2 = 1/13, \tau_1 = 1/7, \tau_2 = \pi/22$. Also, $(\sigma_1+\sigma_3)u_1 + (\sigma_2+\sigma_4)v_1 + (\sigma_7+\sigma_9)\omega_1 + (\sigma_8+\sigma_{10})\tau_1 + (\sigma_5+\sigma_6) \approx 0.6728 < 1$, and $(\sigma_1+\sigma_3)u_2 + (\sigma_2+\sigma_4)v_2 + (\sigma_7+\sigma_9)\omega_2 + (\sigma_8+\sigma_{10})\tau_2 + (\sigma_5+\sigma_6) \approx 0.8412 < 1$. As the hypothesis of Theorem 3.3 is verified, therefore we deduce by its conclusion that there exists at least one solution of the system (4.1) with f_1, f_2, g_1 and g_2 given by (4.4).

5. Conclusions

In the present research work, we investigated the existence and uniqueness of solutions for a new coupled system of multi-term Hilfer fractional differential equations of different orders involving non-integral and autonomous type Riemann-Liouville mixed integral nonlinearities equipped with nonlocal coupled multi-point and Riemann-Liouville integral boundary conditions. Firstly, we proved an auxiliary result concerning the linear variant of the given problem, helping us to transform the problem at hand into a fixed point problem. Then we proved the existence of a unique solution for the given problem by applying Banach's contraction mapping principle and derived the existence results by means of Krasnosel'skiǐ's fixed point theorem and Leray-Schauder nonlinear alternative. All the obtained results are well illustrated by numerical examples. Our results are new and enrich the literature on nonloacl nonlinear integral boundary value problems for Hilfer fractional differential equations.

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-PHD-80-130-42). The authors, therefore, acknowledge with thanks DSR technical and financial support.

Conflict of interest

The authors declare no conflict of interest.

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