AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 12566–12586.
DOI: 10.3934/math.2022696
Received: 21 January 2022
Revised: 12 April 2022
Accepted: 18 April 2022
Published: 28 April 2022

Research article

Double Ore extensions of anti-angle type for Hopf algebras

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Abstract: The aim of this article is to extend the structure of a bialgebra (Hopf algebra) which is connected graded as an algebra and generated in degree 1 to its double Ore extensions of anti-angle type. We construct two non-commutative and non-cocommutative Hopf algebras of infinite dimension and investigate the lifting of homological properties.

Keywords: connected graded algebra; Hopf double Ore extension; anti-angle type
Mathematics Subject Classification: 16S36, 16W30, 16W50

1. Introduction

In 2003, Panov studied Ore extensions in the class of Hopf algebras [4]. In 2017, Pansera gave necessary and sufficient conditions to extend the structure of a bialgebra R to the skew polynomial ring of automorphism type $R[z;\sigma]$, and to define a Hopf algebra structure on the quotient $R[z;\sigma]/I$ for I a certain bi-ideal of $R[z;\sigma]$ and construct a class of non-commutative, non-cocommutative, semisimple Hopf algebras of dimension $2n^2$ through Hopf-Ore extension [5]. In 2008, Zhang and Zhang introduced a new construction for extending a given algebra A, called a double Ore extension, which resembles that of an Ore extension [7]. However, to the best knowledge of the author, there are no studies on extending the structure of a bialgebra (Hopf algebra) to its double Ore extensions which can not be presented as iterated Ore extensions yet.

In this article, we focus on a special type of right double Ore extensions, which is defined in Section 2 and we call it the double Ore extensions of anti-angle type. Our main aim is to extend the structure of a bialgebra (Hopf algebra) which is connected graded as an algebra and generated in degree 1 to its double Ore extensions of anti-angle type. This type of double Ore extensions can not be presented as iterated Ore extensions, thus the extensions are non-trivial. As an example, we construct a non-commutative and non-cocommutative Hopf algebra of infinite dimension. We also investigate the lifting of homological properties from this type of a bialgebra to its double Ore extensions of anti-angle type for a specific Drinfel'd twist and parameter. Furthermore, we extend the construction of double Ore extensions of anti-angle type to multiple Ore extensions and find a class of multiple Ore extensions with good properties, we call it the multiple Ore extensions of circular type. Last but not least, we modified the definition of the comultiplication and get another type, called the double Ore extensions of anti-angle-primitive type. We construct another example and lift other homological properties.

Throughout this article, $\mathbf{k}$ is an algebraically closed field of characteristic 0. All vector spaces, algebras, coalgebras and unadorned tensors are over $\mathbf{k}$. H is a Hopf algebra, μ is its multiplication, υ is its unit, Δ is its comultiplication, ϵ is its counit and S is its antipode. For $h \in H$, we always write $\Delta(h) = h_{(1)} \otimes h_{(2)}$ with the summation omitted.

The main theorems in this article are as follows:

Theorem 1.1. *Let R be a bialgebra which is connected graded as an algebra and generated in degree 1, (s, J) be a twisted homomorphism for R. Let $H = R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$ be a double Ore extension of anti-angle type. Then the bialgebra structure of R can be extended to H such that $\Delta(y_i) = J(y_i \otimes y_i)$ and $\epsilon(y_i) = 1 (i = 1, 2)$. Conversely, if there exist an invertible element $J \in R \otimes R$, $s \in \mathrm{Aut}(R)$ which preserves degree such that $R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$ is a bialgebra with $\Delta(y_i) = J(y_i \otimes y_i)$ and $\epsilon(y_i) = 1 (i = 1, 2)$, then (s, J) is a twisted homomorphism for R.*

Let $H' = H/\left\langle y_1 y_2 - y_2^2 \right\rangle$, it also extends the bialgebra structure of R as in Theorem 1.1 and we have that H'/I' is a bialgebra for the bi-ideal $I' = \left\langle y_1^2 - t, y_2^2 - t \right\rangle$, where $0 \neq t \in R$ and $\Delta(t) = J(s \otimes s)(J)(t \otimes t)$.

Theorem 1.2. *Let R be a Hopf algebra which is connected graded as an algebra and generated in degree 1 with antipode S, (s, J) be a twisted homomorphism for R. Suppose also that $s \circ S = S \circ s$ and $s^2 = \mathrm{id}$. If there exists $0 \neq t \in R$, with $\Delta(t) = J(s \otimes s)(J)(t \otimes t)$, and such that*

(1) $tJ^1 S(J^2) = 1$;
(2) $ts(S(J^1)J^2) = 1$,

where $J = J^1 \otimes J^2$ with the summation omitted, then there exists a Hopf algebra structure on H'/I' with $S(y_i) = y_i (i = 1, 2)$. Conversely, if there exists a Hopf structure on H'/I' with $S(y_i) = y_i (i = 1, 2)$, then $tJ^1 S(J^2) = ts(S(J^1)J^2) = 1$.

Theorem 1.3. *Let R be a bialgebra which is connected graded as an algebra and generated in degree 1, $s \in \mathrm{Aut}(R)$ be a coalgebra homomorphism for R and $H = R^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ be a double Ore extension of anti-angle type, then as algebras:*

(1) If R is a domain, then H is a domain;
(2) If R is prime (semiprime), then H is prime (semiprime).

Theorem 1.4. *Let A be a Hopf algebra, which is connected graded as an algebra and generated in degree 1, the antipode of A preserves degree, then $B = A^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ is a Hopf double Ore extension of anti-angle-primitive type if and only if there is a character $\chi : A \to \boldsymbol{k}$ such that $s(a) = \chi(a_{(1)})a_{(2)} = a_{(1)}\chi(a_{(2)})$ for all $a \in A$.*

In Section 2, we give necessary and sufficient conditions to extend the structure of a bialgebra R which is connected graded as an algebra and generated in degree 1 to the double Ore extensions of anti-angle type $H = R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$, and to define a Hopf algebra structure on the quotient H'/I' for a certain bi-ideal $I' = \left\langle y_1^2 - t, y_2^2 - t \right\rangle$ of $H' = H/\left\langle y_1y_2 - y_2^2 \right\rangle$. At last, we give the definition of a Hopf double Ore extension of anti-angle-grouplike type.

In Section 3, we give an example for the extending of bialgebra and Hopf algebra structures from a bialgebra R which is connected graded as an algebra and generated in degree 1 and construct a non-commutative and non-cocommutative Hopf algebra of infinite dimension. Unfortunately, we find that it's hard to give the classification of twisted homomorphisms for a given bialgebra R relative to the double Ore extension of anti-angle type. The difficulty of this problem is that we can't provide the form of $\Phi : R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma] \to R'^{\vee}_{\{0,1\}}[y'_1, y'_2; \sigma']$ such that $\Phi(R) = R'$ as bialgebras, since it is related to the Jacobian conjecture.

In Section 4, we investigate the lifting of some homological properties (being a domain, being prime (semiprime)) from a bialgebra R which is connected graded as an algebra and generated in degree 1 to its double Ore extension of anti-angle type $R^{\vee}_P[y_1, y_2; \sigma]$ for a specific Drinfel'd twist J and parameter P. Finally, we find that the above homological properties can't lift to the Hopf algebra $H/I = R^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]/\left\langle y_1^2 - t, y_2^2 - t \right\rangle$, since the relations $y_1^2 = t$ and $y_2^2 = t$ produced zero divisors.

In Section 5, we extend the construction of double Ore extensions of anti-angle type to multiple Ore extensions and find a class of multiple Ore extensions with good properties, called the multiple Ore extensions of circular type. We get a similar conclusion as in Section 4.

In Section 6, we modify the definition of the comultiplication and get another type of Hopf double Ore extensions of anti-angle type, called an anti-angle-primitive type. We give the basic properties of this type and the necessary and sufficient conditions for a Hopf algebra $B = A^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ to be this type. At the end, we construct another non-commutative and non-cocommutative Hopf algebra of infinite dimension and lift other homological properties (being right noetherian, having finite right Krull dimension, having finite global dimension) for it.

2. Fundamental results of DOE of anti-angle type

In this section, we try to give necessary and sufficient conditions to extend the structure of a bialgebra R which is connected graded as an algebra and generated in degree 1 to the double Ore extensions of anti-angle type $H = R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$, and to define a Hopf algebra structure on the quotient H'/I' for $I' = \left\langle y_1^2 - t, y_2^2 - t \right\rangle$ a certain bi-ideal of $H' = H/\left\langle y_1y_2 - y_2^2 \right\rangle$.

Let's recall the definition of a twist for a bialgebra, as given in Pansera [5], and we will start with it.

Definition 2.1. Let R be a bialgebra and J be an invertible element in $R \otimes R$. J is called a *Drinfel'd twist* for R if J satisfies:

(1) $(\mathrm{id} \otimes \Delta)(J)(1 \otimes J) = (\Delta \otimes \mathrm{id})(J)(J \otimes 1)$;
(2) $(\mathrm{id} \otimes \epsilon)(J) = (\epsilon \otimes \mathrm{id})(J) = 1$.

