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Research article

Circular surfaces and singularities in Euclidean 3-space $\mathbb{E}^3$

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Abstract: The approach of the paper is on circular surfaces. A circular surface is a one-parameter family of standard circles with fixed radius regarding a curve, which acts as the spine curve. In the study, we have parametrized circular surfaces and have provided its geometric properties like singularities and striction curves comparing with those of ruled surfaces. Furthermore, we have addressed the conditions of minimality of roller coaster surfaces. Meanwhile, we support the results of the approach by some examples.

Keywords: lines of curvature and singularity; roller coaster surface
Mathematics Subject Classification: 53A04, 53A05, 53A17

1. Introduction

In Euclidean 3-space $\mathbb{E}^3$, as a set of points swept out by a moving oriented line, ruled surface has been studied thoroughly (see, e.g., [2, 4, 6]). The moving frame of a ruled surface has been built, so we can obtain the curvature functions, or invariants, which determine the local shape of ruled surface. Therefore, it has been utilized in computer aided design, and architecture (see [7, 8]). With an analogous idea, Izumiya et al. studied one-parameter smooth families of circles in $\mathbb{E}^3$ [5]. Circular surface with constant radius is a surface put up with such family. They addressed some geometric aspects and singular points of circular surfaces by comparing with corresponding aspects of ruled surfaces. Ruled surfaces, circular surfaces are significant topics in various areas [1, 3, 9–12].

However, at present there are a wide number of various names of the circular surface in earlier materials, that is , canal surface of a space curve, torus and cylindrical surface are special circular surfaces. Ruled surfaces and circular surfaces are closely linked. The tangential developable ruled surfaces are the envelopes of a family of circle planes, the tangential ruled surfaces are characterized by tangents to the regression edge and singularities of tangential developable ruled surfaces are composed by the edge of regression. Therefore, it is natural to approach circular surfaces in a similar way to ruled surfaces.

The present work includes the following: Section 3 gives definitions and studies the geometry of circular surfaces with constant radii that can be swept along a curve (the spine curve) via changing the position of a circle with its center as a generating circle. Then, in analogy to [3, 5], we address the characterization of curves to be striction curves on circular surfaces. Moreover, we provide an investigation on local singular points of circular surfaces. By considering that all generating circles are curvature lines except at singular or umbilical points, a classification of such circular surfaces into spheres, canal surfaces, special type of surfaces, or surfaces smoothly connected the three surfaces. Finally, we give some examples with figures supporting our idea of how to form roller coaster and circular surfaces. This approach is set up to analyze geometrically circular surfaces in terms of spine curves and the spherical indicatrices of normal vectors of circle planes.

2. Preliminaries

Let the Euclidean 3-space $\mathbb{E}^3$ be the whole space. For this approach, we use [2, 5, 6] as general references. Let

$$\mathbf{P}(u,\theta) = (p_1(u,\theta), p_2(u,\theta), p_3(u,\theta)), \ (u,\theta) \in D \subseteq \mathbb{R}^2,$$

represent a regular surface M. The tangents of M are

$$\mathbf{P}_u(u,\theta) = \frac{\partial \mathbf{P}}{\partial u}, \ \mathbf{P}_\theta(u,\theta) = \frac{\partial \mathbf{P}}{\partial \theta}.$$

The unit normal vector to M is

$$\mathbf{U}(u,\theta)=\frac{\mathbf{P}_u \times \mathbf{P}_\theta}{\|\mathbf{P}_u \times \mathbf{P}_\theta\|}, \tag{1}$$

where $\times$ denotes the cross product in $\mathbb{E}^3$. The metric (first fundamental form) I is defined by

$$I(u,\theta) = g_{11}du^2 + 2g_{12}dud\theta + g_{22}d\theta^2,$$

where $g_{11} =< \mathbf{P}_u, \mathbf{P}_u >$, $g_{12} =< \mathbf{P}_u, \mathbf{P}_\theta >$, $g_{22} =< \mathbf{P}_\theta, \mathbf{P}_\theta >$. We define the second fundamental form II of M by

$$II(u,\theta) = h_{11}du^2 + 2h_{12}dud\theta + h_{22}d\theta^2,$$

where $h_{11} =< \mathbf{P}_{uu}, \mathbf{U} >$, $h_{12} =< \mathbf{P}_{u\theta}, \mathbf{U} >$, $h_{22} =< \mathbf{P}_{\theta\theta}, \mathbf{U} >$. The mean and Gaussian curvatures are respectively

$$H(u,\theta) = \frac{h_{11}g_{11} - 2h_{12}g_{12} + h_{22}g_{22}}{2(g_{11}g_{22} - g_{12}^2)}, \text{ and } \quad K(u,\theta) = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}. \tag{2}$$

The unit sphere with center in the origin is defined by:

$$\mathbb{S}^2 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{E}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}. \tag{3}$$

3. Circular surfaces

Here, we present the notion of circular surfaces in Euclidean 3-space $\mathbb{E}^3$. Given a spherical curve $\mathbf{e}_1 : I \to \mathbb{S}^2 \subseteq \mathbb{E}^3$ parameterized by its arc-length u, then there exists a unique Blaschke frame

$\{\mathbf{e}_1 = \mathbf{e}_1(u),\ \mathbf{e}_2(u) = \frac{\mathbf{e}_1^{'}(u)}{\|\mathbf{e}_1^{'}(u)\|},\ \mathbf{e}_3(u) = \mathbf{e}_1 \times \mathbf{e}_2\}$ along $\mathbf{e}_1(u) \in \mathbb{S}^2$ satisfying the Blaschke formulae

$$\begin{pmatrix} \mathbf{e}_1^{'} \\ \mathbf{e}_2^{'} \\ \mathbf{e}_3^{'} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma \\ 0 & -\gamma & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix};\ \left(' = \frac{d}{du} \right), \tag{4}$$

where $\gamma(u)$ is called the spherical (geodesic) curvature function of $\mathbf{e}_1(u) \in \mathbb{S}^2$. Let $\alpha(u)$ be a regular space curve parametrized by

$$\alpha^{'}(u) = \delta(u)\mathbf{e}_1(u) + \sigma(u)\mathbf{e}_2(u) + \eta(u)\mathbf{e}_3(u), \tag{5}$$

where $\sigma(u)$, $\delta(u)$ and $\eta(u)$ are called curvature functions (invariants) of the curve $\alpha(u)$. Thus, for a positive number $r > 0$, and in terms of the solutions of the differential system (4), M can be defined as follows:

$$M : \mathbf{P}(u, \theta) = \alpha(u) + r\left(\cos\theta\mathbf{e}_2(u) + \sin\theta\mathbf{e}_3(u)\right),\ u \in I,\ \theta \in \mathbb{R}, \tag{6}$$

where $\alpha(u)$ is called spine curve, and $\theta \to \alpha(u) + r(\cos\theta\mathbf{e}_2(u) + \sin\theta\mathbf{e}_3(u))$ are called generating circles (see Figure 1) [3]. It is clear that Eq (5) offers a way to describe circular surfaces with a known radius $r > 0$ using the equation

$$\alpha(u) = \alpha_0 + \left(\int_0^u \delta(u)\mathbf{e}_1(u) + \sigma(u)\mathbf{e}_2(u) + \eta(u)\mathbf{e}_3(u) \right) du. \tag{7}$$

