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Research article

Parametric generalized (p, q)-integral inequalities and applications

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Abstract: A new generalized (p, q)-integral identity is derived. Using this new identity as an auxiliary result, we derive new parametric generalizations of certain integral inequalities using the class of s-preinvex functions. We discuss several new and known special cases of the obtained results. This shows that our results are quite unifying. To demonstrate the significance of the main results, we also present some interesting applications.

Keywords: convex; preinvex; s-preinvex; post quantum calculus; integral inequalities
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1. Introduction

A set $C \subseteq \mathbb{R}$ is said to be convex, if

$$(1-\tau)x + \tau y \in C, \quad \forall x, y \in C, t \in [0, 1].$$

Similarly, a function $\mathcal{F} : C \to \mathbb{R}$ is said to be convex, if

$$\mathcal{F}((1-\tau)x + \tau y) \leq (1-\tau)\mathcal{F}(x) + \tau\mathcal{F}(y), \quad \forall x, y \in C, \tau \in [0, 1].$$

The theory of convexity, is a study of the properties of convex sets and convex functions. These classical concepts are simple in nature but are very useful from the application's point of view. These concepts have many applications in different fields of pure and applied sciences such as in optimization theory, operations research, numerical analysis and theory of means. In recent years these classical concepts have been generalized in different ways according to the need of the problem. Resultantly one can see a variety of new and significant generalizations of classical convexity in the literature. For useful details, see [1]. Besides applications, these classical concepts of the theory of convexity have also played a tremendous role in the development of the theory of inequalities. A wide class of integral inequalities are direct consequences of the applications of the convexity property of the functions. One of the most studied results regarding the convexity property of the functions is Hermite-Hadamard's inequality. This inequality was obtained by Hermite and Hadamard independently. This result provides us with a necessary and sufficient condition for a function to be convex. In recent years several new generalizations and variants of Hermite-Hadamard's inequality have been derived. The classical form of Hermite-Hadamard's inequality is:

Let $f : \mathcal{I} = [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

Dragomir and Pearce [2] have written a very useful and informative monograph on recent developments of Hermite-Hadamard's inequality and its applications. From the application's point of view, this double inequality has been studied intensively and one can see that this result has wide applications in numerical analysis and in the theory of means. Another significant result pertaining to the convex functions is Simpson's inequality. This result reads as:

Let $\mathcal{F} : [a, b] \to \mathbb{R}$ be a four times continuously differentiable function on (a, b) and $\|\mathcal{F}^4\|_\infty = \sup_{x\in(a,b)} |\mathcal{F}^4(x)| < \infty$, then

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(a)+\mathcal{F}(b)}{2} + 2\mathcal{F}\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b \mathcal{F}(x)\mathrm{d}x\right| \leq \frac{1}{2880}\|\mathcal{F}^4\|_\infty (b-a)^4.$$

This integral inequality has also been extended and generalized in different directions using novel and innovative approaches. For example, Sarikaya et al. [3] derived Simpson's inequality using differentiable convex functions.

Let $\mathcal{F} : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) such that $\mathcal{F}' \in L[a, b]$. If $|\mathcal{F}'|^q$ is convex, then

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathcal{F}(a) + 4\mathcal{F}\left(\frac{a+b}{2}\right) + \mathcal{F}(b)\right] - \frac{1}{b-a}\int_a^b \mathcal{F}(x)\mathrm{d}x\right| \\ &\leq \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left\{\left(\frac{|\mathcal{F}'(a)|^q + 3|\mathcal{F}'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|\mathcal{F}'(a)|^q + |\mathcal{F}'(b)|^q}{4}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Dragomir et al. [4] has written a very informative paper regarding the developments and applications of Simpson's inequality.

Like convexity, these inequalities have also been extended and generalized in different ways. One of the most significant and interesting ways to obtain a new variant is to use the concepts of quantum calculus instead of ordinary calculus.

Quantum calculus which is commonly known as q-calculus is a branch of mathematics in which we obtain q-analogues of mathematical objects, which can be recaptured by taking $q \to 1^-$. It is also known as calculus without limits. This fascinating branch of mathematics has attracted many researchers because of its numerous applications in both mathematics and physics. It also serves as the bridge between mathematics and physics. q-calculus is also considered a subfield of time scale calculus. Time scale calculus provides a unified framework for studying dynamic equations in discrete and continuous domains. In q-calculus, we are concerned with a specific times scale, called the q-time scale. Over the years, the classical concepts of q-calculus have been modified and generalized according to the need of the problems.

Tariboon and Ntouyas [5] introduced the q-calculus concepts on finite intervals and obtained several new q-analogues of classical mathematical objects. The ideas presented in their paper have attracted many researchers; thus, it opened a new dimension of research. Another interesting generalization of q-calculus is post quantum calculus, also known as (p, q)-calculus. In q-calculus, we deal with a q–number with one base q; however, (p, q)-calculus includes p- and q-numbers with two independent variables p and q. Chakarabarti and Jagannathan first considered this in [6].

As discussed earlier, convexity and its link with inequalities is a fascinating area of mathematics. Many inequalities, particularly integral inequalities known to us, can be derived by using the convexity property of the functions. Tariboon and Ntouyas [7] used the concepts of q-calculus and obtained various q-analogues of several classical inequalities. Sudsutad et al. [8] and Noor et al. [9] obtained some q-analogues of Hermite-Hadamard like inequalities. Alp et al. [10] obtained a corrected q-analogue of Hermite-Hadamard's inequality. Noor et al. [11] obtained q-analogues of Hermite-Hadamard like inequalities using the concept of preinvex functions. Noor et al. [12] also obtained q-Ostrowski-type of inequalities. Zhang et al. [13] derived a generalized q-integral identity and, using this identity, obtained several new q-analogues of integral inequalities. Recently, Du et al. [14] derived another new generalized q-integral identity and obtained quantum estimates of parameterized integral inequalities. Deng et al. [15] obtained the quantum analogues of Simpson type of integral inequalities.

Kunt et al. [16] derived (p, q)-analogues of Hermite–Hadamard's inequality. Awan et al. [17] recently derived a new (p, q)-integral identity using twice (p, q)-differentiable functions and obtained new associated (p, q)-inequalities.

Since it has been discussed that integral inequalities play a fundamental role in different fields of pure and applied sciences, for recent and interesting studies, see [18, 19]. Thus, the main purpose of this study was to derive new (p, q)-analogues of integral inequalities of Simpson's type by essentially using the s-preinvexity property of the function. For this, we first derive a new generalized (p, q)-integral identity. Utilizing this new identity as an auxiliary result, we establish our main results. We also discuss several special cases, which shows that our results are quite unifying. In the last section, we also present some applications of the main results to show the significance of the results. We hope that the ideas and techniques of this paper will attract interested readers.

2. Preliminaries

In this section, we discuss some previously known concepts that will be helpful in obtaining the main results of the paper.

Definition 1 ([20]). *A function $\mathcal{F} : \mathcal{B} \to \mathbb{R}$ is said to be preinvex with respect to bifunction $\theta(.,.)$:*

$\mathcal{B} \times \mathcal{B} \to \mathbb{R}$, *if*

$$\mathcal{F}(x + \tau\theta(y, x)) \leq (1 - \tau)\mathcal{F}(x) + \tau\mathcal{F}(y)$$

for all $x, y \in \mathcal{B}$ *and* $\tau \in [0, 1]$.

Here, $\mathcal{B} \subset \mathbb{R}^n$ is an invex set introduced and studied by B-Israel and Mond [21].

Another important generalization of preinvex functions is s-preinvex functions. The class of s-preinvex functions is defined as:

Definition 2 ([22]). *A function* $\mathcal{F} : \mathcal{B} \to \mathbb{R}$ *is said to be s-preinvex with respect to bifunction* $\theta(.,.) : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$, *if*

$$\mathcal{F}(x + \tau\theta(y, x)) \leq (1 - \tau)^s\mathcal{F}(x) + \tau^s\mathcal{F}(y)$$

for all $x, y \in \mathcal{B}$ *and* $\tau \in [0, 1]$, $s \in (0, 1]$.

In order to obtain some of the main results of the paper, we need the famous condition C, which was introduced and studied by Mohan and Neogy [23].

