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Research article

Monotonicity, convexity and inequalities involving zero-balanced Gaussian hypergeometric function

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Abstract: We generalize several monotonicity and convexity properties as well as sharp inequalities for the complete elliptic integrals to the zero-balanced Gaussian hypergeometric function $F(a, b; a + b; x)$.

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1. Introduction

For $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined by [1, 3, 4, 25]

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \quad x \in (-1, 1), \tag{1.1}$$

where (a, n) denotes the shifted factorial function $(a, n) \equiv a(a + 1) \cdots (a + n - 1)$ for $n \in \mathbb{N}$, and $(a, 0) = 1$ for $a \neq 0$. It is well known that $F(a, b; c; x)$ is widely applied in geometric function theory, theory of mean values as well as in many other fields of mathematics and some other disciplines. Many elementary functions and special functions in mathematical physics are particular or limiting cases of the Gaussian hypergeometric function. $F(a, b; c; x)$ is said to be zero-balanced if $c = a + b$. For the known properties of $F(a, b; c; x)$, the readers are referred to [1, 3, 4, 7, 8, 13, 15, 18, 20, 21, 23, 26].

As the special cases of the Gaussian hypergeometric function, for $r \in (0, 1), a \in (0, 1)$, the generalized elliptic integrals of the first and the second kinds are defined by [5, 6, 24].

$$\mathscr{K}_a(r) = \frac{\pi}{2} F\left(a, 1 - a; 1; r^2\right), \tag{1.2}$$

$$\mathscr{E}_a(r) = \frac{\pi}{2} F\left(a-1, 1-a; 1; r^2\right), \tag{1.3}$$
$$\mathscr{K}_a(0) = \frac{\pi}{2},\ \mathscr{K}_a(1^-) = \infty,\ \ \mathscr{E}_a(0) = \frac{\pi}{2},\ \mathscr{E}_a(1^-) = \frac{\sin(\pi a)}{2(1-a)}.$$

Set $\mathscr{K}_a'(r) = \mathscr{K}_a(r')$, $\mathscr{E}_a'(r) = \mathscr{E}_a(r')$. Here and hereafter we always let $r' = \sqrt{1-r^2}$ for $r \in [0, 1]$. Note that, when $a = 1/2$, the functions $\mathscr{K}_a(r)$ and $\mathscr{E}_a(r)$ reduce to the classical complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ of the first and second kind [3,4].

$$\mathscr{K}(r) = \frac{\pi}{2} F\left(1/2, 1/2; 1; r^2\right) \ \text{ for } \ r \in (0,1),\ \mathscr{K}(0) = \frac{\pi}{2},\ \mathscr{K}(1^-) = \infty, \tag{1.4}$$

$$\mathscr{E}(r) = \frac{\pi}{2} F\left(-1/2, 1/2; 1; r^2\right) \ \text{ for } \ r \in (0,1),\ \mathscr{E}(0) = \frac{\pi}{2},\ \mathscr{E}(1^-) = 1. \tag{1.5}$$

Set $\mathscr{K}'(r) = \mathscr{K}(r')$, $\mathscr{E}'(r) = \mathscr{E}(r')$.

Let $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.577156649\cdots$ be the Euler-Mascheroni constant. For $a > 0, b > 0$, the Gamma, Psi and Beta functions are defined respectively by [1,3,4]

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt, \quad \psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}, \quad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

The so-called Ramanujan R-function $R(a)$ is defined by [11,13]

$$R(a) = -2\gamma - \psi(a) - \psi(1-a),\ a \in (0,1), \tag{1.6}$$

which is the special case of the following function of two parameters a and b.

$$R(a,b) = -2\gamma - \psi(a) - \psi(b),\ \text{for } a, b \in (0, +\infty),$$

which is sometimes called the Ramanujan constant although it is a function of a and b ([11]).

For $a \in (0, 1)$, we set [5,12]

$$B(a) = B(a, 1-a) = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}. \tag{1.7}$$

By the symmetry, we can sometimes assume that $a \in (0, 1/2]$ in (1.6) and (1.7).