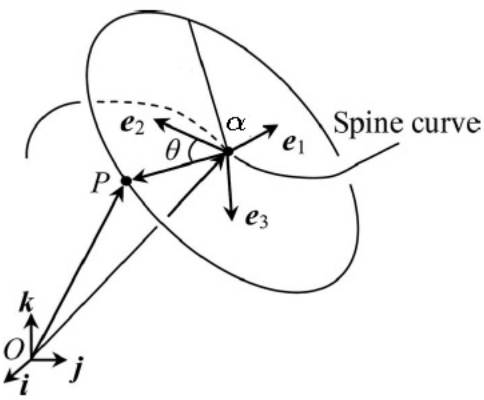

Figure 1. A cross section of M [3].

In this work we we do not consider circular surfaces with constant vector $\mathbf{e}_1$. The following definition will be useful:

Definition 1. Suppose that M is a circular surface of Eq (6). Then, at $u \in I \subseteq \mathbb{R}$,

1) M is called canal (tubular) if the spine curve is orthogonal to the circular plane, that is, $\alpha^{'}(u)$, $\mathbf{e}_1(u)$, $\mathbf{e}_2(u)$, and $\mathbf{e}_3(u)$ satisfy

$$\delta(u) = < \mathbf{e}_1, \alpha^{'} > \neq 0, \text{ and } < \mathbf{e}_2, \alpha^{'} > = < \mathbf{e}_3, \alpha^{'} > = 0 \Leftrightarrow \sigma(u) = \eta(u) = 0. \tag{8}$$

2) M is said to be non-canal (tangential or roller coaster) circular surface if the spine curve is tangent to the circular plane, that is, α', $\mathbf{e}_1(u)$, $\mathbf{e}_2(u)$, and $\mathbf{e}_3(u)$ satisfy

$$\delta(u) =< \mathbf{e}_1, \alpha' >= 0, \text{ and } \sigma(u) =< \mathbf{e}_2, \alpha' >= 0 \text{ or } \eta(u) =< \mathbf{e}_3, \alpha' >= 0. \tag{9}$$

It is well known that lines are considered as the simplest examples of curves, and circles with a fixed radius give another simple example of curves. Ruled surfaces are described out by a family of lines and circular surfaces by a series of circles with a fixed radius. Ruled surfaces have spine curves, circular surfaces and striction curves. Therefore, it is normal to study circular surfaces comparing with the class of ruled surfaces.

Now, we investigate the geometric properties of M. Thus, direct computation gives

$$\begin{gathered} \mathbf{P}_u(u,\theta) = (\delta - r\cos\theta)\mathbf{e}_1 + \gamma\mathbf{P}_\theta + \sigma\mathbf{e}_2 + \eta\mathbf{e}_3, \\ \mathbf{P}_\theta(u,\theta) = r(-\sin\theta\mathbf{e}_2 + \cos\theta\mathbf{e}_3). \end{gathered} \tag{10}$$

The coefficients of the first fundamental form are

$$\begin{gathered} g_{11} = (\delta - r\cos\theta)^2 + (\sigma - r\gamma\sin\theta)^2 + (\eta + r\gamma\cos\theta)^2, \\ g_{12} = r(r\gamma - \sigma\sin\theta + \eta\cos\theta), \; g_{22} = r^2. \end{gathered} \tag{11}$$

Hence, the unit normal is found by

$$\mathbf{U}(u,\theta) = \frac{(\sigma\cos\theta + \eta\sin\theta)\mathbf{e}_1 + (r\cos\theta - \delta)(\cos\theta\mathbf{e}_2 + \sin\theta\mathbf{e}_3)}{\sqrt{(r\cos\theta - \delta)^2 + (\sigma\cos\theta + \eta\sin\theta)^2}}. \tag{12}$$

This leads to

$$\begin{aligned} h_{11} &= \frac{\begin{gathered} r(-\delta + r\cos\theta)[(\gamma\sigma + \eta')\sin\theta + \left(\delta + \sigma' - \eta\gamma - r\cos\theta\right)\cos\theta - r\gamma^2] \\ +r(\delta' - \sigma + r\gamma\sin\theta)(\gamma\cos\theta + \eta\sin\theta) \end{gathered}}{\sqrt{(r\cos\theta - \delta)^2 + (\sigma\cos\theta + \eta\sin\theta)^2}}, \\ h_{12} &= \frac{r\left[(\sigma\cos\theta + \eta\sin\theta)\sin\theta + \gamma(\delta - r\cos\theta)\right]}{\sqrt{(r\cos\theta - \delta)^2 + (\sigma\cos\theta + \eta\sin\theta)^2}}, \\ h_{22} &= \frac{r(\delta - r\cos\theta)}{\sqrt{(r\cos\theta - \delta)^2 + (\sigma\cos\theta + \eta\sin\theta)^2}}. \end{aligned} \tag{13}$$

3.0.1. Striction curves

In $\mathbb{E}^3$, striction curves of circular surfaces are defined by Izumiya et al. [5]. We put a new definition of striction curves as follows: Suppose that the position vector of M,

$$\zeta(u) = \alpha(u) + r(\cos\theta(u)\mathbf{e}_2(u) + \sin\theta(u)\mathbf{e}_3(u)), \tag{14}$$

is the striction curve of $\zeta(u)$, then it satisfies

$$< \zeta', \cos\theta(u)\mathbf{e}_2(u) + \sin\theta(u)\mathbf{e}_3(u) >= 0.$$

Equivalently with,

$$\sigma(u)\cos\theta(u) + \eta(u)\sin\theta(u) = 0. \tag{15}$$

Using Eq (15), striction points only occur iff

$$\sin\theta(u) = \mp\frac{\sigma(u)}{\sqrt{\sigma^2+\eta^2}}, \text{ and } \cos\theta(u) = \pm\frac{\eta(u)}{\sqrt{\sigma^2+\eta^2}}. \tag{16}$$

Hence, there are two striction curves parametrized as

$$\left.\begin{aligned} \zeta_1(u) &= \alpha(u) + \tfrac{r}{\sqrt{\sigma^2+\eta^2}}\left(\eta(u)\mathbf{e}_2(u) - \sigma(u)\mathbf{e}_3(u)\right), \\ \zeta_2(u) &= \alpha(u) + \tfrac{r}{\sqrt{\sigma^2+\eta^2}}\left(-\eta(u)\mathbf{e}_2(u) + \sigma(u)\mathbf{e}_3(u)\right). \end{aligned}\right\} \tag{17}$$

According to Eqs (8) and (17), any curve on the canal surface is transversal to the generating circle satisfying the condition $\zeta_1(u) = \zeta_2(u) = \alpha(u)$. Hence, the family of canal surfaces is comparable to the cylindrical surfaces class.