Definition 2.2. Let R be a bialgebra. Let J be a Drinfel'd twist for R and $s \in \mathrm{End}(R)$. We say that the pair (s, J) is a *twisted homomorphism* for R if s satisfies:

(1) $J(s \otimes s)\Delta(h) = \Delta(s(h))J$ for all $h \in R$;
(2) $\epsilon \circ s = \epsilon$.

Note that, for any homomorphism of coalgebras $s \in \mathrm{End}(R)$, the pair $(s, 1 \otimes 1)$ is a twisted homomorphism for R.

We will now recall the definition of a double Ore extension, as given in Zhang and Zhang [7].

Definition 2.3. Let A be an algebra and B be another algebra containing A as a subalgebra. We say B is a *right double Ore extension* of A if the following conditions hold:

(1) B is generated by A and two new variables y_1 and y_2;
(2) $\{y_1, y_2\}$ satisfies a relation

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1y_1 + \tau_2y_2 + \tau_0,$$

where $p_{12}, p_{11} \in \mathbf{k}$ and $\tau_1, \tau_2, \tau_0 \in A$;
(3) As a left A-module, $B = \sum\limits_{i,j\geq 0} Ay_1^iy_2^j$ and it is a left free A-module with a basis $\left\{y_1^iy_2^j | i \geq 0, j \geq 0\right\}$;
(4) $y_1A + y_2A + A \subseteq Ay_1 + Ay_2 + A$.

Let P denote the set of scalar parameters $\{p_{12}, p_{11}\}$ and let τ denote the set $\{\tau_1, \tau_2, \tau_0\}$. We call P the *parameter* and τ the *tail*. Similarly, we can define the *left double Ore extension* B of A (see [7]). We say B is a *double Ore extension* (DOE, for short) if it is a left and right double Ore extension of A with the same generating set $\{y_1, y_2\}$.

Remark 2.1. *Condition (4) in Definition 2.3 is equivalent to the existence of two maps*

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} : A \to M_{2\times 2}(A), \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} : A \to M_{2\times 1}(A).$$

Proof. Consider the necessity. If $y_1A + y_2A + A \subseteq Ay_1 + Ay_2 + A$, then for all $a \in A$, there exist maps $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \delta_1, \delta_2 : A \to A$, subject to

$$\begin{cases} y_1a = \sigma_{11}(a)y_1 + \sigma_{12}(a)y_2 + \delta_1(a), \\ y_2a = \sigma_{21}(a)y_1 + \sigma_{22}(a)y_2 + \delta_2(a). \end{cases}$$

It implies the existence of two maps

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} : A \to M_{2\times 2}(A), \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} : A \to M_{2\times 1}(A),$$

which satisfy

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} a = \sigma(a) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \delta(a)$$

for all $a \in A$.

The sufficiency is obvious. □

In [7], the authors have proved that σ is an algebra homomorphism.

Let B be a right double Ore extension of A, we denote $B = A_P[y_1, y_2; \sigma, \delta, \tau]$, where $P = \{p_{12}, p_{11}\} \subseteq \mathbf{k}$, $\tau = \{\tau_1, \tau_2, \tau_0\} \subseteq A$, and σ, δ are as above.

In this article, we focus on a special type of right double Ore extensions.

Let A be a connected graded algebra which is generated in degree 1, that is, A is $\mathbb{N}$-graded, with $A = \mathbf{k} \oplus \bigoplus\limits_{n\geq 1} A_n$, where $A_iA_j = A_{i+j}$ for $i, j \geq 1$. Let B be the right double Ore extension of A, note that B is a $\mathbf{k}$-vector space, so the following fact is automatically established:

$$\sigma(\mathbf{k}) = \begin{pmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{pmatrix} : \mathbf{k} \to M_{2\times 2}(\mathbf{k}).$$

Further we let $\delta = 0, \tau = 0$ and $s \in \mathrm{Aut}(A)$ which preserves degree. Suppose that A_1 is the $\mathbf{k}$-subspace of A generated by $\{x_1, \cdots, x_s, \cdots\}$, where $x_1, \cdots, x_s, \cdots$ are the homogeneous elements of degree 1. For all $a_1 \in A_1$, $a_1 = \sum\limits_i x_i$, we define

$$\sigma(x_i) = \begin{pmatrix} 0 & s(x_i) \\ s(x_i) & 0 \end{pmatrix} : A_1 \to M_{2\times 2}(A_1).$$

Then

$$\sigma(a_1) = \begin{pmatrix} 0 & \sum\limits_i s(x_i) \\ \sum\limits_i s(x_i) & 0 \end{pmatrix} = \begin{pmatrix} 0 & s(a_1) \\ s(a_1) & 0 \end{pmatrix}.$$

Since σ is an algebra homomorphism, the domain can be extended to A_n and then to A. Let's begin with A_2, for all $a_2 \in A_2$, $a_2 = \sum\limits_{i,j} x_ix_j$, now

$$\begin{aligned} \sigma(a_2) &= \sigma\left(\sum_{i,j} x_ix_j\right) = \sum_{i,j} \sigma(x_i)\sigma(x_j) \\ &= \sum_{i,j} \begin{pmatrix} 0 & s(x_i) \\ s(x_i) & 0 \end{pmatrix}\begin{pmatrix} 0 & s(x_j) \\ s(x_j) & 0 \end{pmatrix} \\ &= \sum_{i,j} \begin{pmatrix} s(x_i)s(x_j) & 0 \\ 0 & s(x_i)s(x_j) \end{pmatrix} = \sum_{i,j} \begin{pmatrix} s(x_ix_j) & 0 \\ 0 & s(x_ix_j) \end{pmatrix} \\ &= \begin{pmatrix} s(a_2) & 0 \\ 0 & s(a_2) \end{pmatrix}. \end{aligned}$$

By induction, one can define $\sigma(a_n)$. When n is odd, for all $a_n \in A_n$,

$$\sigma(a_n) = \begin{pmatrix} 0 & s(a_n) \\ s(a_n) & 0 \end{pmatrix} : A_n \to M_{2\times 2}(A_n).$$

When n is even, for all $a_n \in A_n$,

$$\sigma(a_n) = \begin{pmatrix} s(a_n) & 0 \\ 0 & s(a_n) \end{pmatrix} : A_n \to M_{2\times 2}(A_n).$$

Hence, for all $a \in A$, $a = k_0 + \sum\limits_{i=1}^{n} k_ia_i$, where $k_0, k_i \in \mathbf{k}$ and $a_i \in A_i$, then we have

$$\sigma(a) = \sigma(k_0) + \sum_{i=1}^{n} k_i\sigma(a_i).$$

Definition 2.4. We call the above type of right double Ore extension of A as a *double Ore extension of anti-angle type*, denoted by $A_P^{\vee}[y_1, y_2; \sigma]$.

In [2], the authors have proved that this type of double Ore extensions can not be presented as iterated Ore extensions. In [5], the author has extended the structure of a bialgebra R to the skew polynomial ring of automorphism type $R[z, \sigma]$. Now we let R be a bialgebra which is connected graded as an algebra and generated in degree 1. We try to extend the bialgebra structure of R to its double Ore extension of anti-angle type $R_P^{\vee}[y_1, y_2; \sigma]$ for some specific parameter P.

We further assume that the tensor legs of the Drinfel'd twist J must have degrees of the same parity and Δ preserves the parity, that is, for all $h_n \in R_n$, $\Delta(h_n) = h_{(1)_n} \otimes h_{(2)_n}$, where $h_n, h_{(1)_n}$ and $h_{(2)_n}$ have degrees of the same parity.

Theorem 2.1. *Let R be a bialgebra which is connected graded as an algebra and generated in degree* 1*, (s, J) be a twisted homomorphism for R. Let $H = R_{\{0,1\}}^{\vee}[y_1, y_2; \sigma]$ be a double Ore extension of anti-angle type. Then the bialgebra structure of R can be extended to H such that $\Delta(y_i) = J(y_i \otimes y_i)$ and $\epsilon(y_i) = 1 (i = 1, 2)$. Conversely, if there exist an invertible element $J \in R \otimes R$, $s \in \mathrm{Aut}(R)$ which preserves degree such that $R_{\{0,1\}}^{\vee}[y_1, y_2; \sigma]$ is a bialgebra with $\Delta(y_i) = J(y_i \otimes y_i)$ and $\epsilon(y_i) = 1 (i = 1, 2)$, then (s, J) is a twisted homomorphism for R.*

Proof. Since $R = \mathbf{k} \oplus \bigoplus_{n \geq 1} R_n$, we only need to consider the elements from R_n where n is odd. This is because for all $h \in R$, $h = k_0 + \sum_{i=1}^{n} k_i h_i$, where $k_0, k_i \in \mathbf{k}$ and $h_i \in R_i$, then we have $\sigma(h) = \sigma(k_0) + \sum_{i=1}^{n} k_i \sigma(h_i)$. Notice the fact that $y_i \mathbf{k} = \mathbf{k} y_i (i = 1, 2)$ and when n is even, the double Ore extension can be presented as iterated Ore extensions, the proof for the case of the Ore extension was given in [5].

