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Discussion on the hybrid Jaggi-Meir-Keeler type contractions

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Abstract: In this paper, the notion of hybrid Jaggi-Meir-Keeler type contraction is introduced. The existence of a fixed point for such operators is investigated. The derived results combine and extend a number of existing results in the corresponding literature. Examples are established to express the validity of the obtained results.

Keywords: Jaggi contraction; Meir-Keeler contraction; Hyprid contraction; fixed point
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1. Introduction

A century ago, the first metric fixed-point theorem was published by Banach [1]. In fact, before Banach, some famous mathematicians such as Picard and Liouville had used the fixed point approach to solve certain differential equations, more precisely, initial value problems. Inspired by their results, Banach considered it as a separated and independent result in the framework of the nonlinear functional analysis and point-set topology. The statement and the proof of an outstanding work of Banach, also known as contraction mapping principle, can be considered as an art piece: Each contraction, in the setting of a complete metric space, possesses a unique fixed point. Metric fixed point theory has been appreciated and investigated by several researchers. These researchers have different reasons and motivations to study this theory. The most important reason why the researchers find worthful to work and investigate metric fixed point theory is the natural and strong connection of the theoretical result in nonlinear functional analysis with applied sciences. If we look at it with the chronological aspect, we note that the fixed point theory was born as a tool to solve certain differential equations. Banach liberated the theory from being a tool in applied mathematics to an independent work of nonlinear functional analysis. On Picard and Liouville's side, it is a tool to solve the initial value problem. On the other side, from Banach's point of view, the fixed point theory is an independent research topic that has enormous application potential on almost all qualitative sciences, including applied mathematics. Secondly, Banach fixed point theorem not only guarantee the solution (the existence of a fixed point) but also indicate how we reach the mentioned solution (how to find the fixed point). Finally, we need to underline that almost all real world problems can be transferred to a fixed point problem, easily.

With this motivation, several generalizations and extensions of Banach's fixed point theory have been released by introducing new contractions or by changing the structure of the studied abstract space. Among, we shall mention only a few of them that set up the skeleton of the contraction dealt with it. Historically, the first contraction we shall focus on it is the Meir-Keeler contraction [2]. Roughly speaking, the Meir-Keeler contraction can be considered as a uniform contraction. The second contraction that we dealt with is Jaggi contraction [3]. The interesting part of Jaggi's contraction is the following: Jaggi's contraction is one of the first of its kind that involves some rational expression. The last one is called as an interpolative contraction [4]. In the interpolative contraction the terms are used exponentially instead of standard usage of them.

In this paper, we shall introduce a new contraction, hybrid Jaggi-Meir-Keeler type contraction, as a unification and generalization of the Meir-Keeler's contraction, the Jaggi's contraction and interpolative contraction in the setting of a complete metric space. We propose certain assumptions to guarantee the existence of a fixed point for such mappings. In addition, we express some example to indicate the validity of the derived results.

Before going into details, we would like to reach a consensus by explaining the concepts and notations: Throughout the paper, we presume the sets, we deal with, are non-empty. The letter $\mathbb{N}$ presents the set of positive integers. Further, we assume that the pair (X, d) is a complete metric space. This notation is required in each of the following theorems, definitions, lemma and so on. We shall use the pair (X, d) everywhere without repeating that it is a complete metric space.

In what follows we recall the notion of the uniform contraction which is also known as Meir-Keeler contraction:

Definition 1.1. *[2] A mapping* $\mathsf{f} : (\mathsf{X}, \mathsf{d}) \to (\mathsf{X}, \mathsf{d})$ *is said to be a* Meir-Keeler contraction *on* X*, if for every* $\mathcal{E} > 0$*, there exists* $\delta > 0$ *such that*

$$\mathcal{E} \leq \mathsf{d}(\chi, y) < \mathcal{E} + \delta \textit{ implies } \mathsf{d}(\mathsf{f}\chi, \mathsf{f}y) < \mathcal{E}, \tag{1.1}$$

for every $\chi, y \in \mathsf{X}$.

Theorem 1.1. *[2] Any* Meir-Keeler contraction $\mathsf{f} : (\mathsf{X}, \mathsf{d}) \to (\mathsf{X}, \mathsf{d})$ *possesses a unique fixed point.*

Very recently, Bisht and Rakočević [5] suggested the following extension of the uniform contraction:

Theorem 1.2. *[5] Suppose a mapping* $\mathsf{f} : (\mathsf{X}, \mathsf{d}) \to (\mathsf{X}, \mathsf{d})$ *fulfills the following statements:*

(1) for a given $\mathcal{E} > 0$ *there exists a* $\delta(\mathcal{E}) > 0$ *such that*

$$\mathcal{E} < M(\chi, y) < \mathcal{E} + \delta(\mathcal{E}) \text{ implies } \mathsf{d}(\mathsf{f}\chi, \mathsf{f}y) \leq \mathcal{E};$$

(2) $d(f\chi, fy) < M(\chi, y)$*, whenever* $M(\chi, y) > 0$*;*

for any $\chi, y \in X$*, where*

$$M(\chi, y) = \max\Big\{d(\chi, y), \alpha d(\chi, f\chi) + (1-\alpha)d(y, fy), (1-\alpha)d(\chi, f\chi) + \alpha d(y, fy), \frac{\beta[d(\chi, fy) + d(y, f\chi)]}{2}\Big\},$$

with $0 < \alpha < 1$, $0 \leq \beta < 1$. *Then,* f *has a unique fixed point* $u \in X$ *and* $f^n\chi \to u$ *for each* $\chi \in X$.

On the other hand, in 2018, the idea of interpolative contraction was consider to revisit the well-known Kannan's fixed point theorem [6]:

Definition 1.2. *[4] A mapping* $f : (X, d) \to (X, d)$ *is said to be an* interpolative Kannan type contraction *on* X *if there exist* $\kappa \in [0, 1)$ *and* $\gamma \in (0, 1)$ *such that*

$$d(f\chi, fy) \leq \kappa[d(\chi, f\chi)]^{\gamma}[d(y, fy)]^{1-\gamma}, \tag{1.2}$$

for every $\chi, y \in X\setminus \mathrm{Fix}(f)$*, where* $\mathrm{Fix}(f) = \{\chi \in X | f\chi = \chi\}$.

Theorem 1.3. *[4] Any interpolative Kannan-contraction mapping* $f : (X, d) \to (X, d)$ *possesses a fixed point.*

For more interpolative contractions results, we refer to [7–11] and related references therein.

Definition 1.3. *A mapping* $f : (X, d) \to (X, d)$ *is called a Jaggi type hybrid contraction if there is* $\psi \in \Psi$ *so that*

$$d(f\chi, fy) \leq \psi\left(\mathcal{J}_f^s(\chi, y)\right), \tag{1.3}$$

for all distinct $\chi, y \in X$ *where* $p \geq 0$ *and* $\sigma_i \geq 0, i = 1, 2, 3, 4,$ *such that* $\sigma_1 + \sigma_2 = 1$ *and*

$$\mathcal{J}_f^s(\chi, y) = \begin{cases} [\sigma_1 \left(\frac{d(\chi, f\chi)\cdot d(y, fy)}{d(\chi, y)}\right)^s + \sigma_2(d(\chi, y))^s]^{1/p}, & \text{if } \begin{array}{c} p > 0, \\ \chi, y \in X, \chi \neq y \end{array} \\ (d(\chi, f\chi))^{\sigma_1}(d(y, fy))^{\sigma_2}, & \text{if } \begin{array}{c} p = 0, \\ \chi, y \in X\setminus\mathfrak{F}_f(X), \end{array} \end{cases} \tag{1.4}$$

where $\mathfrak{F}_f(X) = \{z \in X : fz = z\}$.