In the past few years, the complete elliptic integrals [5], the generalized elliptic integrals [11,12] and the p-elliptic integrals [14,27] have been studied by many authors. The recent interest is motivated by applications to geometric function theory.

In 2000, Anderson, Qiu, Vamanamurthy, and Vuorinen obtained some results in [6] as follows.

Theorem 1.1 (See [6]). *Let $a \in (0, 1/2]$ be given, and let $b = 1-a$, $c = (\sin(\pi a))/b$. Then the functions*

(1)

$$\hat{f}_1(r) \equiv r'^2 \mathscr{K}_a(r)/\mathscr{E}_a(r)$$

is decreasing from $(0, 1)$ *onto* $(0, 1)$.

(2)

$$\hat{f}_2(r) \equiv [(\pi/2)^2 - (r'\mathscr{K}_a(r))^2]/[\mathscr{E}_a(r) - r'^2\mathscr{K}_a(r)]$$

is increasing from $(0, 1)$ *onto* $(\pi(a^2 + b^2)/(2a), \pi^2/(2c))$.

(3)

$$\hat{f}_3(r) \equiv (\mathscr{E}_a(r) - r'^2\mathscr{K}_a(r))/r^2$$

is increasing and convex from $(0, 1)$ *onto* $(\pi a/2, c/2)$.

In order to estimate the Robin capacity, Anderson, Qiu, and Vamanamurthy in [5] dealt with the convexity properties of the functions $(\mathscr{E}' - r^2\mathscr{K}')/(r')^2$ and $[(\mathscr{E} - r'^2\mathscr{K})/r^2]/[(\mathscr{E}' - r^2\mathscr{K}')/(r')^2]$. They proved the following theorem.

Theorem 1.2 (See [5]). *For $r \in (0,1)$, there exists $r_0 \in (0,1)$ such that the function*

$$F_1(r) \equiv \frac{\mathscr{E}' - r^2\mathscr{K}'}{r'^2},$$

is concave on $(0, r_0)$ and convex on $(r_0, 1)$.

Theorem 1.3 (See [5]). *For $r \in (0,1)$, the function*

$$F_2(r) \equiv \frac{[\mathscr{E} - r'^2\mathscr{K}]/r^2}{[\mathscr{E}' - r^2\mathscr{K}']/(r')^2}$$

is strictly increasing and convex from $(0,1)$ onto $(\pi/4, 4/\pi)$. In particular, for $r \in (0,1)$

$$\frac{\pi}{4} < F_2(r) < \frac{\pi}{4} + \left(\frac{4}{\pi} - \frac{\pi}{4}\right) r.$$

The corresponding properties of the additive counterpart $(\mathscr{E} - r'^2\mathscr{K})/r^2 - (\mathscr{E}' - r^2\mathscr{K}')/(r')^2$ are obtained by Alzer and Richard in [2]. They proved the following theorem.

Theorem 1.4 (See [2]). *For $r \in (0,1)$, the function*

$$F_3(r) \equiv \frac{\mathscr{E} - r'^2\mathscr{K}}{r^2} - \frac{\mathscr{E}' - r^2\mathscr{K}'}{r'^2},$$

is strictly increasing and convex from $(0,1)$ onto $(\pi/4 - 1, 1 - \pi/4)$. Moreover, for all $r \in (0,1)$, the double inequality

$$\frac{\pi}{4} - 1 + \alpha r < F_3(r) < \frac{\pi}{4} - 1 + \beta r,$$

holds for all $r \in (0,1)$ with the best constants

$$\alpha = 0, \quad and \quad \beta = 2 - \pi/2 = 0.42920\cdots.$$

In 2019, Wang et al. [22] generalized Theorem 1.2 to the generalized elliptic integrals and obtained analogous properties. In 2017, Huang et al. [12] generalized Theorem 1.3 and Theorem 1.4 to the generalized elliptic integrals. So it is natural to ask that how to extend these results to zero-balanced Gaussian hypergeometric function $F(a, b; a+b; x)$? The purpose of this paper is to solve this question.

