Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 12458–12470.
DOI:10.3934/math.2022691
Received: 06 January 2022
Revised: 12 March 2022
Accepted: 21 March 2022
Published: 26 April 2022

Research article

Some new results on fuzzy soft r-minimal spaces

I. M. Taha[1,2,*]

[1] Department of Basic Sciences, Higher Institute of Engineering and Technology, Menoufia, Egypt
[2] Department of Mathematics, Faculty of Science, Sohag University, Egypt

* **Correspondence:** Email: imtaha2010@yahoo.com.

Abstract: As a weaker form of fuzzy soft r-minimal continuity by Taha (2021), the notions of fuzzy soft almost (respectively (resp. for short) weakly) r-minimal continuous mappings are introduced, and some properties are given. Also, we show that every fuzzy soft r-minimal continuity is fuzzy soft almost (resp. weakly) r-minimal continuity, but the converse need not be true. After that, we introduce a concept of continuity in a very general setting called fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$-continuous mappings and investigate some properties of these mappings.

Keywords: fuzzy soft r-minimal space; fuzzy soft almost (weakly) r-minimal continuity; fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$-continuity; fuzzy soft r-minimal $\mathcal{C}$-compactness
Mathematics Subject Classification: 06D72, 54A40, 54C05, 54D30

1. Introduction

Zadeh [30] introduced the basic idea of a fuzzy set as an extension of classical set theory. The basic notions of fuzzy sets have been improved and applied in different directions. Along this direction, we can refer to [16, 20, 22, 24–26]. The concept of soft set theory was initiated by Molodtsov [19] in 1999 as a general mathematical tool for modeling uncertainties. Maji et al. [17] introduced the concept of a fuzzy soft set, which combines fuzzy sets [30] and soft sets [19]. Soft set and fuzzy soft set theories have a rich potential for applications in several directions. So far, lots of spectacular and creative studies about the theories of soft sets and fuzzy soft sets have been considered by some scholars (see [2–4, 6, 9, 11–13]). Also, Aygünoğlu et al. [8] studied the topological structure of fuzzy soft sets based on fuzzy topologies in the sense of Šostak [21]. The concept of fuzzy r-minimal structure was introduced by Yoo et al. [29] as an extension of fuzzy topology introduced by Šostak [21]. Also, the concepts of a fuzzy r-minimal space, fuzzy r-minimal continuity, and fuzzy r-minimal compactness were introduced in [15, 29]. Later, Taha [23] introduced the concept of fuzzy soft r-minimal structure, which is an extension of fuzzy soft topology introduced by Aygünoğlu et al. [8]. Also, the concept of fuzzy soft r-minimal continuity and several types of fuzzy soft r-minimal compactness were introduced in [23].

We lay out the remainder of this article as follows. Section 2 contains some basic definitions and results that help in understanding the obtained results. In Section 3, we introduce and study a weaker form of fuzzy soft r-minimal continuous mappings. Additionally, we show that fuzzy soft r-minimal continuity [23] $\Rightarrow$ fuzzy soft almost r-minimal continuity $\Rightarrow$ fuzzy soft weakly r-minimal continuity, but the converse need not be true. In Section 4, we introduce a concept of continuity in a very general setting under the name "fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous mappings". We prove that if $\mathcal{A}$ and $\mathcal{B}$ are operators on $(X, \widetilde{M})$, and C, C^* and $\mathcal{D}$ are operators on $(Y, \widetilde{M}^*)$, then $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ is fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C \sqcap C^*, \mathcal{D})$-continuous iff it is both fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous and fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C^*, \mathcal{D})$-continuous. Finally, Section 5 gives some conclusions and suggests some future works.

2. Preliminary assertions

In this section, we present the basic definitions which we need in the next sections. Throughout this paper, X refers to an initial universe, E is the set of all parameters for X and $A \subseteq E$, the family of all fuzzy sets in X is denoted by I^X (where $I_\circ = (0,1]$, $I = [0,1]$), and for $t \in I$, $\underline{t}(x) = t$, for all $x \in X$.

Definition 2.1. *[1, 8, 17] A fuzzy soft set f_A over X is a mapping from E to I^X such that $f_A(e)$ is a fuzzy set on X, for each $e \in A$ and $f_A(e) = \underline{0}$, if $e \notin A$, where $\underline{0}$ is zero function on X. The fuzzy set $f_A(e)$, for each $e \in E$, is called an element of the fuzzy soft set f_A. $\widetilde{(X,E)}$ denotes the collection of all fuzzy soft sets on X and is called a fuzzy soft universe.*

Definition 2.2. *[18, 28] A fuzzy soft point e_{x_t} over X is a fuzzy soft set over X defined as follows:*

$$e_{x_t}(e^*) = \begin{cases} x_t, & \text{if } e^* = e, \\ \underline{0}, & \text{if } e^* \in E - \{e\}, \end{cases}$$

where x_t is a fuzzy point in X. A fuzzy soft point e_{x_t} is said to belong to a fuzzy soft set f_A, denoted by $e_{x_t} \tilde{\in} f_A$, if $t \leq f_A(e)(x)$. The family of all fuzzy soft points in X is denoted by $\widetilde{P_t(X)}$.

Definition 2.3. *[8] A mapping $\tau : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ is called a fuzzy soft topology on X if it satisfies the following conditions for each $e \in E$.*

(i) $\tau_e(\Phi) = \tau_e(\widetilde{E}) = 1$.

(ii) $\tau_e(f_A \sqcap g_B) \geq \tau_e(f_A) \wedge \tau_e(g_B)$, $\forall f_A, g_B \in \widetilde{(X,E)}$.

(iii) $\tau_e(\bigsqcup_{i\in\Delta}(f_A)_i) \geq \bigwedge_{i\in\Delta} \tau_e((f_A)_i), \forall (f_A)_i \in \widetilde{(X,E)}, i \in \Delta$.

Then, the pair (X, τ_E) is called a fuzzy soft topological space (FSTS, for short).

Definition 2.4. *[23] Let X be a nonempty set and $r \in I_\circ$. A fuzzy soft mapping $\widetilde{M} : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ on X is said to be a fuzzy soft r-minimal structure if the family $\widetilde{M}_{e,r} = \{f_A \in \widetilde{(X,E)} \mid \widetilde{M}_e(f_A) \geq r\}$ for each $e \in E$ contains Φ and $\widetilde{E}$. Then $(X, \widetilde{M})$ is called a fuzzy soft r-minimal space (simply, r-$\widetilde{FMS}$). Every member of $\widetilde{M}_{e,r}$ is called a fuzzy soft r-minimal open set.*

Definition 2.5. *[23] Let $(X, \widetilde{M})$ be an r-$\widetilde{FMS}$, $e \in E$ and $r \in I_\circ$. The fuzzy soft r-minimal interior and fuzzy soft r-minimal closure of f_A, denoted by $I_m(e, f_A, r)$ and $C_m(e, f_A, r)$, resp., are defined as $I_m(e, f_A, r) = \bigsqcup\{g_B \in \widetilde{(X,E)} : g_B \sqsubseteq f_A,\ g_B \in \widetilde{M}_{e,r}\}$ and $C_m(e, f_A, r) = \sqcap\{g_B \in \widetilde{(X,E)} : f_A \sqsubseteq g_B,\ g_B^c \in \widetilde{M}_{e,r}\}$.*