Theorem 1.4. *A continuous mapping* $f : (X, d) \to (X, d)$ *possesses a fixed point* χ *if it forms a Jaggi-type hybrid contraction. In particular, for any* $\chi_0 \in X$*, the sequence* $\{f^n\chi_0\}$ *converges to* χ.

Definition 1.4. *[12] Let* $\alpha : X\times X \to [0, +\infty)$ *be a mapping, where* $X \neq \emptyset$*. A self-mapping* $f : (X, d) \to (X, d)$ *is called* triangular α-orbital admissible *and denote as* $f \in \mathcal{T}_X^{\alpha}$ *if*

$$\alpha(\chi, f\chi) \geq 1 \textit{ implies } \alpha(f\chi, f^2\chi) \geq 1,$$

and

$$\alpha(\chi, y) \geq 1, \textit{and } \alpha(y, fy) \geq 1, \textit{implies } \alpha(\chi, fy) \geq 1$$

for all $\chi, y \in X$.

This concept, was used by many authors, in order to prove variant fixed point results (see, for instance [13–19] and the corresponding references therein).

Lemma 1.1. *[12] Assume that* $\mathsf{f} \in \mathcal{T}_X^\alpha$. *If there exists* $x_0 \in \mathsf{X}$ *such that* $\alpha(x_0, \mathsf{f}x_0) \geq 1$, *then* $\alpha(x_m, x_k) \geq 1$, *for all* $m, n \in \mathbb{N}$, *where the sequence* $\{x_k\}$ *is defined by* $x_{k+1} = x_k$.

The following condition is frequently considered to avoid the continuity of the mappings involved.

(R) if the sequence $\{x_n\}$ in X is such that for each $n \in \mathbb{N}$,

$$\alpha(x_n, x_{n+1}) \geq 1 \qquad \text{and} \qquad \lim_{n\to+\infty} x_n = x \in x,$$

then there exists a subsequence $\left\{x_{n(j)}\right\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(j)}, x) \geq 1, \text{ for each } j \in \mathbb{N}.$$

2. Main results

We start this section by introducing the new contraction, namely, hybrid Jaggi-Meir-Keeler type contraction.

Consider the mapping $\mathsf{f} : (\mathsf{X}, \mathsf{d}) \to (\mathsf{X}, \mathsf{d})$ and the set of fixed point, $\mathfrak{F}_\mathsf{f}(\mathsf{X}) = \{z \in \mathsf{X} : \mathsf{f}z = z\}$. We define the crucial expression $\mathcal{R}_\mathsf{f}^s$ as follows:

$$\mathcal{R}_\mathsf{f}^s(x, y) = \begin{cases} \left[\beta_1\left(\frac{\mathsf{d}(x,\mathsf{f}x)\cdot\mathsf{d}(y,\mathsf{f}y)}{\mathsf{d}(x,y)}\right)^s + \beta_2(\mathsf{d}(x,y))^s + \beta_3\left(\frac{\mathsf{d}(x,\mathsf{f}y)+\mathsf{d}(y,\mathsf{f}x)}{4}\right)^s\right]^{1/s}, \\ \qquad\qquad\qquad\qquad \text{for } s > 0, \quad x, y \in \mathsf{X}, x \neq y \\ \\ (\mathsf{d}(x,\mathsf{f}x))^{\beta_1}(\mathsf{d}(y,\mathsf{f}y))^{\beta_2}\left(\frac{\mathsf{d}(x,\mathsf{f}y)+\mathsf{d}(y,\mathsf{f}x)}{4}\right)^{\beta_3}, \\ \qquad\qquad\qquad\qquad \text{for } s = 0, \quad x, y \in \mathsf{X}, \end{cases} \tag{2.1}$$

where $p \geq 1$ and $\beta_i \geq 0$, $i = 1, 2, 3$ are such that $\beta_1 + \beta_2 + \beta_3 = 1$.

Definition 2.1. *Assume that* $\mathsf{f} \in \mathcal{T}_X^\alpha$. *We say that* $\mathsf{f} : (\mathsf{X}, \mathsf{d}) \to (\mathsf{X}, \mathsf{d})$ *is an* α-hybrid Jaggi-Meir-Keeler type contraction *on* X, *if for all distinct* $x, y \in \mathsf{X}$ *we have:*

(a_1) *for given* $\mathcal{E} > 0$, *there exists* $\delta > 0$ *such that*

$$\mathcal{E} < \max\left\{\mathsf{d}(x, y), \mathcal{R}_\mathsf{f}^s(x, y)\right\} < \mathcal{E} + \delta \implies \alpha(x, y)\mathsf{d}(\mathsf{f}x, \mathsf{f}y) \leq \mathcal{E}; \tag{2.2}$$

(a_2)

$$\alpha(x, y)\mathsf{d}(\mathsf{f}x, \mathsf{f}y) < \max\left\{\mathsf{d}(x, y), \mathcal{R}_\mathsf{f}^s(x, y)\right\}. \tag{2.3}$$

Theorem 2.1. *Any continuous* α*-hybrid Jaggi-Meir-Keeler type contraction* $\mathsf{f} : (\mathsf{X}, \mathsf{d}) \to (\mathsf{X}, \mathsf{d})$ *provide a fixed point if there exists* $x_0 \in \mathsf{X}$, *such that* $\alpha(x_0, \mathsf{f}x_0) \geq 1$ *and* $\alpha(x_0, \mathsf{f}^2 x_0) \geq 1$.

Proof. Let $x_0 \in \mathsf{X}$ be an arbitrary, but fixed point. We form the sequence $\{x_m\}$, as follows:

$$x_m = \mathsf{f}x_{m-1} = \mathsf{f}^m x_0,$$

for all $m \in \mathbb{N}$ and assume that $\mathsf{d}(x_m, x_{m+1}) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Indeed, if for some $l_0 \in \mathbb{N} \cup \{0\}$ we have $\mathsf{d}(x_{l_0}, x_{l_0+1}) = 0$, it follows that $x_{l_0} = x_{l_0+1} = \mathsf{f}x_{l_0}$. Therefore, x_{l_0} is a fixed point of the mapping f and the proof is closed.

Since, by assumption, the mapping f is triangular α-orbital admissible, it follows that

$$\alpha(x_0, \mathsf{f}x_0) \geq 1 \;\Rightarrow\; \alpha(x_1, x_2) = \alpha(\mathsf{f}x_0, \mathsf{f}^2 x_0) \geq 1 \;\Rightarrow ... \Rightarrow$$

$$\alpha(x_n, x_{n+1}) \geq 1, \tag{2.4}$$

for every $n \in \mathbb{N}$.

We shall study two cases; these are $s > 0$ and $s = 0$.