In the remaining part of the proof, we let n be odd. For all $h_n \in R_n$, we have the relations:

$$y_1 h_n = s(h_n) y_2, y_2 h_n = s(h_n) y_1, y_2 y_1 = y_1^2.$$

(1) Let (s, J) be a twisted homomorphism for the bialgebra R, and let $H = R_{\{0,1\}}^{\vee}[y_1, y_2; \sigma]$ be a double Ore extension of anti-angle type.

Since R is a bialgebra, we have a homomorphism of algebras $R \to H \otimes H$. Consider the element $J(y_i \otimes y_i)(i = 1, 2) \in H \otimes H$. Let $J = J^1 \otimes J^2$ with the summation omitted, then J^1 and J^2 are the sum of elements in different grades of R. We only need to consider those elements from R_n where n is odd for the same reason. While it may be an abuse of notation, we just assume J^1 and J^2 are from some R_n where n is odd, then for all $h_n \in R_n$, we have

$$J(y_1 \otimes y_1)\Delta(h_n) = J(s \otimes s)\Delta(h_n)(y_2 \otimes y_2) = \Delta(s(h_n))J(y_2 \otimes y_2),$$

$$J(y_2 \otimes y_2)\Delta(h_n) = J(s \otimes s)\Delta(h_n)(y_1 \otimes y_1) = \Delta(s(h_n))J(y_1 \otimes y_1),$$

$$J(y_2 \otimes y_2)J(y_1 \otimes y_1) = J(s \otimes s)(J)(y_1^2 \otimes y_1^2) = J(s \otimes s)(J)(y_2 y_1 \otimes y_2 y_1) = J(y_1 \otimes y_1)J(y_1 \otimes y_1).$$

Thus Δ satisfies the Ore condition. Hence, there exists a unique algebra homomorphism $\overline{\Delta} : H \to H \otimes H$ such that $\overline{\Delta}|_R = \Delta$ and $\overline{\Delta}(y_i) = J(y_i \otimes y_i)(i = 1, 2)$. While it may be an abuse of notation, we just write $\overline{\Delta} = \Delta$.

Furthermore,

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\Delta(y_i) &= (\mathrm{id}\otimes\Delta)(J)(y_i\otimes\Delta(y_i)) = (\mathrm{id}\otimes\Delta)(J)(1\otimes J)(y_i\otimes y_i\otimes y_i)\\ &= (\Delta\otimes\mathrm{id})(J)(J\otimes 1)(y_i\otimes y_i\otimes y_i) = (\Delta\otimes\mathrm{id})(J)(\Delta(y_i)\otimes y_i)\\ &= (\Delta\otimes\mathrm{id})\Delta(y_i).\end{aligned}$$

This implies that $\Delta : H \to H\otimes H$ is a coassociative map.

Now, since $\epsilon : R \to \mathbf{k}$ is a homomorphism of algebras, then for all $h_n \in R_n$, we have

$$1\cdot\epsilon(h_n) = 1\cdot\epsilon(s(h_n)) = \epsilon(s(h_n))\cdot 1,$$
$$\epsilon(y_2)\epsilon(y_1) = \epsilon(y_1)\epsilon(y_1).$$

Thus ϵ satisfies the Ore condition. So, there exists a unique algebra homomorphism $\overline{\epsilon} : H \to \mathbf{k}$ such that $\overline{\epsilon}|_R = \epsilon$ and $\overline{\epsilon}(y_i) = 1 (i = 1, 2)$. Again, while it may be an abuse of notation, we just write $\overline{\epsilon} = \epsilon$. Moreover, we have

$$(\mathrm{id}\otimes\epsilon)\Delta(y_i) = (\mathrm{id}\otimes\epsilon)(J)y_i = y_i = (\epsilon\otimes\mathrm{id})(J)y_i = (\epsilon\otimes\mathrm{id})\Delta(y_i).$$

Thus ϵ satisfies the counit property in H. Therefore, the bialgebra structure of R extends to H as stated in the theorem.

(2) To prove the converse, suppose that there exist an invertible element $J \in R\otimes R$, $s \in \mathrm{Aut}(R)$ which preserves degree such that $R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$ is a bialgebra with $\Delta(y_i) = J(y_i \otimes y_i)$ and $\epsilon(y_i) = 1 (i = 1, 2)$.

Since $(\mathrm{id}\otimes\epsilon)\Delta(y_i) = (\epsilon\otimes\mathrm{id})\Delta(y_i)$, we must have that $(\mathrm{id}\otimes\epsilon)(J) = (\epsilon\otimes\mathrm{id})(J) = 1$. Also

$$(\mathrm{id}\otimes\Delta)\Delta(y_i) = (\mathrm{id}\otimes\Delta)(J)(y_i\otimes\Delta(y_i)) = (\mathrm{id}\otimes\Delta)(J)(1\otimes J)(y_i\otimes y_i\otimes y_i)$$

and

$$(\Delta\otimes\mathrm{id})\Delta(y_i) = (\Delta\otimes\mathrm{id})(J)(\Delta(y_i)\otimes y_i) = (\Delta\otimes\mathrm{id})(J)(J\otimes 1)(y_i\otimes y_i\otimes y_i).$$

Since $R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$ is a bialgebra, we have that $(\mathrm{id}\otimes\Delta)\Delta(y_i) = (\Delta\otimes\mathrm{id})\Delta(y_i)$ and hence $(\mathrm{id}\otimes\Delta)(J)(1\otimes J) = (\Delta\otimes\mathrm{id})(J)(J\otimes 1)$, that is, J is a Drinfel'd twist for R.

Moreover, for all $h_n \in R_n$, $y_1h_n = s(h_n)y_2$. This implies that

$$\epsilon(h_n) = \epsilon(y_1h_n) = \epsilon(s(h_n)y_2) = \epsilon(s(h_n)),$$

and then for all $h \in R$, $\epsilon(s(h)) = \epsilon(h)$.

That is, $\epsilon\circ s = \epsilon$. Note that for all $h_n \in R_n$, on the one hand

$$\Delta(y_1h_n) = J(y_1\otimes y_1)\Delta(h_n) = J(s\otimes s)\Delta(h_n)(y_2\otimes y_2),$$

on the other hand

$$\Delta(y_1h_n) = \Delta(s(h_n)y_2) = \Delta(s(h_n))J(y_2\otimes y_2).$$

Thus, $J(s\otimes s)\Delta(h_n) = \Delta(s(h_n))J$ and then for all $h \in R$, $J(s\otimes s)\Delta(h) = \Delta(s(h))J$, hence the pair (s, J) is a twisted homomorphism for R. □

As it was said at the beginning of this section, given a Hopf algebra R which is connected graded as an algebra and generated in degree 1, we will find conditions to define a Hopf algebra structure on the quotient $R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]/I$ for some bi-ideal I of $R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$. The following lemma gives us certain conditions to find the bi-ideal on the bialgebra $R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$ which will be used to define the Hopf algebra structure mentioned.

Lemma 2.1. *Let R be a bialgebra which is connected graded as an algebra and generated in degree 1, (s, J) be twisted homomorphism for R. Suppose that there exists $0 \neq t \in R$ such that $\Delta(t) = J(s \otimes s)(J)(t \otimes t)$. Then, for the double Ore extension of anti-angle type $H = R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$ with the bialgebra structure as in Theorem 2.1, $I = \left\langle y_1^2 - t \right\rangle$ is a bi-ideal of H.*

Proof. Let $R = \mathbf{k} \oplus \bigoplus_{n \geq 1} R_n$ be a bialgebra and (s, J) be a twisted homomorphism for R, we only consider the elements from R_n and just assume J^1 and J^2 are from some R_n where n is odd for the same reason. By Theorem 2.1, $H = R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$ is also a bialgebra. Let $0 \neq t \in R$ as in the hypothesis and let I be the ideal in H generated by $y_1^2 - t$. We have to prove that $I = \left\langle y_1^2 - t \right\rangle$ is a coideal of H. We note that t necessarily satisfies

$$\begin{aligned} t &= (\epsilon \otimes \mathrm{id})\Delta(t) = (\epsilon \otimes \mathrm{id})(J)(\epsilon \otimes \mathrm{id})(s \otimes s)(J)\epsilon(t)t \\ &= (\epsilon \otimes \mathrm{id})(s \otimes s)(J)\epsilon(t)t = (\epsilon \otimes s)(J)\epsilon(t)t \\ &= s((\epsilon \otimes \mathrm{id})(J))\epsilon(t)t = s(1)\epsilon(t)t = \epsilon(t)t, \end{aligned}$$

which implies $\epsilon(t) = 1$. So, $\epsilon(y_1^2 - t) = \epsilon(y_1)^2 - \epsilon(t) = 0$. That is, $I \subseteq \ker\epsilon$.