For the purpose, we require some more properties of the zero-balanced Gaussian hypergeometric function, so we will give some lemmas in Section 2. In the last section, we will present our main results and their proofs.

2. Preliminaries

In this section, we give a definition and some lemmas needed in the proofs of our main results in Section 3. Firstly, we recall some formulas below. By [1, 3, 15], the hypergeometric function has the following simple differentiation formula

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x)$$

and it is well known that

$$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x), \tag{2.1}$$

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c > a+b, \tag{2.2}$$

$$B(a,b)F(a,b;a+b;x) + \log(1-x) = R(a,b) + O((1-x)\log(1-x)), \quad x \to 1. \tag{2.3}$$

Definition 2.1. *A function f is said to be strictly completely monotonic on an interval $I \subset \mathbb{R}$ if $(-1)^n f^{(n)}(x) > 0$ for all $x \in I$ and $n = 0, 1, 2, 3\cdots$. If $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in I$ and $n = 0, 1, 2, 3\cdots$, then f is called completely monotonic on I.*

Completely monotonic functions play a dominant role in areas such as numerical analysis [19], probability theory [10], special function theory [1] and physics [9].

The following lemmas will be frequently applied in the sequel.

Lemma 2.1. *(See [15, Lemma 2.1]) Let $-\infty < a < b < \infty$, $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. *(See [16, Lemma 2.4]) Suppose that $r \in (0,\infty)$ is the common radius of convergence of the real power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ with $b_n > 0$, and $\{a_n/b_n\}$ is a non-constant sequence. Let $\varphi(x) = A(x)/B(x)$.*

(1) If there is an $n_0 \in \mathbb{N}$ such that the sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 \leq n \leq n_0$, and decreasing (increasing) for $n \geq n_0$, then φ is increasing (decreasing) on $(0,r)$ if and only if $\varphi'(r^-) \geq 0$ ($\varphi'(r^-) \leq 0$, respectively).

(2) If there is an $n_0 \in \mathbb{N}$ such that the sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 \leq n \leq n_0$, and decreasing (increasing) for $n \geq n_0$, and if $\varphi'(r^-) < 0$ ($\varphi'(r^-) > 0$), then there exists a number $x_0 \in (0,r)$ such that φ is strictly increasing (decreasing) on $(0,x_0]$ and decreasing (increasing, respectively) on $[x_0,r)$.

Lemma 2.3. *(See [17, Lemma 1.1]) Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ and that $b_n > 0$ for all $n \in \{0, 1, 2, \cdots\}$. Let $h(x) = f(x)/g(x)$. If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0,r)$.*

Lemma 2.4. *For $a > 0, b > 0, x \in (0, 1)$ and $a \neq 1$, we define the function*

$$H(x) \equiv \frac{F(a-1, b; a+b; x) - (1-x)F(a, b; a+b; x)}{x},$$

then we get

$$H(x) = \frac{a}{a+b} F(a, b; a+b+1; x),$$

and H is strictly increasing and convex from $(0, 1)$ onto $\left(\frac{a}{a+b}, \frac{1}{bB(a,b)}\right)$.

Proof. By the definition of the Gaussian hypergeometric function in (1.1), we have

$$\begin{aligned} H(x) &= \frac{F(a-1, b; a+b; x) - (1-x)F(a, b; a+b; x)}{x} \\ &= \frac{1}{x}\left[\sum_{n=0}^{\infty} \frac{(a-1, n)(b, n)}{(a+b, n)} \frac{x^n}{n!} - (1-x) \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(a+b, n)} \frac{x^n}{n!}\right] \\ &= \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(a+b, n+1)(n+1)!} \cdot a(n+1)x^n \\ &= \frac{a}{a+b} \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(a+b+1, n)n!} x^n \\ &= \frac{a}{a+b} F(a, b; a+b+1; x). \end{aligned}$$