Based on above, we have the following:

Proposition 1. Any non-canal circular surface has only two striction curves and it intersects generating circles at antipodal points.

3.1. Canal surfaces

Now, we examine the canal surfaces ($\delta(u) \neq 0$, and $\sigma(u) = \eta(u) = 0$) whose parametric curves are curvature lines. Hence, from Eqs (11) and (13), we get $g_{12} = h_{12} = 0 \Leftrightarrow \gamma(u) = 0$. If we substitute this into the Blaschke formulae, we obtain

$$\begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \tag{18}$$

The spherical curve that satisfies the condition in Eq (18) is a great circle on $\mathbb{S}^2$. For instance, a great circle can be given by $\mathbf{e}_1(u) = (\cos u, \sin u, 0)$, whose normal vector can be constructed using $\mathbf{e}_2 = \mathbf{e}_1'$ as $\mathbf{e}_2(u) = (-\sin u, \cos u, 0)$. Therefore, $\mathbf{e}_3(u) = (0, 0, -1)$. Thus, the spine curve is obtained as

$$\alpha(u) = \alpha_0 + \int_0^u \delta(u)(\cos u, \sin u, 0)du.$$

Hence, the canal surfaces family is given as

$$M : \mathbf{P}(u,\theta) = \alpha_0 + \int_0^u \delta(u)(\cos u, \sin u, 0)du + r(-\cos\theta\sin u, \cos\theta\cos u, -\sin\theta),$$

where $u \in I,\ \theta \in \mathbb{R}$. If $\alpha_0 = 0$, $r = 0.5$, $\delta(u) = 1$, $-\pi \leq \theta \leq \pi$, and $0 \leq u \leq 2\pi$, hence we attain a member in the family illustrated in Figure 2. Figure 3 shows the canal surface with $\alpha_0 = 0$, $r = 0.5$, $\delta(u) = u$, $-\pi \leq \theta \leq \pi$, and $0 \leq u \leq 2\pi$.

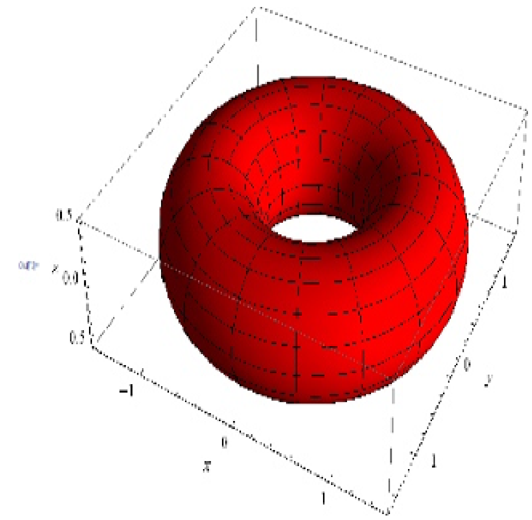

Figure 2. Canal surface with $\alpha_0 = 0$, $r = 0.5$, $\delta(u) = 1$, $-\pi \leq \theta \leq \pi$, and $0 \leq u \leq 2\pi$.

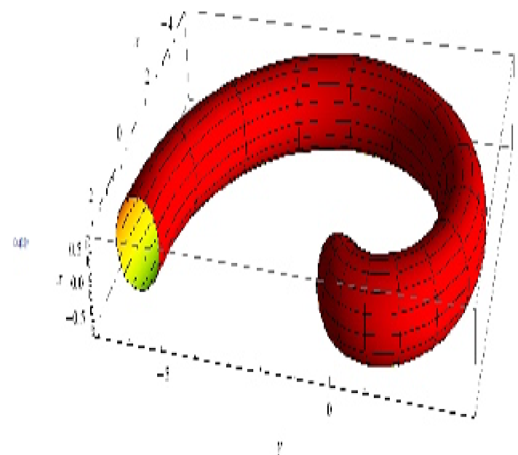

Figure 3. Canal surface with $\alpha_0 = 0$, $r = 0.5$, $\delta(u) = u$, $-\pi \leq \theta \leq \pi$, and $0 \leq u \leq 2\pi$.

Let s denote the arc-length parameter of $\alpha(u)$, and $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be Serret-Frenet frame. Therefore, we have

$$\mathbf{T}(s) = \frac{\alpha'}{\|\alpha'\|} = \mathbf{e}_1, \ \mathbf{N}(s) = \frac{d\mathbf{T}}{ds}\left\|\frac{d\mathbf{T}}{ds}\right\|^{-1} = \mathbf{e}_2, \ \mathbf{B}(s) = \mathbf{e}_3. \tag{19}$$

Differentiating Eq (19) regarding s, we obtain:

$$\frac{d}{ds}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{20}$$

where the curvature $\kappa(s)$, and torsion $\tau(s)$ of $\alpha(s)$ are given by:

$$\kappa(s) = \frac{1}{|\delta|},\ \tau(s) = \frac{\gamma}{\delta}, \text{ with } \delta(s) \neq 0. \tag{21}$$

Note that $\mathbf{T}(s)$ is orthogonal to the circle planes at each point of $\alpha(s)$. The canal surface can thus be parametrized as

$$M : \mathbf{P}(s,\theta) = \alpha(s) + r\left(\cos\theta\mathbf{N} + \sin\theta\mathbf{B}\right),\ s \in I,\ \theta \in \mathbb{R}. \tag{22}$$

The partial derivatives $\mathbf{P}_s(s,\theta)$ and $\mathbf{P}_\theta(s,\theta)$ are obtained as follows:

$$\mathbf{P}_s(s,\theta) = \lambda\mathbf{T} + \tau\mathbf{P}_\theta, \text{ and } \mathbf{P}_\theta(s,\theta) = r\left(-\sin\theta\mathbf{N} + \cos\theta\mathbf{B}\right), \tag{23}$$

where $\lambda = 1 - r\kappa\cos\theta$. The elements of the first fundamental form are

$$g_{11}(s,\theta) = \lambda^2 + r^2\tau^2,\ g_{12}(s,\theta) = r^2\tau,\ g_{22}(s,\theta) = r^2.$$

The unit normal vector is

$$\mathbf{U}(s,\theta) = \cos\theta\mathbf{N} + \sin\theta\mathbf{B}. \tag{24}$$

This leads to

$$h_{11} = -r\tau^2 + \kappa\lambda\cos\theta,\ h_{12} = -r\tau,\ h_{22} = -r.$$

The mean and Gaussian curvature functions of the canal surface are

$$H(s,\theta) = \frac{1}{2}\left(\frac{1}{r} + rK\right), \text{ and } K(s,\theta) = -\frac{\kappa\cos\theta}{\lambda r}. \tag{25}$$

Since each generating circle is a curvature line, the value of one principal curvature is

$$\chi_1(u,\theta) := \left\|\frac{d\mathbf{P}}{d\theta} \times \frac{d^2\mathbf{P}}{d\theta^2}\right\| \left\|\frac{d\mathbf{P}}{d\theta}\right\|^{-3} = \frac{1}{r}. \tag{26}$$

The second principal curvature is

$$\chi_2(u,\theta) = \frac{K(u,\theta)}{\chi_1} = -\frac{\kappa\cos\theta}{\lambda}. \tag{27}$$

Hence, we record the following results:

Theorem 1. In $\mathbb{E}^3$, suppose that we have a family of canal surfaces sharing the same radius r, and scalar function $\delta(u)$. Then, the mean and Gaussian curvatures at corresponding points have the same value.