Case (A). For the case $s > 0$, letting $x = x_{n-1}$ and $y = x_n = \mathsf{f}x_{n-1}$ in (a_2), we get

$$\mathsf{d}(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n)\mathsf{d}(\mathsf{f}x_{n-1}, \mathsf{f}x_n) < \max\left\{\mathsf{d}(x_{n-1}, x_n), \mathcal{R}^s_{\mathsf{f}}(x_{n-1}, x_n)\right\}, \tag{2.5}$$

where

$$\begin{aligned}
\mathcal{R}^s_{\mathsf{f}}(x_{n-1}, x_n) &= [\beta_1\left(\frac{\mathsf{d}(x_{n-1},\mathsf{f}x_{n-1})\cdot\mathsf{d}(x_n,\mathsf{f}x_n)}{\mathsf{d}(x_{n-1},x_n)}\right)^s + \beta_2(\mathsf{d}(x_{n-1}, x_n))^s + \\
&\qquad +\beta_3\left(\frac{\mathsf{d}(x_{n-1},\mathsf{f}x_n)+\mathsf{d}(x_n,\mathsf{f}x_{n-1})}{4}\right)^s]^{1/s} \\
&= [\beta_1\left(\frac{\mathsf{d}(x_{n-1},x_n)\cdot\mathsf{d}(x_n,x_{n+1})}{\mathsf{d}(x_{n-1},x_n)}\right)^s + \beta_2(\mathsf{d}(x_{n-1}, x_n))^s + \\
&\qquad +\beta_3\left(\frac{\mathsf{d}(x_{n-1},x_{n+1})+\mathsf{d}(x_n,x_n)}{4}\right)^s]^{1/s} \\
&\leq [\beta_1(\mathsf{d}(x_n, x_{n+1}))^s + \beta_2(\mathsf{d}(x_{n-1}, x_n))^s + \\
&\qquad +\beta_3\left(\frac{\mathsf{d}(x_{n-1},x_n)+\mathsf{d}(x_n,x_{n+1})}{4}\right)^s]^{1/s}.
\end{aligned}$$

If we can find $n_0 \in \mathbb{N}$ such that $\mathsf{d}(x_{n_0}, x_{n_0+1}) \geq \mathsf{d}(x_{n_0-1}, x_{n_0})$, we have

$$\begin{aligned}
\mathcal{R}^s_{\mathsf{f}}(x_{n_0-1}, x_{n_0}) &\leq [\beta_1(\mathsf{d}(x_{n_0}, x_{n_0+1}))^s + \beta_2(\mathsf{d}(x_{n_0}, x_{n_0+1}))^s + \\
&\qquad +\beta_3(\mathsf{d}(x_{n_0}, x_{n_0+1}))^s]^{1/s} \\
&= \mathsf{d}(x_{n_0}, x_{n_0+1})(\beta_1 + \beta_2 + \beta_3)^{1/s} \\
&= \mathsf{d}(x_{n_0}, x_{n_0+1}).
\end{aligned}$$

Then, $\max\left\{\mathsf{d}(x_{n_0}, x_{n_0+1}), \mathcal{R}^s_{\mathsf{f}}(x_{n_0-1}, x_{n_0})\right\} = \mathsf{d}(x_{n_0}, x_{n_0+1})$, and using (2.4), respectively (2.5) we get

$$\begin{aligned}
\mathsf{d}(x_{n_0}, x_{n_0+1}) &\leq \alpha(x_{n_0-1}, x_{n_0})\mathsf{d}(\mathsf{f}x_{n+0-1}, \mathsf{f}x_{n_0}) \\
&< \max\left\{\mathsf{d}(x_{n_0}, x_{n_0+1}), \mathcal{R}^s_{\mathsf{f}}(x_{n_0-1}, x_{n_0})\right\} \leq \mathsf{d}(x_{n_0}, x_{n_0+1}),
\end{aligned}$$

which is a contradiction. Therefore, $\mathsf{d}(x_n, x_{n+1}) < \mathsf{d}(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$ and (2.5) becomes

$$\mathsf{d}(x_n, x_{n+1}) < \mathsf{d}(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. Consequently, there exists $b \geq 0$ such that $\lim_{n\to+\infty} \mathsf{d}(x_{n-1}, x_n) = b$. If $b > 0$, we have

$$\mathsf{d}(x_m, x_{m+1}) \geq b > 0,$$

for any $m \in \mathbb{N}$. On the one hand, since (2.2) holds for every given $\mathcal{E} > 0$, it is possible to choose $\mathcal{E} = b$ and let $\delta > 0$ be such that (2.2) is satisfied. On the other hand, since, also, $\lim_{n\to+\infty} \max\left\{\mathsf{d}(x_{n-1}, x_n), \mathcal{R}^s_{\mathsf{f}}(x_{n-1}, x_n)\right\} = b$, there exists $m_0 \in \mathbb{N}$ such that

$$b < \max\left\{\mathsf{d}(x_{m_0-1}, x_{m_0}), \mathcal{R}^s_{\mathsf{f}}(x_{m_0-1}, x_{m_0})\right\} < b + \delta.$$

Thus, by (2.2), together with (2.4) we obtain

$$\mathsf{d}(x_{m_0}, x_{m_0+1}) \leq \alpha(x_{m_0}, x_{m_0+1})\mathsf{d}(\mathsf{f}x_{m_0-1}, \mathsf{f}x_{m_0}) < b,$$

which is a contradiction. Therefore,

$$\lim_{n\to+\infty} \mathsf{d}(x_n, x_{n+1}) = b = 0. \tag{2.6}$$

We claim now that $\{x_n\}$ is a Cauchy sequence. Let $\mathcal{E} > 0$ be fixed and we can choose that $\delta' = \min\{\delta(\mathcal{E}), \mathcal{E}, 1\}$. Thus, from (2.6) it follows that there exists $j_0 \in \mathbb{N}$ such that

$$\mathsf{d}(x_n, x_{n+1}) < \frac{\delta'}{2}, \tag{2.7}$$

for all $n \geq j_0$. Now, we consider the set

$$\mathcal{A} = \left\{x_l \,|\, l \geq j_0,\ \mathsf{d}(x_l, x_{j_0}) < \mathcal{E} + \frac{\delta'}{2}\right\}. \tag{2.8}$$

We claim that $\mathsf{f}y \in \mathcal{A}$ whenever $y = x_l \in \mathcal{A}$. Indeed, in case of $l = j_0$, we have $\mathsf{f}x_l = \mathsf{f}x_{j_0} = x_{j_0+1}$, and taking (2.7) into account, we get

$$\mathsf{d}(x_{j_0}, x_{j_0+1}) < \frac{\delta'}{2} < \mathcal{E} + \frac{\delta'}{2}. \tag{2.9}$$

Thus, we will assume that $l > j_0$, and we distinguish two cases, namely:

Case 1. Suppose that

$$\mathcal{E} < \mathsf{d}(x_l, x_{j_0}) < \mathcal{E} + \frac{\delta'}{2}. \tag{2.10}$$