It is clear that $H(0^+) = \frac{a}{a+b}$. By (2.2), $H(1^-) = \frac{1}{bB(a,b)}$. The remainings conclusion is clear. ◻

Remark 2.5. *Let $a \in (0, 1/2], b = 1 - a, x = r^2, r \in (0, 1)$. From Lemma 2.4, we can obtain the properties of the function $\hat{f}_3(r)$ in Theorem 1.1 which is proved by Anderson et al. in [6].*

3. Main results and proofs

Theorem 3.1. *For $a > 0, b > 0, x \in (0, 1)$ and $a \neq 1$, the function*

$$f_1(x) \equiv \frac{(1-x)F(a, b; a+b; x)}{F(a-1, b; a+b; x)}$$

is strictly decreasing from $(0, 1)$ onto $(0, 1)$.

Proof. We can write $f_1(x)$ as

$$f_1(x) = 1 - \frac{F(a-1, b; a+b; x) - (1-x)F(a, b; a+b; x)}{F(a-1, b; a+b; x)}. \tag{3.1}$$

Clearly, $f_1(0^+) = 1$ and by (2.3), $f_1(1^-) = 0$. By (1.1),

$$\begin{aligned} &F(a-1, b; a+b; x) - (1-x)F(a, b; a+b; x) \\ &= \sum_{n=0}^{\infty} \frac{a-1}{n+a-1} a_n x^n - \left[\sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n\right] \end{aligned}$$

$$= \sum_{n=1}^{\infty} \left(\frac{-n}{n+a-1} a_n + a_{n-1} \right) x^n$$
$$= \sum_{n=1}^{\infty} \frac{a}{n+a+b-1} a_{n-1} x^n, \tag{3.2}$$

where $a_n = [(a,n)(b,n)]/[(a+b,n)n!]$. By (3.2), $F(a-1,b;a+b;x) - (1-x)F(a,b;a+b;x)$ is positive and strictly increasing on $(0,1)$. It is easy to see that $F(a-1,b;a+b;x)$ is positive for $a > 0$ and is strictly decreasing (increasing) for $a \in (0,1) (a \in (1,\infty)$, respectively). Hence, for $a \in (0,1)$, f_1 is strictly decreasing on $(0,1)$.

Next, we will study the monotonicity of f_1 for $a \in (1,\infty)$. By (1.1),

$$f_1(x) = \frac{(1-x)F(a,b;a+b;x)}{F(a-1,b;a+b;x)} = \frac{\sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n}{\sum_{n=0}^{\infty} \frac{a-1}{n+a-1} a_n x^n} = \frac{\sum_{n=0}^{\infty} b_n x^n}{\sum_{n=0}^{\infty} c_n x^n},$$

where $b_0 = 1, b_n = a_n - a_{n-1}, n \ge 1$ and $c_n = \frac{a-1}{n+a-1} a_n$. Since

$$\frac{b_0}{c_0} - \frac{b_1}{c_1} = 1 - \frac{a_1 - a_0}{\frac{a-1}{a} a_1} = 1 - \frac{ab - (a+b)}{(a-1)b} = \frac{a}{(a-1)b} > 0, \tag{3.3}$$

so by (3.3), $\frac{b_0}{c_0} > \frac{b_1}{c_1}$. For $n \ge 1$,

$$\frac{b_n}{c_n} = \frac{a_n - a_{n-1}}{\frac{a-1}{n+a-1} a_n} = \frac{n+a-1}{a-1} \left(1 - \frac{a_{n-1}}{a_n} \right)$$
$$= \frac{n+a-1}{a-1} \left[1 - \frac{(a+b+n-1)n}{(a+n-1)(b+n-1)} \right]$$
$$= \frac{n+a-1}{a-1} \cdot \frac{ab-a-b-n+1}{(a+n-1)(b+n-1)}$$
$$= \frac{1}{a-1} \left[\frac{a(b-1)}{b+n-1} - 1 \right], \tag{3.4}$$
$$\frac{b_{n+1}}{c_{n+1}} - \frac{b_n}{c_n} = \frac{a(b-1)}{a-1} \left(\frac{1}{b+n} - \frac{1}{b+n-1} \right). \tag{3.5}$$

Therefore, for $a > 1, b > 1$, by (3.3) and (3.5), $\{b_n/c_n\}$ is strictly decreasing for $n = 1, 2, 3, \cdots$. By Lemma 2.3, the function f_1 is strictly decreasing.