Corollary 1. The principal curvatures of a canal surface are constant along every generating circle.

Example 1. Here, we construct a family of canal surfaces sharing a circular helix with the expression

$$\alpha(s) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}),$$

where $a, b, c \in \mathbb{R}$, $a^2 + b^2 = c^2$ and $0 \leq s \leq 2\pi$. It is readily to attain that

$$\left.\begin{array}{c} \mathbf{T}(s) = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}), \\ \mathbf{N}(s) = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0), \\ \mathbf{B}(s) = (\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}), \\ \kappa(s) = \frac{a}{a^2+b^2}, \text{ and } \tau(s) = \frac{b}{a^2+b^2}. \end{array}\right\}$$

Hence, we construct canal surface family passing the helix as

$$\mathbf{P}(s,\theta)=(a\cos\frac{s}{c},a\sin\frac{s}{c},\frac{bs}{c})+r\,(0,\cos\theta,\sin\theta)\begin{pmatrix}-\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c}\\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0\\ \frac{b}{c}\sin\frac{s}{c} & -\frac{b}{c}\cos\frac{s}{c} & \frac{a}{c}\end{pmatrix},$$

once $a=b=1$, $r=0.3$, and $0\leq\theta\leq 2\pi$, we get a member of such family depicted in Figure 4. Figure 5 provides another member of the family in which $a=b=1$, $r=0.2$, and $0\leq\theta\leq 2\pi$

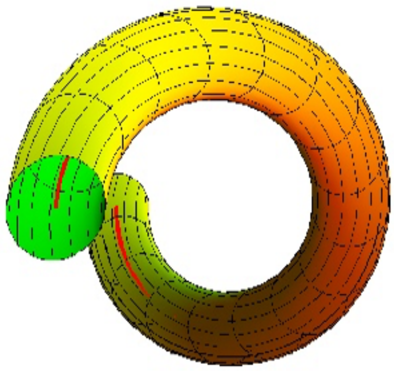

Figure 4. Canal surface passing helix with $a=b=1$, $r=0.3$, and $0\leq\theta\leq 2\pi$.

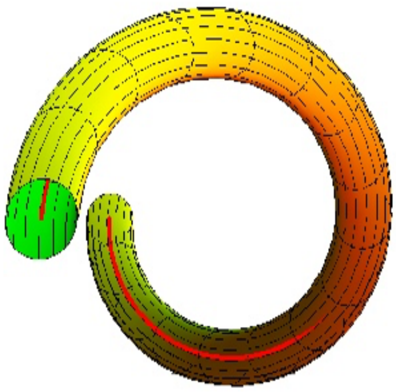

Figure 5. Canal surface passing helix in which $a=b=1$, $r=0.2$, and $0\leq\theta\leq 2\pi$.

3.2. Singularities

Singularities are fundamental for aspects of circular surfaces and are addressed as follows: It is interesting to note that the singularities on a generating circle are generically at two points.. Therefore, by using Eqs (4)–(6), we can show that M has a singular point at (u_0,θ_0) iff

$$<\mathbf{e}_1,\alpha^{'}+r\cos\theta\mathbf{e}_2^{'}+r\sin\theta\mathbf{e}_3^{'}>=0,\ \text{and}\ <\alpha^{'},r\cos\theta\mathbf{e}_2+r\sin\theta\mathbf{e}_3>=0,$$

which yields

$$\delta(u_0) - r\cos\theta = 0, \text{ and } \sigma(u_0)\cos\theta + \eta(u_0)\sin\theta = 0. \tag{28}$$

Hence, we take in consideration the following two main cases:
(A) Let $\cos\theta = \delta(u_0)/r$. Then we have two subclasses:
(1) If $\delta(u_0) = 0$, then we have $\theta = \pi/2$ and $\theta = 3\pi/2$. Also, from Eq (28), we get $\eta(u_0) = 0$ ($\sigma(u_0) \neq 0$), since $r \neq 0$. In consequence, there are two singularities on the generating circle, resp., at $\theta = 3\pi/2$ and $\theta = \pi/2$.
(2) If $\delta(u) \neq 0$, then by substituting $\cos\theta = \delta/r$ into the second equation in Eq (28), we find $-\sigma(u_0)\delta(u_0)/r = \eta(u_0)\sin\theta$, and we have two more cases:

(i) If $\eta(u_0) = 0$, then we find $\sigma(u_0) = 0$. Thus, the singular point of M is at (u_0, θ_0) such that $\eta(u_0) = \sigma(u_0) = 0$, and $\theta_0 = \cos^{-1}(\delta(u_0)/r)$.

(ii) If $\eta(u_0) \neq 0$, then we have $\sigma(u) \neq 0$. So, from the second equation in Eq (28), one can easily see that

$$\theta_0 = -\sin^{-1}(\sigma(u_0)\delta(u_0)/\eta(u_0)r), \text{ and } \theta_0 = \pi - \sin^{-1}(\sigma(u_0)\delta(u_0)/\eta(u_0)r).$$

Case (B) Let $\sigma(u_0)\cos\theta + \eta(u_0)\sin\theta = 0$, which yields $\sigma(u_0) = -\eta(u_0)\tan\theta$. Then, from $\cos\theta = \delta(u_0)/r$ we get: If $|\delta(u_0)| > r$, then there is no singular point. If $|\delta(u_0)| < r$, then we have singular points on the generating circle at $\theta = \pm\cos^{-1}(\delta(u_0)/r)$. If $\delta(u_0) = \pm r$, then the singular points occurring at $\theta = \pi/2$ and $\theta = 3\pi/2$.

Now, let us provide some examples on singular points of circular surfaces.

Example 2. Put $\mathbf{e}_1(u) = (\sin u, 0, \cos u)$, then $\mathbf{e}_2(u) = (\cos u, 0, -\sin u)$, and $\mathbf{e}_3(u) = (0, 1, 0)$.
(1) If $\sigma(u) = u$, and $\delta(u) = \eta(u) = 0$, then $\alpha'(u) = (u\cos u, 0, -u\sin u)$. Hence,

$$\alpha(u) = (u\sin u + \cos u, 0, u\cos u - \sin u),$$

in which constants of integration are chosen to be zero. Evidently, $\alpha(u)$ has a cusp at $u_0 = 0$, see Figure 6. One can easily see that the striction curves are

$$\begin{aligned} \zeta_1(u) &= (u\sin u + \cos u, -r, u\cos u - \sin u), \\ \zeta_2(u) &= (u\sin u + \cos u, r, u\cos u - \sin u). \end{aligned}$$

Therefore, the surface M with the spine curve $\alpha(u)$ is

$$M : \mathbf{P}(u, \theta) = (u\sin u + (1 + r\cos\theta)\cos u, r\sin\theta, u\cos u - (1 + r\cos\theta)\sin u),$$

which satisfies of (A), that is, M has singular points at $(u, \frac{\pi}{2})$, and $(u, \frac{3\pi}{2})$ along the striction curves (blue), and the spine curve $\alpha(u)$ (green), see Figure 7; $r = 1$, $u \in [-\pi, \pi]$, and $\theta \in [0, 2\pi]$.