Furthermore,

$$\begin{aligned} \Delta(y_1^2 - t) &= \Delta(y_1)^2 - \Delta(t) = J(y_1 \otimes y_1)J(y_1 \otimes y_1) - \Delta(t) \\ &= J(s \otimes s)(J)(y_2 y_1 \otimes y_2 y_1) - \Delta(t) \\ &= J(s \otimes s)(J)(y_1^2 \otimes y_1^2) - \Delta(t) \\ &= J(s \otimes s)(J)((y_1^2 - t) \otimes y_1^2) + J(s \otimes s)(J)(t \otimes (y_1^2 - t)) + J(s \otimes s)(J)(t \otimes t) - \Delta(t) \\ &= J(s \otimes s)(J)((y_1^2 - t) \otimes y_1^2) + J(s \otimes s)(J)(t \otimes (y_1^2 - t)), \end{aligned}$$

which belongs to $I \otimes H + H \otimes I$. Therefore, I is a bi-ideal of H. □

Remark 2.2. *Note that, conversely, if $I = \left\langle y_1^2 - t \right\rangle$ is a bi-ideal of H, then $\Delta(t) - J(s \otimes s)(J)(t \otimes t) \in I \otimes H + H \otimes I$.*

Hence, given a bialgebra R which is connected graded as an algebra and generated in degree 1, (s, J) a twisted homomorphism for R and an element t that satisfies the hypothesis of Lemma 2.1, we have that H/I is a bialgebra, for the double Ore extension of anti-angle type $H = R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$ and $I = \left\langle y_1^2 - t \right\rangle$.

Remark 2.3. *By symmetry, one can check that $H' = H/\left\langle y_1 y_2 - y_2^2 \right\rangle$ also extended the bialgebra structure of R as in Theorem 2.1 and further we have that H'/I' is a bialgebra for the bi-ideal $I' = \left\langle y_1^2 - t, y_2^2 - t \right\rangle$ as in Lemma 2.1.*

The next theorem presents conditions to extend a Hopf algebra structure from a Hopf algebra R which is connected graded as an algebra and generated in degree 1 to the quotient bialgebra H'/I'.

Theorem 2.2. *Let R be a Hopf algebra which is connected graded as an algebra and generated in degree 1 with antipode S, (s, J) be a twisted homomorphism for R. Suppose also that $s \circ S = S \circ s$ and $s^2 = \mathrm{id}$. If there exists $0 \neq t \in R$, with $\Delta(t) = J(s \otimes s)(J)(t \otimes t)$, and such that*

(1) $tJ^1 S(J^2) = 1$;
(2) $ts(S(J^1)J^2) = 1$,

where $J = J^1 \otimes J^2$ with the summation omitted, then there exists a Hopf algebra structure on H'/I' with $S(y_i) = y_i (i = 1, 2)$. Conversely, if there exists a Hopf structure on H'/I' with $S(y_i) = y_i (i = 1, 2)$, then $tJ^1 S(J^2) = ts(S(J^1)J^2) = 1$.

Proof. Let $R = \mathbf{k} \oplus \bigoplus_{n \geq 1} R_n$ be a Hopf algebra which is connected graded as an algebra and generated in degree 1 with antipode S and (s, J) a twisted homomorphism for R such that $s^2 = \mathrm{id}$, we only consider the elements from R_n and just assume J^1 and J^2 are from some R_n where n is odd for the same reason. By Remark 2.3, $H' = H/\left\langle y_1 y_2 - y_2^2 \right\rangle$ is a bialgebra, and $I' = \left\langle y_1^2 - t, y_2^2 - t \right\rangle$ is the bi-ideal of H'.

We just write h for the element $h + I'$ of H'/I'. And to define the antipode, we just extend the antipode S of R to H'/I' defining $S(y_i) = y_i (i = 1, 2)$. We note that $S : H'/I' \to H'/I'$ is well defined, since $S(y_i^2 - t) = S(y_i)^2 - S(t) = y_i^2 - t \in I$ and, using that $s \circ S = S \circ s$, $S(a_n)y_1 = y_2 S(s(a_n))$, $S(a_n)y_2 = y_1 S(s(a_n))$ for all $a_n \in R_n$. Also, we have that

$$\begin{aligned} \mu(\mathrm{id} \otimes S)\Delta(y_1) &= \mu(\mathrm{id} \otimes S)(J(y_1 \otimes y_1)) = J^1 y_1 S(y_1) S(J^2) \\ &= y_2^2 s^2(J^1) S(J^2) = tJ^1 S(J^2) = 1, \end{aligned}$$

$$\begin{aligned} \mu(\mathrm{id} \otimes S)\Delta(y_2) &= \mu(\mathrm{id} \otimes S)(J(y_2 \otimes y_2)) = J^1 y_2 S(y_2) S(J^2) \\ &= y_1^2 s^2(J^1) S(J^2) = tJ^1 S(J^2) = 1, \end{aligned}$$

and

$$\begin{aligned} \mu(S \otimes \mathrm{id})\Delta(y_1) &= \mu(S \otimes \mathrm{id})(J(y_1 \otimes y_1)) = S(y_1) S(J^1) J^2 y_1 \\ &= y_1^2 s(S(J^1)) s(J^2) = ts(S(J^1)J^2) = 1, \end{aligned}$$

$$\begin{aligned} \mu(S \otimes \mathrm{id})\Delta(y_2) &= \mu(S \otimes \mathrm{id})(J(y_2 \otimes y_2)) = S(y_2) S(J^1) J^2 y_2 \\ &= y_2^2 s(S(J^1)) s(J^2) = ts(S(J^1)J^2) = 1. \end{aligned}$$

Since S is an antipode for R, the antipode property is verified for R as well. Therefore, S is an antipode of H'/I' and so H'/I' is a Hopf algebra.

The converse follows from the above four equations. □

Definition 2.5. The right double Ore extension of a Hopf algebra stated in Theorem 2.2 is called a *Hopf double Ore extension of anti-angle-grouplike type* (HDOE, for short).

Remark 2.4. *The condition $\mathbb{N}$-graded in this section is not essential. In fact, we can replace $\mathbb{N}$-graded with G-graded, where G is a cyclic group. In particular, $G = \mathbb{Z}_n$ for some $n \in \mathbb{N}$.*

3. Examples

In this section, we will give an example for the extending of bialgebra and Hopf algebra structures from a bialgebra R which is connected graded as an algebra and generated in degree 1 and construct a non-commutative and non-cocommutative Hopf algebra of infinite dimension.

Example 3.1. Let $\mathcal{O}(M_2(\mathbf{k})) = \mathbf{k}[a,b,c,d]$, the polynomial functions on 2×2 matrices. It is a connected graded algebra which is generated in degree 1 and can be described as follows:

As an algebra, $\mathcal{O}(M_2(\mathbf{k}))$ is simply the commutative polynomial ring.

Define a comultiplication and counit on $\mathcal{O}(M_2(\mathbf{k}))$ by

$$\begin{aligned}\Delta(a) &= a\otimes a + b\otimes c, \epsilon(a) = 1,\\ \Delta(b) &= a\otimes b + b\otimes d, \epsilon(b) = 0,\\ \Delta(c) &= c\otimes a + d\otimes c, \epsilon(c) = 0,\\ \Delta(d) &= c\otimes b + d\otimes d, \epsilon(d) = 1.\end{aligned}$$

Then $\mathcal{O}(M_2(\mathbf{k}))$ becomes a bialgebra. If we write

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

the comultiplication and counit are given by $\Delta(X_{ij}) = \sum\limits_k X_{ik}\otimes X_{kj}$ and $\epsilon(X_{ij}) = \delta_{ij}$.

The bialgebra $\mathcal{O}(M_2(\mathbf{k}))$ is not a Hopf algebra, since the group-like element $\det X$ is not invertible in $\mathcal{O}(M_2(\mathbf{k}))$.

However, there is a Hopf algebra closely related to $\mathcal{O}(M_2(\mathbf{k}))$, that is $\mathcal{O}(GL_2(\mathbf{k}))$, which equals to $\mathcal{O}(M_2(\mathbf{k}))[(\det X)^{-1}]$. It is a Hopf algebra by defining $S(X) = X^{-1}$, that is, $S(X_{ij})$ is the ij^{th} entry of X^{-1}.

For the Hopf algebra $R = \mathcal{O}(GL_2(\mathbf{k}))$, we let

$$s(a) = a, s(b) = b, s(c) = c, s(d) = d, J = 1\otimes 1.$$

The following claims hold:

(1) The $\mathbf{k}$-map s keeps the defining relations of R. Hence, $s\in \mathrm{Aut}(R)$;
(2) J is a Drinfel'd twist for R, and
$$J(s\otimes s)\Delta(z) = \Delta(s(z))J$$
for $z = a,b,c,d$. Hence, (s,J) is a twisted homomorphism for R;
(3) Setting $t = 1$ and we have $\Delta(t) = J(s\otimes s)(J)(t\otimes t) = 1\otimes 1$;
(4) The relation $s\circ S(z) = S\circ s(z)$ and $s^2(z) = z$ hold for $z = a,b,c,d$ and $tJ^1S(J^2) = ts(S(J^1)J^2) = 1$.

Then the double Ore extension of anti-angle type $R^{\vee}_{\{0,1\}}[y_1,y_2;\sigma]$ is a bialgebra with $\Delta(y_i) = y_i\otimes y_i$ and $\epsilon(y_i) = 1 (i = 1,2)$ by Theorem 2.1.

