AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 12689–12701.
DOI:10.3934/math.2022702
Received: 05 January 2022
Revised: 07 April 2022
Accepted: 17 April 2022
Published: 29 April 2022

Research article

On cubic semisymmetric bi-Cayley graphs on nonabelian simple groups

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Abstract: In this paper, we classify cubic semisymmetric bi-Cayley graphs on nonabelian simple groups, extending a remarkable classification of cubic nonnormal symmetric Cayley graphs on nonabelian simple groups.

Keywords: semisymmetric graph; bi-Cayley graph; nonabelian simple group
Mathematics Subject Classification: 20B15, 20B30, 05C25

1. Introduction

Graphs considered in this paper are finite, simple and undirected. For a graph Γ, we denote by $V\Gamma$, $E\Gamma$ and $A\Gamma$ the vertex set, the edge set and the arc set of Γ respectively, and by $\mathsf{Aut}\Gamma$ the full automorphism group of Γ. If some $G \leq \mathsf{Aut}\Gamma$ is transitive on $V\Gamma$, $E\Gamma$ or $A\Gamma$, then Γ is called *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive*, respectively. An arc-transitive graph is also called *symmetric*. If G is transitive on $E\Gamma$ but intransitive on $V\Gamma$, then Γ is called *G-semisymmetric*, and Γ is simply called *semisymmetric* while Γ is $\mathsf{Aut}\Gamma$-semisymmetric. It is easily known that connected G-semisymmetric graphs are bipartite.

Suppose G is a group acting on a set Ω. The *stabilizer* of G on a point $\alpha \in \Omega$ is the subgroup $G_\alpha = \{g \in G \mid \alpha^g = \alpha\}$. Then G is said to be *semiregular* on Ω if $G_\alpha = 1$ for each point α in Ω, and *regular* on Ω if G is semiregular and transitive on Ω. A graph Γ is said to be a *Cayley graph* on a group G if $G \leq \mathsf{Aut}\Gamma$ is regular on $V\Gamma$, and a *bi-Cayley graph* on a group G if $G \leq \mathsf{Aut}\Gamma$ is semiregular and has exactly two orbits (with the same length) on $V\Gamma$. In particular, Γ is called *normal* on G if G is normal in $\mathsf{Aut}\Gamma$, and *nonnormal* on G otherwise. Moreover, follow the notion in [5], a semisymmetric bi-Cayley graph Γ on a group G is called *almost-normal* if the normalizer $N_{\mathsf{Aut}\Gamma}(G)$ is semisymmetric on Γ.

Cayley graphs on nonabelian simple groups have received much attention in the literature, see [8,10, 11,16–20,25–29] and references therein. Here a remarkable contribution is the complete classification of connected cubic nonnormal symmetric Cayley graphs on nonabelian simple groups, which was first studied by Li [15] in 1996 and completed by Xu et al. [25, 26] in 2007. It finally turns out that such graphs are only two $(A_{48}, 5)$-arc-transitive Cayley graphs on A_{47}. Bi-Cayley graphs can be viewed as natural generalizations of Cayley graphs, and involve many interesting examples that are not Cayley graphs such as the Petersen graph, the Gray graph [2] (which is the smallest cubic semisymmetric graph and a bi-Cayley graph on the extraspecial metacyclic group $\mathbb{Z}_9 : \mathbb{Z}_3$ of order 27) and Bouwer graph [3]. Thus the above remarkable classification naturally motivates us to classify connected cubic (normal and nonnormal) semisymmetric bi-Cayley graphs on nonabelian simple groups.

Theorem 1.1. *Let Γ be a connected cubic semisymmetric bi-Cayley graph on a nonabelian simple group T. Then either*

(1) Γ is normal on T, and $\mathsf{Aut}\Gamma = T : \mathbb{Z}_3$ or $T : S_3$; or
(2) Γ is nonnormal on T, and $\mathsf{Aut}\Gamma = (R \times S).O$, where $S > T$, $(S, T) = (\mathsf{M}_{24}, \mathsf{M}_{23})$ or (A_n, A_{n-1}) with $n \in \{12, 24, 48, 96, 192, 384\}$, $|R||O|$ divides $384/|S : T|$ and $O \leq \mathsf{Out}(S)$.

Applying Theorem 1.1, the connected cubic nonnormal semisymmetric bi-Cayley graphs on nonabelian simple groups may be determined by computation with Magma package [1], see Example 3.5 for a specific example with $T = \mathsf{M}_{23}$.

Normal semisymmetric bi-Cayley graphs are definitely almost-normal, but the converse is not necessarily true. In fact, determining the normality of Cayley or bi-Cayley graphs is difficult in general for finding the full automorphism groups. However, Theorem 1.1 has the following interesting consequence which shows that, for connected cubic semisymmetric bi-Cayley graphs on nonabelian simple groups, the normality and the almost-normality are equivalent.

Corollary 1.2. *A connected cubic semisymmetric bi-Cayley graph on a nonabelian simple group T is normal on T if and only if it is almost-normal on T.*

The importance of Corollary 1.2 is that it may lead to an explicit characterization of normal semisymmetric bi-Cayley graphs on nonabelian simple groups. To express the result, we first introduce a general construction of bi-Cayley graphs. Let G be a group, R and L be inverse-closed subsets (may empty) of $G \setminus \{1\}$, and let S be a nonempty subset of G. Define a graph $\Gamma = \mathsf{BiCay}(G, R, L, S)$ with vertex set

$$\{g_0 \mid g \in G\} \cup \{g_1 \mid g \in G\},$$

and edge set

$$\{\{g_0, h_0\} \mid hg^{-1} \in R\} \cup \{\{g_1, h_1\} \mid hg^{-1} \in L\} \cup \{\{g_0, h_1\} \mid hg^{-1} \in S\}.$$

It is easily known that Γ is a bi-Cayley graph on a group isomorphic to G, and each bi-Cayley graph can be constructed in this way (refer to [29]). Also, one may assume $1 \in S$ up to isomorphism.