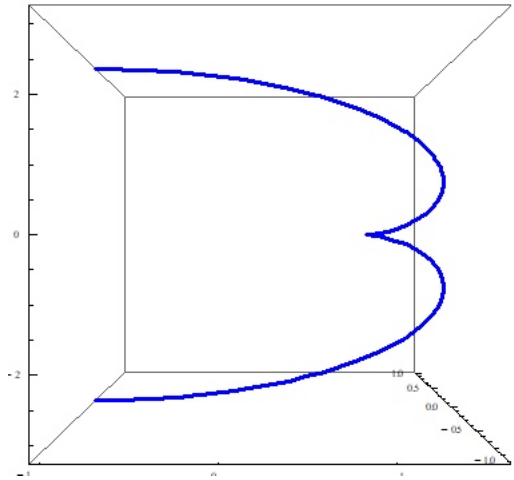

Figure 6. $\alpha(u)$ has a cusp at $u_0 = 0$.

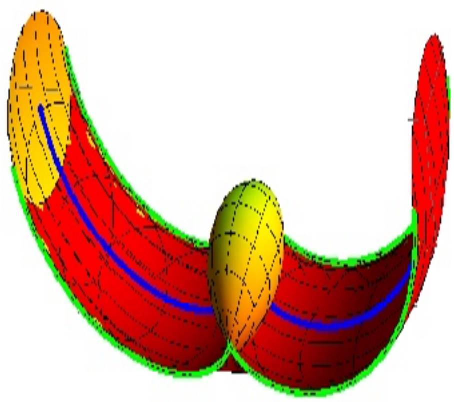

Figure 7. Circular surface has singularities at $(u, \frac{\pi}{2})$, and $(u, \frac{3\pi}{2})$.

(2) If $\sigma(u) = 1$, $\delta(u) = u$, and $\eta(u) = 0$, then $\alpha^{'}(u) = (u \sin u, 0, u \cos u) + (\cos u, 0, -\sin u)$. Taking integral with zero integration constants yields

$$\alpha(u) = (2 \sin u - u \cos u, 0, 2 \cos u + u \sin u).$$

The spine curve is not singular at $u \in [-\pi, \pi]$ (Figure 8). It is easy to see that the striction curves are

$$\begin{aligned} \zeta_1(u) &= (2 \sin u - u \cos u, 1, 2 \cos u + u \sin u)), \\ \zeta_2(u) &= (2 \sin u - u \cos u, -1, 2 \cos u + u \sin u). \end{aligned}$$

Hence, M with the given curve $\alpha(u)$ is

$$M : \mathbf{P}(u, \theta) = (2 \sin u - \cos u(u - r \cos \theta), r \sin \theta, 2 \cos u + \sin u(u - r \cos \theta)),$$

which satisfies the case (A) of singularity, that is, there are different singularities appear on the striction curves (blue), and the spine curve $\alpha(u)$ (green), see Figure 9; $r = 1$, $u \in [-\pi, \pi]$, and $\theta \in [0, 2\pi]$.

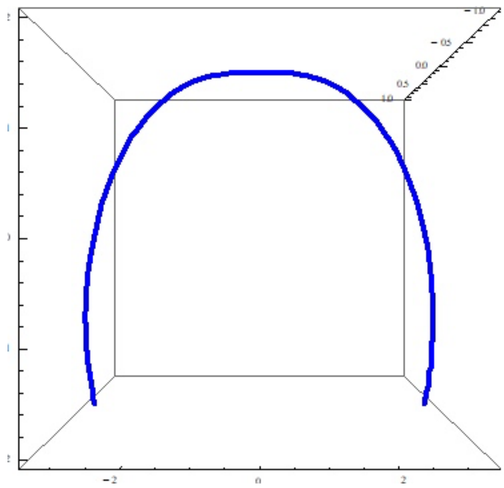

Figure 8. $\alpha(u)$ has no singulaity.

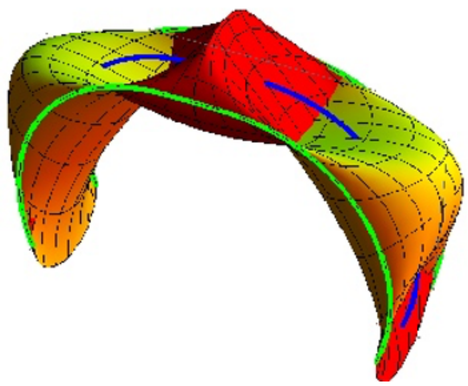

Figure 9. M has different singularities.

(3) If we choose $\delta(u) = -1$, and $\eta(u) = \sigma(u) = 1$, then

$$\alpha^{'}(u) = (\cos u - \sin u, 1, \cos u - \sin u),$$

from which, and taking integral with zero integration constants, it follows that

$$\alpha(u) = (\cos u + \sin u, u, \sin u + \cos u).$$

The spine curve is regular at $u \in [-\pi, \pi]$ (Figure 10). An analogous arguments show that:

$$\zeta_1(u) = \left(\sin u + (1 + \frac{r}{\sqrt{2}})\cos u, u - \frac{r}{\sqrt{2}}, \cos u - (1 + \frac{r}{\sqrt{2}})\sin u\right),$$

$$\zeta_2(u) = \left(\sin u + (1 - \frac{r}{\sqrt{2}})\cos u, u + \frac{r}{\sqrt{2}}, \cos u + (1 + \frac{r}{\sqrt{2}})\sin u\right).$$

In similar manner, the circular surface with the given $\alpha(u)$ is

$$M : \mathbf{P}(u,\theta) = (\sin u + (1 + r\cos\theta)\cos u, u - r\sin\theta, (1 - r\cos\theta)\sin u + \cos u),$$

which satisfies the case (A) of singularity, that is, there are different singularities appear on the striction curves (blue), and the spine curve $\alpha(u)$ (green), see Figure 11; $r = 1$, $u \in [-\pi, \pi]$, and $\theta \in [0, 2\pi]$.