The quotient bialgebra

$$H_\infty := R^{\vee}_{\{0,1\}}[y_1,y_2;\sigma]/\left\langle y_1y_2 - y_2^2, y_1^2 - 1, y_2^2 - 1\right\rangle$$

is a bialgebra by Remark 2.3.

By setting $S(y_i) = y_i (i = 1,2)$, H_∞ has a Hopf algebra structure by Theorem 2.2. It is generated by a,b,c,d,y_1,y_2 with relations

$$\begin{aligned} y_1^2 = y_2^2 = y_1y_2 = y_2y_1 = 1,\\ y_1z = zy_2, y_2z = zy_1, (z = a,b,c,d).\end{aligned}$$

The coalgebra structure is

$$\begin{aligned}\Delta(a) &= a \otimes a + b \otimes c, \epsilon(a) = 1,\\ \Delta(b) &= a \otimes b + b \otimes d, \epsilon(b) = 0,\\ \Delta(c) &= c \otimes a + d \otimes c, \epsilon(c) = 0,\\ \Delta(d) &= c \otimes b + d \otimes d, \epsilon(d) = 1,\\ \Delta(y_1) &= y_1 \otimes y_1, \epsilon(y_1) = 1,\\ \Delta(y_2) &= y_2 \otimes y_2, \epsilon(y_2) = 1.\end{aligned}$$

The antipode is

$$S(X) = X^{-1}, S(y_1) = y_1, S(y_2) = y_2.$$

H_∞ is a non-commutative and non-cocommutative Hopf algebra of infinite dimension.

Remark 3.1. *In fact, y_1, y_2 are invertible in H since $y_1^2 = y_2^2 = y_1y_2 = y_2y_1 = 1$, then $y_1 = y_2$, hence $H_\infty = R[y; s]$ is a Hopf-Ore extension, where $y = y_1 = y_2$.*

The above example shows that Hopf double Ore extensions also play a key role in the theory of constructing the examples of Hopf algebras which are non-commutative and non-cocommutative.

However, it's hard to give the classification of twisted homomorphisms for a given bialgebra R relative to the double Ore extension of anti-angle type.

Let (s, J) and (s', J') be twisted homomorphisms for two bialgebras R and R' respectively, and $R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$ and $R'^{\vee}_{\{0,1\}}[y'_1, y'_2; \sigma']$ the corresponding double Ore extensions of anti-angle type. The datum (s, J) is said to be equivalent to (s', J'), denoted by $(s, J) \approx (s', J')$, if there is a bialgebra isomorphism $\Phi : R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma] \to R'^{\vee}_{\{0,1\}}[y'_1, y'_2; \sigma']$ such that $\Phi(R) = R'$ as bialgebras.

The difficulty of this problem is that we can't provide the form of Φ, since it is related to the Jacobian conjecture.

4. The homological properties of a special HDOE

In this section, we will investigate the lifting of homological properties from a bialgebra R which is connected graded as an algebra and generated in degree 1 to its double Ore extensions of anti-angle type $R^{\vee}_{P}[y_1, y_2; \sigma]$ for a specific Drinfel'd twist J and parameter P.

In fact, we have noted that $H = R^{\vee}_{\{0,1\}}[y_1, y_2; \sigma]$ in Theorem 2.1 is not even a domain since the relation $y_2y_1 = y_1^2$ produced zero divisors, so we can't lift most of the homological properties from R to H. However, if we choose a specific Drinfel'd twist $J = 1 \otimes 1$ and a specific parameter $P = \{1, 0\}$, this problem can be solved.

Since the parameter is changed, the related theorems need to be modified. As mentioned in Section 2, for any homomorphism of coalgebras $s \in \mathrm{End}(R)$, the pair $(s, 1 \otimes 1)$ is automatically a twisted homomorphism for R, so we don't need the condition of the twisted homomorphism in the next theorems.

Theorem 4.1. *Let R be a bialgebra which is connected graded as an algebra and generated in degree 1, $s \in \mathrm{Aut}(R)$ be a coalgebra homomorphism for R. Let $H = R^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ be a double Ore extension of anti-angle type. Then the bialgebra structure of R can be extended to H such that $\Delta(y_i) = y_i \otimes y_i$ and $\epsilon(y_i) = 1 (i = 1, 2)$.*

Proof. In this theorem, for all $h_n \in R_n$ where n is odd, we have the relations:

$$y_1 h_n = s(h_n)y_2, y_2 h_n = s(h_n)y_1, y_2 y_1 = y_1 y_2.$$

The remaining part of the proof is similar to that of Theorem 2.1, so we omit it. □

The following lemma gives us certain conditions to find the bi-ideal on the bialgebra $R^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ which will be used to define the Hopf algebra structure on the quotient $R^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]/I$ for some bi-ideal I of $R^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$.

Lemma 4.1. *Let R be a bialgebra which is connected graded as an algebra and generated in degree* 1*, $s \in \mathrm{Aut}(R)$ be a coalgebra homomorphism for R. Suppose that there exists $0 \neq t \in R$ such that $\Delta(t) = t \otimes t$. Then, for the double Ore extension of anti-angle type $H = R^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ with the bialgebra structure as in Theorem 4.1, $I = \left\langle y_1^2 - t, y_2^2 - t \right\rangle$ is a bi-ideal of H.*

Proof. The proof is similar to that of Lemma 2.1, so we omit it. □

Hence, given a bialgebra R which is connected graded as an algebra and generated in degree 1, $s \in \mathrm{Aut}(R)$ a coalgebra homomorphism for R and an element t that satisfies the hypothesis of Lemma 4.1, we have that H/I is a bialgebra, for the double Ore extension of anti-angle type $H = R^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ and $I = \left\langle y_1^2 - t, y_2^2 - t \right\rangle$.

The next theorem presents conditions to extend a Hopf algebra structure from a Hopf algebra R which is connected graded as an algebra and generated in degree 1 to the quotient bialgebra H/I.

Theorem 4.2. *Let R be a Hopf algebra which is connected graded as an algebra and generated in degree* 1 *with antipode S, $s \in \mathrm{Aut}(R)$ be a coalgebra homomorphism for R. Suppose also that $s \circ S = S \circ s$ and $s^2 = \mathrm{id}$. If there exists $0 \neq t \in R$, with $\Delta(t) = t \otimes t$, then there exists a Hopf algebra structure on H/I with $S(y_i) = y_i (i = 1, 2)$.*

Proof. The proof is similar to that of Theorem 2.2, so we omit it. □

Now we begin to lift some of the homological properties from a bialgebra R which is connected graded as an algebra and generated in degree 1 to its double Ore extension of anti-angle type.

Theorem 4.3. *Let R be a bialgebra which is connected graded as an algebra and generated in degree* 1*, $s \in \mathrm{Aut}(R)$ be a coalgebra homomorphism for R and $H = R^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ be a double Ore extension of anti-angle type, then as algebras:*

(1) If R is a domain, then H is a domain;
(2) If R is prime (semiprime), then H is prime (semiprime).

Proof. (1) Let $R = \mathbf{k} \oplus \bigoplus_{q \geq 1} R_q$. Suppose $f(y_1, y_2) = \sum_{i+j \leq n} a_{ij} y_1^i y_2^j, g(y_1, y_2) = \sum_{k+l \leq m} b_{kl} y_1^k y_2^l$, they are polynomials of degree n and m in H respectively, and a_{ij}, b_{kl} are the nonzero leading coefficients. Consider the monomials $a y_1^{s_1} y_2^{t_1}$ and $b y_1^{s_2} y_2^{t_2}$, where $a, b \in R$, the product of the monomials is a map $\phi : H \times H \to H$, now we explain that ϕ is injective. We make a convention:

(i) In the first position of the Cartesian product, if $s_1 < s_2$, then $a y_1^{s_1} y_2^{t_1}$ is the required unique monomial.

(ii) In the second position of the Cartesian product, if $t_1 < t_2$, then $ay_1^{s_1}y_2^{t_1}$ is the required unique monomial.