Theorem 1.3. *A graph Γ is a connected cubic normal semisymmetric bi-Cayley graph on a nonabelian simple group T if and only if that*

$$\Gamma \cong \mathsf{BiCay}(T, \emptyset, \emptyset, \{1, s, ss^g\})$$

for some $s \in T$ and $g \in \mathsf{Aut}(T)$ satisfying

$$o(g) = 3, ss^g s^{g^2} = 1, \langle s, s^g \rangle = T,$$

and there exists no element $h \in \mathsf{Aut}(T)$ satisfying $\{s, ss^g\}^h = \{s^{-1}, (ss^g)^{-1}\}$.

Theorem 1.3 helps us finding many specific examples or proving the nonexistence of examples for certain nonabelian simple groups via searching in Magma [1]. For example, computation shows that there is no example for $T = A_5$ and A_6, but exist examples for $T = A_7$ and A_8, see Examples 4.1 and 4.2.

We remark that a result of Tutte [24, P.59] states that an edge-transitive graph of odd valency is either symmetric or semisymmetric, thus this paper together [9] and [21] completes the characterization of cubic edge-transitive bi-Cayley graphs on nonabelian simple groups.

After making some preparations in Section 2, we prove Theorem 1.1 and Corollary 1.2 in Section 3, and Theorem 1.3 in Section 4.

2. Preliminaries

2.1. Background results

Bi-coset graph is an important tool for understanding semi-symmetric graphs. Let G be a group and L, R be subgroups of G such that $L \cap R$ is core-free in G (namely $L \cap R$ does not contain any nontrivial normal subgroup of G). Define a *bi-coset graph* with vertex set $[G : L] \cup [G : R]$, and Lx is adjacent to Ry with $x, y \in G$ if and only if $yx^{-1} \in RL$. This bi-coset graph is denoted by $\mathsf{Cos}(G, L, R)$.

The following lemma is known, refer to [13, Section 3].

Lemma 2.1. *Using notation as above, then* $\mathsf{Cos}(G, L, R)$ *is G-semisymmetric. Conversely, each G-semisymmetric graph is isomorphic to* $\mathsf{Cos}(G, G_\alpha, G_\beta)$*, where α and β are adjacent vertices.*

In particular, $\mathsf{Cos}(G, G_\alpha, G_\beta)$ is connected cubic G-semisymmetric if and only if G, G_α, G_β satisfy the following conditions:

- $\langle G_\alpha, G_\beta \rangle = G$,
- $|G_\alpha : G_\alpha \cap G_\beta| = |G_\beta : G_\alpha \cap G_\beta| = 3$, and
- $G_\alpha \cap G_\beta$ is core-free in G.

In 1980, a landscape work of Goldschmidt [14] determined all the triples $(G_\alpha, \mathrm{G}_\beta, G_{\alpha\beta})$ satisfying the above conditions (refer to [22, Table 3]), which are called *Goldschmidt amalgams* after him.

Theorem 2.2 ([14]). *Let Γ be a connected cubic G-semisymmetric graph. Then there are exactly fifteen possible amalgams $(G_\alpha, \mathrm{G}_\beta, G_{\alpha\beta})$ with α, β adjacent vertices. In particular, $|G_\alpha| = |G_\beta| = 3 \cdot 2^i$ where* $0 \le i \le 7$.

Investigating normal quotients of graphs has been very successful in studying various families of graphs. Let Γ be a G-edge-transitive graph and let N be an intransitive normal subgroup of G. The *normal quotient Γ_N* of Γ with respect to N is defined to be the graph with the set of N-orbits as its vertex set and two N-orbits B_1, B_2 are adjacent if and only if some vertex in B_1 is adjacent in Γ to some vertex in B_2. The original graph is said to be a *normal cover* of Γ_N if $|\Gamma(\alpha) \cap B_2| = 1$ for each edge $\{B_1, B_2\}$ in Γ_N and $\alpha \in B_1$.

Lemma 2.3. *Let Γ be a connected cubic G-semisymmetric graph with bipartitions Δ_1 and Δ_2. Let $N \lhd G$ and $\alpha \in V\Gamma$. Then either*

(1) N acts transitively on at least one of Δ_1 and Δ_2; or

(2) N is semiregular and intransitive on both $\varDelta_1$ and $\varDelta_2$, and the following statements are true:

(i) $\varGamma$ is a normal cove of $\varGamma_N$.

(ii) $G^{V\varGamma_N} = G/N$, $(G/N)_{\alpha^N} = G_\alpha N/N$ and $\varGamma_N$ is a connected cubic G/N-semisymmetric graph.

Proof. We only need to prove that $(G/N)_{\alpha^N} = G_\alpha N/N$ because the other statements are known, see [13, Lemma 5.1] or [19, Lemma 2.3]. For $Ng \in G/N$, one has the following equivalences:

$$\begin{aligned} Ng \in (G/N)_{\alpha^N} &\Longleftrightarrow (\alpha^g)^N = (a^N)^{Ng} = \alpha^N \\ &\Longleftrightarrow \alpha^g = \alpha^n \ for\ some\ n \in N \\ &\Longleftrightarrow g \in NG_\alpha \\ &\Longleftrightarrow Ng \in NG_\alpha/N. \end{aligned}$$

Therefore $(G/N)_{\alpha^N} = G_\alpha N/N$. In particular, if $N_\alpha = 1$, then $(G/N)_{\alpha^N} \cong G_\alpha/(N \cap G_\alpha) = G_\alpha/N_\alpha = G_\alpha$. □

For a bi-Cayley graph $\mathsf{BiCay}(T, \emptyset, \emptyset, S)$ and $\tau \in \mathsf{Aut}(T)$, with bipartitions $T_0 := \{t_0 \mid t \in T\}$ and $T_1 := \{t_1 \mid t \in T\}$. Define

$$\sigma_{\tau,s}:\ t_0 \to (t^\tau)_0,\ t_1 \to (st^\tau)_1,\ \ for\ each\ t \in T.$$

From [29, Theorem 1.1], we have the following assertion which is important for characterizing normal bi-Cayley graphs.

Lemma 2.4. *Let $\varGamma = \mathsf{BiCay}(T, \emptyset, \emptyset, S)$ be a normal bi-Cayley graph on a group T. Then*

$$\mathrm{A}_{1_0} = \langle \sigma_{\tau,s} \mid \tau \in \mathsf{Aut}(T), s \in S, S^\tau = s^{-1}S \rangle.$$

The next lemma is similar to [13, Lemma 6.2], we include a proof here for the convenience of the readers.