Condition C. A set $\mathcal{B} \subset \mathbb{R}$ is said to be an invex set with respect to bifunction $\theta(.,.)$ if and only if for any $x, y \in \mathcal{B}$ and $\tau \in [0, 1]$, we have

(i) $\theta(x, x + \tau\theta(y, x)) = -\tau\theta(y, x)$;

(ii) $\theta(y, x + \tau\theta(y, x)) = (1 - \tau)\theta(y, x)$.

Note that for any $x, y \in \mathcal{B}$, $\tau_1, \tau_2 \in [0, 1]$ and from condition C, we can deduce

$$\theta(x + \tau_2\theta(y, x), x + \tau_1\theta(y, x)) = (\tau_2 - \tau_1)\theta(y, x).$$

For the sake of completeness, we will now recall the some basic concepts from quantum and post quantum calculus.

Definition 3 ([5, 7]). *Let* $\mathcal{F} : J := [a, b] \subset \mathbb{R} \to \mathbb{R}$ *be an arbitrary function. Then, the q–derivative of* $\mathcal{F}$ *on* J *at* τ *is defined by*

$${}_aD_q\mathcal{F}(\tau) = \begin{cases} \dfrac{\mathcal{F}(\tau) - \mathcal{F}(q\tau + (1 - q)a)}{(1 - q)(\tau - a)}, & \text{if } \tau \neq a; \\ \lim\limits_{\tau \to a} D_q\mathcal{F}(\tau), & \text{if } \tau = a, \end{cases}$$

where $0 < q < 1$ *is a constant.*

Definition 4 ([5, 7]). *Let* $\mathcal{F} : J := [a, b] \to \mathbb{R}$ *be an arbitrary function. Then, q–integral on* J *is defined as*

$$\int_a^x \mathcal{F}(\delta)\mathrm{d}_q\delta = (1 - q)(x - a)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n x + (1 - q^n)a)$$

for $x \in J$ *and* $0 < q < 1$.

Definition 5 ([24]). *Let $\mathcal{X} : J \to \mathbb{R}$ be a continuous function and let $x \in J$ and $0 < q < p \leq 1$. Then the (p, q)-derivative on J of function $\mathcal{X}$ at x is defined as*

$$ {}_a\mathrm{D}_{p,q}\mathcal{X}(x) = \frac{\mathcal{X}(px + (1-p)a)) - \mathcal{X}(qx + (1-q)a)}{(p-q)(x-a)}, \quad x \neq a. \tag{2.1}$$

Definition 6 ([24]). *Let $\mathcal{X} : J \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the (p, q)-integral on J is defined as:*

$$\int_a^x \mathcal{X}(\tau)\mathrm{d}_{p,q}\tau = (p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\mathcal{X}\left(\frac{q^n}{p^{n+1}}x + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right)$$

for $x \in J$.

Note that, if we take $p = 1$ in the above definitions, then we get the concepts for quantum calculus. For arbitrary real numbers α and β, where $\alpha \neq \beta$, we define

1) Arithmetic Mean
$$A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$
for $\alpha, \beta \in \mathbb{R}$.
2) Logarithmic Mean
$$L(\alpha, \beta) = \frac{\beta - \alpha}{\ln|\beta| - \ln|\alpha|}$$
for $\alpha, \beta \in \mathbb{R}\backslash\{0\}$.
3) Generalized Log-Mean
$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)}\right]^{\frac{1}{n}}$$
for $n \in \mathbb{N}$ with $n \geq 1$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$.

For more information, see [25].

3. A key lemma

In this section, we present an identity associated with the (p, q)-integral, which plays an important role in establishing our main results.

Lemma 1. *Let $\mathcal{F} : \mathcal{B} = [a, a + \theta(b, a)] \to \mathbb{R}$ be a (p, q)-differentiable function on $\mathcal{B}^\circ$ with $\theta(b, a) > 0$. If ${}_a\mathrm{D}_{p,q}\mathcal{F}$ is integrable on $\mathcal{B}$ and $0 < q < p \leq 1$, then*

$$\begin{aligned} &\varepsilon\mathcal{F}(a) + (\delta - \varepsilon)\mathcal{F}\left(\frac{2a + \theta(b, a)}{2}\right) + (1 - \delta)\mathcal{F}(a + \theta(b, a)) - \frac{1}{p\theta(b, a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x \\ &= \theta(b, a)\left[\int_0^{\frac{1}{2}}(q\tau - \varepsilon)\,{}_a\mathrm{D}_{p,q}\mathcal{F}(a + \tau\theta(b, a))\,{}_0\mathrm{d}_{p,q}\tau + \int_{\frac{1}{2}}^1(q\tau - \delta)\,{}_a\mathrm{D}_{p,q}\mathcal{F}(a + \tau\theta(b, a))\,{}_0\mathrm{d}_{p,q}\tau\right]. \end{aligned} \tag{3.1}$$

Proof. Let

$$\mathcal{K}_1^{\diamond} = \int_0^{\frac{1}{2}} (q\tau - \varepsilon)\ {}_a\mathrm{D}_{p,q}\mathcal{F}(a + \tau\theta(b,a))\, {}_0\mathrm{d}_{p,q}\tau$$

and

$$\mathcal{K}_2^{\diamond} = \int_{\frac{1}{2}}^{1} (q\tau - \delta)\ {}_a\mathrm{D}_{p,q}\mathcal{F}(a + \tau\theta(b,a))\, {}_0\mathrm{d}_{p,q}\tau.$$

A direct computation gives

$$\begin{aligned}
\mathcal{K}_1^{\diamond} &= \int_0^{\frac{1}{2}} q\tau\, {}_a\mathrm{D}_{p,q}\mathcal{F}(a + \tau\theta(b,a))\, {}_0\mathrm{d}_{p,q}\tau - \varepsilon \int_0^{\frac{1}{2}} {}_a\mathrm{D}_{p,q}\mathcal{F}(a + \tau\theta(b,a))\, {}_0\mathrm{d}_{p,q}\tau \\
&= \int_0^{\frac{1}{2}} q\frac{\mathcal{F}(a + p\tau\theta(b,a)) - \mathcal{F}(a + q\tau\theta(b,a))}{(p-q)\theta(b,a)}\, {}_0\mathrm{d}_{p,q}\tau \\
&\quad - \varepsilon \int_0^{\frac{1}{2}} \frac{\mathcal{F}(a + p\tau\theta(b,a)) - \mathcal{F}(a + q\tau\theta(b,a))}{\tau(p-q)\theta(b,a)}\, {}_0\mathrm{d}_{p,q}\tau \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{q^{n+1}}{p^{n+1}} \frac{\mathcal{F}\left(a + p\frac{q^n}{2p^{n+1}}\theta(b,a)\right) - \mathcal{F}\left(a + \frac{q^{n+1}}{2p^{n+1}}\theta(b,a)\right)}{\theta(b,a)} \\
&\quad - \varepsilon \sum_{n=0}^{\infty} \frac{\mathcal{F}\left(a + p\frac{q^n}{2p^{n+1}}\theta(b,a)\right) - \mathcal{F}\left(a + \frac{q^{n+1}}{2p^{n+1}}\theta(b,a)\right)}{\theta(b,a)} \\
&= \frac{q\sum\limits_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\mathcal{F}\left(a + p\frac{q^n}{2p^{n+1}}\theta(b,a)\right) - p\sum\limits_{n=1}^{\infty} \frac{q^n}{p^{n+1}}\mathcal{F}\left(a + p\frac{q^n}{2p^{n+1}}\theta(b,a)\right)}{2\theta(b,a)} \\
&\quad - \varepsilon \frac{\sum\limits_{n=0}^{\infty} \mathcal{F}\left(a + p\frac{q^n}{2p^{n+1}}\theta(b,a)\right) - \sum\limits_{n=1}^{\infty} \mathcal{F}\left(a + p\frac{q^n}{2p^{n+1}}\theta(b,a)\right)}{\theta(b,a)} \\
&= \frac{1}{2}\left[\frac{\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{\theta(b,a)} - (p-q)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \frac{\mathcal{F}\left(a + p\frac{q^n}{2p^{n+1}}\theta(b,a)\right)}{\theta(b,a)}\right] - \varepsilon\frac{\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right) - \mathcal{F}(a)}{\theta(b,a)} \\
&= \left(\frac{1}{2} - \varepsilon\right)\frac{\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{\theta(b,a)} + \frac{\varepsilon}{\theta(b,a)}\mathcal{F}(a) - \frac{1}{2}(p-q)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \frac{\mathcal{F}\left(a + p\frac{q^n}{2p^{n+1}}\theta(b,a)\right)}{\theta(b,a)}
\end{aligned}$$

$$= \left(\frac{1}{2} - \varepsilon\right)\frac{\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{\theta(b,a)} + \frac{\varepsilon}{\theta(b,a)}\mathcal{F}(a) - \frac{1}{p\theta^2(b,a)}\int_a^{a+\frac{p}{2}\theta(b,a)} \mathcal{F}(x)\, {}_0\mathrm{d}_{p,q}x.$$