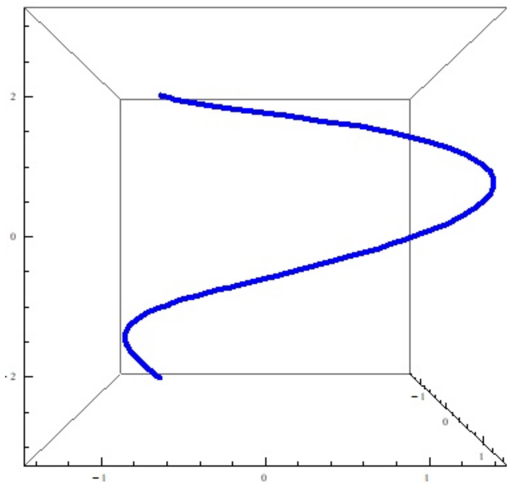

Figure 10. $\alpha(u)$ has no singulaity.

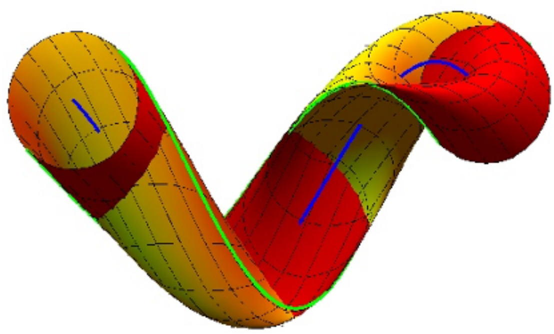

Figure 11. M has different singularities.

3.3. Roller coaster surfaces

This subsection devotes to discuss circular surfaces whose generating circles are curvature lines except at singular or umbilici points. Using Eqs (10) and (12), then all generating circles can be seen as curvature lines iff $\mathbf{U}_\theta(u,\theta)$ is parallel to $\mathbf{P}_\theta(u,\theta)$, which is equivalent to

$$2r\,\|\mathbf{P}_u \times \mathbf{P}_\theta\|^2\,(\delta\sigma\sin\theta - \delta\eta\cos\theta + r\eta) = 0. \tag{29}$$

We now address such equation in details: if $r = 0$, then we cannot generate the surface(hence, we suppose $r > 0$). In addition, if $\|\mathbf{P}_u \times \mathbf{P}_\theta\| = 0$, the circular surface M is singular. Thus, we assume that the surface M is regular, hence

$$\delta\sigma\sin\theta - \delta\eta\cos\theta + r\eta = 0. \tag{30}$$

Hence, we have three cases:
Case (1)- Let $\delta(u) = \eta(u) = \sigma(u) = 0$. Utilizing Eq (5), we conclude that $\alpha'(u) = \mathbf{0}$. Hence,

$$M = \{\mathbf{P} \in \mathbb{E}^3 \mid \|\mathbf{P} - \alpha_0\|^2 = r^2\},$$

which is a sphere with fixed radius r.
Case (2)- Let $\sigma(u) = \eta(u) = 0$. Utilizing Eq (5), we get $\alpha'(u) = \delta(u)\mathbf{e}_1(u)$, that is, the tangent vector of α is orthogonal to circle plane. Then, the surface is canal.
Case (3)- Let $\delta(u) = \eta(u) = 0$. Therefore, $\alpha'(u) = \sigma(u)\mathbf{e}_2(u)$, and the circular surface becomes a roller coaster surface [5]. Further, if $\sigma(u)$ is a constant, then from Eqs (4) and (5), we have $\alpha(u) = \alpha_0 + \sigma\mathbf{e}_2$. The roller coaster surface become a sphere with fixed radius $\sqrt{r^2+\sigma^2} > 0$, that is,

$$M = \{\mathbf{P} \in \mathbb{E}^3 \mid \|\mathbf{P} - \alpha_0\|^2 = r^2 + \sigma^2\},$$

so all circle points are on a sphere of radius $\sqrt{r^2+\sigma^2}$ with center point α_0.

As we know, a developable surface is a cone, a cylinder or a tangent developable surface. Hence, the set of developable surfaces are in correspondence with the set of the circular surfaces as follows: Roller coaster surfaces correspond to tangential developable surfaces, Spheres to cones, canal surfaces to cylinder. Hence, we obtain the following proposition:
Proposition 2. Suppose M is a circular surface whose generating circles are curvature lines except at singularities or umbilical points. Then M is a part of a sphere, a tangential circular (roller coaster) surface, or a canal surface.

We now consider the properties and relations of the curvature functions of non-canal surfaces which are characterized by $\delta(u) = 0$, and $\sigma(u)$, $\eta(u)$ are not simultaneously equal to zero in the following.

As stated in Case (3), a non-canal surface with $\delta(u) = 0$, $\sigma(u) \neq 0$, and $\eta(u) = 0$ is named as roller coaster surface. Thus, from Eq (17), the two striction curves are

$$\begin{cases} \zeta_1(u) = \alpha(u) - r\mathbf{e}_3(u), \\ \zeta_2(u) = \alpha(u) + r\mathbf{e}_3(u). \end{cases} \tag{31}$$

Further, from Eq (28), we get:

$$r\cos\theta = 0, \text{ and } \sigma(u)\cos\theta = 0.$$

From both equations given above, it is easy to see that M is non-singular at (u, θ) iff $\theta \neq \pi/2$ and $\theta \neq 3\pi/2$ since $r \neq 0$ and $\sigma(u) \neq 0$. Thus, there are two sets of singularities giving two curves of all singular points on the generating circle. We have found, by substituting $\theta = \pi/2$ and $\theta = 3\pi/2$ into Eq (6)

$$\begin{cases} \mathbf{S}_1(u) := \mathbf{P}(s, \pi/2) = \alpha(u) - r\mathbf{e}_3(u), \\ \mathbf{S}_2(u) := \mathbf{P}(s, 3\pi/2) = \alpha(u) + r\mathbf{e}_3(u). \end{cases} \tag{32}$$

From the Eqs (31) and (32) the following proposition can be reached.

Proposition 3. In the Euclidean 3-space $\mathbb{E}^3$, any roller coaster surface has only two striction curves coinciding with singular locus and intersecting each generating circle at antipodal points.

Since $\delta(u) = \eta(u) = 0$, the curvatures $\kappa_i(u)$, and the torsions $\tau_i(u)$ of the striction curves $\zeta_i(u)$ can be obtained as (i=1, 2):

$$\begin{aligned} \kappa_1(u) &= \frac{\sqrt{\gamma^2 + 1}}{|\sigma - r\gamma|},\ \tau_1(u) = \frac{\gamma'}{(\sigma - r\gamma)(1 + \gamma')}, \\ \kappa_2(u) &= \frac{\sqrt{1 + \gamma^2}}{|\sigma + r\gamma|},\ \tau_2(u) = \frac{\gamma'}{(\sigma + r\gamma)(1 + \gamma')}. \end{aligned}$$

It is obvious that if γ is constant, the torsion of the striction curves vanishes, that is, the striction curves are planar curves. Furthermore, from the Eqs (11) and (13), we obtain

$$K(u, \theta) = \frac{\left(\sigma^2 + r^2\right)\cos\theta - r\sigma'}{\left(\sigma^2 + r^2\right)^2 \cos\theta}, \text{ and } H(u, \theta) = \frac{\left(\sigma^2 + r^2\right)(1 + r^2)\cos\theta - r^2\sigma'}{2r\left(\sigma^2 + r^2\right)\cos\theta}.$$

Hence, the mean and Gaussian curvatures do not depend on γ. Also, it can be seen that they satisfy the relation

$$2rH(u, \theta) - rK(u, \theta) = 1.$$

Proposition 4. Suppose that we have a family of roller coaster surfaces sharing the same radius r, scalar function $\sigma(u)$ and $\sigma'(u)$. Then, the mean and Gaussian curvatures at corresponding points are the same value independent of the geodesic curvature of the spherical curve $\mathbf{e}_1(u) \in \mathbb{S}^2$.