Lemma 2.5. *Let $\varGamma$ be a connected cubic G-semisymmetric graph with bipartitions $\varDelta_1$ and $\varDelta_2$, where G is faithful on both $\varDelta_1$ and $\varDelta_2$. Suppose further that $K \lhd G$ is transitive on $\varDelta_1$ and $\varDelta_2$, and $K_\alpha \neq 1$ for a vertex α of $\varGamma$. Then $K_v^{\varGamma(v)} \geq \mathbb{Z}_3$ is transitive for each vertex v of $\varGamma$.*

Proof. Assume $\alpha \in \varDelta_i$ with $i \in \{1, 2\}$. Since G is faithful on $\varDelta_{3-i}$, there exists $\beta \in \varDelta_{3-i}$ such that K_α does not fix β. By the connectivity of $\varGamma$, there exists a path $(\alpha_0, \alpha_1, \cdots, \alpha_t)$ with $\alpha_0 = \alpha$ such that K_α fixes $\alpha_0, \ldots, \alpha_{t-1}$ but moves α_t. It follows that

$$1 \neq K_{\alpha_{t-1}}^{\varGamma(\alpha_{t-1})} \lhd G_{\alpha_{t-1}}^{\varGamma(\alpha_{t-1})}.$$

Since $\varGamma$ is a G-semisymmetric cubic graph, $G_{\alpha_{t-1}}^{\varGamma(\alpha_{t-1})}$ is primitive, hence $K_{\alpha_{t-1}}^{\varGamma(\alpha_{t-1})}$ is transitive.

Without loss of generality, we may assume $\alpha_{t-1} \in \varDelta_1$ and $\alpha_t \in \varDelta_2$. Let $\{\beta_1, \beta_2\}$ be an edge of $\varGamma$ with $\beta_1 \in \varDelta_1$ and $\beta_2 \in \varDelta_2$. Since K is transitive on $\varDelta_1$, there is $x \in K$ such that $\beta_1^x = \alpha_{t-1}$, hence $\{\beta_1, \beta_2\}^x = \{\alpha_{t-1}, \beta_2^x\}$ is an edge of $\varGamma$. Now both α_t and β_2^x are in $\varGamma(\alpha_{t-1})$, the transitivity of $K_{\alpha_{t-1}}^{\varGamma(\alpha_{t-1})}$ implies that there is $y \in K_{\alpha_{t-1}}$ such that $(\beta_2^x)^y = \alpha_t$. It follows that $y^{-1}x^{-1} \in K$ such

$$\{\alpha_{t-1}, \alpha_t\}^{y^{-1}x^{-1}} = \{\beta_1, \beta_2\},$$

namely $\varGamma$ is K-edge-transitive. Thereby $K_v^{\varGamma(v)} \geq \mathbb{Z}_3$ is transitive for each vertex $v \in V\varGamma$. □

Let $G = N.H$ be a group extension. If N is contained in $Z(G)$, the center of G, then the extension is called a *central extension*, and if further G is perfect, that is, the commutator subgroup G' equals to G, we call G a *covering group* of H. Schur [23] proved that every nonabelian simple group T admits a unique maximal covering group M such that each covering group of T is a homomorphic image of M; the center $Z(M)$ is called the *Schur multiplier* of T, denoted by $\mathsf{Mult}(T)$. The next result states that each central extension $N.T$ with T nonabelian simple can derive a covering group of T, refer to [20, Lemma 2.11].

Lemma 2.6. *Let $G = N.T$ be a central extension with T a nonabelian simple group. Then $G = NG'$ and $G' = Z(G').T$ is a covering group of T, where $Z(G') = N \cap G'$ is a quotient of $\mathsf{Mult}(T)$.*

The next lemma can be read out from [11, Lemma 3.3].

Lemma 2.7. *Let $T < S$ be nonabelian simple groups. If $|S : T| \mid 384$, then either*

(1) $(S, T) = (\mathrm{A}_n, \mathrm{A}_{n-1})$ *with $n \geq 6$ dividing 384; or*
(2) $(S, T) = (\mathsf{M}_{11}, \mathrm{PSL}_2(11)), (\mathsf{M}_{12}, \mathsf{M}_{11})$ *or* $(\mathsf{M}_{24}, \mathsf{M}_{23})$.

With similar arguments as in [8, P. 143], the following lemma is easy to prove by checking the nonabelian simple groups contained in $\mathrm{GL}(e, 2)$ for $3 \leq e \leq 7$.

Lemma 2.8. *Let $L = R : T$ be a split extension, where $|R| \mid 384$ and T is a nonabelian simple group. Then either*

(a) $L = R \times T$; *or*
(b) $L \neq R \times T$, $T \leq \mathrm{GL}(e, 2)$ *with* $2^e \mid |R|$, *and e and T satisfy one of the following:*
  (b.1) $e = 3$ *and* $T = \mathrm{PSL}_3(2)$.
  (b.2) $e = 4$ *and* $T \in \{\mathrm{PSL}_3(2), \mathrm{PSL}_4(2), \mathrm{A}_5, \mathrm{A}_6, \mathrm{A}_7\}$.
  (b.3) $e = 5$ *and* $T \in \{\mathrm{PSL}_3(2), \mathrm{PSL}_4(2), \mathrm{PSL}_5(2), \mathrm{A}_5, \mathrm{A}_6, \mathrm{A}_7\}$.
  (b.4) $e = 6$ *and* $T \in \{\mathrm{PSL}_k(2)\ (3 \leq k \leq 6), \mathrm{PSL}_2(8), \mathrm{A}_5, \mathrm{A}_6, \mathrm{A}_7, \mathrm{PSU}_3(3), \mathrm{PSU}_4(2), \mathrm{PSp}_6(2)\}$.
  (b.5) $e = 7$ *and* $T \in \{\mathrm{PSL}_k(2)\ (3 \leq k \leq 7), \mathrm{PSL}_2(8), \mathrm{A}_5, \mathrm{A}_6, \mathrm{A}_7, \mathrm{PSU}_3(3), \mathrm{PSU}_4(2), \mathrm{PSp}_6(2)\}$.

2.2. Preparatory lemmas

The *solvable radical* of a group is its largest solvable normal subgroup.