In this sense, we get a unique monomial through ϕ. Then we multiply the monomials and get

(a) If $b_{kl} \in \mathbf{k}$, note that $\sigma_{12}(b_{kl}) = \sigma_{21}(b_{kl}) = 0, p_{12} = 1$, the leading term of fg is $a_{ij}b_{kl}y_1^{k+i}y_2^{l+j}$. Since $a_{ij} \neq 0, b_{kl} \neq 0$, then the leading coefficient is nonzero, so the degree of fg is $n+m$. Hence H is a domain.
(b) If $b_{kl} \in R_n$ where n is odd, note that $\sigma_{11}(b_{kl}) = \sigma_{22}(b_{kl}) = 0, p_{12} = 1$, the leading term of fg is $a_{ij}s^{i+j}(b_{kl})y_1^{k+j}y_2^{l+i}$. Since $a_{ij} \neq 0, b_{kl} \neq 0$ and $s \in \mathrm{Aut}(R)$, then the leading coefficient is nonzero, so the degree of fg is $n+m$. Hence H is a domain.
(c) If $b_{kl} \in R_n$ where n is even, note that $\sigma_{12}(b_{kl}) = \sigma_{21}(b_{kl}) = 0, p_{12} = 1$, the leading term of fg is $a_{ij}s^{i+j}(b_{kl})y_1^{k+i}y_2^{l+j}$. Since $a_{ij} \neq 0, b_{kl} \neq 0$ and $s \in \mathrm{Aut}(R)$, then the leading coefficient is nonzero, so the degree of fg is $n+m$. Hence H is a domain.
(d) If $b_{kl} \in R$, without loss of generality, suppose $b_{kl} = b_{kl}^0 + b_{kl}^1 + \cdots + b_{kl}^q$, where $b_{kl}^0 \in \mathbf{k}, b_{kl}^p \in R_p (p = 1, 2, \cdots, q)$, based on the discussions of (a) (b) (c) above, the leading term of fg is

$$a_{ij}s^{i+j}\left(\sum_{p=0}^{q} b_{kl}^{2p}\right)y_1^{k+i}y_2^{l+j} + a_{ij}s^{i+j}\left(\sum_{p=0}^{q} b_{kl}^{2p+1}\right)y_1^{k+j}y_2^{l+i}.$$

Since $a_{ij} \neq 0, b_{kl} \neq 0$ and $s \in \mathrm{Aut}(R)$, then the leading coefficient is nonzero, so the degree of fg is $n+m$. Hence H is a domain.

(2) Each element of the leading coefficient in (1) is the leading coefficient of some element of fRg. Thus $fRg \neq 0$ and so $fHg \neq 0$. Hence H is prime (semiprime). □

However, the above homological properties of a Hopf algebra R which is connected graded as an algebra and generated in degree 1 can't lift to the Hopf algebra $H/I = R^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]/\left\langle y_1^2 - t, y_2^2 - t\right\rangle$, since the relations $y_1^2 = t$ and $y_2^2 = t$ produced zero divisors.

5. Hopf multiple Ore extensions of circular type

In this section, we extend the construction of double Ore extensions of anti-angle type to multiple Ore extensions. We find a class of multiple Ore extensions with good properties, called the multiple Ore extensions of circular type.

To begin our work, let's recall the definition of a multiple Ore extension introduced in [6].

Definition 5.1. Let A be an algebra and B be another algebra containing A as a subalgebra. We say B is a *n-Ore extension* of A if the following conditions hold:

(1) B is generated by A and n new variables $y_1, y_2, \cdots, y_n$;
(2) For any $1 \leq i < j \leq n$, $\{y_1, y_2, \cdots, y_n\}$ satisfies a relation

$$y_jy_i = \sum_{1\leq l\leq m\leq n} p_{ij}^{lm}y_ly_m + \sum_{i=1}^{n} t_{ij}^l y_l + t_0,$$

where $p_{ij}^{lm} \in \mathbf{k}$ and $t_{ij}^l \in A$;

(3) As a left A-module, $B = \sum\limits_{i_1,i_2,\cdots,i_n\geq 0} Ay_1^{i_1}y_2^{i_2}\cdots y_n^{i_n}$ and it is a left free A-module with a basis $\left\{y_1^{i_1}y_2^{i_2}\cdots y_n^{i_n}|i_1,i_2,\cdots,i_n\geq 0\right\}$;

(4) There exist a $\mathbf{k}$-linear map $\sigma : A \to M_n(A)$ and a $\mathbf{k}$-linear map $\delta : A \to A^{\oplus n}$ such that

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} a = \sigma(a) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \delta(a),$$

where $M_n(A)$ is the $n\times n$ matrix algebra with entries in A.

Set

$$P = \left\{p_{ij}^{lm} \in \mathbf{k}|1\leq i<j\leq n, 1\leq l\leq m\leq n\right\},$$
$$T = \left\{t_{ij}^{l} \in A|1\leq i<j\leq n, 1\leq l\leq n\right\},$$

then we denote the n-Ore extension B of A by $B = A[y_1, y_2, \cdots, y_n; \sigma, \delta, P, T]$.

When $n\geq 2$, the n-Ore extension is called a *multiple Ore extension*.

Now we extend the construction of double Ore extensions of anti-angle type to multiple Ore extensions.

Let $A = \mathbf{k}\oplus\bigoplus\limits_{n\geq 1} A_n$ be a connected graded algebra generated in degree 1 and B be the m-Ore extension of A, note that B is a $\mathbf{k}$-vector space, so the following fact is automatically established:

$$\sigma(\mathbf{k}) = \begin{pmatrix} \mathbf{k} & 0 & \cdots & 0 \\ 0 & \mathbf{k} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mathbf{k} \end{pmatrix} : \mathbf{k} \to M_{m\times m}(\mathbf{k}).$$

Further we let $s \in \mathrm{Aut}(A)$ which preserves degree. Suppose that A_1 is the $\mathbf{k}$-subspace of A generated by $\{x_1, \cdots, x_s, \cdots\}$, where $x_1, \cdots, x_s, \cdots$ are the homogeneous elements of degree 1. For all $a_1 \in A_1$, $a_1 = \sum\limits_i x_i$, we define

$$\sigma(x_i) = \begin{pmatrix} 0 & s(x_i) & 0 & \cdots & 0 \\ 0 & 0 & s(x_i) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & s(x_i) \\ s(x_i) & 0 & 0 & \cdots & 0 \end{pmatrix} : A_1 \to M_{m\times m}(A_1).$$

Since s is an algebra homomorphism, we have

$$\sigma(a_1) = \begin{pmatrix} 0 & s(a_1) & 0 & \cdots & 0 \\ 0 & 0 & s(a_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & s(a_1) \\ s(a_1) & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since σ is an algebra homomorphism, the domain can be extended to A_n and then to A. Let's begin with A_2, for all $a_2 \in A_2$, $a_2 = \sum\limits_{i,j} x_i x_j$, now

$$\begin{aligned}\sigma(a_2) &= \sigma\left(\sum_{i,j} x_i x_j\right) = \sum_{i,j} \sigma(x_i)\sigma(x_j) \\ &= \begin{pmatrix} 0 & 0 & s(a_2) & 0 & \cdots & 0 \\ 0 & 0 & 0 & s(a_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ s(a_2) & 0 & 0 & 0 & \cdots & 0 \\ 0 & s(a_2) & 0 & 0 & \cdots & 0 \end{pmatrix}.\end{aligned}$$

By induction, one can define $\sigma(a_n)$.

In fact, the matrix of σ act on a_i is the $s(a_i)$ times of the corresponding basic circular matrix.

Note that for all $a \in A$, $a = k_0 + \sum\limits_{i=1}^{n} k_i a_i$, where $k_0, k_i \in \mathbf{k}$ and $a_i \in A_i$, then we have

$$\sigma(a) = \sigma(k_0) + \sum_{i=1}^{n} k_i \sigma(a_i).$$

Assume $a_0 = 1$, then the matrix of $\sigma(a)$ is:

$$\begin{pmatrix} \sum\limits_{i=0}^{n} k_{im} s(a_{im}) & \sum\limits_{i=0}^{n} k_{im+1} s(a_{im+1}) & \cdots & \sum\limits_{i=0}^{n} k_{im+m-1} s(a_{im+m-1}) \\ \sum\limits_{i=0}^{n} k_{im+m-1} s(a_{im+m-1}) & \sum\limits_{i=0}^{n} k_{im} s(a_{im}) & \cdots & \sum\limits_{i=0}^{n} k_{im+m-2} s(a_{im+m-2}) \\ \vdots & \vdots & & \vdots \\ \sum\limits_{i=0}^{n} k_{im+1} s(a_{im+1}) & \sum\limits_{i=0}^{n} k_{im+2} s(a_{im+2}) & \cdots & \sum\limits_{i=0}^{n} k_{im} s(a_{im}) \end{pmatrix}.$$

We find that the matrix of σ act on the element of A is a circular matrix.

Definition 5.2. We call the above type of multiple Ore extension of A as a *multiple Ore extension of circular type*, denoted by $A^{\epsilon}[y_1, y_2, \cdots, y_m; \sigma, \delta, P, T]$.

It is easy to check that this type of multiple Ore extensions can not be presented as iterated Ore extensions.

Here we can get a similar conclusion as in Section 4.

Theorem 5.1. *Let R be a bialgebra which is connected graded as an algebra and generated in degree* 1*, $s \in \mathrm{Aut}(R)$ be a coalgebra homomorphism for R and $H = R^{\epsilon}[y_1, y_2, \cdots, y_m; \sigma, \delta, P, T]$ be the multiple Ore extension of circular type, then as algebras:*

(1) If R is a domain, then H is a domain;
(2) If R is prime (semiprime), then H is prime (semiprime).

Proof. The proof is similar to that of Theorem 4.3, so we omit it. □

6. Anti-angle-primitive type

If we modified the definition of the comultiplication, we can get another type of Hopf double Ore extensions of anti-angle type.