On the other hand, one has

$$\begin{aligned}
\mathcal{K}_2^{\diamond} &= \int_{\frac{1}{2}}^{1} q\tau\, {}_a\mathrm{D}_{p,q}\mathcal{F}(a + \tau\theta(b,a))\, {}_0\mathrm{d}_{p,q}\tau - \delta \int_{\frac{1}{2}}^{1} {}_a\mathrm{D}_{p,q}\mathcal{F}(a + \tau\theta(b,a))\, {}_0\mathrm{d}_{p,q}\tau \\
&= \int_0^{1} q\tau\, {}_a\mathrm{D}_{p,q}\mathcal{F}(a + \tau\theta(b,a))\, {}_0\mathrm{d}_{p,q}\tau - \delta \int_0^{1} {}_a\mathrm{D}_{p,q}\mathcal{F}(a + \tau\theta(b,a))\, {}_0\mathrm{d}_{p,q}\tau
\end{aligned}$$

$$-\left(\int_0^{\frac{1}{2}} q\tau\, {}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau - \delta\int_0^{\frac{1}{2}} {}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau\right).$$

Since

$$\begin{aligned}
&\int_0^1 q\tau\, {}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau - \delta\int_0^1 {}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau\\
&\quad= \int_0^1 q\frac{\mathcal{F}(a+p\tau\theta(b,a)) - \mathcal{F}(a+q\tau\theta(b,a))}{(p-q)\theta(b,a)}\,{}_0\mathrm{d}_{p,q}\tau\\
&\qquad - \delta\int_0^1 \frac{\mathcal{F}(a+p\tau\theta(b,a)) - \mathcal{F}(a+q\tau\theta(b,a))}{\tau(p-q)\theta(b,a)}\,{}_0\mathrm{d}_{p,q}\tau\\
&\quad= \sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+1}}\frac{\mathcal{F}\left(a+p\frac{q^n}{p^{n+1}}\theta(b,a)\right) - \mathcal{F}\left(a+\frac{q^{n+1}}{p^{n+1}}\theta(b,a)\right)}{\theta(b,a)}\\
&\qquad - \delta\sum_{n=0}^{\infty}\frac{\mathcal{F}\left(a+p\frac{q^n}{p^{n+1}}\theta(b,a)\right) - \mathcal{F}\left(a+\frac{q^{n+1}}{p^{n+1}}\theta(b,a)\right)}{\theta(b,a)}\\
&\quad= \frac{q\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\mathcal{F}\left(a+p\frac{q^n}{p^{n+1}}\theta(b,a)\right) - p\sum\limits_{n=1}^{\infty}\frac{q^n}{p^{n+1}}\mathcal{F}\left(a+p\frac{q^n}{p^{n+1}}\theta(b,a)\right)}{\theta(b,a)}\\
&\qquad - \delta\frac{\sum\limits_{n=0}^{\infty}\mathcal{F}\left(a+p\frac{q^n}{p^{n+1}}\theta(b,a)\right) - \sum\limits_{n=0}^{\infty}\mathcal{F}\left(a+p\frac{q^n}{p^{n+1}}\theta(b,a)\right)}{\theta(b,a)}\\
&\quad= \frac{\mathcal{F}(a+\theta(b,a))}{\theta(b,a)} - \sum_{n=0}^{\infty}(p-q)\frac{q^n}{q^{n+1}}\frac{\mathcal{F}\left(a+p\frac{q^n}{p^{n+1}}\theta(b,a)\right)}{\theta(b,a)} - \delta\frac{\mathcal{F}(a+\theta(b,a)) - \mathcal{F}(a)}{\theta(b,a)}\\
&\quad= \frac{(1-\delta)}{\theta(b,a)}\cdot\mathcal{F}(a+\theta(b,a)) + \frac{\delta}{\theta(b,a)}\mathcal{F}(a) - \sum_{n=0}^{\infty}(p-q)\frac{q^n}{p^{n+1}}\frac{\mathcal{F}\left(a+p\frac{q^n}{p^{n+1}}\theta(b,a)\right)}{\theta(b,a)}\\
&\quad= \frac{(1-\delta)}{\theta(b,a)}\mathcal{F}(a+\theta(b,a)) + \frac{\delta}{\theta(b,a)}\mathcal{F}(a) - \frac{1}{p\theta^2(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_0\mathrm{d}_{p,q}x
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}} q\tau\, {}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau - \delta\int_0^{\frac{1}{2}} {}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau\\
&\quad= \frac{1}{2}\left[\frac{\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{\theta(b,a)} - (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\frac{\mathcal{F}\left(a+p\frac{q^n}{p^{n+1}}\theta(b,a)\right)}{\theta(b,a)}\right] - \delta\frac{\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right) - \mathcal{F}(a)}{\theta(b,a)}\\
&\qquad = \left(\frac{1}{2}-\delta\right)\frac{\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{\theta(b,a)} + \frac{\delta}{\theta(b,a)}\mathcal{F}(a) - \frac{1}{p\theta^2(b,a)}\int_a^{a+\frac{p}{2}\theta(b,a)}\mathcal{F}(x)\,{}_0\mathrm{d}_{p,q}x.
\end{aligned}$$

Then, we obtain

$$\mathcal{K}_2^{\diamond} = \frac{(1-\delta)}{\theta(b,a)}\mathcal{F}(a+\theta(b,a)) - \left(\frac{1}{2}-\delta\right)\frac{\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{\theta(b,a)} - \frac{1}{p\theta^2(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_0\mathrm{d}_{p,q}x$$

$$+\frac{1}{p\theta^2(b,a)}\int_a^{a+\frac{p}{2}\theta(b,a)}\mathcal{F}(x)\,{}_0\mathrm{d}_{p,q}x,$$

which leads to the desired identity (3.1). □

Note that if we take $\theta(b,a)=b-a$ and $p=1$ in Lemma 1, then we recaptures Lemma 2.1 [14].

We will now discuss some more special cases of Lemma 1. This shows that by setting different suitable values for the parameters ε and δ, we can obtain new integral identities.

Corollary 1. *Under the assumptions of Lemma 1, taking $\varepsilon=\frac{1}{6}$ and $\delta=\frac{5}{6}$, we obtain*

$$\begin{aligned}&\frac{\mathcal{F}(a)}{6}+\frac{2\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{3}+\frac{\mathcal{F}(a+\theta(b,a))}{6}-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x\\&=\theta(b,a)\left[\int_0^{\frac{1}{2}}\left(q\tau-\frac{1}{6}\right){}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau+\int_{\frac{1}{2}}^1\left(q\tau-\frac{5}{6}\right){}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau\right].\end{aligned}$$

Corollary 2. *Under the assumptions of Lemma 1, taking $\varepsilon=\delta=\frac{q}{p+q}$, we obtain*

$$\begin{aligned}&\frac{q}{p+q}\mathcal{F}(a)+\frac{p}{p+q}\mathcal{F}(a+\theta(b,a))-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x\\&=q\theta(b,a)\int_0^1\left(\tau-\frac{1}{p+q}\right){}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau.\end{aligned}$$

Corollary 3. *Under the assumptions of Lemma 1, taking $\varepsilon=\frac{1}{4}$ and $\delta=\frac{3}{4}$, we obtain*

$$\begin{aligned}&\frac{\mathcal{F}(a)}{4}+\frac{2\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{4}+\frac{\mathcal{F}(a+\theta(b,a))}{2}-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x\\&=\theta(b,a)\left[\int_0^{\frac{1}{2}}\left(q\tau-\frac{1}{4}\right){}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau+\int_{\frac{1}{2}}^1\left(q\tau-\frac{3}{4}\right){}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau\right].\end{aligned}$$

Corollary 4. *Under the assumptions of Lemma 1, if we take $p=1$, we obtain*

$$\begin{aligned}&\varepsilon\mathcal{F}(a)+(\delta-\varepsilon)\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)+(1-\delta)\mathcal{F}(a+\theta(b,a))-\frac{1}{\theta(b,a)}\int_a^{a+\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_qx\\&=\theta(b,a)\left[\int_0^{\frac{1}{2}}(q\tau-\varepsilon)\,{}_a\mathrm{D}_q\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_q\tau+\int_{\frac{1}{2}}^1(q\tau-\delta)\,{}_a\mathrm{D}_q\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_q\tau\right].\end{aligned}$$

4. Main results and discussion

In this section, we will discuss our main results.