Lemma 2.9. *Let T be a nonabelian simple group, and let Γ be a connected cubic semisymmetric bi-Cayley graph on T. Suppose R is the solvable radical of $\mathsf{Aut}\Gamma$ and $3 \mid |R|$. Then $RT = R \times T$.*

Proof. Write $L = RT$, $\mathrm{A} = \mathsf{Aut}\Gamma$ and let $\alpha \in V\Gamma$. As $R \cap T$ is solvable and normal in T, we have $R \cap T = 1$ and $L = R : T$ is a split extension. Since Γ is a semisymmetric bi-Cayley graph on T, we obtain $|L : L_\alpha| = |V\Gamma|/2 = |T|$, so $|L_\alpha| = |R|$ is divisible by 3. By Theorem 2.2, $|R| = |L_\alpha|$ divides $3 \cdot 2^7 = 384$, and the kernel of A_α acting on $\Gamma(\alpha)$ is a 2-group, we deduce $3 \,\big|\, |L_\alpha^{\Gamma(\alpha)}|$, and Γ is L-semisymmetric. If $2^3 \nmid |R|$, Lemma 2.8 leads to $L = R \times T$, as required. Thus assume in the following that $2^3 \mid |R|$. Then $24 \mid |R|$ as $3 \mid |R|$.

Let R_2 be a Sylow 2-subgroup of R and let $S = \{R_2^r \mid r \in R\}$, the set of the conjugate subgroups of R_2 in R. Since $|R|$ divides 384, $|S| = |R : N_R(R_2)|$ divides 3, then the nonabelian simplicity of T implies that T acts trivially on S by conjugation, that is, T normalizes R_2. Thereby $P := R_2T = R_2 : T \leq L$,

$|P : P_\alpha| = |T|$ and $|L : P| = 3$. Let K be the kernel of L acting on $[L : P]$ by right multiplication. Then K is the largest normal subgroup of L contained in P ([7, Example 1.3.4]) and $\mathbb{Z}_3 \le L/K \le S_3$, implying $K \ge T$. Since $|P| = |R_2||T| = |T||P_\alpha|$, $K_\alpha \le P_\alpha$ is a 2-groups.

If A acts unfaithfully on the bipartitions of Γ, by [13, Lemma 5.2], $\Gamma = \mathsf{K}_{3,3}$ is arc-transitive, a contradiction. Hence A acts faithfully on the bipartitions of Γ. If $K_\alpha \neq 1$, then Lemma 2.5 implies that $K_\alpha^{\Gamma(\alpha)} \ge \mathbb{Z}_3$, contradicting that K_α is a 2-group. Thus $K_\alpha = 1$. Consequently, $T = K \lhd L$, and so $L = R \times T$. □

Denote by $\mathsf{soc}(G)$ the socle of a group G, that is, the product of all minimal normal subgroups of G.

Lemma 2.10. *Let Γ be a connected cubic G-semisymmetric graph, and let G have a nonabelian simple subgroup T which has two orbits of the same length on $V\Gamma$. Suppose further G has a trivial solvable radical. Then G is almost simple, and either*

(a) $\mathsf{soc}(G) = T$; *or*

(b) $\mathsf{soc}(G) > T$, $\mathsf{soc}(G)_\alpha \ge \mathbb{Z}_3$ *for* $\alpha \in V\Gamma$, *and* Γ *is* $\mathsf{soc}(G)$*-semisymmetric. Further, one of the following is true:*

(b.1) $(\mathsf{soc}(G), T) = (A_n, A_{n-1})$ *with* $n = 6$ *or* $n \ge 12$ *dividing* 384.

(b.2) $(\mathsf{soc}(G), T) = (\mathsf{M}_{24}, \mathsf{M}_{23})$.

Proof. By Theorem 2.2, $|G_\alpha| \mid 384$. Since $|G : G_\alpha| = |V\Gamma|/2 = |T : T_\alpha|$, it follows that $|G|$ divides $384|T|$.

Let N be a minimal normal subgroup of G. Since G has a trivial solvable radical, $N = S_1 \times S_2 \cdots \times S_d = S^d$, where each $S_i \cong S$ is nonabelian simple and $d \ge 1$. Since T is nonabelian simple, either $T \cap N = 1$ or $N \ge T$. For the former, $|N||T|$ divides $384|T|$, so $|N|$ divides 384, hence N is a $\{2,3\}$-group and thus solvable, a contradiction. Therefore $N \ge T$. As distinct minimal normal subgroups of G intersect trivially, by the arbitrariness of T, we obtain that N is the unique minimal normal subgroup of G, namely $\mathsf{soc}(G) = N$. If $T = N$, then $\mathsf{soc}(G) = T$, as in Lemma 2.10(a).

Now suppose $T < N$. Then $|N : N_\alpha| = |T : T_\alpha|$, hence $|N| = |T||N_\alpha : T_\alpha|$ divides $384|T|$. Assume $d \ge 2$. Since $S_1 \lhd N$ and $T < N$, we have $S_1 \cap T \lhd T$, so $S_1 \cap T = 1$ or T. For the former, then $|S_1 T| = |S_1||T|$ divides $|N|$, and so divides $384|T|$. It follows that S_1 is a $\{2,3\}$-group and solvable, a contradiction. If the latter occurs, then $T \le S_1$ and $|S|^d = |N|$ divides $384|S|$, implying $|S|^{d-1} \mid 384$, also a contradiction.

The above contradiction shows that $d = 1$, namely $\mathsf{soc}(G) = S > T$. Since $|S : T| = |S_\alpha : T_\alpha|$ divides 384, the pair (S, T) satisfies Lemma 2.7. In particular $\mathsf{Out}(S) \le \mathbb{Z}_2^2$. Notice that $G_\alpha \ge \mathbb{Z}_3$ and

$$G_\alpha/S_\alpha = G_\alpha/(S \cap G_\alpha) \cong SG_\alpha/S = G/S \le \mathsf{Out}(S) \le \mathbb{Z}_2^2,$$

we conclude that $S_\alpha \ge \mathbb{Z}_3$ is transitive on $\Gamma(\alpha)$, thus Γ is S-semisymmetric. Now to complete the proof, we only need to prove $(S, T) \neq (A_8, A_7)$, $(\mathsf{M}_{11}, \mathrm{PSL}_2(11))$ and $(\mathsf{M}_{12}, \mathsf{M}_{11})$.

