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Research article

Blow-up of solutions to the coupled Tricomi equations with derivative type nonlinearities

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Abstract: This paper is concerned with blow-up results of solutions to coupled system of the Tricomi equations with derivative type nonlinearities. Upper bound lifespan estimates of solutions to the Cauchy problem with small initial values are derived by using the test function method (see the proof of Theorem 1.1) and iteration argument (see the proof of Theorem 1.2), respectively. Our main new contribution is that lifespan estimates of solutions to the problem in the sub-critical and critical cases which are connected with the Glassey conjecture are established. To the best knowledge of authors, the results in Theorems 1.1 and 1.2 are new.

Keywords: coupled Tricomi equations; derivative type nonlinearities; test function method; iteration method; blow-up; lifespan estimates
Mathematics Subject Classification: 35L70, 58J45

1. Introduction and main results

In this work, we consider the following Cauchy problem of coupled Tricomi equations with derivative type nonlinearities

$$\begin{cases} u_{tt} - t^{2m}\Delta u = |v_t|^p, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ v_{tt} - t^{2m}\Delta v = |u_t|^q, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(0,x) = \varepsilon u_0(x), \ \ u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \\ v(0,x) = \varepsilon v_0(x), \ \ v_t(0,x) = \varepsilon v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $m > 0$, $n \geq 1$. The initial values possess compact supports

$$\mathrm{supp}(u_0, u_1, v_0, v_1) \subset \{x \in \mathbb{R}^n \big| |x| \leq R\}, \tag{1.2}$$

where $R \geq 2$ is a constant.

Firstly, we recall some known results related to the single Tricomi equation

$$\begin{cases} u_{tt} - t^{2m}\Delta u = f(u, u_t), & (t, x) \in [0, T) \times \mathbb{R}^n, \\ u(0, x) = \varepsilon u_0(x), \ u_t(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $f(u, u_t) = |u|^p, |u_t|^p, |u_t|^p + |u|^q$. Lin and Tu [27] study blow-up dynamics of problem (1.3) with $f(u, u_t) = |u|^p$ by using the test function method and iteration argument in the sub-critical case. An iteration procedure together with slicing method is employed to prove formation of singularity of solution in the critical case. He et al. [11] investigate blow-up result of solution by constructing the Riccati type differential inequality. Lucentea and Palmieri [29] establish blow-up dynamics for the Tricomi equation with derivative type nonlinearity $f(u, u_t) = |u_t|^p$ in the sub-critical and critical cases by utilizing the integral representation formula. Upper bound lifespan estimates of solution are obtained by using the test function method (see [22]). For the Tricomi equation with combined nonlinearities $f(u, u_t) = |u_t|^p + |u|^q$, blow-up results and lifespan estimates of solution are established by employing the iteration argument (see [1]). Hamouda and Hamza [6] study properties of solution to the liner Cauchy problem corresponding to problem (1.3). Formation of singularities of solution are obtained by constructing ordinary differential inequality. More detailed illustration related to the study of the Tricomi equation can be found in [10, 12, 13, 33, 40].

Equation (1.3) reduces to the well known semilinear wave equation when $m = 0$, namely

$$\begin{cases} u_{tt} - \Delta u = f(u, u_t), & (t, x) \in [0, T) \times \mathbb{R}^n, \\ u(0, x) = \varepsilon u_0(x), \ u_t(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.4}$$

Problem (1.4) is concerned with the Strauss conjecture when $f(u, u_t) = |u|^p$ (see [41]). We note the critical exponent $p_S(1) = \infty$. For $n \geq 2$, $p_S(n)$ is the positive root of quadratic equation

$$-(n-1)p^2 + (n+1)p + 2 = 0.$$

The critical exponent $p_S(n)$ for $n \geq 2$ divides the interval $(1, \infty)$ into two parts. For $p \in (1, p_S(n)]$, the solution blows up in finite time (see [7, 9, 16, 17, 20, 21, 25, 30, 31, 42–44]). While the solution exists globally (in time) when $p \in (p_S(n), \infty)$ (see [2, 4, 23, 24, 26, 28, 45]). For $f(u, u_t) = |u_t|^p$, Eq (1.4) is related to the Glassey conjecture [3], where the critical exponent is characterized by $p_G(n) = \frac{n+1}{n-1}$. Concretely, Lai and Tu [19] investigate problem (1.4) with space dependent damping $\frac{\mu}{(1+|x|)^\beta} u_t$ $(\beta > 2)$ and $f(u, u_t) = |u|^p, |u_t|^p$, respectively. Blow-up results and lifespan estimates of solutions are obtained by using the test function method. For problem (1.4) involving mixed nonlinearities $f(u, u_t) = |u_t|^p + |u|^q$, Han and Zhou [8] obtain upper bound lifespan estimates of solution to the Cauchy problem by constructing proper test function and the ordinary differential inequalities. Lai and Takamura [18] illustrate blow-up results and upper bound lifespan estimates of solution to the problem with time dependent damping term $\frac{\mu}{(1+t)^\beta} u_t$ $(\beta > 1)$ by making use of a multiplier and iteration argument. Formation of singularities for solution to problem (1.4) with scale invariant damping $\frac{\mu}{1+t} u_t$ are investigated by applying the test function approach (see [5]).

Recently, blow-up dynamics of the Cauchy problem for coupled system of semilinear wave equations

$$\begin{cases} u_{tt} - \Delta u = f_1(v, v_t), & (t,x) \in [0,T) \times \mathbb{R}^n, \\ v_{tt} - \Delta v = f_2(u, u_t), & (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(0,x) = \varepsilon u_0(x),\ u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \\ v(0,x) = \varepsilon v_0(x),\ v_t(0,x) = \varepsilon v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

attracts extensive attention (see [15, 32, 35–39]). Using the test function method, Ikeda et al. [15] obtain blow-up results and lifespan estimates of solutions to problem (1.5) with power nonlinear terms $f_1(v, v_t) = |v|^p$, $f_2(u, u_t) = |u|^q$, derivative nonlinear terms $f_1(v, v_t) = |v_t|^p$, $f_2(u, u_t) = |u_t|^q$ and mixed nonlinear terms $f_1(v, v_t) = |v|^p$, $f_2(u, u_t) = |u_t|^q$, respectively. Palmieri and Takamura [38] discuss the coupled system of semilinear wave equations (1.5) with time dependent weak damping terms and power nonlinearities. Upper bound lifespan estimates of solutions are derived by utlizing the iteration argument. Palmieri and Takamura [35] consider the coupled system of semilinear time dependent damped wave equations of derivative type nonlinearities in the scattering case. Upper bound lifespan estimates of solutions in the sub-critical case are obtained by taking advantage of the Kato lemma. While in the critical case, an iteration procedure based on the slicing method is employed. Formation of singularities of solutions to problem (1.5) with time dependent damping and mixed nonlinear terms in the scattering case are established through the iteration argument (see [37]). Ikeda et al. [14] study the Cauchy problem for coupled system of semilinear Tricomi equations

$$\begin{cases} u_{tt} - t^{2m_1}\Delta u = f_1(v, v_t), & (t,x) \in [0,T) \times \mathbb{R}^n, \\ v_{tt} - t^{2m_2}\Delta v = f_2(u, u_t), & (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(0,x) = \varepsilon u_0(x),\ u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \\ v(0,x) = \varepsilon v_0(x),\ v_t(0,x) = \varepsilon v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

where the nonlinear terms are power type nonlinearities $f_1(v, v_t) = |v|^p$ and $f_2(u, u_t) = |u|^q$. Blow-up dynamics and lifespan estimates of solutions are obtained by constructing test functions which are related to the Gauss hypergeometric functions.

Inspired by the results in [14, 15, 22, 29], we consider blow-up dynamics of the Cauchy problem for coupled Tricomi equations (1.1). For the Cauchy problem of single Tricomi equation with derivative nonlinear term, upper bound lifespan estimates of solution are established by using the test function method (see [22]) and integral representation formula (see [29]), respectively. Concerning the Cauchy problem for coupled semilinear wave equations and coupled Tricomi equations with power type nonlinearities, formation of singularities are established by employing the test function method (see [14, 15]). We obtain blow-up results and upper bound lifespan estimates of solutions to problem (1.1) by using test function method and iteration argument, respectively. It is worth to mention that the test function utilized in the proof of Theorem 1.1 is different from the test functions employed in [14, 15], which are related to the Gauss hypergeometric functions. Lucente and Palmieri [29] establish upper bound lifespan estimate of solution to the Cauchy problem for single Tricomi equation by using integral representation formula and constructing ordinary differential inequality. While in this paper, we combine the integral representation formula with iteration method to present the proof of Theorem 1.2. The results obtained in this paper can be regarded as an extended work in [14, 15, 22, 29]. To our best knowledge, the results in Theorems 1.1 and 1.2 are new. In addition, we present a comparison for lifespan estimates in Theorems 1.1 and 1.2 in a special case (see Remark 1.1).