In order to describe the kinematic-geometric properties of roller coaster surface, the Serret–Frenet of $\alpha(u)$ is needed to be built. So, let υ denote the arc-length of $\alpha(u)$ and $\{\mathbf{t}(\upsilon), \mathbf{n}(\upsilon), \mathbf{b}(\upsilon)\}$ be the moving Serret–Frenet frame along $\alpha(u)$. Therefore, we have

$$\mathbf{t}(\upsilon) = \frac{\alpha'}{\|\alpha'\|} = \mathbf{e}_2,\ \mathbf{b}(\upsilon) = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|},\ \mathbf{n}(\upsilon) = \mathbf{b}(\upsilon) \times \mathbf{t}(\upsilon) = \frac{-\mathbf{e}_1 + \gamma\mathbf{e}_3}{\sqrt{1 + \gamma^2}}. \tag{33}$$

Let

$$\cos\varphi(\upsilon) = \frac{\gamma}{\sqrt{1 + \gamma^2}},\ \sin\varphi(\upsilon) = \frac{1}{\sqrt{1 + \gamma^2}},$$

it follows that

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \tag{34}$$

Thus, the following are satisfied:

$$\frac{d}{d\upsilon}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}=\begin{pmatrix}0 & \widetilde{\kappa}(\upsilon) & 0\\ -\widetilde{\kappa}(\upsilon) & 0 & \widetilde{\tau}(\upsilon)\\ 0 & -\widetilde{\tau}(\upsilon) & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}, \tag{35}$$

where the curvature $\widetilde{\kappa}(\upsilon)$, and torsion $\widetilde{\tau}(\upsilon)$ of $\alpha(\upsilon)$ are given by

$$\widetilde{\kappa}(\upsilon):=\frac{1}{\sigma}\sqrt{\gamma^2+1}=\frac{1}{\sigma\sin\varphi},\ \widetilde{\tau}(\upsilon)+\frac{d\varphi}{d\upsilon}=0,\ \frac{d\varphi}{d\upsilon}=\pm\frac{d\gamma/d\upsilon}{\sigma(1+\gamma^2)}. \tag{36}$$

Since every generating circle is a curvature line, the value of one principal curvature is

$$\chi_1(u,\theta):=\left\|\frac{d\mathbf{P}}{d\theta}\times\frac{d^2\mathbf{P}}{d\theta^2}\right\|\left\|\frac{d\mathbf{P}}{d\theta}\right\|^{-3}=\frac{1}{r}.$$

The second principal curvature is

$$\chi_2(u,\theta)=\frac{K(u,\theta)}{\chi_1}=\frac{\left(\sigma^2+r^2\right)\cos\theta-r\sigma'}{r\left(\sigma^2+r^2\right)\cos\theta}.$$

Corollary 2. The principal curvature of a roller coaster is constant on every generating circle.

3.3.1. Flat and minimal roller coaster surfaces

A surface with zero Gaussian curvature is named as a flat surface. Evidently, M is flat iff $K(u,\theta)=0$, that is,

$$\left(\sigma^2+r^2\right)\cos\theta-r\sigma'=0.$$

Hence, for all $\theta\in I\subseteq\mathbb{R}$, we have

$$\frac{\partial^2K(u,\theta)}{\partial^2\theta}+K(u,\theta)=0\Leftrightarrow\sigma'(u)=0. \tag{37}$$

From Eqs (36) and (37) the expression of $\sigma'(u)$ in terms of the Serret–Frenet's invariants is

$$\sigma'(u)=0\Leftrightarrow\widetilde{\kappa}\widetilde{\tau}\cos\varphi-\frac{d\widetilde{\kappa}}{d\upsilon}\sin\varphi=0.$$

Then, $\frac{d\widetilde{\kappa}}{d\upsilon}=\widetilde{\tau}=0$ in a neighborhood of each point of M with $\widetilde{\kappa}\neq0$. Thus, roller coaster surfaces of zero Gaussian curvature are parts of plane. Similarly, we attain that M is minimal flat.

On the other hand, from Eqs (6) and (34), it follows that

$$M:\mathbf{P}(\upsilon,\theta)=\alpha(\upsilon)+r\cos\theta\mathbf{t}+\sin\theta(\cos\varphi\mathbf{n}+\sin\varphi\mathbf{b})r. \tag{38}$$

where $\varphi=\varphi(\upsilon)$. Furthermore, the expression of the two striction curves is

$$\begin{aligned}\zeta_1(\upsilon)&=\alpha(\upsilon)-r(\cos\varphi(\upsilon)\mathbf{n}+\sin\varphi(\upsilon)\mathbf{b}),\\ \zeta_2(\upsilon)&=\alpha(\upsilon)+r(\cos\varphi(\upsilon)\mathbf{n}+\sin\varphi(\upsilon)\mathbf{b}).\end{aligned}$$

Direct computation gives

$$\begin{cases} \mathbf{P}_\upsilon(\upsilon,\theta) = (1-\widetilde{\kappa}r\cos\varphi\sin\theta)\mathbf{t}+\widetilde{\kappa}r\cos\theta\mathbf{n}, \\ \mathbf{P}_\theta(\upsilon,\theta) = -r\sin\theta\mathbf{t}+r\cos\theta(\cos\varphi\mathbf{n}+\sin\varphi\mathbf{b}). \end{cases}$$

This leads to

$$g_{11} = (1-\widetilde{\kappa}r\cos\varphi\sin\theta)^2 + (\widetilde{\kappa}r\cos\theta)^2,$$
$$g_{12} = -r\sin\theta + r^2\widetilde{\kappa}\cos\varphi,\ g_{22} = r^2.$$

The unit normal is

$$\mathbf{U}(\upsilon,\theta) = \frac{a_2b_3\mathbf{t}-a_1b_3\mathbf{n}+(a_1b_2+a_2b_1)\,\mathbf{b}}{\sqrt{\left(a_1^2+a_2^2\right)b_3^2+(a_1b_2+a_2b_1)^2}},$$

where

$$\left.\begin{array}{c} a_1 = (1-r\widetilde{\kappa}\cos\varphi\sin\theta)\,, a_2 = \widetilde{\kappa}r\cos\theta, \\ b_1 = -r\sin\theta,\ b_2 = r\cos\theta\cos\varphi,\ b_3 = r\cos\theta\sin\varphi. \end{array}\right\}$$