If $(S, T) = (\mathsf{M}_{11}, \mathrm{PSL}_2(11))$, by Frattini argument, $S = TS_\alpha$. Notice that $|S_\alpha| \mid 384$, by [12, Theorem 1.1], no such factorization exists, a contradiction.

Let α, β be adjacent vertices of Γ. Then $|S_\alpha| = |S_\beta|$, $|T_\alpha| = |T_\beta|$, $|S_\alpha : S_\alpha \cap S_\beta| = 3$, and $\langle S_\alpha, S_\beta \rangle = S$. Recall $|S_\alpha| = |S : T||T_\alpha|$ divides 384.

Set $k = |T_\alpha|$, and define

$$\mathcal{S}_k = \{H \mid H \leq S, |H| = k|S:T|\}, \quad \mathcal{T} = \{H \mid H \leq S, H \cong T\},$$

$$\mathcal{P}_k = \{(H_1, H_2) \mid H_1, H_2 \in \mathcal{S}_k, |H_1 : H_1 \cap H_2| = 3, \langle H_1, H_2\rangle = S\},$$

$$\mathcal{Q}_k = \{(|H_1 \cap T|, |H_2 \cap T|) \mid T \in \mathcal{T}, (H_1, H_2) \in \mathcal{P}_k\}.$$

Then $(S_\alpha, S_\beta) \in \mathcal{P}_k$ and $(|T_\alpha|, |T_\beta|) \in \mathcal{Q}_k$.

Assume $(S, T) = (\mathrm{A}_8, \mathrm{A}_7)$. By [6], A_8 has no subgroup with order 384, so $|S_\alpha| \neq 384$. Then since $3 \mid |S_\alpha|$ and $|S_\alpha| = |S:T||T_\alpha| = 8k$ divides 384, we conclude $k \in \{3, 6, 12, 24\}$. But searching in Magma [1] shows $|\mathcal{P}_k| = 0$ for each $k \in \{3, 6, 12, 24\}$, a contradiction.

Now assume $(S, T) = (\mathsf{M}_{12}, \mathsf{M}_{11})$. Then $|S_\alpha| \neq 384$ as M_{12} has no subgroup with order 384 by [6]. Since $|S_\alpha| = |S:T||T_\alpha| = 12k$ divides 384, we deduce $k \in \{1, 2, 4, 8, 16\}$. However, computation with Magma [1] shows $|\mathcal{P}_4| = |\mathcal{P}_8| = 0$, so $k \neq 4$ and 8. For $k = 1, 2, 16$, computation shows $\mathcal{Q}_1 = \{(1, 3), (3, 1)\}$, $\mathcal{Q}_2 = \{(2, 6), (2, 3), (3, 2), (6, 2)\}$ and $\mathcal{Q}_{16} = \{(24, 16), (48, 16), (16, 24), (16, 48)\}$, contradicting $|T_\alpha| = |T_\beta|$. □

3. Proof of Theorem 1.1

For convenience, we make the following hypothesis in this section.

Hypothesis 3.1. *Let T be a nonabelian simple group, and let Γ be a connected cubic semisymmetric bi-Cayley graph on T. Denote by $\varDelta_1$ and $\varDelta_2$ the bipartitions of Γ. Set* $\mathrm{A} = \mathsf{Aut}\Gamma$ *and $\alpha \in V\Gamma$.*

Lemma 3.2. *Under Hypothesis 3.1, then $T \neq \mathrm{A}_5$.*

Proof. If $T = \mathrm{A}_5$, then Γ is of order 120. Checking the census of connected semisymmetric cubic graphs on up to 768 vertices ([4]), there is no such graph of order 120 admitting a semiregular automorphism group A_5, a contradiction. □

Lemma 3.3. *Under Hypothesis 3.1, and suppose that* A *has a trivial solvable radical, and T is not normal in* A. *Then part* (2) *of Theorem 1.1 is true with $R = 1$.*

Proof. Set $S = \mathsf{soc}(\mathrm{A})$. By Lemma 3.2, $T \neq \mathrm{A}_5$. Then we see from Lemma 2.10 that $S > T$ is a simple group, $S_\alpha \geq \mathbb{Z}_3$ and either $(S, T) = (\mathsf{M}_{24}, \mathsf{M}_{23})$ or (A_n, A_{n-1}) with $n \geq 12$ a divisor of 384. If the latter case occurs, by Frattini argument, $S = TS_\alpha$, then the semiregularity of T derives that $n = |S:T| = |S_\alpha|$ is divisible by 3. This together with that $n \geq 12$ divides 384 implies $n = 12, 24, 48, 96, 192$ or 384, as required. □

Lemma 3.4. *Under Hypothesis 3.1, and suppose that* A *has a nontrivial solvable radical R, and T is not normal in* A. *Then part* (2) *of Theorem 1.1 is true with $R \neq 1$ and $n \neq 384$.*

Proof. If A acts unfaithfully on $\varDelta_1$ or $\varDelta_2$, by [13, Lemma 5.2], $\Gamma = \mathsf{K}_{3,3}$ is arc-transitive, a contradiction. Thus A acts faithfully on $\varDelta_1$ and $\varDelta_2$. Write $L = RT$. As $R \cap T$ is solvable and normal in T, we obtain that $R \cap T = 1$ and $|L_\alpha| = |R| \neq 1$. By Theorem 2.2, $|\mathrm{A}| = |T||\mathrm{A}_\alpha|$ divides $384|T|$, so $|L|$ divides $384|T|$. Hence $|R| \mid 384$. If R is transitive on $\varDelta_1$ or $\varDelta_2$, then $|T|$ divides $|R|$, thus T is a $\{2, 3\}$-group and solvable, a contradiction. Therefore R is not transitive on both $\varDelta_1$ and $\varDelta_2$. It then follows from Lemma 2.3 that

R is semiregular on both $\varDelta_1$ and $\varDelta_2$, $A/R \le \mathsf{Aut}(\varGamma_R)$, and the quotient $\varGamma_R$ is a connected cubic A/R-semisymmetric graph. Moreover, notice that A/R has a trivial solvable radical, and $L/R \cong T$ has two orbits with equal length on $V\varGamma_R$, hence the triple $(\varGamma_R, G/R, L/R)$ (as $(\varGamma, G, T)$ there) satisfies Lemma 3.3. Consequently (note $T \ne \mathrm{A}_5$ by Lemma 3.2), A/R is almost simple, and either

(a) $\mathsf{soc}(\mathrm{A}/R) = L/R$; or

(b) $\mathsf{soc}(\mathrm{A}/R) > L/R \cong T$, $\mathsf{soc}(A/R)_\alpha \ge \mathbb{Z}_3$ and $\varGamma_R$ is $\mathsf{soc}(\mathrm{A}/R)$-semisymmetric. Further, one of the following is true:

  (b.1) $(\mathsf{soc}(\mathrm{A}/R), L/R) = (\mathrm{A}_n, \mathrm{A}_{n-1})$ where $n \ge 12$ divides 384.