Throughout this paper, we use the following expressions

$$\begin{aligned} F_{GG}(n,m,p,q) &= -\frac{(n-1)(m+1)}{2} + \frac{m}{2} + \frac{p+1}{pq-1}, \\ \Lambda_{GG}(n,m,p,q) &= \frac{p+1}{pq-1} - (\frac{n-1}{2} + \frac{m}{2(m+1)}) - \frac{m(p+1)}{(pq-1)(m+1)}, \\ \Omega_{GG}(n,p,q) &= \max\{F_{GG}(n,0,p,q), F_{GG}(n,0,q,p)\}. \end{aligned}$$

C denotes the positive constant independent of ε, which may vary from line to line. $A \lesssim B$ stands for $A \le CB$, where C is a positive constant.

From the local existence result of solution to the Cauchy problem of single Tricomi equation with derivative type nonlinearity $|u_t|^p$ in Theorem 2 in [22], we can obtain the existence and uniqueness of solutions to the coupled system (1.1) by using the Banach fixed point theorem. We omit the details for simplicity.

The main results in this paper are described as follows.

Theorem 1.1. *Assume that the initial data $u_0, v_0 \in H^1(\mathbb{R}^n)$, $u_1, v_1 \in H^{1-\frac{1}{m+1}}(\mathbb{R}^n)$ satisfy*

$$(2m+1)^{\frac{m}{m+1}} \frac{\Gamma(\frac{1}{2} + \frac{m}{2(m+1)})}{\Gamma(\frac{1}{2} - \frac{m}{2(m+1)})} u_0 + u_1 > 0,$$

$$(2m+1)^{\frac{m}{m+1}} \frac{\Gamma(\frac{1}{2} + \frac{m}{2(m+1)})}{\Gamma(\frac{1}{2} - \frac{m}{2(m+1)})} v_0 + v_1 > 0,$$

where $\Gamma(s) = \int_0^{+\infty} z^{s-1}e^{-z}dz$ is the Gamma function for $s > 0$. Suppose that (u, v) are a pair of solutions to problem (1.1) *which satisfy*

$$\mathrm{supp}\,(u,v) \subset \{(t,x) \in [0,T) \times \mathbb{R}^n \,\Big|\, |x| \le R + \frac{t^{m+1}}{m+1}\}.$$

Then, there exists a small positive constant $\varepsilon_0 = \varepsilon_0(n, m, p, q, R, u_0, u_1, v_0, v_1)$ such that the lifespan estimates satisfy

$$T(\varepsilon) \le \begin{cases} C\varepsilon^{-\widetilde{F}_{GG}^{-1}(n,m,p,q)}, & \widetilde{F}_{GG}(n,m,p,q) > 0, \\ \exp(C\varepsilon^{-(pq-1)}), & \widetilde{F}_{GG}(n,m,p,q) = 0, \ p \ne q, \\ \exp(C\varepsilon^{-(p-1)}), & \widetilde{F}_{GG}(n,m,p,q) = 0, \ p = q, \end{cases} \tag{1.7}$$

where $\widetilde{F}_{GG} = \max\{F_{GG}(n,m,p,q), F_{GG}(n,m,q,p)\}$ and $\varepsilon \in (0, \varepsilon_0]$.

Theorem 1.2. *Assume that the initial data $u_0, v_0 \in C_0^2(\mathbb{R}^n)$, $u_1, v_1 \in C_0^1(\mathbb{R}^n)$ are non-negative functions. Suppose that (u, v) are a pair of solutions to problem* (1.1) *which satisfy*

$$\mathrm{supp}\,(u,v) \subset \{(t,x) \in [0,T) \times \mathbb{R}^n \,\Big|\, |x| \le R + \frac{t^{m+1}}{m+1}\}.$$

Then, the lifespan estimates satisfy

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\widetilde{\Lambda}_{GG}^{-1}(n,m,p,q)}, & \widetilde{\Lambda}_{GG}(n,m,p,q) > 0, \\ \exp(C\varepsilon^{-(pq-1)}), & \widetilde{\Lambda}_{GG}(n,m,p,q) = 0, \ p \neq q, \\ \exp(C\varepsilon^{-(p-1)}), & \widetilde{\Lambda}_{GG}(n,m,p,q) = 0, \ p = q, \end{cases} \tag{1.8}$$

where $\widetilde{\Lambda}_{GG} = \max\{(m+1)\Lambda_{GG}(n,m,p,q), (m+1)\Lambda_{GG}(n,m,q,p\}$.

Remark 1.1. *Direct calculation shows* $\widetilde{F}_{GG}(n,m,p,q) = m + \widetilde{\Lambda}_{GG}(n,m,p,q)$. *Therefore, the first lifespan estimate in* (1.7) *is better than the the first lifespan estimate in* (1.8) *in the sub-critical case. We conjecture that the curve in the* $p-q$ *plane which satisfies* $\widetilde{F}_{GG}(n,m,p,q) = 0$ *is the critical curve. We will verify this conjecture in our future work. We observe that Ikeda et al. [15] obtain the following upper bound lifespan estimates of solutions to problem* (1.1) *with* $m = 0$, *namely*

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\Omega_{GG}^{-1}(n,p,q)}, & \Omega_{GG}(n,p,q) > 0, \\ \exp(C\varepsilon^{-(pq-1)}), & \Omega_{GG}(n,p,q) = 0, \ p \neq q, \\ \exp(C\varepsilon^{-(p-1)}), & \Omega_{GG}(n,p,q) = 0, \ p = q. \end{cases} \tag{1.9}$$

It is worth to mention that the lifespan estimates in (1.7) *and the lifespan estimates in* (1.8) *are coincide with the lifespan estimates in* (1.9) *when* $m = 0$.

Remark 1.2. *Problem* (1.1) *is equivalent to the Cauchy problem for single Tricomi equation when* $p = q$, *which has been studied in [22, 29]. Lifespan estimates of solutions in* (1.7) *and* (1.8) *in the case* $p = q$ *are coincide with the lifespan estimates of solutions in [22, 29].*

2. Proof of Theorem 1.1

2.1. Basic definition and related lemmas

Firstly, we illustrate the definition of weak solutions.

Definition 2.1. *Assume that the pair of functions* (u, v) *satisfy*

$$(u,v) \in (C([0,T); H^1(\mathbb{R}^n)) \cap C^1([0,T); H^{1-\frac{1}{m+1}}(\mathbb{R}^n)))^2,$$

$$(u_t, v_t) \in L^q_{loc}([0,T) \times \mathbb{R}^n) \times L^p_{loc}([0,T) \times \mathbb{R}^n).$$

Then, (u, v) *are called weak solutions of problem* (1.1) *on* $[0, T)$ *if*

$$\begin{aligned} &\varepsilon \int_{\mathbb{R}^n} u_1(x)\Psi(0,x)dx + \int_0^T \int_{\mathbb{R}^n} |v_t|^p \Psi(t,x)dxdt \\ =& \int_0^T \int_{\mathbb{R}^n} (-\partial_t u(t,x)\partial_t \Psi(t,x) + t^{2m}\nabla u(t,x)\nabla\Psi(t,x))dxdt, \\ &\varepsilon \int_{\mathbb{R}^n} v_1(x)\Psi(0,x)dx + \int_0^T \int_{\mathbb{R}^n} |u_t|^q \Psi(t,x)dxdt \\ =& \int_0^T \int_{\mathbb{R}^n} (-\partial_t v(t,x)\partial_t \Psi(t,x) + t^{2m}\nabla v(t,x)\nabla\Psi(t,x))dxdt, \end{aligned} \tag{2.1}$$

where $\Psi(t,x) \in C_0^\infty([0,T)\times\mathbb{R}^n)$ *and* $T \in (1, T(\varepsilon))$. *Here,* $T(\varepsilon)$ *represents the upper bound lifespan estimate of solutions to problem* (1.1)*, which satisfies*

$$T(\varepsilon) = \sup\{T > 0, \ \textit{there exist a pair of energy solutions to problem}\ (1.1)\}.$$

Lemma 2.2. *The cut off function* $\eta(t) \in C^\infty([0,\infty))$ *is defined by*

$$\eta(t) = \begin{cases} 1, & t \le \frac{1}{2}, \\ decreasing, & t \in (\frac{1}{2}, 1), \\ 0, & t \ge 1, \end{cases}$$

which satisfies $|\eta'(t)|, |\eta''(t)| < C$. *Let* $\eta_T(t) = \eta(t/T)$ *and* $k > 1$. *It holds that*

$$\partial_t \eta_T^{2k} = \frac{2k}{T}\eta_T^{2k-1}\eta',$$

$$\partial_t^2 \eta_T^{2k} = \frac{2k(2k-1)}{T^2}\eta_T^{2k-2}|\eta'|^2 + \frac{2k}{T^2}\eta_T^{2k-1}\eta''.$$

Suppose that the function $\theta(t) \in C^\infty([0,\infty))$ *satisfies*

$$\theta(t) = \begin{cases} 0, & t < \frac{1}{2}, \\ \eta(t), & t \ge \frac{1}{2}, \end{cases} \qquad \theta_M(t) = \theta(\frac{t}{M}).$$

The proof of Lemma 2.2 is easy, we omit the details.