Theorem 1. *Let $\mathcal{F}:\mathcal{B}=[a,a+\theta(b,a)]\to\mathbb{R}$ be a (p,q)-differentiable function on $\mathcal{B}^\circ$ with $\theta(b,a)>0$. If $|{}_a\mathrm{D}_q\mathcal{F}|$ is integrable and an s-preinvex function with $0<q<p\le 1$, then*

$$\left|\varepsilon\mathcal{F}(a)+(\delta-\varepsilon)\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)+(1-\delta)\mathcal{F}(a+\theta(b,a))-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x\right|$$

$$\leq \theta(b,a)\Bigg[|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(a)|\Bigg(2^{1-s}(L_1^{\diamond}(\varepsilon;p,q)+L_2^{\diamond}(\delta;p,q))-(L_3^{\diamond}(s,\varepsilon;p,q)+L_4^{\diamond}(s,\delta;p,q))\Bigg)$$
$$+|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(b)|\Bigg(L_3^{\diamond}(s,\varepsilon;p,q)+L_4^{\diamond}(s,\delta;p,q)\Bigg)\Bigg], \tag{4.1}$$

where

$$L_1^{\diamond}(\varepsilon;p,q)=\int\limits_0^{\frac{1}{2}}|q\tau-\varepsilon|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{8\varepsilon^2(p+q-1)-2\varepsilon q(p+q)+q^2}{4q(p+q)}, & \text{for } 0\leq\frac{\varepsilon}{q}\leq\frac{1}{2};\\ \frac{2\varepsilon(p+q)-q}{4(p+q)}, & \text{for } \frac{1}{2}<\frac{\varepsilon}{q},\end{cases}$$

$$L_2^{\diamond}(\delta;p,q)=\int\limits_{\frac{1}{2}}^{1}|q\tau-\delta|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{3q-2\delta(p+q)}{4(p+q)}, & \text{for } 0\leq\frac{\delta}{q}\leq\frac{1}{2};\\ \frac{8\delta^2(p+q-1)-6q\delta(p+q)+5q^2}{4q(p+q)}, & \text{for } \frac{1}{2}<\frac{\delta}{q}\leq 1;\\ \frac{2\delta(p+q)-3q}{4(p+q)}, & \text{for } 1<\frac{\delta}{q},\end{cases}$$

$$L_3^{\diamond}(s,\varepsilon;p,q)=\int_0^{\frac{1}{2}}\tau^s\,|q\tau-\varepsilon|\,{}_0\mathrm{d}_{q}\tau=\begin{cases}\frac{2\varepsilon^{s+2}(p-q)}{q^{s+1}}\left(\frac{1}{p^{s+1}-q^{s+1}}-\frac{1}{p^{s+2}-q^{s+2}}\right)\\ +\frac{(p-q)(q-2\varepsilon)+(q-p)(1-2\varepsilon)q^{s+2}}{2^{s+2}(p^{s+1}-q^{s+1})(p^{s+2}-q^{s+2})}, & \text{for } 0\leq\frac{\varepsilon}{q}\leq\frac{1}{2};\\ -\frac{(p-q)(q-2\varepsilon)+(q-p)(1-2\varepsilon)q^{s+2}}{2^{s+2}(p^{s+1}-q^{s+1})(p^{s+2}-q^{s+2})}, & \text{for } \frac{1}{2}<\frac{\varepsilon}{q},\end{cases}$$

$$L_4^{\diamond}(s,\delta;p,q)=\int_{\frac{1}{2}}^{1}\tau^s\,|q\tau-\delta|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{\delta(p-q)(1-2^{s+1})}{2^{s+1}(p^{s+1}-q^{s+1})}+\frac{q(p-q)(2^{s+2}-1)}{2^{s+2}(p^{s+2}-q^{s+2})}, & \text{for } 0\leq\frac{\delta}{q}\leq\frac{1}{2},\\ -\frac{\delta(p-q)(1+2^{s+1})}{2^{s+1}(p^{s+1}-q^{s+1})}+\frac{q(p-q)(1+2^{s+2})}{2^{s+2}(p^{s+2}-q^{s+2})}\\ \frac{2\delta^{s+2}(p-q)}{q^{s+1}}\left(\frac{1}{p^{s+1}-q^{s+1}}-\frac{1}{p^{s+2}-q^{s+2}}\right), & \text{for } \frac{1}{2}<\frac{\delta}{q}\leq 1;\\ -\frac{\delta(p-q)(1-2^{s+1})}{2^{s+1}(p^{s+1}-q^{s+1})}+\frac{q(p-q)(2^{s+2}-1)}{2^{s+2}(p^{s+2}-q^{s+2})}, & \text{for } 1<\frac{\delta}{q}.\end{cases}$$

Proof. Using Lemma 1 and the given condition that $|{}_{a}\mathrm{D}_{p,q}\mathcal{F}|$ is an s-preinvex function, we have

$$\left|\varepsilon\mathcal{F}(a)+(\delta-\varepsilon)\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)+(1-\delta)\mathcal{F}(a+\theta(b,a))-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x\right|$$
$$=\theta(b,a)\left|\int_0^{\frac{1}{2}}(q\tau-\varepsilon)\;{}_{a}\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau+\int_{\frac{1}{2}}^{1}(q\tau-\delta)\;{}_{a}\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau\right|$$
$$\leq\theta(b,a)\left[\int_0^{\frac{1}{2}}|q\tau-\varepsilon|\,|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))|\,{}_0\mathrm{d}_{p,q}\tau+\int_{\frac{1}{2}}^{1}|q\tau-\delta|\,|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))|\,{}_0\mathrm{d}_{p,q}\tau\right]$$
$$\leq\theta(b,a)\left[\int_0^{\frac{1}{2}}|q\tau-\varepsilon|\left((1-\tau)^s|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(a)|+\tau^s|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(b)|\right){}_0\mathrm{d}_{p,q}\tau\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}|q\tau-\delta|\left((1-\tau)^s|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(a)|+\tau^s|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(b)|\right){}_0\mathrm{d}_{p,q}\tau\right]$$
$$=\theta(b,a)\left[|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(a)|\left(\int_0^{\frac{1}{2}}(1-\tau)^s\,|q\tau-\varepsilon|\,{}_0\mathrm{d}_{p,q}\tau+\int_{\frac{1}{2}}^{1}(1-\tau)^s\,|q\tau-\delta|\,{}_0\mathrm{d}_{p,q}\tau\right)\right.$$
$$\left.+|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(b)|\left(\int_0^{\frac{1}{2}}\tau^s\,|q\tau-\varepsilon|\,{}_0\mathrm{d}_{p,q}\tau+\int_{\frac{1}{2}}^{1}\tau^s\,|q\tau-\delta|\,{}_0\mathrm{d}_{p,q}\tau\right)\right]$$
$$\leq\theta(b,a)\left[|{}_{a}\mathrm{D}_{p,q}\mathcal{F}(a)|\left(\int_0^{\frac{1}{2}}(2^{1-s}-\tau^s)\,|q\tau-\varepsilon|\,{}_0\mathrm{d}_{p,q}\tau+\int_{\frac{1}{2}}^{1}(2^{1-s}-\tau^s)\,|q\tau-\delta|\,{}_0\mathrm{d}_{p,q}\tau\right)\right.$$

$$
\begin{aligned}
&+|{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|\left(\int_0^{\frac{1}{2}} \tau^s\,|q\tau-\varepsilon|\;{}_0\mathrm{d}_{p,q}\tau+\int_{\frac{1}{2}}^1 \tau^s\,|q\tau-\delta|\;{}_0\mathrm{d}_{p,q}\tau\right)\Bigg]\\
&=\theta(b,a)\Big[|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|\left(2^{1-s}(L_1^\diamond(\varepsilon;p,q)+L_2^\diamond(\delta;p,q))-(L_3^\diamond(s,\varepsilon;p,q)+L_4^\diamond(s,\delta;p,q))\right)\\
&\quad+|{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|\left(L_3^\diamond(s,\varepsilon;p,q)+L_4^\diamond(s,\delta;p,q)\right)\Big].
\end{aligned}
$$

This completes the proof. □

If we take $\theta(b,a)=b-a$ and $p=1$ in Theorem 1, then we have Theorem 2.1 [14].