  (b.2) $(\mathsf{soc}(\mathrm{A}/R), L/R) = (\mathsf{M}_{24}, \mathsf{M}_{23})$.

Assume (a) occurs. Then $L \lhd \mathrm{A}$. Since A acts faithfully on $\varDelta_1$ and $\varDelta_2$, and $|L_\alpha| = |R| \ne 1$, by Lemma 2.5, $L_\alpha^{\varGamma(\alpha)} \ge \mathbb{Z}_3$, so $3 \mid |R|$. It then follows from Lemma 2.9 that $L = R \times T$, hence $T \lhd \mathrm{A}$, a contradiction.

Assume (b) occurs. If $L \ne R \times T$, as $|R| \mid 384$, T satisfies part (b) of Lemma 2.8, which violates that $T \cong L/R$ satisfies (b.1) or (b.2) above, a contradiction. Thus $L = R \times T$. Suppose $S = M/R = \mathsf{soc}(\mathrm{A}/R)$ and $C = C_M(R)$. Then $T \le C \lhd M$ and $C \cap R = Z(R) \le Z(C)$. Observe $1 \ne C/(C \cap R) \cong CR/R \lhd M/R = S$, we derive that $M = CR$, $C \cap R = Z(C)$ and $C = (C \cap R).S$ is a central extension. Now Lemma 2.6 implies that $C' = (C' \cap Z(C)).S$ is a covering group of S, and $C' \cap Z(C) = Z(C')$ is a quotient of $\mathsf{Mult}(S)$. Notice that $\mathsf{Mult}(\mathsf{M}_{24}) = 1$ and $\mathsf{Mult}(\mathrm{A}_n) \cong \mathbb{Z}_2$ for $n \ge 12$.

Suppose $Z(C') > 1$. Then $Z(C') = \mathsf{Mult}(S) = \mathbb{Z}_2$, $C' \cong \mathbb{Z}_2.\mathrm{A}_n$ and $(S, T) = (\mathrm{A}_n, \mathrm{A}_{n-1})$ with $n \ge 12$ dividing 384. Since $Z(C') \lhd M$, $M/R = S$ and $RT = R \times T$, we deduce that $Z(C') \le R$ and C' has a subgroup $TZ(C') = T \times Z(C') \cong T \times \mathbb{Z}_2$. However, by [8, Proposition 2.6], the covering group $C' = \mathbb{Z}_2.\mathrm{A}_n$ with $n \ge 8$ has no subgroup isomorphic to $\mathbb{Z}_2 \times \mathrm{A}_{n-1}$, a contradiction.

Therefore $Z(C') = 1$, and $C' \cong S$ is normal in M, so $C' \cap R = 1$ and $M = R \times C'$ as $M/R = S$. It follows $\mathrm{A} = (R \times C').\mathcal{O}$ with $\mathcal{O} \le \mathsf{Out}(S)$. Since $|\mathrm{A} : \mathrm{A}_\alpha| = |M : M_\alpha| = |T|$, we deduce $|R||\mathcal{O}| = |\mathrm{A}_\alpha|/|S : T|$ divides $384/|S : T|$. Finally, for the case $(S, T) = (\mathrm{A}_n, \mathrm{A}_{n-1})$ with $n \ge 12$ dividing 384, as $|R| > 1$, $n \ne 384$. Since $C' \lhd \mathrm{A}$, by Lemma 2.5, $C'_\alpha \ge \mathbb{Z}_3$, then the semiregularity of T implies that $n = |S : T| = |C' : T| = |C'_\alpha|$ is divisible by 3. Now since $n \ge 12$ divides 384, we derive that $n \in \{12, 24, 28, 96, 192\}$, as in part (2) of Theorem 1.1. □

Now we are ready to prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. Let T be a nonabelian simple group, and let $\varGamma$ be a connected cubic semisymmetric bi-Cayley graph on T. Set $\mathrm{A} = \mathsf{Aut}\varGamma$. If $\varGamma$ is nonnormal on T, by Lemmas 3.3 and 3.4, part (2) of Theorem 1.1 holds.

Assume $\varGamma$ is normal on T. Since $\varGamma$ is semisymmetric, $\varGamma$ is bipartite, and so we may suppose that $\varGamma = \mathsf{BiCay}(T, \emptyset, \emptyset, S)$, with bipartitions $T_0 := \{t_0 \mid t \in T\}$ and $T_1 := \{t_1 \mid t \in T\}$, and $S = \{1, a, b\}$. By Lemma 2.4,

$$\mathrm{A}_{1_0} = \langle \sigma_{\tau,s} \mid \tau \in \mathsf{Aut}(T), s \in S, S^\tau = s^{-1}S \rangle,$$

which is transitive on $\varGamma(1_0) = \{1_1, a_1, b_1\}$ by the semisymmetry of $\varGamma$. If $\sigma_{\tau,s}$ fixes each vertex in $\varGamma(1_0)$, then

$$1_1 = (1_1)^{\sigma_{\tau,s}} = (s1^\tau)_1 = s_1, a_1 = (a_1)^{\sigma_{\tau,s}} = (sa^\tau)_1, b_1 = (b_1)^{\sigma_{\tau,s}} = (sb^\tau)_1.$$

It follows $s = 1$, $a^\tau = a$ and $b^\tau = b$. Since $\varGamma$ is connected, $\langle a, b\rangle = \langle S \rangle = T$, hence $\tau = 1$ and $\sigma_{\tau,s} = \sigma_{1,1}$ is the identity automorphism of $\varGamma$, namely A_{1_0} acts faithfully on $\varGamma(1_0)$. Consequently $\mathbb{Z}_3 \le \mathrm{A}_{1_0} \le \mathrm{S}_3$. Now by Frattini argument, we deduce $\mathrm{A} = T : \mathrm{A}_{1_0} = T : \mathbb{Z}_3$ or $T : \mathrm{S}_3$, part (1) of Theorem 1.1 holds. □

Proof of Corollary 1.2. We only need to prove the sufficiency.