Definition 2.6. *[23] Let $(X, \widetilde{M})$ and $(Y, \widetilde{M}^*)$ be r-$\widetilde{FMS}$s. Then, a fuzzy soft mapping φ_ψ from $\widetilde{(X,E)}$ into $\widetilde{(Y,F)}$ is called fuzzy soft r-minimal continuous if $\varphi_\psi^{-1}(g_B) \in \widetilde{M}_{e,r}$ for every $g_B \in \widetilde{M}^*_{k,r}$, $e \in E$ and $(\psi(e) = k) \in F$.*

Definition 2.7. *[23] Let X be a nonempty set and $\widetilde{M} : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$. Then, $\widetilde{M}$ is said to have property (P) if*

$$\widetilde{M}_e(\sqcup_{j\in J}(f_A)_j) \geq \sqcap_{j\in J}\widetilde{M}_e((f_A)_j)$$

for $(f_A)_j \in \widetilde{(X,E)}$, $j \in J$ and $e \in E$.

Definition 2.8. *[23] Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, $e \in E$ and $r \in I_\circ$. Then, φ_ψ is called fuzzy soft r-minimal open if $\varphi_\psi(f_A) \in \widetilde{M}^*_{k,r}$ for every $f_A \in \widetilde{M}_{e,r}$. Also, φ_ψ is called fuzzy soft r-minimal closed if $(\varphi_\psi(f_A))^c \in \widetilde{M}^*_{k,r}$ for every $f_A^c \in \widetilde{M}_{e,r}$.*

Definition 2.9. *[23] Let $(X, \widetilde{M})$ be an r-$\widetilde{FMS}$, $g_B \in \widetilde{(X,E)}$, $e \in E$ and $r \in I_\circ$. Then, g_B is called fuzzy soft r-minimal compact (resp. fuzzy soft r-minimal almost compact and fuzzy soft r-minimal nearly compact) iff for every family $\{(f_A)_i \in \widetilde{(X,E)} \mid (f_A)_i \in \widetilde{M}_{e,r}\}_{i\in\Gamma}$ such that $g_B \sqsubseteq \bigsqcup_{i\in\Gamma}(f_A)_i$, there exists a finite subset $\Gamma_\circ$ of Γ such that $g_B \sqsubseteq \bigsqcup_{i\in\Gamma_\circ}(f_A)_i$ (resp. $g_B \sqsubseteq \bigsqcup_{i\in\Gamma_\circ} C_m(e,(f_A)_i,r)$ and $g_B \sqsubseteq \bigsqcup_{i\in\Gamma_\circ} I_m(e, C_m(e,(f_A)_i,r),r)$).*

Main properties of fuzzy soft sets and soft topology are found in [1, 8, 10, 14, 17, 27].

3. Weaker forms of fuzzy soft r-minimal continuity

In this section, we introduce a weaker form of fuzzy soft r-minimal continuity called fuzzy soft almost (resp. weakly) r-minimal continuous mappings and investigate some properties of these mappings. Also, we show that fuzzy soft r-minimal continuity [23] ⇒ fuzzy soft almost r-minimal continuity ⇒ fuzzy soft weakly r-minimal continuity, but the converse need not be true.

Definition 3.1. *A fuzzy soft mapping $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ is called fuzzy soft almost (resp. weakly) r-minimal continuous if, for fuzzy soft point e_{x_t} over X and each $g_B \in \widetilde{M}^*_{k,r}$ containing $\varphi_\psi(e_{x_t})$, there is $f_A \in \widetilde{M}_{e,r}$ containing e_{x_t} such that $\varphi_\psi(f_A) \sqsubseteq I_{m^*}(k, C_{m^*}(k, g_B, r), r)$ (resp. $\varphi_\psi(f_A) \sqsubseteq C_{m^*}(k, g_B, r)$).*

Theorem 3.1. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping. Suppose that one of the following properties holds:*
(i) $\varphi_\psi^{-1}(g_B) \sqsubseteq I_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)), r)$, if $g_B \in \widetilde{M}^*_{k,r}$.
(ii) $C_m(e, \varphi_\psi^{-1}(C_{m^*}(k, I_{m^*}(k, g_B, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$, if $g_B^c \in \widetilde{M}^*_{k,r}$.
Then, φ_ψ is fuzzy soft almost r-minimal continuous.

Proof. (i) ⇒ (ii) Let $g_B^c \in \widetilde{M}^*_{k,r}$. Then, from (i), it follows

$$\begin{aligned}\varphi_\psi^{-1}(g_B^c) &\sqsubseteq I_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B^c, r), r)), r) = I_m(e, \varphi_\psi^{-1}((C_{m^*}(k, I_{m^*}(k, g_B, r), r))^c), r) \\ &= I_m(e, (\varphi_\psi^{-1}(C_{m^*}(k, I_{m^*}(k, g_B, r), r)))^c, r) = (C_m(e, \varphi_\psi^{-1}(C_{m^*}(k, I_{m^*}(k, g_B, r), r)), r))^c.\end{aligned}$$

Hence, $C_m(e, \varphi_\psi^{-1}(C_{m^*}(k, I_{m^*}(k, g_B, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$.
Similarly, we get (ii) ⇒ (i).

Suppose that (i) holds. Let $e_{x_t} \in \widetilde{P_t(X)}$, and $g_B \in \widetilde{M}^*_{k,r}$ containing $\varphi_\psi(e_{x_t})$. Then, by (i),

$$e_{x_t} \tilde{\in} I_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)), r),$$

and so there exists $f_A \in \widetilde{M}_{e,r}$ containing e_{x_t} such that $f_A \sqsubseteq \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r))$. It follows that $\varphi_\psi(f_A) \sqsubseteq I_{m^*}(k, C_{m^*}(k, g_B, r), r)$. Hence, φ_ψ is fuzzy soft almost r-minimal continuous.

In a similar way, one can prove the following corollary.

Corollary 3.1. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and Y has the property (P). Suppose that one of the following properties holds for $g_B \in \widetilde{(Y, F)}$, $e \in E$ and $r \in I_\circ$:*

(i) $\varphi_\psi^{-1}(g_B) \sqsubseteq I_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)), r)$, if $g_B \in \widetilde{M}^*_{k,r}$.

(ii) $\varphi_\psi^{-1}(I_{m^*}(k, g_B, r)) \sqsubseteq I_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, I_{m^*}(k, g_B, r), r), r)), r)$.

(iii) $C_m(e, \varphi_\psi^{-1}(C_{m^*}(k, I_{m^*}(k, C_{m^*}(k, g_B, r), r), r)), r) \sqsubseteq \varphi_\psi^{-1}(C_{m^*}(k, g_B, r))$.

Then, φ_ψ is fuzzy soft almost r-minimal continuous.

Lemma 3.1. *Every fuzzy soft r-minimal continuous mapping [23] is fuzzy soft almost r-minimal continuous.*

Proof. It follows from Theorem 3.1.

In general, the converse of Lemma 3.1 is not true, as shown by Example 3.1.