We will now discuss some more special cases of Theorem 1.

Corollary 5. *Under the assumptions of Theorem 1, taking $\varepsilon=\frac{1}{6}$ and $\delta=\frac{5}{6}$, we obtain*

$$
\begin{aligned}
&\left|\frac{\mathcal{F}(a)}{6}+\frac{2\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{3}+\frac{\mathcal{F}(a+\theta(b,a))}{6}-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x\right|\\
&\le\theta(b,a)\Big[|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|\left(2^{1-s}\left(M_1^\diamond(p,q)+M_2^\diamond(p,q)\right)-\left(M_3^\diamond(s;p,q)+M_4^\diamond(s;p,q)\right)\right)\\
&\quad+|{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|\left(M_3^\diamond(s;p,q)+M_4^\diamond(s;p,q)\right)\Big],
\end{aligned}
\tag{4.2}
$$

where

$$
M_1^\diamond(p,q)=\int_0^{\frac{1}{2}}\left|q\tau-\frac{1}{6}\right|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{2(p+q-1)-3q(p+q)+9q^2}{36q(p+q)}, & \textit{for } 0\le\frac{1}{6q}\le\frac{1}{2};\\ \frac{p-2q}{12(p+q)}, & \textit{for } \frac{1}{2}<\frac{1}{6q},\end{cases}
$$

$$
M_2^\diamond(p,q)=\int_{\frac{1}{2}}^1\left|q\tau-\frac{5}{6}\right|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{4q-5p}{12(p+q)}, & \textit{for } 0\le\frac{5}{6q}\le\frac{1}{2};\\ \frac{50(p+q-1)-45q(p+q)+45q^2}{36q(p+q)}, & \textit{for } \frac{1}{2}<\frac{5}{6q}\le 1;\\ \frac{5p-4q}{12(1+q)}, & \textit{for } 1<\frac{5}{6q},\end{cases}
$$

$$
M_3^\diamond(s;p,q)=\int_0^{\frac{1}{2}}\tau^s\left|q\tau-\frac{1}{6}\right|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{2(p-q)}{6^{s+2}q^{s+1}}\left(\frac{1}{p^{s+1}-q^{s+1}}-\frac{1}{p^{s+2}-q^{s+2}}\right)\\ +\frac{(p-q)(3q-1)+2(q-p)q^{s+2}}{3.2^{s+2}(p^{s+1}-q^{s+1})(p^{s+2}-q^{s+2})}, & \textit{for } 0\le\frac{1}{6q}\le\frac{1}{2};\\ -\frac{(p-q)(3q-1)+2(q-p)q^{s+2}}{3.2^{s+2}(p^{s+1}-q^{s+1})(p^{s+2}-q^{s+2})}, & \textit{for } \frac{1}{2}<\frac{1}{6q},\end{cases}
$$

$$
M_4^\diamond(s;p,q)=\int_{\frac{1}{2}}^1\tau^s\left|q\tau-\frac{5}{6}\right|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{5(p-q)(1-2^{s+1})}{6.2^{s+1}(p^{s+1}-q^{s+1})}+\frac{q(p-q)(2^{s+2}-1)}{2^{s+2}(p^{s+2}-q^{s+2})}, & \textit{for } 0\le\frac{5}{6q}\le\frac{1}{2};\\ -\frac{5(p-q)(1+2^{s+1})}{6.2^{s+1}(p^{s+1}-q^{s+1})}+\frac{q(p-q)(1+2^{s+2})}{2^{s+2}(p^{s+2}-q^{s+2})}\\ \frac{2.5^{s+2}(p-q)}{5^{s+2}.q^{s+1}}\left(\frac{1}{p^{s+1}-q^{s+1}}-\frac{1}{p^{s+2}-q^{s+2}}\right), & \textit{for } \frac{1}{2}<\frac{5}{6q}\le 1;\\ -\frac{5(p-q)(1-2^{s+1})}{6.2^{s+1}(p^{s+1}-q^{s+1})}+\frac{q(p-q)(2^{s+2}-1)}{2^{s+2}(p^{s+2}-q^{s+2})}, & \textit{for } 1<\frac{5}{6q}.\end{cases}
$$

Example 1. *Let $\mathcal{F}:[0,1]\to\mathbb{R}$ be defined by $\mathcal{F}(x)=x^2$. From Corollary 5 with $q=\frac{1}{2}$, $p=\frac{3}{4}$, $s=1$, and $\theta(b,a)=b-a$, the left side of* (4.2) *becomes*

$$
\left|\frac{\mathcal{F}(a)}{6}+\frac{2\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{3}+\frac{\mathcal{F}(a+\theta(b,a))}{6}-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x\right|
$$

$$= \left|\frac{1}{3} - \frac{4}{3}\int_0^{\frac{3}{4}} x^2 \, d_{\frac{1}{2},\frac{3}{4}}x\right| = \left|\frac{1}{3} - \frac{27}{76}\right| \approx 0.0219,$$

and the right side of (4.2) *becomes*

$$\begin{aligned}
&\theta(b,a)\Big[|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|\left(2^{1-s}\left(M_1^\diamond(p,q)+M_2^\diamond(p,q)\right)-\left(M_3^\diamond(s;p,q)+M_4^\diamond(s;p,q)\right)\right)\\
&\quad+|{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|\left(M_3^\diamond(s;p,q)+M_4^\diamond(s;p,q)\right)\Big]\\
&=\left[|{}_0\mathrm{D}_{\frac{3}{4},\frac{1}{2}}\mathcal{F}(0)|\left(\left(M_1^\diamond\left(\frac{3}{4},\frac{1}{2}\right)+M_2^\diamond\left(\frac{3}{4},\frac{1}{2}\right)\right)-\left(M_3^\diamond\left(1;\frac{3}{4},\frac{1}{2}\right)+M_4^\diamond\left(1;\frac{3}{4},\frac{1}{2}\right)\right)\right)\right.\\
&\left.+|{}_0\mathrm{D}_{\frac{3}{4},\frac{1}{2}}\mathcal{F}(1)|\left(M_3^\diamond\left(1;\frac{3}{4},\frac{1}{2}\right)+M_4^\diamond\left(1;\frac{3}{4},\frac{1}{2}\right)\right)\right]\\
&=\frac{5}{4}\left(\frac{68}{2565}+\frac{33}{38}\right)\approx 1.1187.
\end{aligned}$$

It is clear that

$$0.0219 \le 1.1187,$$

which demonstrates the result described in Corollary 5.

Corollary 6. *Under the assumptions of Theorem 1, taking* $\varepsilon = \delta = \frac{q}{p+q}$*, we obtain*

$$\begin{aligned}
&\left|\frac{q}{p+q}\mathcal{F}(a)+\frac{p}{p+q}\mathcal{F}(a+\theta(b,a))-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x\right|\\
&\le\theta(b,a)\Big[|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|\left(2^{1-s}\left(N_1^\diamond(p,q)+N_2^\diamond(p,q)\right)-\left(N_3^\diamond(s;p,q)+N_4^\diamond(s;p,q)\right)\right)\\
&\quad+|{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|\left(N_3^\diamond(s;p,q)+N_4^\diamond(s;p,q)\right)\Big],
\end{aligned} \tag{4.3}$$

where

$$N_1^\diamond(p,q)=\int_0^{\frac{1}{2}}\left|q\tau-\frac{q}{p+q}\right|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{8q^2(p+q-1)-q^2(p+q)^2}{4q(p+q)^3}, & \textit{for } 0\le\frac{1}{1+q}\le\frac{1}{2};\\ \frac{q}{4(p+q)}, & \textit{for } \frac{1}{2}<\frac{1}{1+q},\end{cases}$$