Suppose on the contrary that T is a nonabelian simple group, and there is a connected cubic semisymmetric bi-Cayley graph $\varGamma$ on T such that $\varGamma$ is almost-normal but not normal on T. Set $\mathrm{A} = \mathsf{Aut}\varGamma$ and $G = N_{\mathrm{A}}(T)$. Then $\varGamma$ is G-semisymmetric, and with the same reason (view G as A there) as in the previous paragraph, we derive that $G = T : \mathbb{Z}_3$ or $T : \mathrm{S}_3$. Since, by assumption, T is not normal in A, we see from Theorem 1.1(2) that

$$\mathrm{A} = (R \times S).O,$$

where $S > T$, $(S, T) = (\mathsf{M}_{24}, \mathsf{M}_{23})$ or $(\mathrm{A}_n, \mathrm{A}_{n-1})$ with $n \in \{12, 24, 48, 96, 192, 384\}$, $|R||O|$ divides $384/|S : T|$ and $O \leq \mathsf{Aut}(S) \leq \mathbb{Z}_2$. If $R = 1$, as T is maximal in S, we obtain $G \leq T.\mathbb{Z}_2$, which is impossible as $\varGamma$ is G-semisymmetric. Thus $R \neq 1$. Since R centralizes T, $R \leq N_{\mathrm{A}}(T) = G$ and so $R \times T \lhd G$. This together with $T : \mathbb{Z}_3 \leq G \leq T : \mathrm{S}_3$ implies that $R \geq \mathbb{Z}_3$. Notice that $3 \mid |S : T|$, we derive that $|\mathrm{A}_\alpha| = |\mathrm{A} : T|$ is divisible by 3^2, which is a contradiction by Theorem 2.2. □

We end this section with a specific example of cubic nonnormal semisymmetric bi-Cayley graph on M_{23}.

Example 3.5. *Let*

$$\begin{aligned}
G =&\langle(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,24),(2,16,9,6,8)\\
&(3,12,13,18,4)(7,17,10,11,22)(14,19,21,20,15),(1,22)(2,11)(3,15)(4,17)(5,9)\\
&(6,19)(7,13)(8,20)(10,16)(12,21)(14,18)(23,24)\rangle,\\
L =&\langle(1,6,17,11)(2,15,8,4)(3,23,22,5)(7,16,9,21)(10,13,18,24)(12,19,14,20),\\
&(1,2)(3,19)(4,6)(5,14)(7,24)(8,17)(9,13)(10,21)(11,15)(12,23)(16,18)(20,22),\\
&(1,17)(2,8)(3,22)(4,15)(5,23)(6,11)(7,9)(10,18)(12,14)(13,24)(16,21)(19,20),\\
&(1,10,5,17,18,23)(2,21,14,8,16,12)(3,13,6,22,24,11)(4,20,7,15,19,9)\rangle,\\
R =&\langle(1,2)(2,3)(4,6)(21,10)(16,18)(8,17)(7,24)(12,23)(13,9)(20,22)(5,23)(11,15),\\
&(1,6,24)(19,11,14)(4,2,7)(21,13,20)(16,12,17)(8,23,18)(22,9,10)(5,15,3),\\
&(1,17)(19,20)(4,15)(21,16)(6,11)(8,2)(18,10)(7,9)(12,14)(13,24)(22,3)(5,23),\\
&(1,19)(4,18)(21,6)(16,11)(8,3)(7,5)(12,24)(17,20)(13,14)(22,2)(9,23)(10,15)\rangle.
\end{aligned}$$

Let $\Gamma = \mathsf{Cos}(G, L, R)$. *Then* Γ *is a connected cubic* M_{24}*-semisymmetric bi-Cayley graph on* M_{23}.

Proof. A direct computation with Magma [1] shows that $L \cong \mathbb{Z}_3 : \mathrm{D}_8$, $R \cong \mathrm{S}_4$, $\langle L, R\rangle = G \cong \mathsf{M}_{24}$ and $|L : L \cap R| = 3$. By Lemma 2.1, Γ is a connected cubic M_{24}-semisymmetric graph. Notice that both L and R are transitive on the set $\{1, 2, ..., 24\}$ and $|L| = |R| = 24$, they are regular on $\{1, 2, \ldots, 24\}$. Let $T = G_1$, the stabilizer of G on the point 1. Then $T \cong \mathsf{M}_{23}$, $G = TL = TR$ and $|G| = |T||L| = |T||R|$. It follows that T is regular on the bipartitions of $\varGamma$, namely $\varGamma$ is a bi-Cayley graph on M_{23}. Further, by [9, Lemma 7.1], Γ is not arc-transitive, hence Γ is semisymmetric. Finally as $|G| = 24|\mathsf{M}_{23}|$, by Theorem 1.1, $\varGamma$ is nonnormal on M_{23}. □

4. Proof of Theorem 1.3

For a nonabelian simple group T, we may view T as a normal subgroup of $\mathsf{Aut}(T)$ up to isomorphism, and then each element in $\mathsf{Aut}(T)$ may acts on T by conjugate action.