Definition 6.1. Let A be a Hopf algebra, which is connected graded as algebra and generated in degree 1, a *Hopf double Ore extension of anti-angle-primitive type* (HDOE, for short) of A is an algebra B such that

(1) B is a Hopf algebra with Hopf subalgebra A;
(2) $B = A^{\vee}_{\{p_{12},0\}}[y_1, y_2; \sigma]$ is the double Ore extension of anti-angle type;
(3) y_1, y_2 are primitive elements, that is:

$$\Delta(y_1) = y_1 \otimes 1 + 1 \otimes y_1, \Delta(y_2) = y_2 \otimes 1 + 1 \otimes y_2.$$

The following discussion is closely connected to [1] and [4].

Remark 6.1. *Note that the more general definition of the comultiplications in Definition 6.1 should be*

$$\Delta(y_1) = y_1 \otimes r_1 + r_2 \otimes y_1, \Delta(y_2) = y_2 \otimes r_1 + r_2 \otimes y_2,$$

for some $r_1, r_2 \in A$. Indeed, since Δ is a coassociative map, comparing

$$(\mathrm{id} \otimes \Delta)\Delta(y_i) = y_i \otimes \Delta(r_1) + r_2 \otimes y_i \otimes r_1 + r_2 \otimes r_2 \otimes y_i, (i = 1, 2)$$

with

$$(\Delta \otimes \mathrm{id})\Delta(y_i) = y_i \otimes r_1 \otimes r_1 + r_2 \otimes y_i \otimes r_1 + \Delta(r_2) \otimes y_i, (i = 1, 2).$$

We obtain $\Delta(r_i) = r_i \otimes r_i$, which means r_1 and r_2 are grouplike elements. Furthermore, by the main property of the antipode, r_1 and r_2 need to be invertible. Note that A is connected graded as algebra, hence $r_i \in \boldsymbol{k}$, then $\Delta(r_i) = r_i \otimes 1$, which requires that $r_i = 1$.

Now we give the basic properties of the Hopf double Ore extensions of anti-angle-primitive type.

Theorem 6.1. *Let A be a Hopf algebra, which is connected graded as an algebra and generated in degree 1, $B = A^{\vee}_{\{p_{12},0\}}[y_1, y_2; \sigma]$ be the Hopf double Ore extension of anti-angle-primitive type, then*

(1) $p_{12} = 1$;
(2) B is a double Ore extension of A;
(3) $\epsilon(y_i) = 0, S(y_i) = -y_i (i = 1, 2)$;
(4) $\Delta \circ s = s \otimes \mathrm{id} = \mathrm{id} \otimes s$ for all $a \in A$, that is

$$\Delta(s(a)) = s(a_{(1)}) \otimes a_{(2)} = a_{(1)} \otimes s(a_{(2)}).$$

Proof. Let $A = \mathbf{k} \oplus \bigoplus_{n \geq 1} A_n$ be the Hopf algebra. Assume that the comultiplication $\Delta|_A$ can be extended to $B = A^{\vee}_{\{p_{12},0\}}[y_1, y_2; \sigma]$. Then the homomorphism Δ preserves relation $y_2 y_1 = p_{12} y_1 y_2$, that is, $\Delta(y_2)\Delta(y_1) = p_{12}\Delta(y_1)\Delta(y_2)$.

(1) On the one hand,

$$\begin{aligned}\Delta(y_2)\Delta(y_1) &= (y_2\otimes 1+1\otimes y_2)(y_1\otimes 1+1\otimes y_1)\\ &= p_{12}y_1y_2\otimes 1+y_2\otimes y_1+y_1\otimes y_2+p_{12}\otimes y_1y_2.\end{aligned}$$

On the other hand,

$$\begin{aligned}p_{12}\Delta(y_1)\Delta(y_2) &= p_{12}(y_1\otimes 1+1\otimes y_1)(y_2\otimes 1+1\otimes y_2)\\ &= p_{12}y_1y_2\otimes 1+p_{12}y_1\otimes y_2+p_{12}y_2\otimes y_1+p_{12}\otimes y_1y_2.\end{aligned}$$

Compare the coefficients of $y_1\otimes y_2$ and $y_2\otimes y_1$ on the both sides, we have $p_{12}=1$.

(2) Since $s\in \mathrm{Aut}(A), p_{12}=1$, then B is a free right A-module with a basis $\left\{y_2^i y_1^j | i\geq 0, j\geq 0\right\}$ and $y_1A+y_2A+A=Ay_1+Ay_2+A$. Hence B is a double Ore extension of A.

(3) Applying $\epsilon\otimes \mathrm{id}$ to $\Delta(y_i)=y_i\otimes 1+1\otimes y_i$ and using the main property of the counit, we see that $y_i=\epsilon(y_i)+y_i$. This implies $\epsilon(y_i)=0$. Applying $\mu(\mathrm{id}\otimes S)$ to $\Delta(y_i)=y_i\otimes 1+1\otimes y_i$ and by the main property of the antipode, we see that $0=\epsilon(y_i)=y_i+S(y_i)$. This implies $S(y_i)=-y_i$.

(4) For any $a=\sum\limits_{i=0}^{n} a_i\in A$, where $a_i\in A_i$. Since Δ is an algebra homomorphism, we have

$$\Delta(y_1)\Delta(a)=\Delta\left(\sum_{i=0}^{n} s(a_{2i})\right)\Delta(y_1)+\Delta\left(\sum_{i=0}^{n} s(a_{2i+1})\right)\Delta(y_2),$$
$$\Delta(y_2)\Delta(a)=\Delta\left(\sum_{i=0}^{n} s(a_{2i+1})\right)\Delta(y_1)+\Delta\left(\sum_{i=0}^{n} s(a_{2i})\right)\Delta(y_2).$$

Compute in detail and compare the coefficients of $y_i\otimes 1$ and $1\otimes y_i$, we have

$$\Delta\left(\sum_{i=0}^{n} s(a_{2i})\right)=\sum_{i=0}^{n} s(a_{(1)2i})\otimes a_{(2)}=a_{(1)}\otimes\sum_{i=0}^{n} s(a_{(2)2i}),$$
$$\Delta\left(\sum_{i=0}^{n} s(a_{2i+1})\right)=\sum_{i=0}^{n} s(a_{(1)2i+1})\otimes a_{(2)}=a_{(1)}\otimes\sum_{i=0}^{n} s(a_{(2)2i+1}).$$

Add up the above two equations, we get

$$\Delta(s(a))=s(a_{(1)})\otimes a_{(2)}=a_{(1)}\otimes s(a_{(2)}).$$

□

Furthermore, we give necessary and sufficient conditions for a Hopf algebra $B=A^{\vee}_{\{1,0\}}[y_1,y_2;\sigma]$ to be a Hopf double Ore extension of anti-angle-primitive type.

Theorem 6.2. *Let A be a Hopf algebra, which is connected graded as an algebra and generated in degree 1, the antipode of A preserves degree, then $B=A^{\vee}_{\{1,0\}}[y_1,y_2;\sigma]$ is a Hopf double Ore extension of anti-angle-primitive type if and only if there is a character $\chi: A\to \boldsymbol{k}$ such that $s(a)=\chi(a_{(1)})a_{(2)}=a_{(1)}\chi(a_{(2)})$ for all $a\in A$.*

Proof. Necessity. For any $a \in A$, put $\chi(a) := s(a_{(1)})S(a_{(2)}) \in A$, then we have

$$\begin{aligned}\Delta(\chi(a)) &= \Delta(s(a_{(1)}))\Delta(S(a_{(2)})) = s(a_{(1)(1)})S(a_{(2)(2)}) \otimes a_{(1)(2)}S(a_{(2)(1)}) \\ &= s(a_{(1)})S(a_{(4)}) \otimes a_{(2)}S(a_{(3)}) = s(a_{(1)})S(a_{(3)}) \otimes \epsilon(a_{(2)}) \\ &= s(a_{(1)}\epsilon(a_{(2)}))S(a_{(3)}) \otimes 1 = s(a_{(1)})S(a_{(2)}) \otimes 1 = \chi(a) \otimes 1.\end{aligned}$$

Using the main property of the counit, we see that $\chi(a) \in \mathbf{k}$. One can regard χ as a map $\chi : A \to \mathbf{k}$. Since $s \in \mathrm{Aut}(A)$ and χ is a character, it follows that for all $a, b \in A$,

$$\begin{aligned}\chi(ab) &= s(a_{(1)}b_{(1)})S(a_{(2)}b_{(2)}) = s(a_{(1)})s(b_{(1)})S(b_{(2)})S(a_{(2)}) \\ &= s(a_{(1)})\chi(b)S(a_{(2)}) = \chi(a)\chi(b),\end{aligned}$$

$$\begin{aligned}\chi(a+b) &= s((a+b)_{(1)})S((a+b)_{(2)}) \\ &= s(a_{(1)})S(a_{(2)}) + s(b_{(1)})S(b_{(2)}) = \chi(a) + \chi(b).\end{aligned}$$

One can recover s from χ,

$$\chi(a_{(1)})a_{(2)} = s(a_{(1)})S(a_{(2)})a_{(3)} = s(a_{(1)})\epsilon(a_{(2)}) = s(a_{(1)}\epsilon(a_{(2)})) = s(a).$$

On the one hand,

$$\Delta(s(a)) = \Delta(\chi(a_{(1)})a_{(2)}) = \chi(a_{(1)})\Delta(a_{(2)}) = \chi(a_{(1)})a_{(2)} \otimes a_{(3)}.$$