Then, we study the monotonicity of g_1 for $a \in (1,\infty), b \in (0,1]$.

Differentiation of $f_1(x)$ gives

$$f_1'(x) = -\frac{F(a,b;a+b;x)}{F(a-1,b;a+b;x)} + \frac{ab}{a+b} \frac{F(a,b;a+b+1;x)}{F(a-1,b;a+b;x)} - \frac{(a-1)b}{a+b} \frac{(1-x)F(a,b;a+b;x)F(a,b+1;a+b+1;x)}{(F(a-1,b;a+b;x))^2}.$$

For $a > 1, b > 0$, $x \in (0,1)$, we have

$$\frac{(1-x)F(a,b;a+b;x)F(a,b+1;a+b+1;x)}{(F(a-1,b;a+b;x))^2} > 0$$

and

$$\lim_{x\to 1^-} -\frac{F(a,b;a+b;x)}{F(a-1,b;a+b;x)} = -\infty,$$
$$\lim_{x\to 1^-} \frac{ab}{a+b}\frac{F(a,b;a+b+1;x)}{F(a-1,b;a+b;x)} = b.$$

Hence $f_1'(1^-) = -\infty$. By (3.5), for $b \in (0,1]$, $\{b_n/c_n\}$ is strictly increasing for $n = 1,2,3,\cdots$, and since $\{b_n/c_n\}$ is strictly decreasing on $0 \le n \le 1$, and $f_1'(1^-) = -\infty < 0$. Hence by Lemma 2.2(1), f_1 is strictly decreasing on $(0,1)$. □

Theorem 3.2. *For $a > 0, b > 0, x \in (0,1)$ and $a \neq 1$, the function*

$$f_2(x) \equiv \frac{1-\left[\sqrt{1-x}F(a,b;a+b;x)\right]^2}{F(a-1,b;a+b;x)-(1-x)F(a,b;a+b;x)}$$

is strictly increasing on $(0,1)$ if $0 < a+b-2ab < 1$.

Proof. Let $h_1(x) = 1-[\sqrt{1-x}F(a,b;a+b;x)]^2$ and $h_2(x) = F(a-1,b;a+b;x)-(1-x)F(a,b;a+b;x)$, since $h_1(0^+) = h_2(0^+) = 0$, then

$$\begin{aligned}\frac{h_1'(x)}{h_2'(x)} &= -2[\sqrt{1-x}F(a,b;a+b;x)]\\ &\times \frac{\left[-\frac{1}{2\sqrt{1-x}}F(a,b;a+b;x)+\sqrt{1-x}\frac{ab}{a+b}(1-x)^{-1}F(a,b;a+b+1;x)\right]}{\frac{(a-1)b}{a+b}F(a,b+1;a+b+1;x)+F(a,b;a+b;x)-\frac{ab}{a+b}F(a,b;a+b+1;x)}\\ &= F(a,b;a+b;x)\frac{\sum_{n=0}^{\infty}\left(1-\frac{2ab}{a+b+n}\right)a_nx^n}{\sum_{n=0}^{\infty}\frac{a(n+1)}{a+b+n}a_nx^n}\\ &= F(a,b;a+b;x)G(x) \qquad (3.6)\end{aligned}$$

where $a_n = [(a,n)(b,n)]/[(a+b,n)n!]$ and

$$G(x) = \frac{\sum_{n=0}^{\infty}\left(1-\frac{2ab}{a+b+n}\right)a_nx^n}{\sum_{n=0}^{\infty}\frac{a(n+1)}{a+b+n}a_nx^n}. \qquad (3.7)$$