$$N_2^\diamond(p,q)=\int_{\frac{1}{2}}^{1}\left|q\tau-\frac{q}{p+q}\right|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{q}{4(p+q)}, & \textit{for } 0\le\frac{1}{1+q}\le\frac{1}{2};\\ \frac{8q^2(p+q-1)-q^2(p+q)^2}{4q(p+q)^3}, & \textit{for } \frac{1}{2}<\frac{1}{1+q}\le 1;\\ \frac{-q}{4(p+q)}, & \textit{for } 1<\frac{1}{1+q},\end{cases}$$

$$N_3^\diamond(s;p,q)=\int_0^{\frac{1}{2}}\tau^s\left|q\tau-\frac{q}{p+q}\right|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{2q(p-q)}{(p+q)^{s+2}}\left(\frac{1}{p^{s+1}-q^{s+1}}-\frac{1}{p^{s+2}-q^{s+2}}\right)\\ +\frac{q(p-q)(p+q-2)-(p-q)^2q^{s+2}}{2^{s+2}(p+q)(p^{s+1}-q^{s+1})(p^{s+2}-q^{s+2})}, & \textit{for } 0\le\frac{1}{1+q}\le\frac{1}{2};\\ -\frac{q(p-q)(p+q-2)-(p-q)^2q^{s+2}}{2^{s+2}(p+q)(p^{s+1}-q^{s+1})(p^{s+2}-q^{s+2})}, & \textit{for } \frac{1}{2}<\frac{1}{1+q},\end{cases}$$

$$N_4^\diamond(s;p,q)=\int_{\frac{1}{2}}^{1}\tau^s\left|q\tau-\frac{q}{p+q}\right|\,{}_0\mathrm{d}_{p,q}\tau=\begin{cases}\frac{q(p-q)(1-2^{s+1})}{2^{s+1}(p+q)(p^{s+1}-q^{s+1})}+\frac{q(p-q)(2^{s+2}-1)}{2^{s+2}(p^{s+2}-q^{s+2})}, & \textit{for } 0\le\frac{1}{1+q}\le\frac{1}{2};\\ -\frac{q(p-q)(1+2^{s+1})}{2^{s+1}(p+q)(p^{s+1}-q^{s+1})}+\frac{q(p-q)(1+2^{s+2})}{2^{s+2}(p^{s+2}-q^{s+2})}\\ \frac{2q(p-q)}{(p+q)^{s+2}}\left(\frac{1}{p^{s+1}-q^{s+1}}-\frac{1}{p^{s+2}-q^{s+2}}\right), & \textit{for } \frac{1}{2}<\frac{1}{1+q}\le 1;\\ -\frac{q(p-q)(1-2^{s+1})}{2^{s+1}(p+q)(p^{s+1}-q^{s+1})}+\frac{q(p-q)(2^{s+2}-1)}{2^{s+2}(p^{s+2}-q^{s+2})}, & \textit{for } 1<\frac{1}{1+q}.\end{cases}$$

Example 2. *Let's reconsider the function $\mathcal{F} : [0,1] \to \mathbb{R}$ defined by $\mathcal{F}(x) = x^2$. From Corollary 6 with $q = \frac{1}{2}$, $p = \frac{3}{4}$, $s = 1$, and $\theta(b,a) = b - a$, the left side of* (4.3) *becomes*

$$\left|\frac{q}{p+q}\mathcal{F}(a) + \frac{p}{p+q}\mathcal{F}(a+\theta(b,a)) - \frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)} \mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x\right|$$

$$= \left|\frac{3}{5} - \frac{4}{3}\int_0^{\frac{3}{4}} x^2\, d_{\frac{1}{2},\frac{3}{4}}x\right| = \left|\frac{3}{5} - \frac{27}{76}\right| \approx 0.2447,$$

and the right side of (4.3) *becomes*

$$\begin{aligned}
&\theta(b,a)\Big[|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|\left(2^{1-s}\left(N_1^\diamond(p,q) + N_2^\diamond(p,q)\right) - \left(N_3^\diamond(s;p,q) + N_4^\diamond(s;p,q)\right)\right) \\
&\quad + |{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|\left(N_3^\diamond(s;p,q) + N_4^\diamond(s;p,q)\right)\Big] \\
&= \Bigg[|{}_0\mathrm{D}_{\frac{3}{4},\frac{1}{2}}\mathcal{F}(0)|\left(\left(N_1^\diamond\left(\frac{3}{4},\frac{1}{2}\right) + N_2^\diamond\left(\frac{3}{4},\frac{1}{2}\right)\right) - \left(N_3^\diamond\left(1;\frac{3}{4},\frac{1}{2}\right) + N_4^\diamond\left(1;\frac{3}{4},\frac{1}{2}\right)\right)\right) \\
&\quad + |{}_0\mathrm{D}_{\frac{3}{4},\frac{1}{2}}\mathcal{F}(1)|\left(N_3^\diamond\left(1;\frac{3}{4},\frac{1}{2}\right) + N_4^\diamond\left(1;\frac{3}{4},\frac{1}{2}\right)\right)\Bigg] \\
&= \frac{5}{4}\left(\frac{208}{475} + \frac{619}{11875}\right) \approx 0.6125.
\end{aligned}$$

It is clear that

$$0.2447 \le 0.6125,$$

which demonstrates the result described in Corollary 6.

Corollary 7. *Under the assumptions of Theorem 1, taking $\varepsilon = \frac{1}{4}$ and $\delta = \frac{3}{4}$, we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(a)}{4} + \frac{2\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{4} + \frac{\mathcal{F}(a+\theta(b,a))}{2} - \frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)} \mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x\right| \\
&\le \theta(b,a)\Big[|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|\left(2^{1-s}\left(Q_1^\diamond(p,q) + Q_2^\diamond(p,q)\right) - \left(Q_3^\diamond(s;p,q) + Q_4^\diamond(s;p,q)\right)\right) \\
&\quad + |{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|\left(Q_3^\diamond(s;p,q) + Q_4^\diamond(s;p,q)\right)\Big],
\end{aligned} \tag{4.4}$$

where

$$Q_1^\diamond(p,q) = \int_0^{\frac{1}{2}} \left|q\tau - \frac{1}{4}\right| {}_0\mathrm{d}_{p,q}\tau = \begin{cases} \frac{2q^2 - q(p+q) + p + q - 1}{8q(p+q)}, & \text{for } 0 \le \frac{1}{4q} \le \frac{1}{2}; \\ \frac{p-q}{8(p+q)}, & \text{for } \frac{1}{2} < \frac{1}{4q}, \end{cases}$$

$$Q_2^\diamond(p,q) = \int_{\frac{1}{2}}^{1} \left|q\tau - \frac{3}{4}\right| {}_0\mathrm{d}_{p,q}\tau = \begin{cases} \frac{3(q-p)}{8(p+q)}, & \text{for } 0 \le \frac{3}{4q} \le \frac{1}{2}; \\ \frac{9(p+q-1) - 9pq + q^2}{8q(p+q)}, & \text{for } \frac{1}{2} < \frac{3}{4q} \le 1; \\ \frac{3(p-q)}{8(p+q)}, & \text{for } 1 < \frac{3}{4q}, \end{cases}$$

$$Q_3^\diamond(s;p,q) = \int_0^{\frac{1}{2}} \tau^s \left|q\tau - \frac{1}{4}\right| {}_0\mathrm{d}_{p,q}\tau = \begin{cases} \frac{p-q}{2^{2s+3}q^{s+1}}\left(\frac{1}{p^{s+1}-q^{s+1}} - \frac{1}{p^{s+2}-q^{s+2}}\right) \\ + \frac{(p-q)(2q-1) + (q-p)q^{s+2}}{2^{s+3}(p^{s+1}-q^{s+1})(p^{s+2}-q^{s+2})}, & \text{for } 0 \le \frac{1}{4q} \le \frac{1}{2}; \\ -\frac{(p-q)(2q-1) + (q-p)q^{s+2}}{2^{s+3}(p^{s+1}-q^{s+1})(p^{s+2}-q^{s+2})}, & \text{for } \frac{1}{2} < \frac{1}{4q}, \end{cases}$$

$$Q_4^{\diamond}(s;p,q)=\int_{\frac{1}{2}}^{1}\tau^s\left|q\tau-\frac{3}{4}\right|\,{}_0d_{p,q}\tau=\begin{cases}\frac{3(p-q)(1-2^{s+1})}{2^{s+3}(p^{s+1}-q^{s+1})}+\frac{q(p-q)(2^{s+2}-1)}{2^{s+2}(p^{s+2}-q^{s+2})}, & \textit{for } 0\le\frac{3}{4q}\le\frac{1}{2};\\ -\frac{3(p-q)(1+2^{s+1})}{2^{s+3}(p^{s+1}-q^{s+1})}+\frac{q(p-q)(1+2^{s+2})}{2^{s+2}(p^{s+2}-q^{s+2})} & \\ \frac{3^{s+2}(p-q)}{2^{2s+3}q^{s+1}}\left(\frac{1}{p^{s+1}-q^{s+1}}-\frac{1}{p^{s+2}-q^{s+2}}\right), & \textit{for } \frac{1}{2}<\frac{3}{4q}\le 1;\\ -\frac{3(p-q)(1-2^{s+1})}{2^{s+3}(p^{s+1}-q^{s+1})}+\frac{q(p-q)(2^{s+2}-1)}{2^{s+2}(p^{s+2}-q^{s+2})}, & \textit{for } 1<\frac{3}{4q}.\end{cases}$$