Lemma 2.3. *(Lemma 4 in [22]) Let* $m > -\frac{1}{2}$ *and* $\gamma = \frac{m}{2(m+1)}$. *Assume that*

$$y(t) = t^{m+1/2}K_{\gamma+1/2}(\frac{t^{m+1}}{m+1}),$$

where $K_\nu(\cdot)$ *stands for modified Bessel function. It holds that* $y(t) \in C^1([0,\infty)) \cap C^\infty(0,\infty)$ *which satisfies*

$$y''(t) - \frac{2m}{t}y'(t) - t^{2m}y(t) = 0. \tag{2.2}$$

Moreover, $y(t)$ *possesses the following properties:*

(1) $y(t) > 0$, $y'(t) < 0$,
(2) $\lim_{t\to 0^+} y(t) = 2^{\gamma-\frac{1}{2}}(m+1)^{\gamma+\frac{1}{2}}\Gamma(\gamma+\frac{1}{2}) = c_0(\gamma) > 0$,
(3) $\lim_{t\to 0^+} \frac{y'(t)}{t^{2m}} = -c_0(-\gamma) < 0$,
(4) $y(t) = \sqrt{\frac{(m+1)\pi}{2}} t^{\frac{m}{2}} \exp(-\frac{t^{m+1}}{m+1}) \times (1 + O(t^{-(m+1)}))$, for large $t > 0$,
(5) $y'(t) = -\sqrt{\frac{(m+1)\pi}{2}} t^{\frac{3m}{2}} \exp(-\frac{t^{m+1}}{m+1}) \times (1 + O(t^{-(m+1)}))$, for large $t > 0$.

2.2. Proof of Theorem 1.1

We introduce the following test function

$$\Psi(t,x) = -t^{-2m}\partial_t(\eta_M^{2k}(t)y(t))\phi(x), \tag{2.3}$$

where

$$\phi(x) = \begin{cases} \int_{S^{n-1}} e^{x\cdot w}dw, & n \geq 2, \\ e^x + e^{-x}, & n = 1. \end{cases}$$

It holds that $\Delta\phi(x) = \phi(x)$ and

$$0 < \phi(x) \lesssim (1+|x|)^{-\frac{n-1}{2}}e^{|x|}. \tag{2.4}$$

We note $\Psi(t,x) \in C_0^\infty([0,\infty)\times\mathbb{R}^n)$ and

$$\Psi(0,x) = \lim_{t\to 0^+}\Psi(t,x) = c_0(-\gamma)\phi(x) \geq 0.$$

Applying the first equality in (2.1) and (2.3), we obtain

$$\begin{aligned}
&\varepsilon\int_{\mathbb{R}^n} u_1(x)c_0(-\gamma)\phi dx + \int_0^T\int_{\mathbb{R}^n}|v_t|^p(-t^{-2m})\eta_M^{2k}\partial_t y\phi dxdt \\
&+ \int_0^T\int_{\mathbb{R}^n}|v_t|^p(-t^{-2m})\partial_t\eta_M^{2k}y\phi dxdt \\
&= \int_0^T\int_{\mathbb{R}^n}\partial_t u\partial_t(t^{-2m}\partial_t(\eta_M^{2k}y)\phi)dxdt - \int_0^T\int_{\mathbb{R}^n}\nabla u\nabla\phi\partial_t(\eta_M^{2k}y)dxdt \\
&= \int_0^T\int_{\mathbb{R}^n}u_t t^{-2m}(-2mt^{-1}\partial_t\eta_M^{2k}y - 2mt^{-1}\eta_M^{2k}\partial_t y + \partial_t^2\eta_M^{2k}y \\
&+ 2\partial_t\eta_M^{2k}\partial_t y + \eta_M^{2k}\partial_t^2 y)\phi dxdt - \int_0^T\int_{\mathbb{R}^n}\nabla u\nabla\phi\partial_t(\eta_M^{2k}y)dxdt \\
&= -\varepsilon c_0(\gamma)\int_{\mathbb{R}^n}u_0(x)\phi dx - 2m\int_0^T\int_{\mathbb{R}^n}u_t t^{-2m-1}\partial_t\eta_M^{2k}y\phi dxdt \\
&+ \int_0^T\int_{\mathbb{R}^n}u_t t^{-2m}(\partial_t^2\eta_M^{2k}y + 2\partial_t\eta_M^{2k}\partial_t y)\phi dxdt \\
&+ \int_0^T\int_{\mathbb{R}^n}u_t t^{-2m}\eta_M^{2k}(\partial_t^2 y - \frac{2m}{t}\partial_t y - t^{2m}y)\phi dxdt.
\end{aligned} \tag{2.5}$$

Using (2.2), (2.5) and the fact $\partial_t\eta_M(t) < 0$, we conclude

$$\begin{aligned}
&\varepsilon c_1 + \int_0^T\int_{\mathbb{R}^n}|v_t|^p t^{-2m}\eta_M^{2k}|\partial_t y|\phi dxdt \\
&\leq -2m\int_0^T\int_{\mathbb{R}^n}u_t t^{-2m-1}\partial_t\eta_M^{2k}y\phi dxdt + \int_0^T\int_{\mathbb{R}^n}u_t t^{-2m}\partial_t^2\eta_M^{2k}y\phi dxdt \\
&+ \int_0^T\int_{\mathbb{R}^n}2u_t t^{-2m}\partial_t\eta_M^{2k}\partial_t y\phi dxdt \\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{2.6}$$

where $c_1 = c_0(\gamma)\int_{\mathbb{R}^n} u_0(x)\phi dx + c_0(-\gamma)\int_{\mathbb{R}^n} u_1(x)\phi dx > 0$.

Taking $k \geq q' = \frac{q}{q-1}$, we obtain

$$
\begin{aligned}
I_1 &= -2m\int_0^T\int_{\mathbb{R}^n} u_t t^{-2m-1}\partial_t\eta_M^{2k} y\phi dxdt \\
&\lesssim M^{-1}\int_{\frac{M}{2}}^M\int_{\{|x|\leq R+\frac{t^{m+1}}{m+1}\}} |u_t t^{-\frac{2m}{q}}\theta_M^{2k-2}|\partial_t y|^{\frac{1}{q}}\phi^{\frac{1}{q}}||t^{-\frac{2m(q-1)}{q}}|\partial_t y|^{-\frac{1}{q}} y\phi^{\frac{q-1}{q}}|dxdt \\
&\lesssim M^{-1}(\int_0^T\int_{\mathbb{R}^n}|u_t|^q t^{-2m}\theta_M^{2k}|\partial_t y|\phi dxdt)^{\frac{1}{q}}(\int_{\frac{M}{2}}^M\int_{\{|x|\leq R+\frac{t^{m+1}}{m+1}\}} t^{-2m}|\partial_t y|^{-\frac{1}{q-1}}|y|^{\frac{q}{q-1}}\phi dxdt)^{\frac{q-1}{q}} \\
&\lesssim M^{-1}(\int_0^T\int_{\mathbb{R}^n}|u_t|^q t^{-2m}\theta_M^{2k}|\partial_t y|\phi dxdt)^{\frac{1}{q}} \\
&\times(\int_{\frac{M}{2}}^M\int_0^{R+\frac{t^{m+1}}{m+1}} t^{-2m-\frac{3m}{2(q-1)}+\frac{mq}{2(q-1)}}(1+r)^{\frac{n-1}{2}}\exp(r-\frac{t^{m+1}}{m+1})drdt)^{\frac{q-1}{q}} \\
&\lesssim M^{-1}[M^{\frac{(n-1)(m+1)}{2}+1-2m-\frac{3m}{2(q-1)}+\frac{mq}{2(q-1)}}]^{\frac{q-1}{q}}(\int_0^T\int_{\mathbb{R}^n}|u_t|^q t^{-2m}\theta_M^{2k}|\partial_t y|\phi dxdt)^{\frac{1}{q}} \\
&\lesssim M^{-1-\frac{3}{2}m+\frac{m}{2q}+(\frac{(m+1)(n-1)}{2}+1)\frac{q-1}{q}}(\int_0^T\int_{\mathbb{R}^n}|u_t|^q t^{-2m}\theta_M^{2k}|\partial_t y|\phi dxdt)^{\frac{1}{q}},
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
I_2 &= \int_0^T\int_{\mathbb{R}^n} u_t t^{-2m}\partial_t^2\eta_M^{2k} y\phi dxdt \\
&\lesssim M^{-2}\int_{\frac{M}{2}}^M\int_{\{|x|\leq R+\frac{t^{m+1}}{m+1}\}} |u_t t^{-\frac{2m}{q}}\theta_M^{2k-2}|\partial_t y|^{\frac{1}{q}}\phi^{\frac{1}{q}}||t^{-\frac{2m(q-1)}{q}}|\partial_t y|^{-\frac{1}{q}} y\phi^{\frac{q-1}{q}}|dxdt \\
&\lesssim M^{-2}(\int_0^T\int_{\mathbb{R}^n}|u_t|^q t^{-2m}\theta_M^{2k}|\partial_t y|\phi dxdt)^{\frac{1}{q}}(\int_{\frac{M}{2}}^M\int_{\{|x|\leq R+\frac{t^{m+1}}{m+1}\}} t^{-2m}|y|^{\frac{q}{q-1}}|\partial_t y|^{-\frac{1}{q-1}}\phi dxdt)^{\frac{q-1}{q}} \\
&\lesssim M^{-2-\frac{3m}{2}+\frac{m}{2q}+(\frac{(m+1)(n-1)}{2}+1)\frac{q-1}{q}}(\int_0^T\int_{\mathbb{R}^n}|u_t|^q t^{-2m}\theta_M^{2k}|\partial_t y|\phi dxdt)^{\frac{1}{q}},
\end{aligned} \tag{2.8}
$$