Example 3.1. Let $X = \{x, y\}$ and $E = \{e_1, e_2\}$ be the parameter set of X. Define f_E and $g_E \in \widetilde{(X, E)}$ as follows: $f_E = \{(e_1, \{\frac{x}{0.2}, \frac{y}{0.4}\}), (e_2, \{\frac{x}{0.2}, \frac{y}{0.4}\})\}$, $g_E = \{(e_1, \{\frac{x}{0.1}, \frac{y}{0.5}\}), (e_2, \{\frac{x}{0.1}, \frac{y}{0.5}\})\}$. Define fuzzy soft r-minimal structures $\widetilde{M}_E, \widetilde{W}_E : E \longrightarrow [0, 1]^{\widetilde{(X,E)}}$ as follows: $\forall\ e \in E$,

$$\widetilde{M}_e(h_E) = \begin{cases} \frac{1}{2}, & \text{if} \quad h_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3}, & \text{if} \quad h_E \in \{f_E, g_E\}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\widetilde{W}_e(h_E) = \begin{cases} \frac{1}{2}, & \text{if} \quad h_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if} \quad h_E \in \{f_E, g_E\}, \\ \frac{1}{3}, & \text{if} \quad h_E = f_E \sqcup g_E, \\ 0, & \text{otherwise}. \end{cases}$$

Then, the identity fuzzy soft mapping $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (X, \widetilde{W})$ is fuzzy soft almost $\frac{1}{3}$-minimal continuous, but it is not fuzzy soft $\frac{1}{3}$-minimal continuous.

Definition 3.2. *Let $(X, \widetilde{M})$ be an r-$\widetilde{FMS}$, $f_A \in \widetilde{(X, E)}$, $e \in E$ and $r \in I_\circ$. Then,*

(i) f_A is fuzzy soft r-minimal semiopen if $f_A \sqsubseteq C_m(e, I_m(e, f_A, r), r)$,

(ii) f_A is fuzzy soft r-minimal preopen if $f_A \sqsubseteq I_m(e, C_m(e, f_A, r), r)$,

(iii) f_A is fuzzy soft r-minimal regularly open if $f_A = I_m(e, C_m(e, f_A, r), r)$,

(iv) f_A is fuzzy soft r-minimal β-open if $f_A \sqsubseteq C_m(e, I_m(e, C_m(e, f_A, r), r), r)$.

A fuzzy soft set f_A is called a fuzzy soft r-minimal semiclosed (resp., fuzzy soft r-minimal preclosed, fuzzy soft r-minimal regularly closed and fuzzy soft r-minimal β-closed) set if the complement of f_A is a fuzzy soft r-minimal semiopen (resp., fuzzy soft r-minimal preopen, fuzzy soft r-minimal regularly open and fuzzy soft r-minimal β-open) set.

Theorem 3.2. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and Y has the property (P). Suppose that one of the following properties holds for $g_B \in \widetilde{(Y, F)}$, $e \in E$ and $r \in I_\circ$:*

(i) $\varphi_\psi^{-1}(g_B) = C_m(e, \varphi_\psi^{-1}(g_B), r)$, if g_B is fuzzy soft r-minimal regularly closed.

(ii) $\varphi_\psi^{-1}(g_B) = I_m(e, \varphi_\psi^{-1}(g_B), r)$, if g_B is fuzzy soft r-minimal regularly open.

Then, φ_ψ is fuzzy soft almost r-minimal continuous.

Proof. (i) ⇒ (ii) is obvious.

Suppose that (ii) holds. Let $e_{x_t} \in \widetilde{P_t(X)}$ and $g_B \in \widetilde{M}^*_{k,r}$ containing $\varphi_\psi(e_{x_t})$. Since $I_{m^*}(k, C_{m^*}(k, g_B, r), r)$ is fuzzy soft r-minimal regularly open, then by (ii),

$$\varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)) = I_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)), r),$$

there is $f_A \in \widetilde{M}_{e,r}$ containing e_{x_t} such that $f_A \sqsubseteq \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r))$. This implies $\varphi_\psi(f_A) \sqsubseteq I_{m^*}(k, C_{m^*}(k, g_B, r), r)$. Hence, φ_ψ is fuzzy soft almost r-minimal continuous.

In a similar way, one can prove the following corollary.

Corollary 3.2. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and X has the property (P). Suppose that one of the following properties holds for $g_B \in \widetilde{(Y, F)}$, $e \in E$ and $r \in I_\circ$:*

(i) $\varphi_\psi^{-1}(g_B) \in \widetilde{M}_{e,r}$, if g_B is fuzzy soft r-minimal regularly open.

(ii) $(\varphi_\psi^{-1}(g_B))^c \in \widetilde{M}_{e,r}$, if g_B is fuzzy soft r-minimal regularly closed.

Then, φ_ψ is fuzzy soft almost r-minimal continuous.

Theorem 3.3. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and X has the property (P). Suppose that one of the following properties holds for $g_B \in \widetilde{(Y, F)}$, $e \in E$ and $r \in I_\circ$:*

(i) $C_m(e, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_{m^*}(k, g_B, r))$, if g_B is fuzzy soft r-minimal β-open.

(ii) $C_m(e, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_{m^*}(k, g_B, r))$, if g_B is fuzzy soft r-minimal semiopen.

(iii) $\varphi_\psi^{-1}(g_B) \sqsubseteq I_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)), r)$, if g_B is fuzzy soft r-minimal preopen.

(iv) $C_m(e, \varphi_\psi^{-1}(C_{m^*}(k, I_{m^*}(k, C_{m^*}(k, g_B, r), r), r)), r) \sqsubseteq \varphi_\psi^{-1}(C_{m^*}(k, g_B, r))$, if g_B is fuzzy soft r-minimal preopen.

Then, φ_ψ is fuzzy soft almost r-minimal continuous.

Proof. (i) ⇒ (ii) Since every fuzzy soft r-minimal semiopen set is fuzzy soft r-minimal β-open, it is obvious.

Suppose that (ii) holds. Let g_B be a fuzzy soft r-minimal regular closed set. Then, g_B is fuzzy soft r-minimal semiopen, and so from (ii), we have

$$C_m(e, \varphi_\psi^{-1}(g_B), r) \sqsubseteq \varphi_\psi^{-1}(C_{m^*}(k, g_B, r)) = \varphi_\psi^{-1}(g_B).$$

This implies $\varphi_\psi^{-1}(g_B) = C_m(e, \varphi_\psi^{-1}(g_B), r)$, and hence from Theorem 3.2, φ_ψ is fuzzy soft almost r-minimal continuous.

Suppose that (iii) holds. Let g_B be a fuzzy soft r-minimal regular open set. Then, g_B is fuzzy soft r-minimal preopen, and so from (iii), we have

$$\varphi_\psi^{-1}(g_B) \sqsubseteq I_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)), r) = I_m(e, \varphi_\psi^{-1}(g_B), r).$$

This implies $\varphi_\psi^{-1}(g_B) = I_m(e, \varphi_\psi^{-1}(g_B), r)$, and hence by Theorem 3.2, φ_ψ is fuzzy soft almost r-minimal continuous.