Theorem 2. *Let $\mathcal{F}:\mathcal{B}=[a,a+\theta(b,a)]\to\mathbb{R}$ be a (p,q)-differentiable function on $\mathcal{B}^\circ$ with $\theta(b,a)>0$. If $|{}_aD_{p,q}\mathcal{F}|^r$ is integrable and an s-preinvex function with $0<q<p\le 1$ and $m^{-1}+r^{-1}=1$, then*

$$\left|\varepsilon\mathcal{F}(a)+(\delta-\varepsilon)\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)+(1-\delta)\mathcal{F}(a+\theta(b,a))-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_ad_{p,q}x\right|$$
$$\le\theta(b,a)\Big(A_1^{\frac{1}{m}}(\varepsilon,p,q;m)\left(|{}_aD_{p,q}\mathcal{F}(a)|^r(2^{-s}-A_2(s;p,q))+|{}_aD_{p,q}\mathcal{F}(b)|^rA_2(s;p,q)\right)^{\frac{1}{r}}$$
$$+A_3^{\frac{1}{m}}(\delta,p,q;m)\left(|{}_aD_{p,q}\mathcal{F}(a)|^r(2^{-s}-A_4(s;p,q))+|{}_aD_{p,q}\mathcal{F}(b)|^rA_4(s;p,q)\right)^{\frac{1}{r}}\Big),\tag{4.5}$$

where

$$A_1(\varepsilon,p,q;m)=\int_0^{\frac{1}{2}}|q\tau-\varepsilon|^m\,{}_0d_{p,q}\tau$$
$$=\begin{cases}\frac{p-q}{2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{2p^{n+1}}-\varepsilon\right)^m & \\ +\frac{2(p-q)\varepsilon^{m+1}}{q}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^m, & \text{for } 0\le\frac{\varepsilon}{q}\le\frac{1}{2};\\ \frac{p-q}{2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{2p^{n+1}}-\varepsilon\right)^m, & \text{for } \frac{1}{2}<\frac{\varepsilon}{q},\end{cases}$$

$$A_3(\delta,p,q;m)=\int_{\frac{1}{2}}^{1}|q\tau-\delta|^m\,{}_0d_{p,q}\tau$$
$$=\begin{cases}(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{p^{n+1}}-\delta\right)^m-\frac{p-q}{2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{2p^{n+1}}-\delta\right)^m, & \text{for } 0\le\frac{\delta}{q}\le\frac{1}{2};\\ \frac{2(p-q)\delta^{m+1}}{q}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^m & \\ +(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{p^{n+1}}-\delta\right)^m+\frac{p-q}{2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{2p^{n+1}}-\delta\right)^m, & \text{for } \frac{1}{2}<\frac{\delta}{q}\le 1;\\ (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{p^{n+1}}-\delta\right)^m+\frac{p-q}{2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{2p^{n+1}}-\delta\right)^m, & \text{for } 1<\frac{\delta}{q}.\end{cases}$$

$$A_2(s;p,q)=\int_0^{\frac{1}{2}}\tau^s\,{}_0d_{p,q}\tau=\frac{p-q}{2^{s+1}(p^{s+1}-q^{s+1})},$$
$$A_4(s;p,q)=\int_{\frac{1}{2}}^{1}\tau^s\,{}_0d_{p,q}\tau=\frac{(p-q)(2^{s+1}-1)}{2^{s+1}(p^{s+1}-q^{s+1})}.$$

Proof. Using Lemma 1 and the assumption condition that $|{}_aD_{p,q}\mathcal{F}|^r$ is an s-preinvex function, we have

$$\left|\varepsilon\mathcal{F}(a)+(\delta-\varepsilon)\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)+(1-\delta)\mathcal{F}(a+\theta(b,a))-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_ad_{p,q}x\right|$$

$$
\begin{aligned}
&= \theta(b,a)\left|\int_0^{\frac{1}{2}} (q\tau-\varepsilon)\ {}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau + \int_{\frac{1}{2}}^1 (q\tau-\delta)\ {}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau\right| \\
&\le \theta(b,a)\left[\int_0^{\frac{1}{2}} |q\tau-\varepsilon|\,|{}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))|\,{}_0\mathrm{d}_{p,q}\tau + \int_{\frac{1}{2}}^1 |q\tau-\delta|\,|{}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))|\,{}_0\mathrm{d}_{p,q}\tau\right] \\
&\le \theta(b,a)\left[\left(\int_0^{\frac{1}{2}} |q\tau-\varepsilon|^m\ {}_0\mathrm{d}_{p,q}\tau\right)^{\frac{1}{m}}\left(\int_0^{\frac{1}{2}} |{}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))|^r\ {}_0\mathrm{d}_{p,q}\tau\right)^{\frac{1}{r}}\right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 |q\tau-\delta|^m\ {}_0\mathrm{d}_{p,q}\tau\right)^{\frac{1}{m}}\left(\int_{\frac{1}{2}}^1 |{}_a\mathrm{D}_{p,q}\mathcal{F}(a+\tau\theta(b,a))|^r\ {}_0\mathrm{d}_{p,q}\tau\right)^{\frac{1}{r}}\right] \\
&\le \theta(b,a)\left[A_1^{\frac{1}{m}}(\varepsilon,p,q;m)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r\int_0^{\frac{1}{2}}(1-\tau)^s\,{}_0\mathrm{d}_{p,q}\tau + |{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r\int_0^{\frac{1}{2}}\tau^s\,{}_0\mathrm{d}_{p,q}\tau\right)^{\frac{1}{r}}\right. \\
&\quad \left. + A_3^{\frac{1}{m}}(\delta,p,q;m)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r\int_{\frac{1}{2}}^1(1-\tau)^s\,{}_0\mathrm{d}_{p,q}\tau + |{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r\int_{\frac{1}{2}}^1\tau^s\,{}_0\mathrm{d}_{p,q}\tau\right)\right] \\
&\le \theta(b,a)\left[A_1^{\frac{1}{m}}(\varepsilon,p,q;m)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r\int_0^{\frac{1}{2}}(2^{1-s}-\tau^s)\,{}_0\mathrm{d}_{p,q}\tau + |{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r\int_0^{\frac{1}{2}}\tau^s\,{}_0\mathrm{d}_{p,q}\tau\right)^{\frac{1}{r}}\right. \\
&\quad \left. + A_3^{\frac{1}{m}}(\delta,p,q;m)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r\int_{\frac{1}{2}}^1(2^{1-s}-\tau^s)\,{}_0\mathrm{d}_{p,q}\tau + |{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r\int_{\frac{1}{2}}^1\tau^s\,{}_0\mathrm{d}_{p,q}\tau\right)\right] \\
&= \theta(b,a)\left[A_1^{\frac{1}{m}}(\varepsilon,p,q;m)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r(2^{-s}-A_2(s;p,q)) + |{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r A_2(s;p,q)\right)^{\frac{1}{r}}\right. \\
&\quad \left. + A_3^{\frac{1}{m}}(\delta,p,q;m)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r(2^{-s}-A_4(s;p,q)) + |{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r A_4(s;p,q)\right)^{\frac{1}{r}}\right].
\end{aligned}
$$

This completes the proof. □

If we take $\theta(b,a) = b-a$ and $p = 1$ in Theorem 2, then we have Theorem 2.2 [14].