$$
\begin{aligned}
I_3 &= \int_0^T\int_{\mathbb{R}^n} 2u_t t^{-2m}\partial_t\eta_M^{2k}\partial_t y\phi dxdt \\
&\lesssim M^{-1}\int_{\frac{M}{2}}^M\int_{\{|x|\leq R+\frac{t^{m+1}}{m+1}\}} |u_t t^{-\frac{2m}{q}}\theta_M^{2k-2}|\partial_t y|^{\frac{1}{q}}\phi^{\frac{1}{q}}||t^{-\frac{2m(q-1)}{q}}|\partial_t y|^{\frac{q-1}{q}}\phi^{\frac{q-1}{q}}|dxdt \\
&\lesssim M^{-1}(\int_0^T\int_{\mathbb{R}^n}|u_t|^q t^{-2m}\theta_M^{2k}|\partial_t y|\phi dxdt)^{\frac{1}{q}}(\int_{\frac{M}{2}}^M\int_{\{|x|\leq R+\frac{t^{m+1}}{m+1}\}} |\partial_t y|\phi t^{-2m}dxdt)^{\frac{q-1}{q}} \\
&\lesssim M^{-1-\frac{m}{2}+\frac{m}{2q}+(\frac{(m+1)(n-1)}{2}+1)\frac{q-1}{q}}(\int_0^T\int_{\mathbb{R}^n}|u_t|^q t^{-2m}\theta_M^{2k}|\partial_t y|\phi dxdt)^{\frac{1}{q}},
\end{aligned} \tag{2.9}
$$

where we have used the estimates in Lemma 2.3 and (2.4).

From (2.6)–(2.9), we arrive at

$$\begin{aligned}&\varepsilon c_1 + \int_0^T \int_{\mathbb{R}^n} |v_t|^p t^{-2m} \eta_M^{2k} |\partial_t y| \phi dx dt \\ &\lesssim M^{-1-\frac{m}{2}+\frac{m}{2q}+(\frac{(m+1)(n-1)}{2}+1)\frac{q-1}{q}} (\int_0^T \int_{\mathbb{R}^n} |u_t|^q t^{-2m} \theta_M^{2k} |\partial_t y| \phi dx dt)^{\frac{1}{q}}.\end{aligned} \tag{2.10}$$

Similarly, making use of the second equality in (2.1) and (2.3) with $k \geq p'$ yields

$$\begin{aligned}&\varepsilon c_2 + \int_0^T \int_{\mathbb{R}^n} |u_t|^q t^{-2m} \eta_M^{2k} |\partial_t y| \phi dx dt \\ &\lesssim M^{-1-\frac{m}{2}+\frac{m}{2p}+(\frac{(m+1)(n-1)}{2}+1)\frac{p-1}{p}} (\int_0^T \int_{\mathbb{R}^n} |v_t|^p t^{-2m} \theta_M^{2k} |\partial_t y| \phi dx dt)^{\frac{1}{p}},\end{aligned} \tag{2.11}$$

where $c_2 = c_0(\gamma) \int_{\mathbb{R}^n} v_0(x)\phi dx + c_0(-\gamma) \int_{\mathbb{R}^n} v_1(x)\phi dx > 0$.

Combining (2.10) and (2.11), we have

$$\begin{aligned}&(\varepsilon c_1 + \int_0^T \int_{\mathbb{R}^n} |v_t|^p t^{-2m} \eta_M^{2k} |\partial_t y| \phi dx dt)^{pq} \\ &\lesssim M^{pq(\frac{(n-1)(m+1)}{2}-\frac{m}{2})-p+\frac{m}{2}-\frac{(n-1)(m+1)}{2}-1} \int_0^T \int_{\mathbb{R}^n} |v_t|^p t^{-2m} \eta_M^{2k} |\partial_t y| \phi dx dt,\end{aligned} \tag{2.12}$$

$$\begin{aligned}&(\varepsilon c_2 + \int_0^T \int_{\mathbb{R}^n} |u_t|^q t^{-2m} \eta_M^{2k} |\partial_t y| \phi dx dt)^{pq} \\ &\lesssim M^{pq(\frac{(n-1)(m+1)}{2}-\frac{m}{2})-q+\frac{m}{2}-\frac{(n-1)(m+1)}{2}-1} \int_0^T \int_{\mathbb{R}^n} |u_t|^q t^{-2m} \eta_M^{2k} |\partial_t y| \phi dx dt.\end{aligned} \tag{2.13}$$

We set

$$Y[w](M) = \int_1^M (\int_0^T \int_{\mathbb{R}^n} w(t,x) \theta_\sigma^{2k}(t) dx dt) \sigma^{-1} d\sigma.$$

It holds that

$$\begin{cases} Y[w](M) \lesssim \int_0^T \int_{\mathbb{R}^n} w(t,x) \theta_M^{2k}(t) dx dt, \\ \dfrac{dY[w](M)}{dM} = M^{-1} \int_0^T \int_{\mathbb{R}^n} w(t,x) \theta_M^{2k}(t) dx dt. \end{cases}$$

Let $w(t,x) = |v_t|^p t^{-2m} |\partial_t y| \phi$. We define $Y(M) = Y[|v_t|^p t^{-2m} |\partial_t y| \phi](M)$. From (2.12), we have

$$(\varepsilon + Y(M))^{pq} \lesssim M^{pq(\frac{(n-1)(m+1)}{2}-\frac{m}{2})-p+\frac{m}{2}-\frac{(n-1)(m+1)}{2}-1} M \frac{dY(M)}{dM}. \tag{2.14}$$

Solving the ordinary differential inequality (2.14), we have

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-F_{GG}^{-1}(n,m,p,q)}, & F_{GG}(n,m,p,q) > 0, \\ \exp(C\varepsilon^{-(pq-1)}), & F_{GG}(n,m,p,q) = 0. \end{cases} \tag{2.15}$$

For $F_{GG}(n,m,p,q) = 0$ and $p = q$, we obtain

$$\partial_t^2(u+v) - t^{2m}\Delta(u+v) = |u_t|^p + |v_t|^p \geq 2^{-p}|\partial_t(u+v)|^p.$$

Therefore, we derive

$$T(\varepsilon) \leq \exp(C\varepsilon^{-(p-1)}). \tag{2.16}$$

On the other hand, we define $Y_1(M) = Y[|u_t|^q t^{-2m}|\partial_t y|\phi](M)$. From (2.13), we deduce

$$(\varepsilon + Y_1(M))^{pq} \lesssim M^{pq(\frac{(n-1)(m+1)}{2} - \frac{m}{2}) - q + \frac{m}{2} - \frac{(n-1)(m+1)}{2} - 1} M \frac{dY_1(M)}{dM}.$$

Direct computation gives rise to

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-F_{GG}^{-1}(n,m,q,p)}, & F_{GG}(n,m,q,p) > 0, \\ \exp(C\varepsilon^{-(pq-1)}), & F_{GG}(n,m,q,p) = 0,\ p \neq q, \\ \exp(C\varepsilon^{-(p-1)}), & F_{GG}(n,m,q,p) = 0,\ p = q. \end{cases} \tag{2.17}$$

Combining (2.15)–(2.17), we conclude the lifespan estimates in (1.7). This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

In this section, we utilize the iterative method to characterize blow-up results of problem (1.1).