Proof of Theorem 1.3. (Necessity) By assumption, we may assume $\Gamma = \mathsf{BiCay}(T, \emptyset, \emptyset, S)$ with bipartitions $T_0 := \{t_0 \mid t \in T\}$ and $T_1 := \{t_1 \mid t \in T\}$, and $1 \in S$. Then $S_1 = \Gamma(1_0)$ and $|S| = 3$. By [29, Theorem 1.1],

$$\mathrm{A}_{1_0} = \langle \sigma_{\theta,s} \mid \theta \in \mathsf{Aut}(T), s \in S, S^\theta = s^{-1}S \rangle$$

acts faithfully on $\Gamma(1_0) = S_1$, and A_{1_0} contains an element $\sigma_{g,s}$ with order 3 which cyclically permutes all the elements of S_1, where $g \in \mathsf{Aut}(T)$ and $s \in S$. It follows $S_1 = 1_1^{\langle \sigma_{g,s} \rangle} = \{1_1, s_1, (ss^g)_1\}$ and $S = \{1, s, ss^g\}$. The connectivity of Γ forces $\langle s, s^g \rangle = \langle S \rangle = T$. Since $o(\sigma_{g,s}) = 3$, for each $t \in T$, $t_0 = (t_0)^{\sigma_{g,s}^3} = (t^{g^3})_0$ and $t_1 = (t_1)^{\sigma_{g,s}^3} = (ss^g s^{g^2} t^{g^3})_1$, or equivalently $g^3 = 1$ and $ss^g s^{g^2} = 1$. If $g = 1$, then $\langle S \rangle = \langle 1, s, s^2 \rangle = \langle s \rangle \neq T$, contradicting the connectivity of Γ. Hence $o(g) = 3$. Moreover, since Γ is not vertex-transitive, by [5, Proposition 3.3(a)], there is no $h \in \mathsf{Aut}(T)$ such that $S^h = S^{-1}$, namely $\{s, ss^g\}^h = \{s^{-1}, (ss^g)^{-1}\}$.

(Sufficiency) Suppose $\Gamma = \mathsf{BiCay}(T, \emptyset, \emptyset, \{1, s, ss^g\})$, where $s \in T$ and $g \in \mathsf{Aut}(T)$ satisfy the condition in Theorem 1.3. Clearly Γ is a cubic bi-Cayley graph on T. Set $P = \{1, s, ss^g\}$ and $G = \langle T, \sigma_{g,s} \rangle$. By [29, Theorem 1.1], $G = T : \langle \sigma_{g,s} \rangle \leq N_{\mathsf{Aut}\Gamma}(T)$. Since $\langle P \rangle = T$, Γ is connected. Since $P = (1_1)^{\langle \sigma_{g,s} \rangle}$, Γ is G-semisymmetric.

We claim that Γ is semisymmetric (note that a G-semisymmetric graph is not necessarily semisymmetric in general). If not, as Γ is of valency 3, Γ is symmetric and $G < \mathsf{Aut}\Gamma$. If $T \lhd \mathsf{Aut}\Gamma$, by [5, Proposition 3.3(a)], there is an element $h \in \mathsf{Aut}(T)$ such that $\{1, s, ss^g\}^h = S^h = S^{-1} = \{1, s^{-1}, (ss^g)^{-1}\}$, which is not possible by the assumption. If T is not normal in $\mathsf{Aut}\Gamma$, by [21, Theorem 1.1(2)], we would have that $T = \mathrm{A}_{n-1}$ and $\mathsf{Aut}\Gamma = (R \times \mathrm{A}_n).O$, where $n \in \{24, 48, 96\}$ and $|R||O| \mid \frac{96}{n}$. Notice that A_{n-1} is maximal in A_n and $\frac{96}{n} \mid 4$, we deduce that $|N_{\mathsf{Aut}\Gamma}(T) : T|$ divides 4, which is also not possible as $T : \mathbb{Z}_3 \leq G \leq N_{\mathsf{Aut}\Gamma}(T)$.

Therefore, Γ is semisymmetric. Hence Γ is almost-normal as $G \leq N_{\mathsf{Aut}\Gamma}(T)$. Now Corollary 1.2 implies that Γ is normal on T. □

We finally address two examples of normal cubic semisymmetric bi-Cayley graphs on A_7 and A_8, respectively.

Example 4.1. *Let* $s = (1, 6, 2, 5, 4, 3, 7)$*,* $g = (1, 5, 4)$ *and* $\Gamma = \mathsf{BiCay}(\mathrm{A}_7, \emptyset, \emptyset, \{1, s, ss^g\})$*. Then* Γ *is a connected cubic semisymmetric normal bi-Cayley graph on* A_7 *and* $\mathsf{Aut}\Gamma = (\mathbb{Z}_3 \times \mathrm{A}_7) : \mathbb{Z}_2$.

Proof. Obviously $\mathsf{val}(\Gamma) = |\{1, s, ss^g\}| = 3$. With Magma [1], one may check that $\langle s, s^g \rangle = \mathrm{A}_7$, $ss^g s^{g^2} = 1$, $\mathsf{Aut}\Gamma = (\mathbb{Z}_3 \times \mathrm{A}_7) : \mathbb{Z}_2$ and there is no element $h \in \mathrm{S}_7$ such that $\{s, ss^g\}^h = \{s^{-1}, (ss^g)^{-1}\}$. By Theorem 1.3, Γ is a connected cubic semisymmetric normal bi-Cayley graph on A_7. □

Example 4.2. *Let* $s = (1, 6, 8)(2, 7, 3, 5, 4)$*,* $g = (1, 7, 8)(4, 6, 5)$ *and* $\Gamma = \mathsf{BiCay}(\mathrm{A}_8, \emptyset, \emptyset, \{1, s, ss^g\})$*. Then* Γ *is a connected cubic semisymmetric normal bi-Cayley graph on* A_8 *and* $\mathsf{Aut}\Gamma = (\mathbb{Z}_3 \times \mathrm{A}_8) : \mathbb{Z}_2$.

Proof. Clearly $\mathsf{val}(\Gamma) = |\{1, s, ss^g\}| = 3$. A computation with Magma [1] shows that $\langle s, s^g \rangle = \mathrm{A}_8$, $ss^g s^{g^2} = 1$, $\mathsf{Aut}\Gamma = (\mathbb{Z}_3 \times \mathrm{A}_8) : \mathbb{Z}_2$ and there is no element $h \in \mathrm{S}_8$ such that $\{s, ss^g\}^h = \{s^{-1}, (ss^g)^{-1}\}$. By Theorem 1.3, Γ is a connected cubic semisymmetric normal bi-Cayley graph on A_8. □

5. Conclusions

This paper classifies cubic semisymmetric bi-Cayley graphs on nonabelian simple groups, which extends a remarkable classification of cubic nonnormal symmetric Cayley graphs on nonabelian simple groups, and together with former known results, completes the characterization of cubic edge-transitive bi-Cayley graphs on nonabelian simple groups.

Acknowledgments

The authors are very grateful to the referees for the helpful comments. This paper was supported by the National Natural Science Foundation of China (11961076).

Conflict of interest

The authors declare no conflicts of interest.

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