This leads to

$$\begin{aligned} h_{11} &= \frac{a_2b_3\left((a_1)_\upsilon - a_2\widetilde{\kappa}\right) - a_1b_3\left((a_2)_\upsilon + a_1\widetilde{\kappa}\right) + a_2\widetilde{\tau}\,(a_1b_2+a_2b_1)}{\sqrt{\left(a_1^2+a_2^2\right)b_3^2+(a_1b_2+a_2b_1)^2}}, \\ h_{12} &= \frac{b_3\left(a_2\,(a_1)_\theta - a_1\,(a_2)_\theta\right)}{\sqrt{\left(a_1^2+a_2^2\right)b_3^2+(a_1b_2+a_2b_1)^2}}, \\ h_{22} &= \frac{b_3\left(a_2\,(b_1)_\theta - a_1\,(b_2)_\theta\right) + (a_1b_2+a_2b_1)\,(b_3)_\theta}{\sqrt{\left(a_1^2+a_2^2\right)b_3^2+(a_1b_2+a_2b_1)^2}}. \end{aligned}$$

Based on the above analysis, we provide the following theorem.

Theorem 2. Let M be a roller coaster surface of Eq (38). Then:

1- The principal curvatures χ_1 and χ_2 are

$$\begin{aligned} \chi_1(\upsilon,\theta) &= \frac{1}{r}, \\ \chi_2(\upsilon,\theta) &= \frac{\widetilde{\kappa}\sin\varphi[r^2\widetilde{\kappa}^3\cos\theta\sin^3\varphi + (\cos\theta\sin\varphi - \widetilde{\tau}r\cos\varphi)\widetilde{\kappa} + r\frac{d\widetilde{\kappa}}{d\upsilon}\sin\varphi]}{r(r^2\widetilde{\kappa}^2\sin^2\varphi+1)^2\cos\theta}. \end{aligned}$$

2- The Gaussian $K(\upsilon,\theta)$ and mean $H(\upsilon,\theta)$ curvatures can be obtained as:

$$\begin{aligned} K(\upsilon,\theta) &= \frac{\widetilde{\kappa}[r^2\widetilde{\kappa}^3\sin^3\varphi\cos\theta + (\cos\theta\sin\varphi - r\widetilde{\tau}\cos\varphi)\widetilde{\kappa} + r\frac{d\widetilde{\kappa}}{d\upsilon}\sin\varphi]\sin\varphi}{(r^2\widetilde{\kappa}^2\sin^2\varphi+1)^2\cos\theta}, \\ H(\upsilon,\theta) &= \frac{2r^2\widetilde{\kappa}^3\sin^3\varphi\cos\theta + \widetilde{\kappa}(2\sin\varphi\cos\theta + r\widetilde{\tau}\cos\varphi) + r\frac{d\widetilde{\kappa}}{d\upsilon}\sin\varphi}{2(r^2\widetilde{\kappa}^2\sin^2\varphi+1)^{\frac{3}{2}}\cos\theta}. \end{aligned}$$

3- Flat roller coaster surfaces are parts of planes,

4- Minimal roller coaster surfaces are parts of and positioned on planes, 5- The the curvatures $\kappa_i(\upsilon)$, and the torsions $\tau_i(\upsilon)$ of the striction curves $\zeta_i(\upsilon)$ can be obtained as (i=1, 2):

$$\kappa_1(\upsilon) = \frac{\widetilde{\kappa}}{|1 - r\widetilde{\kappa}\cos\varphi|}, \ \tau_1(\upsilon) = \frac{\widetilde{\tau}}{1 - r\widetilde{\kappa}\cos\varphi},$$
$$\kappa_2(\upsilon) = \frac{\kappa}{|1 + \widetilde{\kappa}r\cos\varphi|}, \ \tau_2(\upsilon) = \frac{\widetilde{\tau}}{1 + \widetilde{\kappa}r\cos\varphi}.$$

Example 3. From Example 1, it is easy to show that $\varphi(\upsilon) = \frac{b}{c}\upsilon + \varphi_0$. If $\varphi_0 = 0$, we have $\varphi(\upsilon) = \frac{b}{c}\upsilon$. Thus, we obtain

$$\begin{aligned} M \ : \ \mathbf{P}(\upsilon,\theta) = (&a\cos\frac{b\upsilon}{c} - \frac{ar}{c}\cos\theta\sin\frac{b\upsilon}{c} - r\cos\varphi\sin\theta\cos\frac{b\upsilon}{c} + \frac{br}{c}\sin\varphi\sin\theta\sin\frac{b\upsilon}{c}, \\ &a\sin\frac{b\upsilon}{c} + \frac{ar}{c}\cos\frac{b\upsilon}{c}\cos\theta - r\cos\varphi\sin\theta\sin\frac{b\upsilon}{c} - \frac{br}{c}\sin\varphi\sin\theta\cos\frac{b\upsilon}{c}, \\ &\frac{br}{c} + \frac{r}{c}\cos\theta + \frac{ar}{c}\sin\theta\sin\varphi). \end{aligned}$$

For θ, $\upsilon \in [0, 2\pi]$, and $a = r = b = 1$, the corresponding roller coaster surface is depicted in Figure 12. The singularities appear on the striction curves (blue), and the spine curve $\alpha(u)$ (green).

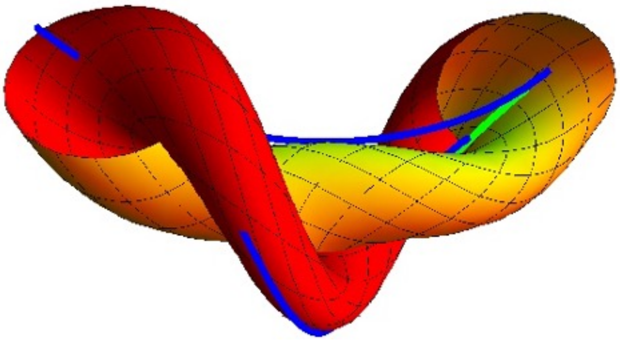

Figure 12. Roller coaster surface with its spine and striction curves.

4. Conclusions

For a given unit circle $\mathbf{C}(\theta)$ and a given space curve $\alpha(u)$, a circular surface $\mathbf{P}(u, \theta)$ can be defined as the system of circles from $\mathbf{C}(\theta)$ with its center points following the curve α. Then, the approach has simplified the study on circular surfaces to investigating two curves: the spherical indicatrices of unit normals of the given space curve and circle planes. Some new findings and theorems relevant to circular surfaces are attained. The study has given some geometric aspects such as striction curves, singularities in comparison with those of ruled surfaces. In addition, the conditions for roller coaster surfaces to be flat or minimal surfaces have been obtained. Lastly, some illustrative examples have been given. The author plans to apply the study in different spaces and investigate the classification of singularities.

Conflict of interest

Author has no conict of interest.

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