3.1. Proof for the sub-critical case

Firstly, for Cauchy problem of the Tricomi equation

$$\begin{cases} u_{tt} - t^{2m}u_{xx} = h(t,x), & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

we have the following integral representation formula.

Lemma 3.1. *(Proposition 2.1 in [29]) Let $n = 1$ and $m > 0$. Suppose that $u_0 \in C_0^2(\mathbb{R})$, $u_1 \in C_0^1(\mathbb{R})$ and $h(t,x) \in C([0,\infty), C^1(\mathbb{R}))$. Then, the solution $u(t,x)$ to problem* (3.1) *can be represented by*

$$\begin{aligned} u(t,x) = {} & a_m \varphi_m^{1-2\gamma}(t) \int_{x-\varphi_m(t)}^{x+\varphi_m(t)} u_0(y)(\varphi_m^2(t) - (y-x)^2)^{\gamma-1} dy \\ & + b_m \int_{x-\varphi_m(t)}^{x+\varphi_m(t)} u_1(y)(\varphi_m^2(t) - (y-x)^2)^{-\gamma} dy \\ & + c_m \int_0^t \int_{x-\varphi_m(t)+\varphi_m(b)}^{x+\varphi_m(t)-\varphi_m(b)} h(b,y) E(t,x;b,y;m) dy db, \end{aligned} \tag{3.2}$$

where

$$\gamma = \frac{m}{2(m+1)}, \quad a_m = 2^{1-2\gamma}\frac{\Gamma(2\gamma)}{\Gamma^2(\gamma)},$$

$$b_m = 2^{2\gamma-1}(m+1)^{1-2\gamma}\frac{\Gamma(2-2\gamma)}{\Gamma^2(1-\gamma)},$$

$$c_m = 2^{2\gamma-1}(m+1)^{-2\gamma}, \quad \varphi_m(t) = \frac{t^{m+1}}{m+1}.$$

The kernel function $E(t,x;b,y;m)$ is defined by

$$E(t,x;b,y;m)=((\varphi_m(t)+\varphi_m(b))^2-(y-x)^2)^{-\gamma}\times F(\gamma,\gamma;1;\frac{(\varphi_m(t)-\varphi_m(b))^2-(y-x)^2}{(\varphi_m(t)+\varphi_m(b))^2-(y-x)^2}),$$

where $F(a,b;c;z)$ stands for the Gauss hypergeometric function $F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$ with the Pochhammer symbol $(d)_0=1$ and $(d)_n=\prod_{k=1}^n(d+k-1)$ for $n\in\mathbb{N}$.

Let (u,v) be a pair of solutions to problem (1.1). We set the space variable $x=(z,w)$, where $z\in\mathbb{R}$ and $w\in\mathbb{R}^{n-1}$. We define $U(t,z)=\int_{\mathbb{R}^{n-1}}u(t,z,w)dw$, $V(t,z)=\int_{\mathbb{R}^{n-1}}v(t,z,w)dw$. Then, $(U(t,z),V(t,z))$ satisfy

$$\begin{cases}U_{tt}-t^{2m}\Delta U=\int_{\mathbb{R}^{n-1}}|v_t(t,z,w)|^p dw, & t>0,\ z\in\mathbb{R},\\ V_{tt}-t^{2m}\Delta V=\int_{\mathbb{R}^{n-1}}|u_t(t,z,w)|^q dw, & t>0,\ z\in\mathbb{R},\\ U(0,z)=\varepsilon U_0(z),\ U_t(0,z)=\varepsilon U_1(z), & z\in\mathbb{R},\\ V(0,z)=\varepsilon V_0(z),\ V_t(0,z)=\varepsilon V_1(z), & z\in\mathbb{R},\end{cases}\tag{3.3}$$

where

$$\begin{aligned}&U_0(z)=\int_{\mathbb{R}^{n-1}}u_0(z,w)dw,\quad U_1(z)=\int_{\mathbb{R}^{n-1}}u_1(z,w)dw,\\ &V_0(z)=\int_{\mathbb{R}^{n-1}}v_0(z,w)dw,\quad V_1(z)=\int_{\mathbb{R}^{n-1}}v_1(z,w)dw,\\ &\operatorname{supp}(U_0(z),U_1(z),V_0(z),V_1(z))\subset(-R,R),\\ &\operatorname{supp}(U(t,\cdot),V(t,\cdot))\subset(-(R+\varphi_m(t)),R+\varphi_m(t)).\end{aligned}$$

From (3.2), we deduce that $U(t,z)$ and $V(t,z)$ can be represented as

$$\begin{aligned}U(t,z)=&\,a_m\varepsilon\varphi_m^{1-2\gamma}(t)\int_{z-\varphi_m(t)}^{z+\varphi_m(t)}U_0(y)(\varphi_m^2(t)-(y-z)^2)^{\gamma-1}dy\\ &+b_m\varepsilon\int_{z-\varphi_m(t)}^{z+\varphi_m(t)}U_1(y)(\varphi_m^2(t)-(y-z)^2)^{-\gamma}dy\\ &+c_m\int_0^t\int_{z-\varphi_m(t)+\varphi_m(b)}^{z+\varphi_m(t)-\varphi_m(b)}\int_{\mathbb{R}^{n-1}}|v_t(b,y,w)|^p dwE(t,z;b,y;m)dydb,\end{aligned}\tag{3.4}$$

$$\begin{aligned}V(t,z)=&\,a_m\varepsilon\varphi_m^{1-2\gamma}(t)\int_{z-\varphi_m(t)}^{z+\varphi_m(t)}V_0(y)(\varphi_m^2(t)-(y-z)^2)^{\gamma-1}dy\\ &+b_m\varepsilon\int_{z-\varphi_m(t)}^{z+\varphi_m(t)}V_1(y)(\varphi_m^2(t)-(y-z)^2)^{-\gamma}dy\\ &+c_m\int_0^t\int_{z-\varphi_m(t)+\varphi_m(b)}^{z+\varphi_m(t)-\varphi_m(b)}\int_{\mathbb{R}^{n-1}}|u_t(b,y,w)|^q dwE(t,z;b,y;m)dydb.\end{aligned}\tag{3.5}$$

Here, we only present the proof of lower bound estimate for $U(t,z)$. The lower bound estimate for $V(t,z)$ can be obtained in an analogous way.

From the fact $\varphi_m(t) - y + z \leq 2\varphi_m(t)$ for $y \in [z - \varphi_m(t), z + \varphi_m(t)]$, we have

$$\begin{aligned} U(t,z) &\geq 2^{\gamma-1} a_m \varepsilon \varphi_m^{-\gamma}(t) \int_{z-\varphi_m(t)}^{z+\varphi_m(t)} U_0(y)(\varphi_m(t) - z + y)^{\gamma-1} dy \\ &+ 2^{-\gamma} b_m \varepsilon \varphi_m^{-\gamma}(t) \int_{z-\varphi_m(t)}^{z+\varphi_m(t)} U_1(y)(\varphi_m(t) - z + y)^{-\gamma} dy \\ &+ c_m \int_0^t \int_{z-\varphi_m(t)+\varphi_m(b)}^{z+\varphi_m(t)-\varphi_m(b)} \int_{\mathbb{R}^{n-1}} |\partial_t v(b,y,w)|^p dw E(t,z;b,y;m) dy db \\ &= \varepsilon I_4 + I_5. \end{aligned} \tag{3.6}$$

We derive lower bound estimates for I_4 and I_5 on the characteristic line $\varphi_m(t) - z = R$ for $z \geq R$. We note $[-R, R] \subset [z - \varphi_m(t), z + \varphi_m(t)]$. It holds that

$$\begin{aligned} I_4(t,z) &= \varphi_m^{-\gamma}(t) \int_{z-\varphi_m(t)}^{z+\varphi_m(t)} 2^{\gamma-1} a_m U_0(y)(\varphi_m(t) - z + y)^{\gamma-1} + 2^{-\gamma} b_m U_1(y)(\varphi_m(t) - z + y)^{-\gamma} dy \\ &\geq \varphi_m^{-\gamma}(t) \int_{-R}^{R} 2^{2(\gamma-1)} a_m U_0(y) R^{\gamma-1} + 2^{-2\gamma} b_m U_1(y) R^{-\gamma} dy \\ &\geq K(z+R)^{-\gamma} \|u_0 + u_1\|_{L^1(\mathbb{R}^n)}, \end{aligned} \tag{3.7}$$

where $K = \min\{2^{2(\gamma-1)} a_m R^{\gamma-1}, 2^{-2\gamma} b_m R^{-\gamma}\}$.