Suppose that (iv) holds. Let g_B be a fuzzy soft r-minimal regular closed set. Then, $I_{m^*}(k, g_B, r)$ is fuzzy soft r-minimal preopen. From hypothesis and $g_B = C_{m^*}(k, I_{m^*}(k, g_B, r), r)$, it follows that

$$\begin{aligned}\varphi_\psi^{-1}(g_B) &= \varphi_\psi^{-1}(C_{m^*}(k, I_{m^*}(k, g_B, r), r)) \sqsupseteq C_m(e, \varphi_\psi^{-1}(C_{m^*}(k, I_{m^*}(k, C_{m^*}(k, I_{m^*}(k, g_B, r), r), r), r)), r)\\ &= C_m(e, \varphi_\psi^{-1}(C_{m^*}(k, I_{m^*}(k, g_B, r), r)), r) = C_m(e, \varphi_\psi^{-1}(g_B), r).\end{aligned}$$

This implies $\varphi_\psi^{-1}(g_B) = C_m(e, \varphi_\psi^{-1}(g_B), r)$. Hence, by Theorem 3.2, φ_ψ is fuzzy soft almost r-minimal continuous.

Theorem 3.4. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping. Suppose that one of the following properties holds:*
(i) $\varphi_\psi^{-1}(g_B) \sqsubseteq I_m(e, \varphi_\psi^{-1}(C_{m^*}(k, g_B, r)), r)$, if $\;g_B \in \widetilde{M}^*_{k,r}$.
(ii) $C_m(e, \varphi_\psi^{-1}(I_{m^*}(k, g_B, r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$, if $\;g_B^c \in \widetilde{M}^*_{k,r}$.
Then, φ_ψ is fuzzy soft weakly r-minimal continuous.

Proof. (i) ⇔ (ii) Let $g_B^c \in \widetilde{M}^*_{k,r}$. Then, from (i), it follows that

$$\begin{aligned}\varphi_\psi^{-1}(g_B^c) &\sqsubseteq I_m(e, \varphi_\psi^{-1}(C_{m^*}(k, g_B^c, r)), r) = I_m(e, \varphi_\psi^{-1}((I_{m^*}(k, g_B, r))^c), r)\\ &= I_m(e, (\varphi_\psi^{-1}(I_{m^*}(k, g_B, r)))^c, r) = (C_m(e, \varphi_\psi^{-1}(I_{m^*}(k, g_B, r)), r))^c.\end{aligned}$$

Hence, $C_m(e, \varphi_\psi^{-1}(I_{m^*}(k, g_B, r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$. Similarly, we get (ii) ⇒ (i).

Suppose that (i) holds. Let $e_{x_t} \in \widetilde{P_t(X)}$ and $g_B \in \widetilde{M}^*_{k,r}$ containing $\varphi_\psi(e_{x_t})$. Then, by (i),

$$e_{x_t} \tilde{\in} I_m(e, \varphi_\psi^{-1}(C_{m^*}(k, g_B, r)), r),$$

and so there exists $f_A \in \widetilde{M}_{e,r}$ containing e_{x_t} such that $f_A \sqsubseteq \varphi_\psi^{-1}(C_{m^*}(k, g_B, r))$. Thus, $\varphi_\psi(f_A) \sqsubseteq C_{m^*}(k, g_B, r)$. Hence, φ_ψ is fuzzy soft weakly r-minimal continuous.

In a similar way, one can prove the following corollary.

Corollary 3.3. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and Y has the property (P). Suppose that one of the following properties holds for $g_B \in \widetilde{(Y, F)}$, $e \in E$ and $r \in I_\circ$:*
(i) $C_m(e, \varphi_\psi^{-1}(I_{m^*}(k, g_B, r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$, if $g_B^c \in \widetilde{M}^*_{k,r}$.
(ii) $C_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(C_{m^*}(k, g_B, r))$.
(iii) $\varphi_\psi^{-1}(I_{m^*}(k, g_B, r)) \sqsubseteq I_m(e, \varphi_\psi^{-1}(C_{m^*}(k, I_{m^*}(k, g_B, r), r)), r)$.
Then, φ_ψ is fuzzy soft weakly r-minimal continuous.

Lemma 3.2. *Every fuzzy soft almost r-minimal continuous mapping is fuzzy soft weakly r-minimal continuous.*

Proof. It follows from Definition 3.1.

In general, the converse of Lemma 3.2 is not true, as shown by Example 3.2.

Example 3.2. Let $X = \{x, y\}$ and $E = \{e_1, e_2\}$ be the parameter set of X. Define f_E and $g_E \in \widetilde{(X, E)}$ as follows: $f_E = \{(e_1, \{\frac{x}{0.5}, \frac{y}{0.5}\}), (e_2, \{\frac{x}{0.5}, \frac{y}{0.5}\})\}$, $g_E = \{(e_1, \{\frac{x}{0.4}, \frac{y}{0.2}\}), (e_2, \{\frac{x}{0.4}, \frac{y}{0.2}\})\}$. Define fuzzy soft r-minimal structures $\widetilde{M}_E, \widetilde{W}_E : E \longrightarrow [0, 1]^{\widetilde{(X,E)}}$ as follows: $\forall\, e \in E$,

$$\widetilde{M}_e(h_E) = \begin{cases} 0.7, & \text{if} \quad h_E \in \{\Phi, \widetilde{E}\},\\ 0.5, & \text{if} \quad h_E = f_E,\\ 0, & \text{otherwise,}\end{cases}$$

$$\widetilde{W}_e(h_E) = \begin{cases} 0.7, & \text{if} \quad h_E \in \{\Phi, \widetilde{E}\}, \\ 0.5, & \text{if} \quad h_E = g_E, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the identity fuzzy soft mapping $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (X, \widetilde{W})$ is fuzzy soft weakly $\frac{1}{2}$-minimal continuous, but it is not fuzzy soft almost $\frac{1}{2}$-minimal continuous.

The following implications are obtained:

$$\text{fuzzy soft } r\text{-minimal continuity}$$
$$\Downarrow$$
$$\text{fuzzy soft almost } r\text{-minimal continuity}$$
$$\Downarrow$$
$$\text{fuzzy soft weakly } r\text{-minimal continuity.}$$

Theorem 3.5. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and X has the property (P). Then, φ_ψ is fuzzy soft weakly r-minimal continuous if $C_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(C_{m^*}(k, g_B, r))$ for each $g_B \in \widetilde{(Y, F)}$ that is a fuzzy soft r-minimal semiopen set, $e \in E$ and $r \in I_\circ$.*

Proof. Let $g_B \in \widetilde{M}^*_{k,r}$. Then, g_B is a fuzzy soft r-minimal semiopen set. From hypothesis and $g_B \sqsubseteq I_{m^*}(k, C_{m^*}(k, g_B, r), r)$, it follows that

$$\varphi_\psi^{-1}(C_{m^*}(k, g_B, r)) \sqsupseteq C_m(e, \varphi_\psi^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)), r) \sqsupseteq C_m(e, \varphi_\psi^{-1}(g_B), r).$$

Hence, by Theorem 3.3, φ_ψ is fuzzy soft almost r-minimal continuous. This implies φ_ψ is fuzzy soft weakly r-minimal continuous.

4. Fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous mappings

In this section, we introduce a concept of continuity in a very general setting called fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous mappings, and we investigate some properties of them.