On the other hand,

$$\Delta(s(a)) = \Delta(\chi(a_{(1)})a_{(2)}) = a_{(1)} \otimes \chi(a_{(2)})a_{(3)} = a_{(1)}\chi(a_{(2)}) \otimes a_{(3)}.$$

We obtain

$$\chi(a_{(1)})a_{(2)} \otimes a_{(3)} = a_{(1)}\chi(a_{(2)}) \otimes a_{(3)}.$$

Then

$$(\chi(a_{(1)})a_{(2)})a_{(3)}S(a_{(4)}) = (a_{(1)}\chi(a_{(2)}))a_{(3)}S(a_{(4)}).$$

Thus

$$(\chi(a_{(1)})a_{(2)})\epsilon(a_{(3)}) = (a_{(1)}\chi(a_{(2)}))\epsilon(a_{(3)}).$$

Hence

$$s(a) = \chi(a_{(1)})a_{(2)} = a_{(1)}\chi(a_{(2)}).$$

Sufficiency. If condition $s(a) = \chi(a_{(1)})a_{(2)} = a_{(1)}\chi(a_{(2)})$ holds, then

$$\Delta(s(a)) = s(a_{(1)}) \otimes a_{(2)} = a_{(1)} \otimes s(a_{(2)}). \tag{6.1}$$

For any $a = \sum\limits_{i=0}^{n} a_i \in A$, where $a_i \in A_i$, we have

$$y_1 a = \sum_{i=0}^{n} s(a_{2i})y_1 + \sum_{i=0}^{n} s(a_{2i+1})y_2, \tag{6.2}$$

$$y_2 a = \sum_{i=0}^{n} s(a_{2i+1})y_1 + \sum_{i=0}^{n} s(a_{2i})y_2. \tag{6.3}$$

It is clear that Δ preserves the above two equations if and only if (6.1) holds.

Since $y_2y_1 = y_1y_2$, we have $\Delta(y_2)\Delta(y_1) = \Delta(y_1)\Delta(y_2)$, then the comultiplication $\Delta|_A$ can be extended to a homomorphism $\Delta : B \to B \otimes B$.

If $\epsilon(y_i) = 0 (i = 1, 2)$, we can prove that ϵ preserves (6.2) and (6.3). Similarly, the counit $\epsilon|_A$ can be extended to a homomorphism $\epsilon : B \to \mathbf{k}$.

Assume that B admits an antipode S which can be extended as an antiautomorphism from A to B by means of $S(y_i) = -y_i (i = 1, 2)$, then S preserves (6.2) and (6.3). This means

$$S(a)S(y_1) = S(y_1)S\left(\sum_{i=0}^{n} s(a_{2i})\right) + S(y_2)S\left(\sum_{i=0}^{n} s(a_{2i+1})\right), \tag{6.4}$$

$$S(a)S(y_2) = S(y_1)S\left(\sum_{i=0}^{n} s(a_{2i+1})\right) + S(y_2)S\left(\sum_{i=0}^{n} s(a_{2i})\right). \tag{6.5}$$

Hence the existence of an antipode B satisfying $S(y_i) = -y_i (i = 1, 2)$ is equivalent to (6.4) and (6.5). It follows that

$$S(a)y_1 = s\left(S\left(\sum_{i=0}^{n} s(a_{2i})\right)\right)y_1 + s\left(S\left(\sum_{i=0}^{n} s(a_{2i+1})\right)\right)y_1, \tag{6.6}$$

$$S(a)y_2 = s\left(S\left(\sum_{i=0}^{n} s(a_{2i+1})\right)\right)y_2 + s\left(S\left(\sum_{i=0}^{n} s(a_{2i})\right)\right)y_2. \tag{6.7}$$

Conditions (6.6) and (6.7) hold if and only if $S(a) = s(S(s(a)))$ holds. Let us prove it, we have

$$\begin{aligned} s(S(s(a))) &= s(S(\chi(a_{(1)})a_{(2)})) = s(\chi(a_{(1)})S(a_{(2)})) = \chi(a_{(1)})s(S(a_{(2)})) \\ &= \chi(a_{(1)})S(a_{(3)})\chi(S(a_{(2)})) = \chi(a_{(1)}S(a_{(2)}))S(a_{(3)}) = S(a). \end{aligned}$$

This proves the existence of an antipode. Hence the Hopf algebra $B = A^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ is a Hopf double Ore extension of anti-angle-primitive type. $\square$

We can construct another non-commutative and non-cocommutative Hopf algebra of infinite dimension through Hopf double Ore extensions of anti-angle-primitive type. It is similar to Example 3.1 but not a Hopf-Ore extension.

Example 6.1. Let $A = \mathcal{O}(GL_2(\mathbf{k}))$, $B = A^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ be the Hopf double Ore extension of anti-angle-primitive type with $s = \mathrm{id}$, then B is a non-commutative and non-cocommutative Hopf algebra of infinite dimension. It is generated by a, b, c, d, y_1, y_2 with relations

$$y_1y_2 = y_2y_1, y_1z = zy_2, y_2z = zy_1, (z = a, b, c, d).$$

The coalgebra structure is

$$\Delta(a) = a \otimes a + b \otimes c, \epsilon(a) = 1,$$
$$\Delta(b) = a \otimes b + b \otimes d, \epsilon(b) = 0,$$
$$\Delta(c) = c \otimes a + d \otimes c, \epsilon(c) = 0,$$
$$\Delta(d) = c \otimes b + d \otimes d, \epsilon(d) = 1,$$

$$\Delta(y_1) = y_1 \otimes 1 + 1 \otimes y_1, \epsilon(y_1) = 0,$$
$$\Delta(y_2) = y_2 \otimes 1 + 1 \otimes y_2, \epsilon(y_2) = 0.$$

The antipode is
$$S(X) = X^{-1}, S(y_1) = -y_1, S(y_2) = -y_2.$$

The following two useful propositions are related to the dimensions. The right Krull dimension of a ring R is denoted by $\mathrm{Kdim}(R)$, the right global dimension of a ring R is denoted by $\mathrm{rgld}(R)$.

Proposition 6.1. *([4, Theorem 6.5.4]) Let R be a ring of finite right Krull dimension, σ an automorphism and δ a σ-derivation, then*

$$\mathrm{Kdim}(R) \leq \mathrm{Kdim}(R[x;\sigma,\delta]) \leq \mathrm{Kdim}(R) + 1.$$

In particular, if $\delta = 0$, then
$$\mathrm{Kdim}(R[x;\sigma]) = \mathrm{Kdim}(R) + 1.$$

Proposition 6.2. *([4, Theorem 7.5.3]) Let R be a ring of finite right global dimension, σ an automorphism and δ a σ-derivation, then*

$$\mathrm{rgld}(R) \leq \mathrm{rgld}(R[x;\sigma,\delta]) \leq \mathrm{rgld}(R) + 1.$$

In particular, if $\delta = 0$ and R is right noetherian, then

$$\mathrm{rgld}(R[x;\sigma]) = \mathrm{rgld}(R) + 1.$$

Now we lift other homological properties for Example 6.1.

Theorem 6.3. *Let $A = \mathcal{O}(GL_2(\boldsymbol{k}))$, $B = A^{\vee}_{\{1,0\}}[y_1, y_2; \sigma]$ be the Hopf double Ore extension of anti-angle-primitive type with $s = \mathrm{id}$, then as an algebra:*

(1) B is right noetherian.
(2) B has finite right Krull dimension.
(3) B has finite right global dimension.

Proof. Recall that as an algebra $A = \mathbf{k}[a, b, c, d]/\langle ad - bc\rangle$ is a finitely generated commutative algebra, then A is right noetherian, has finite right Krull dimension and has finite global dimension.

Since $s = \mathrm{id}$, if we look at it in another light, B is the iterated Ore extensions

$$\mathbf{k}[y_1][y_2;\sigma_0][a;\sigma_1][b;\sigma_2][c;\sigma_3][d;\sigma_4]$$

module the ideal generated by $ad - bc$, where

$$\sigma_0(y_1) = y_1, \sigma_i(y_1) = y_2, \sigma_i(y_2) = y_1, (i = 1, 2, 3, 4).$$

(1) By Hilbert's Basis Theorem, B is right noetherian.
(2) By Proposition 6.1, we have $\mathrm{Kdim}(B) = \mathrm{Kdim}(A) + 2$.
(3) By Proposition 6.2, we have $\mathrm{rgld}(B) = \mathrm{rgld}(A) + 2$. □

7. Conclusions

In this article, we defined the double Ore extensions of anti-angle type, studied Hopf double Ore extensions and focused on three types, they are anti-angle-grouplike type, anti-angle-primitive type and circular type. As examples, we constructed two non-commutative and non-cocommutative Hopf algebras of infinite dimension and investigated the lifting of homological properties.

Acknowledgments

The author thanks the supervisors Z. Wu and D. M. Lu for useful conversations on the subject, providing the references, reading an earlier version of the paper and their useful comments.

Conflict of interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work, there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled.

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