Using the Hölder inequality, we achieve

$$\begin{aligned} |\partial_t V(b,y)| &= |\int_{\mathbb{R}^{n-1}} \partial_t v(b,y,w) dw| \\ &\leq (\int_{\mathbb{R}^{n-1}} |\partial_t v(b,y,w)|^p dw)^{\frac{1}{p}} (\int_{|w| \leq ((R+\varphi_m(b))^2 - y^2)^{\frac{1}{2}}} 1 dw)^{\frac{p-1}{p}} \\ &\leq ((R + \varphi_m(b))^2 - y^2)^{\frac{n-1}{2} \cdot \frac{p-1}{p}} (\int_{\mathbb{R}^{n-1}} |\partial_t v(b,y,w)|^p dw)^{\frac{1}{p}}, \end{aligned}$$

which implies

$$\int_{\mathbb{R}^{n-1}} |\partial_t v(b,y,w)|^p dw \geq |\partial_t V(b,y)|^p ((R + \varphi_m(b))^2 - y^2)^{-\frac{n-1}{2}(p-1)}.$$

It follows that

$$\begin{aligned} I_5 &= c_m \int_0^t \int_{z-\varphi_m(t)+\varphi_m(b)}^{z+\varphi_m(t)-\varphi_m(b)} \int_{\mathbb{R}^{n-1}} |\partial_t v(b,y,w)|^p dw E(t,z;b,y;m) dy db \\ &\geq c_m \Big(\int_{z-\varphi_m(t)}^{z} \int_0^{\varphi_m^{-1}(\varphi_m(t)-(z-y))} ((R + \varphi_m(b))^2 - y^2)^{-\frac{n-1}{2}(p-1)} |\partial_t V(b,y)|^p E(t,z;b,y;m) db dy \\ &+ \int_z^{z+\varphi_m(t)} \int_0^{\varphi_m^{-1}(\varphi_m(t)-(y-z))} ((R + \varphi_m(b))^2 - y^2)^{-\frac{n-1}{2}(p-1)} |\partial_t V(b,y)|^p E(t,z;b,y;m) db dy \Big) \\ &\geq c_m \int_{z-\varphi_m(t)}^{z} \int_0^{\varphi_m^{-1}(\varphi_m(t)-(z-y))} ((R + \varphi_m(b))^2 - y^2)^{-\frac{n-1}{2}(p-1)} |\partial_t V(b,y)|^p E(t,z;b,y;m) db dy. \end{aligned}$$

By shrinking the domain of integration, we obtain

$$I_5 \geq c_m \int_R^z \int_{\varphi_m^{-1}(y-R)}^{\varphi_m^{-1}(y+R)} ((R+\varphi_m(b))^2 - y^2)^{-\frac{n-1}{2}(p-1)} |\partial_t V(b,y)|^p E(t,z;b,y;m) db dy.$$

Since

$$(R+\varphi_m(b))^2 - y^2 = (R+\varphi_m(b)+y)(R+\varphi_m(b)-y) \leq 4R(y+R),$$

we have

$$\begin{aligned} I_5 &\geq c_m \int_R^z \int_{\varphi_m^{-1}(y-R)}^{\varphi_m^{-1}(y+R)} (4R(y+R))^{-\frac{n-1}{2}(p-1)} |\partial_t V(b,y)|^p E(t,z;b,y;m) db dy \\ &= c_m (4R)^{-\frac{n-1}{2}(p-1)} \int_R^z (y+R)^{-\frac{n-1}{2}(p-1)} \int_{\varphi_m^{-1}(y-R)}^{\varphi_m^{-1}(y+R)} |\partial_t V(b,y)|^p E(t,z;b,y;m) db dy. \end{aligned}$$

Employing the fact $F(a,a;c;z) \geq 1$ for $a \in \mathbb{R}$, $c > 0$ and $z \in [0,1)$ (more details can be found in [34]), we derive

$$E(t,z;b,y;m) \geq [(z+R+\varphi_m(b))^2 - (z-y)^2]^{-\gamma} \geq 4^{-\gamma}(z+R)^{-\gamma}(y+R)^{-\gamma}.$$

Therefore, we have

$$I_5 \geq 4^{-\gamma} c_m (z+R)^{-\gamma} (4R)^{-\frac{n-1}{2}(p-1)} \times \int_R^z (y+R)^{-\frac{n-1}{2}(p-1)-\gamma} \int_{\varphi_m^{-1}(y-R)}^{\varphi_m^{-1}(y+R)} |\partial_t V(b,y)|^p db dy.$$

It follows that

$$\begin{aligned} &\left| \int_{\varphi_m^{-1}(y-R)}^{\varphi_m^{-1}(y+R)} \partial_t V(b,y) db \right|^p \\ &\leq (\varphi_m^{-1}(y+R) - \varphi_m^{-1}(y-R))^{p-1} \int_{\varphi_m^{-1}(y-R)}^{\varphi_m^{-1}(y+R)} |\partial_t V(b,y)|^p db \\ &\leq (2R(m+1))^{\frac{p-1}{m+1}} \int_{\varphi_m^{-1}(y-R)}^{\varphi_m^{-1}(y+R)} |\partial_t V(b,y)|^p db. \end{aligned}$$

Making use of $V(\varphi_m^{-1}(y-R), y) = 0$ and the Jensen inequality, we deduce

$$\begin{aligned} I_5 &\geq 4^{-\gamma} c_m (z+R)^{-\gamma} (4R)^{-\frac{n-1}{2}(p-1)} \int_R^z (y+R)^{-\frac{n-1}{2}(p-1)-\gamma} \int_{\varphi_m^{-1}(y-R)}^{\varphi_m^{-1}(y+R)} |\partial_t V(b,y)|^p db dy \\ &\geq C_1 (z+R)^{-\gamma} \int_R^z (y+R)^{-\frac{n-1}{2}(p-1)-\gamma} |V(\varphi_m^{-1}(y+R), y)|^p dy, \end{aligned} \tag{3.8}$$

where $C_1 = 4^{-\gamma} c_m (4R)^{-\frac{n-1}{2}(p-1)} (2R(m+1))^{-\frac{p-1}{m+1}}$.

Combining (3.6)–(3.8), we have

$$(z+R)^\gamma U(t,z) \geq \varepsilon K \|u_0 + u_1\|_{L^1(\mathbb{R}^n)} + C_1 \int_R^z (y+R)^{-\frac{n-1}{2}(p-1)-\gamma} |V(\varphi_m^{-1}(y+R), y)|^p dy.$$

Setting $\widetilde{U}(z) = (z+R)^\gamma U(t,z)$ and $\widetilde{V}(z) = (z+R)^\gamma V(t,z)$, we deduce

$$\begin{aligned}\widetilde{U}(z) &\geq \varepsilon K\|u_0+u_1\|_{L^1(\mathbb{R}^n)} + C_1\int_R^z (y+R)^{-\frac{n-1}{2}(p-1)-\gamma(p+1)}|\widetilde{V}(y)|^p dy\\ &= \varepsilon M + C_1\int_R^z (y+R)^{-\frac{n-1}{2}(p-1)-\gamma(p+1)}|\widetilde{V}(y)|^p dy,\end{aligned} \tag{3.9}$$

where $M = K\|u_0+u_1\|_{L^1(\mathbb{R}^n)}$. Similarly, we obtain

$$\widetilde{V}(z) \geq \varepsilon N + C_2\int_R^z (y+R)^{-\frac{n-1}{2}(q-1)-\gamma(q+1)}|\widetilde{U}(y)|^q dy, \tag{3.10}$$

where $N = K\|v_0+v_1\|_{L^1(\mathbb{R}^n)}$ and $C_2 = 4^{-\gamma}c_m(4R)^{-\frac{n-1}{2}(q-1)}(2R(m+1))^{-\frac{q-1}{m+1}}$.

We are in the position to apply the iteration argument. In the sub-critical case, we assume

$$\widetilde{U}(z) \geq \theta_j(R+z)^{-\alpha_j}(z-R)^{\beta_j}, z \geq R, \tag{3.11}$$

where $\{\alpha_j\}_{j\in\mathbb{N}}$, $\{\beta_j\}_{j\in\mathbb{N}}$ and $\{\theta_j\}_{j\in\mathbb{N}}$ are sequences of non-negative real numbers. We set $\alpha_0 = 0, \beta_0 = 0$, $\theta_0 = M\varepsilon$. From (3.9), we deduce that (3.11) holds with $j = 0$. Plugging (3.11) into (3.10) yields

$$\begin{aligned}\widetilde{V}(z) &\geq C_2\int_R^z (R+y)^{-\frac{n-1}{2}(q-1)-\gamma(q+1)-\alpha_j q}\theta_j^q(y-R)^{\beta_j q}dy\\ &\geq \frac{C_2\theta_j^q}{\beta_j q+1}(R+z)^{-\frac{n-1}{2}(q-1)-\gamma(q+1)-\alpha_j q}(z-R)^{\beta_j q+1}.\end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.9), we have

$$\begin{aligned}\widetilde{U}(z) &\geq C_1\int_R^z (R+y)^{-\frac{n-1}{2}(p-1)-\gamma(p+1)}|\widetilde{V}(y)|^p dy\\ &\geq C_1(\frac{C_2\theta_j^q}{\beta_j q+1})^p\int_R^z (R+y)^{-\frac{n-1}{2}(pq-1)-\gamma(pq+2p+1)-pq\alpha_j}(y-R)^{\beta_j pq+p}dy\\ &\geq C_1(\frac{C_2\theta_j^q}{\beta_j q+1})^p\frac{1}{\beta_j pq+p+1}(R+z)^{-\frac{n-1}{2}(pq-1)-\gamma(pq+2p+1)-pq\alpha_j}(z-R)^{\beta_j pq+p+1}.\end{aligned} \tag{3.13}$$