We have

$$
\begin{aligned}
\mathcal{R}_{\mathsf{f}}^{s}(\chi_l,\chi_{j_0}) &= \left[\beta_1\left(\tfrac{\mathsf{d}(\chi_l,\mathsf{f}\chi_l)\mathsf{d}(\chi_{j_0},\mathsf{f}\chi_{j_0})}{\mathsf{d}(\chi_l,\chi_{j_0})}\right)^s+\beta_2\left(\mathsf{d}(\chi_l,\chi_{j_0})\right)^s+\beta_3\left(\tfrac{\mathsf{d}(\chi_l,\mathsf{f}\chi_{j_0})+\mathsf{d}(\chi_{j_0},\mathsf{f}\chi_l)}{4}\right)^s\right]^{1/s}\\
&= \Bigg[\beta_1\left(\tfrac{\mathsf{d}(\chi_l,\chi_{l+1})\mathsf{d}(\chi_{j_0},\chi_{j_0+1})}{\mathsf{d}(\chi_l,\chi_{j_0})}\right)^s+\beta_2\left(\mathsf{d}(\chi_l,\chi_{j_0})\right)^s+\\
&\qquad +\beta_3\left(\tfrac{\mathsf{d}(\chi_l,\chi_{j_0+1})+\mathsf{d}(\chi_{j_0},\chi_{l+1})}{4}\right)^s\Bigg]^{1/s}\\
&\le \Bigg[\beta_1\left(\tfrac{\mathsf{d}(\chi_l,\chi_{l+1})\mathsf{d}(\chi_{j_0},\chi_{j_0+1})}{\mathsf{d}(\chi_l,\chi_{j_0})}\right)^s+\beta_2\left(\mathsf{d}(\chi_l,\chi_{j_0})\right)^s+\\
&\qquad +\beta_3\left(\tfrac{\mathsf{d}(\chi_l,\chi_{j_0})+\mathsf{d}(\chi_{j_0},\chi_{j_0+1})+\mathsf{d}(\chi_l,\chi_{j_0})+\mathsf{d}(\chi_l,\chi_{l+1})}{4}\right)^s\Bigg]^{1/s}\\
&< \left[\beta_1\left(\mathsf{d}(\chi_l,\chi_{l+1})\right)^s+\beta_2\left(\mathsf{d}(\chi_l,\chi_{j_0})\right)^s+\beta_3\left(\tfrac{2\mathsf{d}(\chi_l,\chi_{j_0})+\mathsf{d}(\chi_{j_0},\chi_{j_0+1})+\mathsf{d}(\chi_l,\chi_{l+1})}{4}\right)^s\right]^{1/s}\\
&< \left[\beta_1\left(\tfrac{\delta'}{2}\right)^s+\beta_2\left(\mathcal{E}+\tfrac{\delta'}{2}\right)^s+\beta_3\left(\tfrac{\mathcal{E}}{2}+\tfrac{\delta'}{4}+\tfrac{\delta'}{4}\right)^s\right]^{1/s}\\
&\le (\beta_1+\beta_2+\beta_3)^{1/s}\left(\mathcal{E}+\tfrac{\delta'}{2}\right)\\
&\le \mathcal{E}+\delta'.
\end{aligned}
$$

In this case,

$$
\mathcal{E} < \mathsf{d}(\chi_l,\chi_{j_0}) \le \max\left\{\mathsf{d}(\chi_l,\chi_{j_0}),\mathcal{R}_{\mathsf{f}}^{s}(\chi_l,\chi_{j_0})\right\} < \max\left\{\left(\mathcal{E}+\frac{\delta'}{2}\right),(\mathcal{E}+\delta')\right\} = (\mathcal{E}+\delta'),
$$

which implies by (a_1) that

$$
\alpha(\chi_l,\chi_{j_0})\mathsf{d}(\mathsf{f}\chi_l,\mathsf{f}\chi_{j_0}) \le \mathcal{E}. \tag{2.11}
$$

But, taking into account that the mapping f is triangular α-orbital admissible, together with (2.4) we have

$$
\alpha(\chi_n,\chi_{n+1}) \ge 1 \text{ and } \alpha(\chi_{n+1},\mathsf{f}\chi_{n+1}) \ge 1 \text{ implies } \alpha(\chi_n,\chi_{n+2}) \ge 1,
$$

and recursively we get that

$$
\alpha(\chi_n,\chi_l) \ge 1, \tag{2.12}
$$

for all $n,l\in\mathbb{N}$. Therefore, from (2.11) and (2.12), we have

$$
\mathsf{d}(\chi_{l+1},\chi_{j_0+1}) = \mathsf{d}(\mathsf{f}\chi_l,\mathsf{f}\chi_{j_0}) \le \mathcal{E}. \tag{2.13}
$$

Now, by the triangle inequality together with (2.7)and (2.13) we get

$$
\mathsf{d}(\chi_{l+1},\chi_{j_0}) \le \mathsf{d}(\chi_{l+1},\chi_{j_0+1}) + \mathsf{d}(\chi_{j_0+1},\chi_{j_0}) < \left(\mathcal{E}+\frac{\delta'}{2}\right),
$$

which means that, indeed $\mathsf{f}\chi_l = \chi_{l+1} \in \mathcal{A}$.
Case 2. Suppose that

$$
\mathsf{d}(\chi_l,\chi_{j_0}) \le \mathcal{E}. \tag{2.14}
$$

Thus,

$$\begin{aligned} \mathsf{d}(\mathsf{f}x_l, x_{j_0}) &\le \mathsf{d}(\mathsf{f}x_l, \mathsf{f}x_{j_0}) + \mathsf{d}(\mathsf{f}x_{j_0}, x_{j_0}) \\ &\le \alpha(x_l, x_{j_0})\mathsf{d}(\mathsf{f}x_l, \mathsf{f}x_{j_0}) + \mathsf{d}(x_{j_0+1}, x_{j_0}) \\ &< \max\left\{\mathsf{d}(x_l, x_{j_0}), \mathcal{R}^s_{\mathsf{f}}(x_l, x_{j_0})\right\} + \mathsf{d}(x_{j_0+1}, x_{j_0}), \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} \mathcal{R}^s_{\mathsf{f}}(x_l, x_{j_0}) &= \left[\beta_1\left(\frac{\mathsf{d}(x_l,x_{l+1})\mathsf{d}(x_{j_0},x_{j_0+1})}{\mathsf{d}(x_l,x_{j_0})}\right)^s + \beta_2\left(\mathsf{d}(x_l, x_{j_0})\right)^s + \right. \\ &\left. +\beta_3\left(\frac{\mathsf{d}(x_l,x_{j_0})+\mathsf{d}(x_{j_0},x_{j_0+1})+\mathsf{d}(x_l,x_{j_0})+\mathsf{d}(x_l,x_{l+1})}{4}\right)^s\right]^{1/s}. \end{aligned}$$

We must consider two subcases

(2a). $\mathsf{d}(x_l, x_{j_0}) \ge \mathsf{d}(x_{j_0}, x_{j_0+1})$. Then,

$$\begin{aligned} \mathcal{R}^s_{\mathsf{f}}(x_l, x_{j_0}) &\le \left[\beta_1\left(\mathsf{d}(x_l, x_{l+1})\right)^s + \beta_2\left(\mathsf{d}(x_l, x_{j_0})\right)^s + \right. \\ &\left. +\beta_3\left(\frac{2\mathsf{d}(x_l,x_{j_0})+\mathsf{d}(x_{j_0},x_{j_0+1})+\mathsf{d}(x_l,x_{l+1})}{4}\right)^s\right]^{1/s} \\ &< \left[\beta_1\left(\frac{\delta'}{2}\right)^s + \beta_2\left(\mathcal{E}\right))^s + \beta_3\left(\frac{2\mathcal{E}+2\frac{\delta'}{2}}{4})\right)^s\right]^{1/s} \\ &< \left[\beta_1 + \beta_2 + \beta_3\right]^{1/s}\left(\frac{\mathcal{E}}{2} + \frac{\delta'}{4}\right). \end{aligned}$$

But, since $\delta' = \min\{\delta, \mathcal{E}, 1\}$, we get

$$\mathcal{R}^s_{\mathsf{f}}(x_l, x_{j_0}) < \frac{3\mathcal{E}}{4},$$

and then

$$\begin{aligned} \mathsf{d}(\mathsf{f}x_l, x_{j_0}) &< \max\left\{\mathsf{d}(x_l, x_{j_0}), \mathcal{R}^s_{\mathsf{f}}(x_l, x_{j_0})\right\} + \mathsf{d}(x_{j_0+1}, x_{j_0}) \\ &< \max\left\{\mathcal{E}, \tfrac{3\mathcal{E}}{4}\right\} + \tfrac{\delta'}{2} \\ &= (\mathcal{E} + \tfrac{\delta'}{2}), \end{aligned}$$

which shows that $\mathsf{f}x_l \in \mathcal{A}$.