First of all, let us introduce a concept of continuity in a very general setting. Let $(X, \widetilde{M})$ and $(Y, \widetilde{M}^*)$ be r-$\widetilde{FMS}$ s, $\mathcal{A}$ and $\mathcal{B} : E \times \widetilde{(X, E)} \times I_\circ \to I^X$ be operators on $(X, \widetilde{M})$, and C and $\mathcal{D} : F \times \widetilde{(Y, F)} \times I_\circ \to I^Y$ be operators on $(Y, \widetilde{M}^*)$, respectively. The difference between two fuzzy soft sets f_A and g_B is a fuzzy soft set, denoted by $f_A \overline{\sqcap}\, g_B$, where

$$(f_A \overline{\sqcap}\, g_B)(e) = \begin{cases} \underline{0}, & \text{if } f_A(e) \leq g_B(e), \\ f_A(e) \wedge (g_B(e))^c, & \text{otherwise,} \end{cases} \quad \text{for each } \; e \in E.$$

Definition 4.1. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and X has the property (P). Then, φ_ψ is called fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous if*

$$\mathcal{A}[e, \varphi_\psi^{-1}(\mathcal{D}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{B}[e, \varphi_\psi^{-1}(C(k, g_B, r)), r] = \Phi$$

for each $g_B \in \widetilde{M}^*_{k,r}$, $e \in E$ and $(\psi(e) = k) \in F$.

In 2021, Taha [23] defined the concept of fuzzy soft r-minimal continuous mappings: $\varphi_{\psi}^{-1}(g_B) \in \widetilde{M}_{e,r}$ for each $g_B \in \widetilde{M}^*_{k,r}$, $e \in E$ and $(\psi(e) = k) \in F$. We can see that the above definition generalizes the concepts of fuzzy soft r-minimal continuous mappings, when we choose $\mathcal{A}$ = identity operator, $\mathcal{B}$ = interior operator, C = identity operator and $\mathcal{D}$ = identity operator.

Let us give a historical justification of the above definition:

I. In Section 3, we defined the concept of fuzzy soft almost r-minimal continuous mappings: $\varphi_{\psi}^{-1}(g_B) \sqsubseteq I_m(e, \varphi_{\psi}^{-1}(I_{m^*}(k, C_{m^*}(k, g_B, r), r)), r)$ for each $g_B \in \widetilde{M}^*_{k,r}$. Here, $\mathcal{A}$ = identity operator, $\mathcal{B}$ = interior operator, C = interior closure operator and $\mathcal{D}$ = identity operator.

II. In Section 3, we defined the concept of fuzzy soft weakly r-minimal continuous mappings: $\varphi_{\psi}^{-1}(g_B) \sqsubseteq I_m(e, \varphi_{\psi}^{-1}(C_{m^*}(k, g_B, r)), r)$ for each $g_B \in \widetilde{M}^*_{k,r}$. Here, $\mathcal{A}$ = identity operator, $\mathcal{B}$ = interior operator, C = closure operator and $\mathcal{D}$ = identity operator.

Definition 4.2. *A fuzzy soft mapping $\varphi_{\psi} : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ is called fuzzy soft r-minimal Ⓢ-continuous iff $\varphi_{\psi}^{-1}(g_B) \in \widetilde{M}_{e,r}$ $\forall$ $g_B \in \widetilde{M}^*_{k,r}$ satisfies property Ⓢ.*

Let $C_{Ⓢ} : F \times \widetilde{(Y, F)} \times I_{\circ} \to I^Y$ be an operator on $(Y, \widetilde{M}^*)$ defined as follows:

$$C_{Ⓢ}(k, g_B, r) = \begin{cases} g_B, & \text{if } g_B \in \widetilde{M}^*_{k,r} \text{ and satisfies property Ⓢ}, \\ \widetilde{E}, & \text{otherwise.} \end{cases}$$

Theorem 4.1. *Let $\varphi_{\psi} : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and X has the property (P). Then, φ_{ψ} is fuzzy soft r-minimal Ⓢ-continuous iff it is fuzzy soft r-minimal $(id, I_m, C_{Ⓢ}, id)$-continuous.*

Proof. (⇒) Let φ_{ψ} be a fuzzy soft r-minimal Ⓢ-continuous and $g_B \in \widetilde{M}^*_{k,r}$.

Case 1. If g_B satisfies property Ⓢ, $C_{Ⓢ}(k, g_B, r) = g_B$, and $\varphi_{\psi}^{-1}(g_B) \in \widetilde{M}_{e,r}$. Thus, we obtain $\varphi_{\psi}^{-1}(g_B) \sqsubseteq I_m(e, \varphi_{\psi}^{-1}(g_B), r) = I_m(e, \varphi_{\psi}^{-1}(C_{Ⓢ}(k, g_B, r)), r)$. Then, $\varphi_{\psi}^{-1}(g_B) \,\overline{\sqcap}\, I_m(e, \varphi_{\psi}^{-1}(C_{Ⓢ}(k, g_B, r)), r) = \Phi$. Hence, φ_{ψ} is fuzzy soft r-minimal $(id, I_m, C_{Ⓢ}, id)$-continuous.

Case 2. If g_B does not satisfy property Ⓢ, $C_{Ⓢ}(k, g_B, r) = \widetilde{E}$. Thus, we obtain $\varphi_{\psi}^{-1}(g_B) \sqsubseteq I_m(e, \varphi_{\psi}^{-1}(\widetilde{E}), r) = I_m(e, \varphi_{\psi}^{-1}(C_{Ⓢ}(k, g_B, r)), r)$. Then, $\varphi_{\psi}^{-1}(g_B) \,\overline{\sqcap}\, I_m(e, \varphi_{\psi}^{-1}(C_{Ⓢ}(k, g_B, r)), r) = \Phi$. Hence, φ_{ψ} is fuzzy soft r-minimal $(id, I_m, C_{Ⓢ}, id)$-continuous.

(⇐) Suppose that $\varphi_{\psi}^{-1}(g_B) \,\overline{\sqcap}\, I_m(e, \varphi_{\psi}^{-1}(C_{Ⓢ}(k, g_B, r)), r) = \Phi$ for each $g_B \in \widetilde{M}^*_{k,r}$. Then, $\varphi_{\psi}^{-1}(g_B) \sqsubseteq I_m(e, \varphi_{\psi}^{-1}(C_{Ⓢ}(k, g_B, r)), r)$. If $g_B \in \widetilde{M}^*_{k,r}$ satisfies property Ⓢ, $C_{Ⓢ}(k, g_B, r) = g_B$, and hence $\varphi_{\psi}^{-1}(g_B) \sqsubseteq I_m(e, \varphi_{\psi}^{-1}(g_B), r)$. Thus, $\varphi_{\psi}^{-1}(g_B) \in \widetilde{M}_{e,r}$. Then, φ_{ψ} is fuzzy soft r-minimal Ⓢ-continuous.