Let

$$\begin{aligned}\alpha_{j+1} &= \frac{n-1}{2}(pq-1)+\gamma(pq+2p+1)+pq\alpha_j,\\ \beta_{j+1} &= \beta_j pq+p+1,\\ \theta_{j+1} &= C_1(\frac{C_2\theta_j^q}{\beta_j q+1})^p\frac{1}{\beta_j pq+p+1}.\end{aligned} \tag{3.14}$$

It is deduced from (3.14) that

$$\alpha_j = A\frac{(pq)^j-1}{pq-1}, \quad \beta_j = B\frac{(pq)^j-1}{pq-1}, \tag{3.15}$$

where $A = \frac{n-1}{2}(pq-1) + \gamma(pq+2p+1)$, $B = p+1$. From $\beta_j \le B\frac{(pq)^j}{pq-1}$, we deduce

$$\theta_j = C_1\Big(\frac{C_2\theta_{j-1}^q}{\beta_{j-1}q+1}\Big)^p \frac{1}{\beta_{j-1}pq+p+1} \ge \widetilde{\theta}(pq)^{-(p+1)j}\theta_{j-1}^{pq}, \tag{3.16}$$

where $\widetilde{\theta} = C_1(C_2)^p(\frac{B}{pq-1})^{-(p+1)}$. From (3.16), we have

$$\begin{aligned}\log\theta_j &\ge pq\log\theta_{j-1} - j\log(pq)^{p+1} + \log\widetilde{\theta}\\ &\ge (pq)^j(\log\theta_0 - \frac{pq}{(pq-1)^2}\log(pq)^{p+1} + \frac{\log\widetilde{\theta}}{pq-1})\\ &\quad + (j+1)\frac{\log(pq)^{p+1}}{pq-1} + \frac{\log(pq)^{p+1}}{(pq-1)^2} - \frac{\log\widetilde{\theta}}{pq-1}.\end{aligned}$$

Choosing $j_0 = \max\{0, \frac{\log\widetilde{\theta}}{\log(pq)^{p+1}} - \frac{pq}{pq-1}\}$, we derive

$$\log\theta_j \ge (pq)^j\log(\hat{\theta}\varepsilon) \tag{3.17}$$

for $j > j_0$, where $\hat{\theta} = M(pq)^{-\frac{pq(p+1)}{(pq-1)^2}}\widetilde{\theta}^{\frac{1}{pq-1}}$. Combining (3.11), (3.15) with (3.17), we have

$$\begin{aligned}\widetilde{U}(z) &\ge \theta_j(R+z)^{-\alpha_j}(z-R)^{\beta_j}\\ &\ge \exp\big((pq)^j[\log(\hat{\theta}\varepsilon) - \frac{A}{pq-1}\log(R+z) + \frac{B}{pq-1}\log(z-R)]\big)\\ &\quad\times(z+R)^{\frac{A}{pq-1}}(z-R)^{-\frac{B}{pq-1}}.\end{aligned} \tag{3.18}$$

We deduce $2(z-R) \ge R+z$ when $z \ge 3R$. Therefore, we obtain

$$\begin{aligned}\widetilde{U}(z) &\ge \exp\big((pq)^j[\log(\hat{\theta}\varepsilon) + \log(R+z)^{-\frac{A}{pq-1}} + \log(\frac{R+z}{2})^{\frac{B}{pq-1}}]\big)\\ &\quad\times(z+R)^{\frac{A}{pq-1}}(z-R)^{-\frac{B}{pq-1}}\\ &= \exp((pq)^j\log[\hat{\theta}2^{-\frac{B}{pq-1}}\varepsilon(R+z)^{\frac{B-A}{pq-1}}])(z+R)^{\frac{A}{pq-1}}(z-R)^{-\frac{B}{pq-1}}.\end{aligned} \tag{3.19}$$

We choose $\varepsilon_1 = \varepsilon_1(n,m,p,q,R,u_0,u_1,v_0,v_1)$ such that $(\hat{\theta}2^{-\frac{B}{pq-1}}\varepsilon_1)^{-\Lambda_{GG}^{-1}} \ge 4R$. For all $\varepsilon \in (0,\varepsilon_1]$ and $\phi_m(t) > (\hat{\theta}2^{-\frac{B}{pq-1}}\varepsilon)^{-\Lambda_{GG}^{-1}}$, we have

$$\phi_m(t) \ge 4R, \quad \hat{\theta}2^{-\frac{B}{pq-1}}\varepsilon\phi_m^{\Lambda_{GG}}(t) > 1.$$

Sending $j \to \infty$ in (3.19), we deduce that $\widetilde{U}(z)$ blows up in finite time. Therefore, we conclude

$$T(\varepsilon) \le C\varepsilon^{-\frac{1}{(m+1)\Lambda_{GG}(n,m,p,q)}}.$$

Similarly, for $\Lambda_{GG}(n,m,q,p) > 0$, we have

$$T(\varepsilon) \le C\varepsilon^{-\frac{1}{(m+1)\Lambda_{GG}(n,m,q,p)}}.$$

In summary, we obtain the upper bound lifespan estimate

$$T(\varepsilon) \le C\varepsilon^{-\widetilde{\Lambda}_{GG}^{-1}(n,m,p,q)}$$

for $\widetilde{\Lambda}_{GG}(n,m,p,q) > 0$.

3.2. Proof for the critical case

Let $\max\{\widetilde{\Lambda}_{GG}(n,m,p,q), \widetilde{\Lambda}_{GG}(n,m,q,p)\} = 0$. Without loss of generality, we set $\widetilde{\Lambda}_{GG}(n,m,p,q) = 0 > \widetilde{\Lambda}_{GG}(n,m,q,p)$. We assume

$$\widetilde{U}(z) \geq D_j(\log(\frac{z}{l_jR}))^{E_j}, \quad \text{for } z \geq l_jR, \tag{3.20}$$

where $l_j = 2 - 2^{-(j+1)}$, $\{D_j\}_{j\in\mathbb{N}}$ and $\{E_j\}_{j\in\mathbb{N}}$ are suitable sequences of non-negative real numbers. From (3.9) and (3.20), we deduce $D_0 = M\varepsilon$ and $E_0 = 0$ when $j = 0$. For $l_j \geq l_0 = \frac{3}{2}$, we have $z \geq \frac{3}{5}(R+z)$. According to (3.10), we derive

$$\begin{aligned}\widetilde{V}(z) &\geq C_2\int_R^z (y+R)^{-\frac{n-1}{2}(q-1)-\gamma(q+1)}|\widetilde{U}(y)|^q dy\\ &\geq C_2D_j^q\int_{l_jR}^z (y+R)^{-\frac{n-1}{2}(q-1)-\gamma(q+1)}(\log(\frac{y}{l_jR}))^{qE_j}dy\\ &\geq \frac{3}{5}C_2D_j^q(z+R)^{-\frac{n-1}{2}(q-1)-\gamma(q+1)+1}(\log(\frac{z}{l_{j+1}R}))^{qE_j}(1-\frac{l_j}{l_{j+1}}).\end{aligned} \tag{3.21}$$

Plugging (3.21) into (3.9), we conclude

$$\begin{aligned}\widetilde{U}(z) &\geq C_1\int_R^z (y+R)^{-\frac{n-1}{2}(p-1)-\gamma(p+1)}|\widetilde{V}(y)|^p dy\\ &\geq C_1(\frac{3}{5}C_2)^pD_j^{pq}(1-\frac{l_j}{l_{j+1}})^p\\ &\times\int_R^z (y+R)^{-\frac{n-1}{2}(pq-1)-\gamma(pq+2p+1)+p}(\log(\frac{y}{l_{j+1}R}))^{pqE_j}dy\\ &\geq C_1(\frac{3}{5})^{p+1}(C_2)^pD_j^{pq}(1-\frac{l_j}{l_{j+1}})^p(pqE_j+1)^{-1}(\log(\frac{z}{l_{j+1}R}))^{pqE_j+1},\end{aligned}$$

where we have used the fact $\widetilde{\Lambda}_{GG}(n,m,p,q) = 0$ and $(R+y)^{-1} \geq \frac{3}{5}y^{-1}$. When $1-\frac{l_j}{l_{j+1}} \geq 2^{-(j+3)}$, we obtain

$$\widetilde{U}(z) \geq C_1(\frac{3}{5})^{p+1}(C_2)^p2^{-p(j+3)}D_j^{pq}(pqE_j+1)^{-1}(\log(\frac{z}{l_{j+1}R}))^{pqE_j+1}. \tag{3.22}$$