(2b). $\mathsf{d}(x_l, x_{j_0}) < \mathsf{d}(x_{j_0}, x_{j_0+1})$. Then,

$$\mathsf{d}(\mathsf{f}x_l, x_{j_0}) \le x_{l+1}, x_l) + \mathsf{d}(x_l, x_{j_0}) < \frac{\delta'}{2} + \frac{\delta'}{2} < \mathcal{E} + \frac{\delta'}{2}.$$

Consequently, choosing some $m, n \in \mathbb{N}$ such that $m > n > j_0$, we can write

$$\mathsf{d}(x_m, x_n) \le \mathsf{d}(x_m, x_{j_0}) + \mathsf{d}(x_{j_0}, x_n) < 2(\mathcal{E} + \frac{\delta'}{2}) < 4\mathcal{E},$$

which leads us to

$$\lim_{m,n\to+\infty} \mathsf{d}(x_m, x_n) = 0.$$

Therefore, $\{x_m\}$ is a Cauchy sequence in a complete metric space. Thus, we can find a point $u \in X$ such that $\lim_{m\to+\infty} x_m = u$. Moreover, since the mapping f is continuous we have

$$u = \lim_{m\to+\infty} f^{m+1}x_0 = \lim_{m\to+\infty} f(f^m x_0) = f(\lim_{m\to+\infty} f^m x_0) = fu,$$

that is, u is a fixed point of f.

Case (B). For the case $s = 0$, letting $x = x_{n-1}$ and $y = x_n = fx_{n-1}$ in (2.2), we get

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) < \max\{d(x_{n-1}, x_n), \mathcal{R}_f(x_{n-1}, x_n)\}, \tag{2.16}$$

where

$$\begin{aligned}
\mathcal{R}_f(x_{n-1}, x_n) &= [d(x_{n-1}, fx_{n-1})]^{\beta_1} [d(x_n, fx_n)]^{\beta_2} \left[\frac{d(x_{n-1}, fx_n) + d(x_n, fx_{n-1}}{4}\right]^{\beta_3} \\
&= [d(x_{n-1}, x_n)]^{\beta_1} [d(x_n, x_{n+1})]^{\beta_2} \left[\frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n}{4}\right]^{\beta_3} \\
&= [d(x_{n-1}, x_n)]^{\beta_1} [d(x_n, x_{n+1})]^{\beta_2} \left[\frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n}{4}\right]^{\beta_3} \\
&\leq [d(x_{n-1}, x_n)]^{\beta_1} [d(x_n, x_{n+1})]^{\beta_2} \left[\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{4}\right]^{\beta_3} \\
&= [d(x_{n-1}, x_n)]^{\beta_1} [d(x_n, x_{n+1})]^{\beta_2} \left[\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{4}\right]^{\beta_3}
\end{aligned}$$

Thus, by (2.3) and taking (2.4) into account we have

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) < \max\{d(x_{n-1}, x_n), \mathcal{R}_f(x_{n-1}, x_n)\}.$$

Now, if there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) \geq d(x_{n_0-1}, x_{n_0})$, we get

$$\begin{aligned}
d(x_{n_0}, x_{n_0+1}) &< \max\{d(x_{n_0-1}, x_{n_0}), \mathcal{R}_f(x_{n_0-1}, x_{n_0})\} \\
&\leq \max\{d(x_{n_0-1}, x_{n_0}), d(x_{n_0}, x_{n_0+1})\} \\
&< d(x_{n_0}, x_{n_0+1}),
\end{aligned}$$

which is a contradiction. Therefore, $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, that is, the sequence $\{x_n\}$ decreasing and moreover, converges to some $b \geq 0$. Moreover, since

$$\mathcal{R}_f(x_{n-1}, x_n) = [d(x_{n-1}, x_n)]^{\beta_1} [d(x_n, x_{n+1})]^{\beta_2} \left[\frac{d(x_{n-1}, x_{n+1})}{4}\right]^{\beta_3},$$

we get that

$$\lim_{n\to+\infty} \max\{d(x_{n-1}, x_n), \mathcal{R}_f(x_{n-1}, x_n)\} = b.$$

If we suppose that $b > 0$, then, $0 < b < d(x_{n-1}, x_n)$ and we can find $\delta > 0$ such that

$$b < \max\{d(x_{n-1}, x_n), \mathcal{R}_f(x_{n-1}, x_n)\} < b + \delta.$$

In this way, taking $\mathcal{E} = b$, we get

$$b = \mathcal{E} < \max\{d(x_{n-1}, x_n), \mathcal{R}_f(x_{n-1}, x_n)\} < \mathcal{E} + \delta,$$

which implies (by (a_1)) that

$$\mathsf{d}(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)\mathsf{d}(\mathsf{f}x_{n-1}, \mathsf{f}x_n)) \leq \mathcal{E} = b,$$

which is a contradiction. We thus proved that

$$\lim_{m\to} \mathsf{d}(x_{n-1}, x_n) = 0. \tag{2.17}$$

We claim now, that the sequence $\{x_n\}$ is Cauchy. Firstly, we remark that, since $\mathsf{d}(x_{n-1}, x_n) = 0$, there exists $j_0 \in \mathbb{N}$, such that

$$\mathsf{d}(x_{n-1}, x_n) < \frac{\delta'}{2}, \tag{2.18}$$

for any $n \geq j_0$, where $\delta' = \min\{\delta, \mathcal{E}, 1\}$. Reasoning by induction, we will prove that the following relation

$$\mathsf{d}(x_{j_0}, x_{j_0+m}) < \mathcal{E} + \frac{\delta'}{2} \tag{2.19}$$

holds, for any $m \in \mathbb{N}$. Indeed, in case of $m = 1$,

$$\mathsf{d}(x_{j_0}, x_{j_0+1}) < \frac{\delta'}{2} < \mathcal{E} + \frac{\delta'}{2},$$

so, (2.19) is true. Now, supposing that (2.19) holds for some l, we shall show that it holds for $l + 1$. We have

$$\begin{aligned}
\mathcal{R}_{\mathsf{f}}(x_{j_0}, x_{j_0+l}) &= \left(\mathsf{d}(x_{j_0}, \mathsf{f}x_{j_0})\right)^{\beta_1} \left(x_{j_0+l}, \mathsf{f}x_{j_0+l}\right)^{s} \left(\tfrac{\mathsf{d}(x_{j_0},\mathsf{f}x_{j_0+l})+\mathsf{d}(x_{j_0+l},\mathsf{f}x_{j_0})}{4}\right)^{\beta_3} \\
&= \left(\mathsf{d}(x_{j_0}, x_{j_0+1})\right)^{\beta_1} \left(x_{j_0+l}, x_{j_0+l+1}\right)^{s} \left(\tfrac{\mathsf{d}(x_{j_0},x_{j_0+l+1})+\mathsf{d}(x_{j_0+l},x_{j_0+1})}{4}\right)^{\beta_3} \\
&\leq \left(\mathsf{d}(x_{j_0}, x_{j_0+1})\right)^{\beta_1} \left(\mathsf{d}(x_{j_0+l}, x_{j_0+l+1})\right)^{s} \\
&\quad \left(\tfrac{\mathsf{d}(x_{j_0},x_{j_0+l})+\mathsf{d}(x_{j_0+l},x_{j_0+l+1})+\mathsf{d}(x_{j_0+l},x_{j_0})+\mathsf{d}(x_{j_0},x_{j_0+1})}{4}\right)^{\beta_3} \\
&< \left(\tfrac{\delta'}{2}\right)^{\beta_1+\beta_2} \left(\left(\tfrac{\mathcal{E}}{2} + \tfrac{\delta'}{4}\right) + \tfrac{\delta'}{4}\right)^{\beta_3} \\
&\leq (\mathcal{E} + \tfrac{\delta'}{2}).
\end{aligned} \tag{2.20}$$