Let $A_n = \left(1-\frac{2ab}{a+b+n}\right)a_n$, $B_n = \frac{a(n+1)}{a+b+n}a_n$, then, $A_n/B_n = [a+b-2ab+n]/a(n+1)$. Simple computations give

$$\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \frac{2ab-a-b+1}{a(n+1)(n+2)}.$$

Since $0 < a+b-2ab < 1$, we can get $G(x) > 0$ and $A_{n+1}/B_{n+1} - A_n/B_n \ge 0$, hence $\{A_n/B_n\}$ is increasing, then $G(x)$ is increasing on $(0,1)$ by Lemma 2.3. Therefore $h_1'(x)/h_2'(x)$ is a product of two positive and increasing functions, the monotonicity of f_2 follows from Lemma 2.1. □

Remark 3.3. *Let $a \in (0,1/2], b = 1-a, x = r^2, r \in (0,1)$.*
(1) The properties of the function $\hat{f}_1(r)$ in Theorem 1.1 follow from Theorem 3.1.
(2) Since $a+b-2ab = 2\left(a-\frac{1}{2}\right)^2+\frac{1}{2} \in (0,1)$, we can obtain the properties of the function $\hat{f}_2(r)$ in Theorem 1.1 by Theorem 3.2.

Theorem 3.4. *For $a > 0, b > 0, x \in (0,1)$ and $a \neq 1$, the function*

$$f_3(x) \equiv \frac{F(a-1,b;a+b;1-x) - xF(a,b;a+b;1-x)}{1-x}$$

is completely monotonic on $(0,1)$. In particular, for $a > 0, b > 0$ and $a \neq 1$, f_3 is decreasing and convex on $(0,1)$.

Proof. By Lemma 2.4, $f_3(x)$ can be written as

$$f_3(x) = \frac{a}{a+b} F(a,b;a+b+1;1-x),$$

differentiation gives

$$f_3^{(n)}(x) = (-1)^n \frac{a(a,n)(b,n)}{(a+b)(a+b+1,n)} F(a+n,b+n;a+b+n+1;1-x), \tag{3.8}$$

for $a > 0, b > 0$, $x \in (0,1)$, it is easy to obtain $(-1)^n f_3^{(n)}(x) > 0$. By Definition 2.1, f_3 is completely monotonic on $(0,1)$. Let $n = 1$ and $n = 2$, so $f_3'(x) < 0, f_3''(x) > 0$, then the monotonicity and convexity of f_3 follow. □

Remark 3.5. *(1) Let $a = b = 1/2, x = r^2, r \in (0,1)$. From Theorem 3.4, we can obtain the properties of the function $(\mathscr{E}' - r^2\mathscr{K}')/(r')^2$ which is proved by Anderson, Qiu, and Vamanamurthy in [5].*

(2) Let $a \in (0,1/2], b = 1-a, x = r^2, r \in (0,1)$. From Theorem 3.4, we can obtain the properties of the function $(\mathscr{E}_a' - r^2\mathscr{K}_a')/(r')^2$ which is proved by Wang, Zhang, and Chu in [22].

Theorem 3.6. *Let $a > 0, b > 0, x \in (0,1)$ and $a \neq 1$, we define the function*

$$\begin{aligned} f_4(x) \equiv & \frac{F(a-1,b;a+b;x) - (1-x)F(a,b;a+b;x)}{x} \\ & - \frac{F(a-1,b;a+b;1-x) - xF(a,b;a+b;1-x)}{1-x}, \end{aligned}$$

then $f_4^{(2n-1)}$ is decreasing(increasing) on $(0,1/2]$ ($[1/2,1)$, respectively) and $f_4^{(2n)}$ is strictly increasing from $(0,1)$ onto $(-\infty,+\infty)$ for $n = 1,2,\cdots$, $n \in \mathbb{N}$.