Definition 4.3. *If $\mathcal{A}$ and $\mathcal{B}$ are operators on $(X, \widetilde{M})$, the intersection operator $\mathcal{A} \sqcap \mathcal{B}$ is defined as follows: $(\mathcal{A} \sqcap \mathcal{B})(e, f_A, r) = \mathcal{A}(e, f_A, r) \sqcap \mathcal{B}(e, f_A, r)$, $\forall$ $f_A \in \widetilde{(X, E)}$ and $e \in E$. $\mathcal{A}$ and $\mathcal{B}$ are called mutually dual if $\mathcal{A} \sqcap \mathcal{B}$ is the identity operator.*

Theorem 4.2. *Let $\varphi_{\psi} : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and X has the property (P). Let $\mathcal{A}$ and $\mathcal{B}$ be operators on $(X, \widetilde{M})$, and C, C^* and $\mathcal{D}$ be operators on $(Y, \widetilde{M}^*)$. Then, φ_{ψ} is fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C \sqcap C^*, \mathcal{D})$-continuous iff it is both fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous and fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C^*, \mathcal{D})$-continuous.*

Proof. If φ_{ψ} is both fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous and fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C^*, \mathcal{D})$-continuous, then for each $g_B \in \widetilde{M}^*_{k,r}$ we have

$$\mathcal{A}[e, \varphi_{\psi}^{-1}(\mathcal{D}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{B}[e, \varphi_{\psi}^{-1}(C(k, g_B, r)), r] = \Phi,$$

$$\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}(C^*(k,g_B,r)),r] = \Phi.$$

Hence,

$$[\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}(C(k,g_B,r)),r]]$$

$$\sqcup$$

$$[\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}(C^*(k,g_B,r)),r]] = \Phi.$$

However,

$$\begin{aligned}
&\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}((C\sqcap C^*)(k,g_B,r)),r]\\
=&\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}(C(k,g_B,r)\sqcap C^*(k,g_B,r)),r]\\
=&\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, (\mathcal{B}[e,\varphi_\psi^{-1}(C(k,g_B,r)),r]\sqcap \mathcal{B}[e,\varphi_\psi^{-1}(C^*(k,g_B,r)),r])\\
=&[\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}(C(k,g_B,r)),r]]\\
&\sqcup [\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}(C^*(k,g_B,r)),r]].
\end{aligned}$$

Thus, $\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}((C\sqcap C^*)(k,g_B,r)),r] = \Phi$. Then, φ_ψ is fuzzy soft r-minimal $(\mathcal{A},\mathcal{B},C\sqcap C^*,\mathcal{D})$-continuous.

Conversely, if φ_ψ is fuzzy soft r-minimal $(\mathcal{A},\mathcal{B},C\sqcap C^*,\mathcal{D})$-continuous, and $g_B\in\widetilde{M}^*_{k,r}$, then

$$\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}((C\sqcap C^*)(k,g_B,r)),r] = \Phi.$$

Now, by the above equalities, we get that

$$[\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}(C(k,g_B,r)),r]]$$

$$\sqcup$$

$$[\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}(C^*(k,g_B,r)),r]] = \Phi.$$

Then, we have

$$\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}(C(k,g_B,r)),r] = \Phi$$

and

$$\mathcal{A}[e,\varphi_\psi^{-1}(\mathcal{D}(k,g_B,r)),r] \,\overline{\sqcap}\, \mathcal{B}[e,\varphi_\psi^{-1}(C^*(k,g_B,r)),r] = \Phi,$$

which means that φ_ψ is both fuzzy soft r-minimal $(\mathcal{A},\mathcal{B},C,\mathcal{D})$-continuous and fuzzy soft r-minimal $(\mathcal{A},\mathcal{B},C^*,\mathcal{D})$-continuous.

Corollary 4.1. *Let $\varphi_\psi : (X,\widetilde{M}) \longrightarrow (Y,\widetilde{M}^*)$ be fuzzy soft almost r-minimal and fuzzy soft r-minimal $(id,I_m,\mathcal{G},id)$-continuous where $\mathcal{G}$ and $I_{m^*}(C_{m^*})$ are mutually dual operators on Y such that $\mathcal{G}(k,g_B,r) = g_B\sqcup(I_{m^*}(k,C_{m^*}(k,g_B,r),r))^c$ for each $g_B\in\widetilde{M}^*_{k,r}$ and $e\in E$. Then, φ_ψ is fuzzy soft r-minimal continuous iff φ_ψ is fuzzy soft almost r-minimal continuous, and $\varphi_\psi^{-1}(g_B)\,\overline{\sqcap}\, I_m(e,\varphi_\psi^{-1}(\mathcal{G}(k,g_B,r)),r) = \Phi$.*

Proof. Fuzzy soft almost r-minimal continuous is equal to fuzzy soft r-minimal $(id,I_m,I_{m^*}(C_{m^*}),id)$-continuous. Since $\mathcal{G}$ and $I_{m^*}(C_{m^*})$ are mutually dual operators on Y, then the result follows from Theorem 4.2.

Definition 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be operators on $(X, \widetilde{M})$. Then, $\mathcal{A} \sqsubseteq \mathcal{B}$ iff $\mathcal{A}(e, f_A, , r) \sqsubseteq \mathcal{B}(e, f_A, r)$, $\forall\ f_A \in \widetilde{(X, E)}$ and $e \in E$.*

Theorem 4.3. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and X has the property (P). Let $\mathcal{A}$ and $\mathcal{B}$ be operators on $(X, \widetilde{M})$, and C, C^* and $\mathcal{D}$ be operators on $(Y, \widetilde{M}^*)$ with $C \sqsubseteq C^*$. If φ_ψ is fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous, then it is fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C^*, \mathcal{D})$-continuous.*

Proof. If φ_ψ is fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous, and $g_B \in \widetilde{M}^*_{k,r}$, thus we obtain

$$\mathcal{A}[e, \varphi_\psi^{-1}(\mathcal{D}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{B}[e, \varphi_\psi^{-1}(C(k, g_B, r)), r] = \Phi.$$

Now, we know $C \sqsubseteq C^*$, for each $g_B \in \widetilde{M}^*_{k,r}$, $\mathcal{B}[e, \varphi_\psi^{-1}(C(k, g_B, r)), r] \sqsubseteq \mathcal{B}[e, \varphi_\psi^{-1}(C^*(k, g_B, r)), r]$. Therefore,

$$\mathcal{A}[e, \varphi_\psi^{-1}(\mathcal{D}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{B}[e, \varphi_\psi^{-1}(C^*(k, g_B, r)), r]$$
$$\sqsubseteq$$
$$\mathcal{A}[e, \varphi_\psi^{-1}(\mathcal{D}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{B}[e, \varphi_\psi^{-1}(C(k, g_B, r)), r].$$

Then, $\mathcal{A}[e, \varphi_\psi^{-1}(\mathcal{D}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{B}[e, \varphi_\psi^{-1}(C^*(k, g_B, r)), r] = \Phi$. Hence, φ_ψ is fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C^*, \mathcal{D})$-continuous.