As in the Case (A), if $\mathsf{d}(x_{j_0}, x_{j_0+l}) > \mathcal{E}$, by (a_2), and keeping in mind the above inequalities, we get

$$\mathcal{E} < \mathsf{d}(x_{j_0}, x_{j_0+l}) \leq \max\left\{\mathsf{d}(x_{j_0}, x_{j_0+l}), \mathcal{R}_{\mathsf{f}}(x_{j_0}, x_{j_0+l})\right\} < \max\left\{\tfrac{\delta'}{2}, (\mathcal{E} + \tfrac{\delta'}{2})\right\} = \mathcal{E} + \delta'$$

$$\text{implies } \alpha(x_{j_0}, x_{j_0+l})\mathsf{d}(\mathsf{f}x_{j_0}, \mathsf{f}x_{j_0+l}) \leq \mathcal{E}.$$

But, since using (2.12), it follows that

$$\mathsf{d}(x_{j_0+1}, x_{j_0+l+1}) = \mathsf{d}(\mathsf{f}x_{j_0}, \mathsf{f}x_{j_0+l}) \leq \mathcal{E},$$

and then, by (b_3) we get

$$\mathsf{d}(x_{j_0}, x_{j_0+l+1}) \leq \mathsf{d}(x_{j_0}, x_{j_0+1}) + \mathsf{d}(x_{j_0+1}, x_{j_0+l+1}) < \frac{\delta'}{2} + \mathcal{E} < \mathcal{E} + \frac{\delta'}{2}.$$

Therefore, (2.19) holds for $(l+1)$. In the opposite situation, if $\mathsf{d}(\chi_{j_0}, \chi_{j_0+l}) \leq \mathcal{E}$, again by the triangle inequality, we obtain

$$\begin{aligned}
\mathsf{d}(\chi_{j_0}, \chi_{j_0+l+1}) &\leq \mathsf{d}(\chi_{j_0}, \chi_{j_0+1}) + \mathsf{d}(\chi_{j_0+1}, \chi_{j_0+l+1}) \\
&\leq \mathsf{d}(\chi_{j_0}, \chi_{j_0+1}) + \alpha(\chi_{j_0}, \chi_{j_0+l})\mathsf{d}(\mathsf{f}\chi_{j_0}, \mathsf{f}\chi_{j_0+l}) \\
&< \frac{\delta'}{2} + \max\left\{\mathsf{d}(\chi_{j_0}, \chi_{j_0+l}), \mathcal{R}_{\mathsf{f}}(\chi_{j_0}, \chi_l)\right\} \\
&< \frac{\delta'}{2} + \max\left\{\mathcal{E}, \frac{\mathcal{E}}{2} + \frac{\delta'}{4}\right\} \\
&= \frac{\delta'}{2} + \mathcal{E}.
\end{aligned}$$

Consequently, the induction is completed. Therefore, $\{\chi_n\}$ is a Cauchy sequence in a complete metric space. Thus, there exists $u \in \mathsf{X}$ such that $\mathsf{f}u = u$. □

In the above Theorem, the continuity condition of the mapping f can be replace by the continuity of f^2.

Theorem 2.2. *Suppose that* $\mathsf{f} : (\mathsf{X}, \mathsf{d}) \to (\mathsf{X}, \mathsf{d})$ *forms an* α*-hybrid Jaggi-Meir-Keeler type contraction such that* f^2 *is continuous. Then,* f *has a fixed point, provided that there exists* $\chi_0 \in \mathsf{X}$*, such that* $\alpha(\chi_0, \mathsf{f}\chi_0) \geq 1$.

Proof. Let $\chi_0 \in \mathsf{X}$ such that $\alpha(\chi_0, \mathsf{f}\chi_0) \geq 1$ and the sequence $\{\chi_n\}$, where $\chi_n = \mathsf{f}\chi_{n-1}$, for any $n \in \mathbb{N}$. Thus, from Theorem 2.3 we know that this is a convergent sequence. Letting $u = \lim_{n\to+\infty} \chi_n$, we claim that $u = \mathsf{f}u$.

Since the mapping f^2 is supposed to be continuous,

$$\mathsf{f}^2 u = \lim_{n\to+\infty} \mathsf{f}^2 \chi_n = u.$$

Assuming on the contrary, that $u \neq \mathsf{f}u$, we have

$$\begin{aligned}
\mathcal{R}^s_{\mathsf{f}}(u, \mathsf{f}u) &= \begin{cases} \left[\beta_1\left(\frac{\mathsf{d}(u,\mathsf{f}u)\cdot\mathsf{d}(\mathsf{f}u,\mathsf{f}^2u)}{\mathsf{d}(u,\mathsf{f}u)}\right)^s + \beta_2(\mathsf{d}(u,\mathsf{f}u))^s + \beta_3\left(\frac{\mathsf{d}(u,\mathsf{f}^2u)+\mathsf{d}(\mathsf{f}u,\mathsf{f}u)}{4}\right)^s\right]^{1/s}, & \text{for } s > 0 \\ (\mathsf{d}(u,\mathsf{f}u))^{\beta_1}(\mathsf{d}(\mathsf{f}u,\mathsf{f}^2u))^{\beta_2}\left(\frac{\mathsf{d}(u,\mathsf{f}^2u)+\mathsf{d}(\mathsf{f}u,\mathsf{f}u)}{4}\right)^{\beta_3}, & \text{for } s = 0 \end{cases} \\
&= \begin{cases} \left[\beta_1\left(\frac{\mathsf{d}(u,\mathsf{f}u)\cdot\mathsf{d}(\mathsf{f}u,u)}{\mathsf{d}(u,\mathsf{f}u)}\right)^s + \beta_2(\mathsf{d}(u,\mathsf{f}u))^s + \beta_3\left(\frac{\mathsf{d}(u,u)+\mathsf{d}(\mathsf{f}u,\mathsf{f}u)}{4}\right)^s\right]^{1/s}, & \text{for } s > 0 \\ (\mathsf{d}(u,\mathsf{f}u))^{\beta_1}(\mathsf{d}(\mathsf{f}u,u))^{\beta_2}\left(\frac{\mathsf{d}(u,u)+\mathsf{d}(\mathsf{f}u,\mathsf{f}u)}{4}\right)^{\beta_3}, & \text{for } s = 0 \end{cases} \\
&= \begin{cases} \left[\beta_1\left(\mathsf{d}(\mathsf{f}u,u)\right)^s + \beta_2(\mathsf{d}(u,\mathsf{f}u))^s\right]^{1/s}, & \text{for } s > 0 \\ 0, & \text{for } s = 0 \end{cases} \\
&= \begin{cases} \left[\beta_1 + \beta_2\right]^{1/s}\mathsf{d}(u,\mathsf{f}u)), & \text{for } s > 0 \\ 0, & \text{for } s = 0 \end{cases}
\end{aligned}$$