Proof. By Lemma 2.4, $f_4(x)$ can be written as

$$f_4(x) = \frac{a}{a+b}\left[F(a,b;a+b+1;x) - F(a,b;a+b+1;1-x)\right]. \tag{3.9}$$

For $n = 1,2,\cdots$, $n \in \mathbb{N}$, differentiation of (3.9) gives

$$\begin{aligned} f_4^{(2n)}(x) =& \frac{a}{a+b} \cdot \frac{(a,2n)(b,2n)}{(a+b+1,2n)} \\ & \times \Big[F(a+2n,b+2n;a+b+2n+1;x) \\ & - F(a+2n,b+2n;a+b+2n+1;1-x)\Big] \\ =& \frac{a}{a+b} \cdot \frac{(a,2n)(b,2n)}{(a+b+1,2n)} \end{aligned}$$

$$\times \sum_{n=0}^{\infty} \frac{(a+2n,n)(b+2n,n)}{(a+b+2n+1,n)n!}\Big[x^n - (1-x)^n\Big], \tag{3.10}$$

$$\begin{aligned} f_4^{(2n-1)}(x) =& \frac{a}{a+b} \cdot \frac{(a,2n-1)(b,2n-1)}{(a+b+1,2n-1)} \\ & \times \Big[F(a+2n-1,b+2n-1;a+b+2n;x) \\ & + F(a+2n-1,b+2n-1;a+b+2n;1-x)\Big]. \end{aligned} \tag{3.11}$$

By (3.10) and (3.11), for $a > 0, b > 0$ and $a \neq 1$, $f_4^{(2n-1)}(x) > 0$ on $(0,1)$, and $f_4^{(2n)}(x) < 0$ on $(0,1/2)$, $f_4^{(2n)}(x) > 0$ on $(1/2,1)$. Therefore, $f_4^{(2n)}$ is strictly increasing on $(0,1)$, and $f_4^{(2n-1)}$ is strictly decreasing on $(0,1/2)$ and increasing on $(1/2,1)$ for $n = 1,2,\cdots$. Clearly, $f_4^{(2n)}(0^+) = -\infty, f_4^{(2n)}(1^-) = +\infty$. □

The following Corollary is obtained by Theorem 3.6.

Corollary 3.7. *Let $a > 0, b > 0, x \in (0,1)$ and $a \neq 1$, the function f_4 is strictly increasing from $(0,1)$ onto $\left(\frac{a}{a+b} - \frac{1}{b}\frac{1}{B(a,b)}, \frac{1}{b}\frac{1}{B(a,b)} - \frac{a}{a+b}\right)$ and f_4 is concave on $(0,1/2]$ and convex on $[1/2,1)$. Moreover, for all $x \in (0,1)$, the double inequality*

$$\frac{a}{a+b} - \frac{1}{b}\frac{1}{B(a,b)} + \alpha_1 x < f_4(x) < \frac{a}{a+b} - \frac{1}{b}\frac{1}{B(a,b)} + \beta_1 x,$$

holds for all $x \in (0,1)$ with the best possible constants

$$\alpha_1 = 0, \ \ and \ \ \beta_1 = \frac{2}{bB(a,b)} - \frac{2a}{a+b}.$$

Proof. Let $n = 1$ in (3.10),

$$f_4'(x) = \frac{a^2 b}{(a+b)(a+b+1)} \times [F(a+1,b+1;a+b+2;x) + F(a+1,b+1;a+b+2;1-x)], \tag{3.12}$$

for $a > 0, b > 0, x \in (0,1)$ and $a \neq 1$, $f_4'(x) > 0$, hence the monotonicity of f_4 follows. By Theorem 3.6, since $f_4^{(2n-1)}$ is decreasing(increasing) on $(0,1/2]$ ($[1/2,1)$, respectively), hence $f_4''(x) < 0$ on $(0,1/2)$ and $f_4''(x) > 0$ on $(1/2,1)$, so the concavity and convexity of f_4 are obtained. The limiting values are easy to know $f_4(0^+) = \frac{a}{a+b} - \frac{1}{b}\frac{1}{B(a,b)}$, $f_4(1^-) = \frac{1}{b}\frac{1}{B(a,b)} - \frac{a}{a+b}$ by (2.2) and (3.9), and thereby the double inequality in Corollary 3.7 follows. □