Definition 4.5. *An operator $\mathcal{B}$ on $(X, \widetilde{M})$ induces another operator $I_m(\mathcal{B})$ defined as follows: $I_m(\mathcal{B})(e, f_A, r) = I_m(e, \mathcal{B}(e, f_A, r), r)$, $\forall\ f_A \in \widetilde{(X, E)}$. Observe that $I_m(\mathcal{B}) \sqsubseteq \mathcal{B}$.*

Definition 4.6. *A fuzzy soft mapping $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ satisfies the openness condition with respect to the operator $\mathcal{B}$ on X if $\mathcal{B}[e, \varphi_\psi^{-1}(f_A), r] \sqsubseteq \mathcal{B}[e, \varphi_\psi^{-1}(I_{m^*}(k, f_A, r)), r]$, $\forall\ f_A \in \widetilde{(X, E)}$.*

Theorem 4.4. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and X has the property (P). Let $\mathcal{A}$ and $\mathcal{B}$ be operators on $(X, \widetilde{M})$, and C and $\mathcal{D}$ be operators on $(Y, \widetilde{M}^*)$. If φ_ψ is fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous and satisfies the openness condition with respect to the operator $\mathcal{B}$, then φ_ψ is fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, I_{m^*}(C), \mathcal{D})$-continuous.*

Proof. If φ_ψ is fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, C, \mathcal{D})$-continuous, and $g_B \in \widetilde{M}^*_{k,r}$, thus

$$\mathcal{A}[e, \varphi_\psi^{-1}(\mathcal{D}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{B}[e, \varphi_\psi^{-1}(C(k, g_B, r)), r] = \Phi.$$

Since φ_ψ satisfies the openness condition with respect to the operator $\mathcal{B}$, then $\mathcal{B}[e, \varphi_\psi^{-1}(C(k, g_B, r)), r] \sqsubseteq \mathcal{B}[e, \varphi_\psi^{-1}(I_{m^*}(k, C(k, g_B, r), r)), r]$, and it follows that

$$\mathcal{A}[e, \varphi_\psi^{-1}(\mathcal{D}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{B}[e, \varphi_\psi^{-1}(I_{m^*}(k, C(k, g_B, r), r)), r]$$
$$\sqsubseteq$$
$$\mathcal{A}[e, \varphi_\psi^{-1}(\mathcal{D}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{B}[e, \varphi_\psi^{-1}(C(k, g_B, r)), r].$$

Thus, we obtain

$$\mathcal{A}[e, \varphi_\psi^{-1}(\mathcal{D}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{B}[e, \varphi_\psi^{-1}(I_{m^*}(k, C(k, g_B, r), r)), r] = \Phi.$$

Then, φ_ψ is fuzzy soft r-minimal $(\mathcal{A}, \mathcal{B}, I_{m^*}(C), \mathcal{D})$-continuous.

Corollary 4.2. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft mapping, and X has the property (P). Let $\mathcal{A}$ and $\mathcal{B}$ be operators on $(X, \widetilde{M})$, and C and $\mathcal{D}$ be operators on $(Y, \widetilde{M}^*)$. If φ_ψ is fuzzy soft weakly r-minimal continuous and satisfies the openness condition with respect to the operator $\mathcal{B}$, then φ_ψ is fuzzy soft almost r-minimal continuous.*

Proof. Let $\mathcal{A}$ = identity operator, $\mathcal{B}$ = interior operator, C = closure operator and $\mathcal{D}$ = identity operator. Then, the result follows from Theorem 4.4.

Theorem 4.5. *If $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ is a fuzzy soft r-minimal $(\mathcal{A}, I_m, C, \mathcal{D})$-continuous mapping, $f_A \sqsubseteq \mathcal{A}(e, f_A, r)$, and $g_B \sqsubseteq \mathcal{D}(k, g_B, r)$ $\forall$ $f_A \in \widetilde{(X, E)}$, $g_B \in \widetilde{(Y, F)}$, then $\varphi_\psi(f_A)$ is fuzzy soft r-minimal C-compact if f_A is fuzzy soft r-minimal compact.*

Proof. Given $\{(g_B)_i \in \widetilde{(Y, F)} | \ (g_B)_i \in \widetilde{M}^*_{k,r}\}_{i \in \Gamma}$ with $\varphi_\psi(f_A) \sqsubseteq \bigsqcup_{i\in\Gamma}(g_B)_i$, then $f_A \sqsubseteq \bigsqcup_{i\in\Gamma} \varphi_\psi^{-1}((g_B)_i)$. Since φ_ψ is fuzzy soft r-minimal $(\mathcal{A}, I_m, C, \mathcal{D})$-continuous, then for each $g_B \in \widetilde{M}^*_{k,r}$,

$$\varphi_\psi^{-1}(g_B) \sqsubseteq \mathcal{A}(e, \varphi_\psi^{-1}(\mathcal{D}(k, g_B, r)), r) \sqsubseteq I_m(e, \varphi_\psi^{-1}(C(k, g_B, r)), r).$$

Then, $f_A \sqsubseteq \bigsqcup_{i\in\Gamma} I_m(e, \varphi_\psi^{-1}(C(k, g_B, r)_i), r)$. Since f_A is fuzzy soft r-minimal compact, there exists a finite subset $\Gamma_\circ$ of Γ such that

$$f_A \sqsubseteq \bigsqcup_{i\in\Gamma_\circ} I_m(e, \varphi_\psi^{-1}(C(k, (g_B)_i, r)), r) \sqsubseteq \bigsqcup_{i\in\Gamma_\circ} \varphi_\psi^{-1}(C(k, (g_B)_i, r)).$$

Thus, $\varphi_\psi(f_A) \sqsubseteq \bigsqcup_{i\in\Gamma_\circ} C(k, (g_B)_i, r)$. Hence, $\varphi_\psi(f_A)$ is fuzzy soft r-minimal C-compact, as required.

The following corollaries are direct results.

Corollary 4.3. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft r-minimal continuous mapping. Then, $\varphi_\psi(f_A)$ is fuzzy soft r-minimal compact if $f_A \in \widetilde{(X, E)}$ is fuzzy soft r-minimal compact.*

Proof. Let $\mathcal{A}$ = identity operator, $\mathcal{B}$ = interior operator, C = identity operator and $\mathcal{D}$ = identity operator. Then, the result follows from Theorem 4.5.

Corollary 4.4. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft almost r-minimal continuous mapping. Then, $\varphi_\psi(f_A)$ is fuzzy soft r-minimal nearly compact if $f_A \in \widetilde{(X, E)}$ is fuzzy soft r-minimal compact.*

Proof. Let $\mathcal{A}$ = identity operator, $\mathcal{B}$ = interior operator, C = interior closure operator and $\mathcal{D}$ = identity operator. Then, the result follows from Theorem 4.5.

Corollary 4.5. *Let $\varphi_\psi : (X, \widetilde{M}) \longrightarrow (Y, \widetilde{M}^*)$ be a fuzzy soft weakly r-minimal continuous mapping. Then, $\varphi_\psi(f_A)$ is fuzzy soft r-minimal almost compact if $f_A \in \widetilde{(X, E)}$ is fuzzy soft r-minimal compact.*

Proof. Let $\mathcal{A}$ = identity operator, $\mathcal{B}$ = interior operator, C = closure operator and $\mathcal{D}$ = identity operator. Then, the result follows from Theorem 4.5.