□

Example 2.1. *Let* $\mathsf{X} = [0, +\infty)$*,* $\mathsf{d} : \mathsf{X} \times \mathsf{X} \to [0, +\infty)$*,* $\mathsf{d}(\chi, y) = |\chi - y|$*, and the mapping* $\mathsf{f} : \mathsf{X} \to \mathsf{X}$*, where*

$$\mathsf{f} = \begin{cases} \frac{1}{2}, & \text{if } \chi \in [0,1] \\ \frac{1}{6}, & \text{if } \chi > 1 \end{cases}.$$

We can easily observe that f *is discontinuous at the point* $\chi = 1$*, but* f^2 *is a continuous mapping. Let also the function* $\alpha : \mathsf{X} \times \mathsf{X} \to [0, +\infty)$*,*

$$\alpha(\chi, y) = \begin{cases} \chi^2 + y^2 + 1, & \text{if } \chi, y \in [0,1] \\ \ln(\chi + y) + 1, & \text{if } \chi, y \in (1, +\infty) \\ 1, & \text{if } \chi = \frac{5}{6}, y = \frac{7}{6} \\ 0, & \text{otherwise} \end{cases},$$

and we choose $\beta_1 = \frac{1}{4}, \beta_2 = \frac{1}{2}, \beta_1 = \frac{1}{4}$ *and* $s = 2$*. The mapping* f *is triangular* α*-orbital admissible and satisfies* (a_2) *in Definition 2.1 for any* $\chi, y \in [0, 1]$*, respectively for* $\chi, y \in (1, +\infty)$*. Taking into account the definition of the function* α*, we have more to check the case* $\chi = \frac{5}{6}$*,* $y = \frac{7}{6}$*. We have*

$$\begin{aligned} \mathcal{R}_{\mathsf{f}}(\tfrac{5}{6}, \tfrac{7}{6}) &= \left[\frac{1}{4} \left(\frac{\mathsf{d}(\frac{5}{6}, \mathsf{f}\frac{5}{6})\mathsf{d}(\frac{7}{6}, \mathsf{f}\frac{7}{6})}{\mathsf{d}(\frac{5}{6}, \frac{7}{6})} \right)^2 + \frac{1}{2}(\mathsf{d}(\tfrac{5}{6}, \tfrac{7}{6}))^2 + \frac{1}{4} \left(\frac{\mathsf{d}(\frac{5}{6}, \mathsf{f}\frac{7}{6}) + \mathsf{d}(\frac{7}{6}, \mathsf{f}\frac{5}{6})}{4} \right)^2 \right]^{1/2} \\ &= \sqrt{\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{9} + \frac{1}{9}} \cdot = \sqrt{\frac{1}{3}}. \end{aligned}$$

Therefore,

$$\alpha(\frac{5}{6}, \frac{7}{6})\mathsf{d}(\mathsf{f}\frac{5}{6}, \mathsf{f}\frac{7}{6}) = \mathsf{d}(\mathsf{f}\frac{5}{6}, \mathsf{f}\frac{7}{6}) = \mathsf{d}(\frac{1}{2}, \frac{1}{6}) = \frac{1}{3} < \frac{1}{\sqrt{3}} = \max\left\{ \mathsf{d}(\frac{5}{6}, \frac{7}{6}), \mathcal{R}_{\mathsf{f}}(\frac{5}{6}, \frac{7}{6}) \right\}.$$

Moreover, since the mapping f *satisfies condition* (a_1) *for*

$$\delta(\mathcal{E}) = \begin{cases} 1 - \mathcal{E}, & \text{for } \mathcal{E} < 1 \\ 1, & \text{for } \mathcal{E} \geq 1, \end{cases}$$

it follows that the assumptions of Theorem 2.3 are satisfied, and $u = \frac{1}{2}$ *ia a fixed point of the mapping* f*.*

Theorem 2.3. *If to the hypotheses of Theorem we add the following assumption*

$$\alpha(u, v) \geq 1 \text{ for any } u, v \in \mathfrak{F}_{\mathsf{f}}(\mathsf{X}),$$

then the mapping f *admits an unique fixed point.*

Proof. Let $u \in \mathsf{X}$ be a fixed point of f. Supposing on the contrary, that we can find $v \in \mathsf{X}$ such that $\mathsf{f}u = u \neq v = \mathsf{f}v$, we have

(i) For $s > 0$,

$$\begin{aligned} \mathcal{R}_{\mathsf{f}}^s &= \left[\beta_1 \left(\frac{\mathsf{d}(u, \mathsf{f}u)\mathsf{d}(v, \mathsf{f}v)}{\mathsf{d}(u, v)} \right)^s + \beta_2 \left(\mathsf{d}(u, v) \right)^s + \beta_3 \left(\frac{\mathsf{d}(u, \mathsf{f}v) + \mathsf{d}(v, \mathsf{f}u)}{4} \right)^s \right]^{1/s} \\ &= \left[\beta_2 \left(\mathsf{d}(u, v) \right)^s + \beta_3 \left(\frac{\mathsf{d}(u, v)}{2} \right)^s \right]^{1/s} \\ &\leq (\beta_2 + \beta_3)^{1/s} \mathsf{d}(u, v) \leq \mathsf{d}(u, v). \end{aligned}$$

Thus, taking $\chi = u$ and $y = v$ in (2.3) we get

$$\mathsf{d}(u, v) \le \alpha(u, v)\mathsf{d}(\mathsf{f}u, \mathsf{f}v) < \max\{\mathsf{d}(u, v), \mathcal{R}_{\mathsf{f}}(u, v)\} = \mathsf{d}(u, v),$$

which is a contradiction.

(ii) For $s = 0$,

$$\begin{aligned}\mathsf{d}(u, v) &\le \alpha(u, v)\mathsf{d}(\mathsf{f}u, \mathsf{f}v) < \max\{\mathsf{d}(u, v), \mathcal{R}_{\mathsf{f}}(u, v)\}\\ &= \max\left\{\mathsf{d}(u, v), (\mathsf{d}(u, \mathsf{f}u))^{\beta_1}(\mathsf{d}(v, \mathsf{f}v))^{\beta_2}\left(\tfrac{\mathsf{d}(u,\mathsf{f}v)+\mathsf{d}(v,\mathsf{f}u)}{4}\right)^{\beta_3}\right\}\\ &= \mathsf{d}(u, v),\end{aligned}$$

which is a contradiction.

Consequently, if there exists a fixed point of the mapping f, under the assumptions of the theorem, this is unique. □

Example 2.2. *Let the set* $\mathsf{X} = [-1, +\infty)$, $\mathsf{d} : \mathsf{X} \times \mathsf{X} \to [0, +\infty)$, $\mathsf{d}(\chi, y) = \left|\chi - y\right|$, *and the mapping* $\mathsf{f} : \mathsf{X} \to \mathsf{X}$, *where*

$$\mathsf{f}\chi = \begin{cases} \frac{\chi}{2} + 1, & \text{if } \chi \in [-1, 0)\\ 1, & \text{if } \chi \in [0, 1]\\ \frac{1}{x}, & \text{if } x > 1\end{cases}.$$

Let also $\alpha : \mathsf{X} \times \mathsf{X} \to [0, +\infty)$ *defined as follows*

$$\alpha(\chi, y) = \begin{cases} \frac{3}{4}, & \text{if } \chi, y \in [-1, 0)\\ \chi^2 + y^2 + 1, & \text{if } \chi, y \in [0, 1]\\ 1, & \text{if } \chi \in [-1, 0), y \in [0, 1]\\ 0, & \text{otherwise}\end{cases}.$$

It is easy to check that, with these chooses, f *is a continuous triangular α-orbital admissible mapping and also, it follows that the mapping* f *satisfies the conditions* (a_2) *from Definition (2.1). Moreover,* f *satisfies the condition* (a_1), *considering* $\delta(\mathcal{E}) = 1 - \mathcal{E}$ *in case of* $\mathcal{E} < 1$ *and* $\delta(\mathcal{E}) = 1$ *for* $\mathcal{E} \ge 1$. *Consequently,* f *satisfies the conditions of Theorem 2.3 and has a unique fixed point,* $u = 0$.