Let

$$\begin{aligned}E_{j+1} &= pqE_j + 1,\\ D_{j+1} &= C_1(\frac{3}{5})^{p+1}(C_2)^p2^{-p(j+3)}D_j^{pq}(pqE_j+1)^{-1}.\end{aligned}$$

Direct computation implies

$$E_j = \frac{(pq)^j-1}{pq-1}. \tag{3.23}$$

Taking advantage of $E_j \leq \frac{(pq)^j}{pq-1}$, we deduce

$$D_j = C_1(\frac{3}{5})^{p+1}(C_2)^p2^{-p(j+2)}D_{j-1}^{pq}(pqE_{j-1}+1)^{-1} \geq \widetilde{D}2^{-pj}D_{j-1}^{pq}(pq)^{-j}, \tag{3.24}$$

where $\widetilde{D} = (pq-1)C_1(\frac{3}{5})^{p+1}(C_2)^p 2^{-2p}$. From (3.24), we derive

$$\begin{aligned}\log D_j &\geq \log \widetilde{D} - j\log(2^p pq) + pq\log D_{j-1}\\ &\geq (pq)^j(\log D_0 - \frac{pq}{(pq-1)^2}\log(2^p pq) + \frac{\log \widetilde{D}}{pq-1})\\ &\quad + \frac{j+1}{pq-1}\log(2^p pq) + \frac{\log(2^p pq)}{(pq-1)^2} - \frac{\log \widetilde{D}}{pq-1}.\end{aligned}$$

Let $j_1 = \max\{0, \frac{\log \widetilde{D}}{\log(2^p pq)} - \frac{pq}{pq-1}\}$. For $j > j_1$, we acquire

$$\log D_j \geq (pq)^j(\log D_0 - \frac{pq}{(pq-1)^2}\log(2^p pq) + \frac{\log \widetilde{D}}{pq-1}) = (pq)^j \log(\hat{D}\varepsilon), \tag{3.25}$$

where $\hat{D} = M(2^p pq)^{-\frac{2}{(pq-1)^2}}\widetilde{D}^{\frac{1}{pq-1}}$.

Applying (3.20), (3.23) and (3.25), for $j > j_1, z \geq l_j R$, we deduce

$$\begin{aligned}\widetilde{U}(z) &\geq \exp((pq)^j\log(\hat{D}\varepsilon))(\log(\frac{z}{2R}))^{\frac{(pq)^j-1}{pq-1}}\\ &= \exp\Big((pq)^j\log\big(\hat{D}\varepsilon\log(\frac{z}{2R})^{\frac{1}{pq-1}}\big)\Big)(\log\frac{z}{2R})^{-\frac{1}{pq-1}}.\end{aligned}$$

We choose $\varepsilon_2 = \varepsilon_2(n, m, p, q, R, u_0, u_1, v_0, v_1)$ such that $\exp((\hat{D}\varepsilon_2)^{-(pq-1)}) \geq 1$. For all $\varepsilon \in (0, \varepsilon_2]$, $z > 2R\exp((\hat{D}\varepsilon)^{-(pq-1)})$, we have $\hat{D}\varepsilon(\log\frac{z}{2R})^{\frac{1}{pq-1}} > 1$. Sending $j \to \infty$, we have $\widetilde{U}(z) \to \infty$. Therefore, we arrive at

$$z \leq \exp(C\varepsilon^{-(pq-1)}).$$

For $\phi_m(t) = z + R < 2z$, we obtain the lifespan estimate

$$T(\varepsilon) \leq \exp(C\varepsilon^{-(pq-1)}).$$

Similarly, we acquire

$$T(\varepsilon) \leq \exp(C\varepsilon^{-(pq-1)})$$

when $\widetilde{\Lambda}_{GG}(n, m, q, p) = 0 > \widetilde{\Lambda}_{GG}(n, m, p, q)$.

For the case $\widetilde{\Lambda}_{GG}(n, m, p, q) = \widetilde{\Lambda}_{GG}(n, m, q, p) = 0$ when $p = q$, following a similar deduction, we prove

$$\widetilde{U}(z) \geq F_j(\log\frac{z}{R})^{H_j}, z \geq R. \tag{3.26}$$

Employing the fact $-\frac{n-1}{2}(p-1) - \gamma(p+1) = -\frac{n-1}{2}(q-1) - \gamma(q+1) = -1$ when $\widetilde{\Lambda}_{GG}(n, m, p, q) = \widetilde{\Lambda}_{GG}(n, m, q, p) = 0$, we deduce

$$H_j = H_{j-1}p^2 + p + 1 = \frac{p^{2j}-1}{p-1},$$

$$\begin{aligned}F_j &= 2^{-(p+1)}C_1C_2^p(H_{j-1}p+1)^{-p}(H_{j-1}p^2+p+1)^{-1}\\ &\geq 2^{-(p+1)}C_1C_2^pH_j^{-(p+1)}F_{j-1}^{p^2} \geq \widetilde{F}p^{-2(p+1)j}F_{j-1}^{p^2},\end{aligned}$$

where $\widetilde{F} = 2^{-(p+1)}C_1C_2^p(p-1)^{p+1}$.

Direct calculation shows

$$\begin{aligned}\log F_j &\geq \log \widetilde{F} - 2(p+1)j\log p + p^2\log F_{j-1}\\ &\geq p^{2j}(\log F_0 - \frac{2p^2(p+1)}{(p^2-1)^2}\log p + \frac{\log \widetilde{F}}{p^2-1})\\ &\quad + \frac{2(p+1)\log p}{p^2-1}(j+1) - \frac{1}{p^2-1}\log \widetilde{F} + \frac{2(p+1)\log p}{(p^2-1)^2}.\end{aligned}$$

Let $j_2 = \max\{0, \frac{\log \widetilde{F}}{2(p+1)\log p} - \frac{p^2}{p^2-1}\}$. For $j > j_2$, we obtain

$$\log F_j \geq p^{2j}(\log F_0 - \frac{2p^2(p+1)}{(p^2-1)^2}\log p + \frac{\log \widetilde{F}}{p^2-1}) = p^{2j}\log(\hat{F}\varepsilon),$$

where $\hat{F} = Mp^{-\frac{2p^2(p+1)}{(p^2-1)^2}}\widetilde{F}^{\frac{1}{p^2-1}}$. Therefore, we acquire

$$\widetilde{U}(z) \geq F_j(\log\frac{z}{R})^{H_j} \geq \exp[p^{2j}\log(\hat{F}\varepsilon(\log\frac{z}{R})^{\frac{1}{p-1}})](\log\frac{z}{R})^{\frac{1}{p-1}}.$$

We choose small positive constant $\varepsilon_3 = \varepsilon_3(n, m, p, q, R, u_0, u_1, v_0, v_1)$ such that

$$\exp[(\hat{F}\varepsilon_3)^{-(p-1)}] \geq 1.$$

Then, for all $\varepsilon \in (0, \varepsilon_3]$, $z > R\exp[(\hat{F}\varepsilon)^{-(p-1)}]$, we have

$$z \geq R, \quad \hat{F}\varepsilon(\log\frac{z}{R})^{\frac{1}{p-1}} > 1.$$

Therefore, $\widetilde{U}(z)$ blows up in finite time when $j \to \infty$. It follows that

$$z \leq R\exp[(\hat{F}\varepsilon)^{-(p-1)}].$$

Bearing in mind $\phi_m(t) = z + R \leq 2z$, we have

$$T(\varepsilon) \leq \exp(C\varepsilon^{-(p-1)}). \tag{3.27}$$

Analogously, we have the same lifespan estimate for $\widetilde{V}(z)$ in (3.27).

Choosing $\varepsilon_4 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, we conclude the lifespan estimates in (1.8). This completes the proof of Theorem 1.2.

4. Conclusions

In this paper, blow-up results of solutions to coupled system of the Tricomi equations with derivative type nonlinearities are studied. Upper bound lifespan estimates of solutions to the Cauchy problem with small initial values are derived. We illustrate the key results by using the test function method (see Theorem 1.1) and integral representation formula together with iteration argument (see Theorem 1.2), respectively. Our main new contribution is that lifespan estimates of solutions to the problem in the sub-critical and critical cases are connected with the Glassey conjecture. To the best of our knowledge, the results in Theorems 1.1 and 1.2 are new. In addition, we present a comparison for lifespan estimates in Theorems 1.1 and 1.2 in a special case (see Remark 1.1).

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Conflict of interest

This work does not have any conflict of interest.

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