5. Conclusions

In this paper, we have defined weaker forms of fuzzy soft r-minimal continuity called fuzzy soft almost r-minimal continuity and fuzzy soft weakly r-minimal continuity. We have investigated the master properties of these continuous forms and provided some illustrative examples to show the relationships between them. Then, we have introduced the concept of fuzzy soft r-minimal $(\mathcal{A},\mathcal{B},C,\mathcal{D})$-continuous mappings and given some characterizations of them. Moreover, we have proved that if $\mathcal{A}$ and $\mathcal{B}$ are operators on $(X,\widetilde{M})$, and C, C^* and $\mathcal{D}$ are operators on $(Y,\widetilde{M}^*)$, then $\varphi_\psi : (X,\widetilde{M}) \longrightarrow (Y,\widetilde{M}^*)$ is fuzzy soft r-minimal $(\mathcal{A},\mathcal{B},C \sqcap C^*,\mathcal{D})$-continuous iff it is both fuzzy soft r-minimal $(\mathcal{A},\mathcal{B},C,\mathcal{D})$-continuous and fuzzy soft r-minimal $(\mathcal{A},\mathcal{B},C^*,\mathcal{D})$-continuous.

In the upcoming work, we will define some new separation axioms in fuzzy soft r-minimal spaces. Also, we shall discuss the concepts given herein in the frames of infra soft topologies [5] and infra fuzzy topologies [7]. We hope that this work will contribute to fuzzy soft r-minimal structure studies.

Conflict of interest

The author declares that there is no conflict of interest.

References

1. B. Ahmad, A. Kharal, On fuzzy soft sets, *Adv. Fuzzy Syst.*, **2009** (2009), 1–6. https://doi.org/10.1155/2009/586507
2. H. Aktaş, N. Çağman, Soft sets and soft groups, *Inform. Sci.*, **177** (2007), 2726–2735. https://doi.org/10.1016/j.ins.2006.12.008
3. J. C. R. Alcantud, T. M. Al-shami, A. A. Azzam, Caliber and chain conditions in soft topologies, *Mathematics*, **9** (2021), 1–15. https://doi.org/10.3390/math9192349
4. T. M. Al-shami, Soft somewhere dense sets on soft topological spaces, *Commun. Korean Math. Soc.*, **33** (2018), 1341–1356. https://doi.org/10.4134/CKMS.c170378
5. T. M. Al-shami, New soft structure: Infra soft topological spaces, *Math. Probl. Eng.*, **2021** (2021), 1–12. https://doi.org/10.1155/2021/3361604
6. T. M. Al-shami, I. Alshammari, B. A. Asaad, Soft maps via soft somewhere dense sets, *Filomat*, **34** (2020), 3429–3440. https://doi.org/10.2298/FIL2010429A
7. Z. A. Ameen, T. M. Al-shami, A. A. Azzam, A. Mhemdi, A novel fuzzy structure: Infra-fuzzy topological spaces, *J. Funct. Space.*, **2022** (2022), 1–11. https://doi.org/10.1155/2022/9778069
8. A. Aygünoğlu, V. Çetkin, H. Aygün, An introduction to fuzzy soft topological spaces, *Hacet. J. Math. Stat.*, **43** (2014), 193–208.
9. N. Çağman, S. Enginoğlu, F. Çitak, Fuzzy soft set theory and its applications, *Iran. J. Fuzzy Syst.*, **8** (2011), 137–147. https://doi.org/10.22111/ijfs.2011.292
10. V. Çetkin, H. Aygün, Fuzzy soft semiregularization spaces, *Ann. Fuzzy Math. Inform.*, **7** (2014), 687–697.
11. D. N. Georgiou, A. C. Megaritis, V. I. Petropoulos, On soft topological spaces, *Appl. Math. Inform. Sci.*, **7** (2013), 1889–1901. https://doi.org/10.12785/amis/070527
12. S. Hussain, B. Ahmad, Soft separation axioms in soft topological spaces, *Hacet. J. Math. Stat.*, **44** (2015), 559–568. https://doi.org/10.15672/HJMS.2015449426
13. Y. B. Jun, Soft BCK/BCI algebras, *Comput. Math. Appl.*, **56** (2008), 1408–1413. https://doi.org/10.1016/j.camwa.2008.02.035

14. A. Kharal, B. Ahmad, Mappings on fuzzy soft classes, *Adv. Fuzzy Syst.*, **2009** (2009), 1–6. https://doi.org/10.1155/2009/407890
15. J. I. Kim, W. K. Min, Y. H. Yoo, Fuzzy r-compactness on fuzzy r-minimal spaces, *Int. J. Fuzzy Logic Intell. Syst.*, **9** (2009), 281–284. https://doi.org/10.5391/IJFIS.2009.9.4.281
16. G. J. Klir, B. Yuan, *Fuzzy sets and fuzzy logic: Theory and applications*, New Jersey: Prentice-Hall, 1995.
17. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *J. Fuzzy Math.*, **9** (2001), 589–602.
18. S. Mishar, R. Srivastava, Hausdorff fuzzy soft topological spaces, *Ann. Fuzzy Math. Inform.*, **9** (2015), 247–260.
19. D. Molodtsov, Soft set theory-First results, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
20. T. Rasham, M. S. Shabbir, P. Agarwal, S. Momani, On a pair of fuzzy dominated mappings on closed ball in the multiplicative metric space with applications, *Fuzzy Sets Syst.*, 2021, 1–16. https://doi.org/10.1016/j.fss.2021.09.002
21. A. P. Šostak, On a fuzzy topological structure, In: *Proceedings of the 13th winter school on abstract analysis*, Section of topology, Palermo: Circolo Matematico di Palermo, 1985, 89–103.
22. I. M. Taha, On fuzzy upper and lower α-ℓ-continuity and their decomposition, *J. Math. Comput. Sci.*, **11** (2021), 427–441. https://doi.org/10.28919/jmcs/5107
23. I. M. Taha, Compactness on fuzzy soft r-minimal spaces, *Int. J. Fuzzy Logic Intell. Syst.*, **21** (2021), 251–258. https://doi.org/10.5391/ijfis.2021.21.3.251
24. I. M. Taha, On r-generalized fuzzy ℓ-closed sets: Properties and applications, *J. Math.*, **2021** (2021), 1–8. https://doi.org/10.1155/2021/4483481
25. I. M. Taha, r-fuzzy δ-ℓ-open sets and fuzzy upper (lower) δ-ℓ-continuity via fuzzy idealization, *J. Math. Comput. Sci.*, **25** (2022), 1–9. https://doi.org/10.22436/jmcs.025.01.01
26. I. M. Taha, On upper and lower generalized semi-continuous fuzzy multifunctions, *J. Math. Comput. Sci.*, **25** (2022), 251–258. https://doi.org/10.22436/jmcs.025.03.04
27. B. P. Varol, H. Aygün, Fuzzy soft topology, *Hacet. J. Math. Stat.*, **41** (2012), 407–419.
28. B. P. Varol, A. Aygünoğlu, H. Aygün, Neighborhood structures of fuzzy soft topological spaces, *J. Int. Fuzzy Syst.*, **27** (2014), 2127–2135. https://doi.org/10.3233/IFS-141177
29. Y. H. Yoo, W. K. Min, J. I. L. Kim, Fuzzy r-minimal structures and fuzzy r-Minimal spaces, *Far East J. Math. Sci.*, **33** (2009), 193–205.
30. L. A. Zadeh, Fuzzy Sets, *Inform. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X