In particular, for the case $s = 0$, the continuity assumption of the mapping f can be replace by the condition (R).

Theorem 2.4. *We presume that* $\mathsf{f} : (\mathsf{X}, \mathsf{d}) \to (\mathsf{X}, \mathsf{d}) \in \mathcal{T}_X^{\alpha}$ *and fulfills*

(a_i) *for given* $\mathcal{E} > 0$, *there exists* $\delta > 0$ *such that*

$$\mathcal{E} < \mathcal{O}(\chi, y) < \mathcal{E} + \delta \ \text{ implies } \alpha(\chi, y)\mathsf{d}(\mathsf{f}\chi, \mathsf{f}y) \le \mathcal{E}, \tag{2.21}$$

with

$$\mathcal{O}(\chi, y) = \max\left\{\mathsf{d}(\chi, y), (\mathsf{d}(\chi, \mathsf{f}\chi))^{\beta_1}\left(\mathsf{d}(y, \mathsf{f}y)\right)^{\beta_2}\left(\frac{\mathsf{d}(\chi, \mathsf{f}y) + \mathsf{d}(y, \mathsf{f}\chi)}{4}\right)^{\beta_3}\right\},$$

for all $\chi, y \in \mathsf{X}$, *where* $\beta_i \ge 0$, $i = 1, 2, 3$ *so that* $\beta_1 + \beta_2 + \beta_3 = 1$;

(a_{ii})
$$\alpha(\chi, y)\mathsf{d}(\mathsf{f}\chi, \mathsf{f}y) < \mathcal{O}(\chi, y). \tag{2.22}$$

The mapping f *has a unique fixed point provided that:*

(α_1) *there exists* $\chi_0 \in \mathsf{X}$ *such that* $\alpha(\chi_0, \mathsf{f}\chi_0) \geq 1$*;*
(α_3) $\alpha(u, v) \geq 1$ *for any* $u, v \in \mathfrak{F}_{\mathsf{f}}(\mathsf{X})$*;*
(α_2) *if the sequence* $\{\chi_n\}$ *in* X *is such that for each* $n \in \mathbb{N}$

$$\alpha(\chi_n, \chi_{n+1}) \geq 1 \qquad \text{and} \qquad \lim_{n\to+\infty} \chi_n = \chi \in \mathsf{X},$$

then there exists a subsequence $\left\{\chi_{n(j)}\right\}$ *of* $\{\chi_n\}$ *such that*

$$\alpha(\chi_{n(j)}, \chi) \geq 1, \text{ for each } j \in \mathbb{N}.$$

Proof. Let $\chi_0 \in \mathsf{X}$ such that $\alpha(\chi_0, \mathsf{f}\chi_0) \geq 1$. Then, we know (following the proof of Theorem 2.3) that the sequence $\{\chi_n\}$, with $\chi_n = \mathsf{f}^n\chi_0$ is convergent; let $u = \lim_{n\to+\infty} \chi_n$. On the other hand, from (α_2), we can find a subsequence $\left\{\chi_{n(j)}\right\}$ of $\{\chi_n\}$ such that

$$\alpha(\chi_{n(j)}, u) \geq 1, \text{ for each } j \in \mathbb{N}.$$

Since we can suppose that $\mathsf{d}(\chi_{n(j)+1}, \mathsf{f}u) > 0$, from (a_{ii}) we have

$$\begin{aligned} \mathsf{d}((\chi_{n(j)+1}, \mathsf{f}u)) &\leq \alpha(\chi_{n(j)}, u)\mathsf{d}(\mathsf{f}\chi_{n(j)}, \mathsf{f}u) < \mathcal{O}(\chi, y) \\ &= \max\left\{\mathsf{d}(\chi_{n(j)}, u), \left(\mathsf{d}(\chi_{n(j)}, \mathsf{f}\chi_{n(j)})\right)^{\beta_1} (\mathsf{d}(u, \mathsf{f}u))^{\beta_2} \left(\tfrac{\mathsf{d}(\chi_{n(j)}, \mathsf{f}u) + \mathsf{d}(u, \mathsf{f}\chi_{n(j)})}{4}\right)^{\beta_3}\right\} \\ &= \max\left\{\mathsf{d}(\chi_{n(j)}, u), \left(\mathsf{d}(\chi_{n(j)}, \chi_{n(j)+1})\right)^{\beta_1} (\mathsf{d}(u, \mathsf{f}u))^{\beta_2} \left(\tfrac{\mathsf{d}(\chi_{n(j)}, \mathsf{f}u) + \mathsf{d}(u, \chi_{n(j)+1})}{4}\right)^{\beta_3}\right\}. \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get $\mathsf{d}(u, \mathsf{f}u) = 0$. Thus, $\mathsf{f}u = u$.
To proof the uniqueness, we consider that we can find another fixed point of f. From (a_{ii}), we have

$$\mathsf{d}(u, v) = \mathsf{d}(\mathsf{f}u, \mathsf{f}v) \leq \alpha(u, v)\mathsf{d}(\mathsf{f}u, \mathsf{f}v) < \mathcal{O}(u, v) = \mathsf{d}(u, v) < \mathsf{d}(u, v),$$

which is a contradiction. Therefore, $u = v$. □

Considering $\alpha(\chi, y) = 1$ in the above theorems, we can easily obtain the following result.

Definition 2.2. *A mapping* $\mathsf{f} : (\mathsf{X}, \mathsf{d}) \to (\mathsf{X}, \mathsf{d})$ *is called hybrid Jaggi-Meir-Keeler type contraction on* X *if for all distinct* $\chi, y \in \mathsf{X}$ *we have:*

(a_1) *for given* $\mathcal{E} > 0$*, there exists* $\delta > 0$ *such that*

$$\mathcal{E} < \max\left\{\mathsf{d}(\chi, y), \mathcal{R}^s_{\mathsf{f}}(\chi, y)\right\} < \mathcal{E} + \delta \implies \mathsf{d}(\mathsf{f}\chi, \mathsf{f}y) \leq \mathcal{E}; \tag{2.23}$$

(a_2) *whenever* $\mathcal{R}_{\mathsf{f}}(\chi, y) > 0$*,*
$$\mathsf{d}(\mathsf{f}\chi, \mathsf{f}y) < \max\left\{\mathsf{d}(\chi, y), \mathcal{R}^s_{\mathsf{f}}(\chi, y)\right\}. \tag{2.24}$$

Corollary 2.1. *Any hybrid Jaggi-Meir-Keeler type contraction* $\mathsf{f} : (\mathsf{X}, \mathsf{d}) \to (\mathsf{X}, \mathsf{d})$ *possesses a unique fixed point provided that* f *is continuous or* f^2 *is continuous.*

Conflicts of interest

The authors declare that they have no conflicts of interest.

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