Remark 3.8. *(1) Let $a = b = 1/2, x = r^2, r \in (0,1)$. From Theorem 3.6, we can obtain the properties of the function $(\mathscr{E} - r'^2\mathscr{K})/r^2 - (\mathscr{E}' - r^2\mathscr{K}')/(r')^2$ which is proved by Alzer and Richard in [2].*

(2) Let $a \in (0,1/2], b = 1-a, x = r^2, r \in (0,1)$. From Theorem 3.6, we can obtain the properties of the function $[\mathscr{E}_a - r'^2\mathscr{K}_a]/r^2 - [\mathscr{E}_a' - r^2\mathscr{K}_a']/(r')^2$ which is proved by Huang, Tan, and Zhang in [12].

Theorem 3.9. *Let $a > 0, b > 0, x \in (0,1)$ and $a \neq 1$, the function*

$$\begin{aligned} f_5(x) \equiv & \frac{F(a-1,b;a+b;x) - (1-x)F(a,b;a+b;x)}{x} \\ & \times \frac{1-x}{F(a-1,b;a+b;1-x) - xF(a,b;a+b;1-x)}, \end{aligned}$$

is strictly increasing from $(0,1)$ *onto* $\left(\frac{abB(a,b)}{a+b}, \frac{a+b}{abB(a,b)}\right)$. *Moreover, for all* $x \in (0,1)$, *the double inequality*

$$\frac{ab}{a+b}B(a,b) < f_5(x) < \frac{a+b}{abB(a,b)},$$

holds for all $x \in (0,1)$.

Proof. By Lemma 2.4, $f_5(x)$ can be written as

$$f_5(x) = \frac{F(a,b;a+b+1;x)}{F(a,b;a+b+1;1-x)}. \tag{3.13}$$

Since $F(a,b;a+b+1;x)$ is positive and strictly increasing on $(0,1)$ and $F(a,b;a+b+1;1-x)$ is positive and strictly decreasing on $(0,1)$. Hence $f_5(x)$ is a product of two positive and increasing functions, then f_5 is strictly increasing on $(0,1)$. The limit values are easy to know $f_5(0^+) = abB(a,b)/(a+b)$, $f_5(1^-) = (a+b)/[abB(a,b)]$ from (2.2) and (3.13), and thereby the double inequality in Theorem 3.9 follows. □

Remark 3.10. *(1) Let* $a = b = 1/2, x = r^2, r \in (0,1)$. *From Theorem 3.9, we can obtain the properties of the function* $[(\mathscr{E} - r'^2\mathscr{K})/r^2]/[(\mathscr{E}' - r^2\mathscr{K}')/(r')^2]$ *are obtained by Anderson, Qiu, and Vamanamurthy in [5].*

(2) Let $a \in (0,1/2], b = 1-a, x = r^2, r \in (0,1)$. *From Theorem 3.9, we can obtain the properties of the function* $[(\mathscr{E}_a - r'^2\mathscr{K}_a)/r^2]/[(\mathscr{E}_a' - r^2\mathscr{K}_a')/(r')^2]$ *which is proved by Huang, Tan, and Zhang in [12].*

4. Conclusions

Many properties of generalized elliptic integrals currently have been published in the literature. Results from zero-balanced Gaussian hypergeometric function $F(a,b;a+b;x)$ are presented in this paper. Firstly we propose some primary properties of the zero-balanced Gaussian hypergeometric function which required in the proofs of main results. Then we generalized the monotonicity and convexity properties of elliptic integrals to the zero-balanced Gaussian hypergeometric function. We obtain some new properties and give sharp inequalities for the zero-balanced Gaussian hypergeometric function $F(a;b;a+b;x)$. These studies will validly improve the theory of special functions and their applications in the natural sciences and engineering.

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Conflict of interest

The authors declare that they have no competing interests.

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