We will now discuss some more special cases of Theorem 2.

Corollary 8. *Under the assumptions of Theorem 2, taking $\varepsilon = \frac{1}{6}$ and $\delta = \frac{5}{6}$, we obtain*

$$
\begin{aligned}
&\frac{\mathcal{F}(a)}{6} + \frac{2\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{3} + \frac{\mathcal{F}(a+\theta(b,a))}{6} - \frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x \\
&\le \theta(b,a)\left(A_1^{\frac{1}{m}}\left(\frac{1}{6},p,q;m\right)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r(2^{-s}-A_2(s;p,q)) + |{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r K_2(s;p,q)\right)^{\frac{1}{r}}\right. \\
&\quad \left. + A_3^{\frac{1}{m}}\left(\frac{5}{6},p,q;m\right)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r(2^{-s}-A_4(s;p,q)) + |{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r A_4(s;p,q)\right)^{\frac{1}{r}}\right). \qquad (4.6)
\end{aligned}
$$

Corollary 9. *Under the assumptions of Theorem 2, taking $\varepsilon = \delta = \frac{q}{p+q}$, we obtain*

$$
\frac{q}{1+q}\mathcal{F}(a) + \frac{p}{p+q}\mathcal{F}(a+\theta(b,a)) - \frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x
$$

$$\leq \theta(b,a)\left(A_1^{\frac{1}{m}}(p,q;m)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r(2^{-s}-A_2(s;p,q))+|{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r A_2(s;p,q)\right)^{\frac{1}{r}}\right.$$
$$\left.+A_3^{\frac{1}{m}}(p,q;m)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r(2^{-s}-A_4(s;p,q))+|{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r A_4(s;p,q)\right)^{\frac{1}{r}}\right). \tag{4.7}$$

Corollary 10. *Under the assumptions of Theorem 2, taking* $\varepsilon=\frac{1}{4}$ *and* $\delta=\frac{3}{4}$*, we obtain*

$$\frac{\mathcal{F}(a)}{4}+\frac{2\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)}{4}+\frac{\mathcal{F}(a+\theta(b,a))}{2}-\frac{1}{p\theta(b,a)}\int_a^{a+p\theta(b,a)}\mathcal{F}(x)\,{}_a\mathrm{d}_{p,q}x$$
$$\leq \theta(b,a)\left(A_1^{\frac{1}{m}}\left(\frac{1}{4},p,q;m\right)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r(2^{-s}-A_2(s;p,q))+|{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r A_2(s;p,q)\right)^{\frac{1}{r}}\right.$$
$$\left.+A_3^{\frac{1}{m}}\left(\frac{3}{4},p,q;m\right)\left(|{}_a\mathrm{D}_{p,q}\mathcal{F}(a)|^r(2^{-s}-A_4(s;p,q))+|{}_a\mathrm{D}_{p,q}\mathcal{F}(b)|^r A_4(s;p,q)\right)^{\frac{1}{r}}\right). \tag{4.8}$$

Theorem 3. *Let* $\mathcal{F},\mathcal{G}:\mathcal{B}\to\mathbb{R}$ *be continuous and nonnegative on* $\mathcal{B}$*. If* $\mathcal{F}$ *and* $\mathcal{G}$ *are* s_1 *and* s_2 *preinvex functions on* $\mathcal{B}$*, then the following* (p,q)*-inequality holds with* $s\in(0,1]$*:*

$$\frac{1}{\theta(b,a)}\int_a^{a+\theta(b,a)}\mathcal{F}(x)\mathcal{G}(x)\,{}_a\mathrm{d}_{p,q}x \leq \mathcal{F}(a)\mathcal{G}(a)\left(2^{1-s_1-s_2}-\frac{p-q}{p^{s_1+s_2+1}-q^{s_1+s_2+1}}\right)$$
$$+\mathcal{F}(a)\mathcal{G}(b)\left(2^{1-s_1}\frac{p-q}{p^{s_2+1}-q^{s_2+1}}-\frac{p-q}{p^{s_1+s_2+1}-q^{s_1+s_2+1}}\right)$$
$$+\mathcal{F}(b)\mathcal{G}(a)\left(2^{1-s_2}\frac{p-q}{p^{s_1+1}-q^{s_1+1}}-\frac{p-q}{p^{s_1+s_2+1}-q^{s_1+s_2+1}}\right)$$
$$+\mathcal{F}(b)\mathcal{G}(b)\frac{p-q}{p^{s_1+s_2+1}-q^{s_1+s_2+1}}. \tag{4.9}$$

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ are s_1 and s_2-preinvex functions, so we have

$$\mathcal{F}(a+\tau\theta(b,a))\leq(1-\tau)^{s_1}\mathcal{F}(a)+\tau^{s_1}\mathcal{F}(b) \tag{4.10}$$

and

$$\mathcal{G}(a+\tau\theta(b,a))\leq(1-\tau)^{s_2}\mathcal{G}(a)+\tau^{s_2}\mathcal{G}(b). \tag{4.11}$$

Multiplying both sides of 4.10 and 4.11, we obtain

$$\mathcal{F}(a+\tau\theta(b,a))\mathcal{G}(a+\tau\theta(b,a))\leq(1-\tau)^{s_1+s_2}\mathcal{F}(a)\mathcal{G}(a)+\tau^{s_2}(1-\tau)^{s_1}\mathcal{F}(a)\mathcal{G}(b)$$
$$+\tau^{s_1}(1-\tau)^{s_2}\mathcal{F}(b)\mathcal{G}(a)+\tau^{s_1+s_2}\mathcal{F}(b)\mathcal{G}(b). \tag{4.12}$$

Taking the (p,q)-integral for (4.12) with respect to τ on $(0,1)$ and using the inequality $(1-\tau)^s\leq 2^{1-s}-\tau^s$, for $\tau\in(0,1)$, we obtain

$$\int_0^1\mathcal{F}(a+\tau\theta(b,a))\mathcal{G}(a+\tau\theta(b,a))\,{}_0\mathrm{d}_{p,q}\tau$$

$$\leq \mathcal{F}(a)\mathcal{G}(a)\int_0^1 (1-\tau)^{s_1+s_2}\ {}_{p,q}\mathrm{d}_{p,q}\tau + \mathcal{F}(a)\mathcal{G}(b)\int_0^1 \tau^{s_2}(1-\tau)^{s_1}\ {}_0\mathrm{d}_{p,q}\tau$$
$$+\mathcal{F}(b)\mathcal{G}(a)\int_0^1 \tau^{s_1}(1-\tau)^{s_2}\ {}_{p,q}\mathrm{d}_{p,q}\tau + \mathcal{F}(b)\mathcal{G}(b)\int_0^1 \tau^{s_1+s_2}\ {}_0\mathrm{d}_{p,q}\tau$$
$$\leq \mathcal{F}(a)\mathcal{G}(a)\int_0^1 (2^{1-s_1-s_2}-\tau^{s_1+s_2})\ {}_0\mathrm{d}_{p,q}\tau + \mathcal{F}(a)\mathcal{G}(b)\int_0^1 \tau^{s_2}(2^{1-s_1}-\tau^{s_1})\ {}_0\mathrm{d}_{p,q}\tau$$
$$+\mathcal{F}(b)\mathcal{G}(a)\int_0^1 \tau^{s_1}(2^{1-s_2}-\tau^{s_2})\ {}_0\mathrm{d}_{p,q}\tau + \mathcal{F}(b)\mathcal{G}(b)\int_0^1 \tau^{s_1+s_2}\ {}_0\mathrm{d}_{p,q}\tau$$
$$= \mathcal{F}(a)\mathcal{G}(a)\left(2^{1-s_1-s_2} - \frac{p-q}{p^{s_1+s_2+1}-q^{s_1+s_2+1}}\right)$$
$$+\mathcal{F}(a)\mathcal{G}(b)\left(2^{1-s_1}\frac{p-q}{p^{s_2+1}-q^{s_2+1}} - \frac{p-q}{p^{s_1+s_2+1}-q^{s_1+s_2+1}}\right)$$
$$+\mathcal{F}(b)\mathcal{G}(a)\left(2^{1-s_2}\frac{p-q}{p^{s_1+1}-q^{s_1+1}} - \frac{p-q}{p^{s_1+s_2+1}-q^{s_1+s_2+1}}\right)$$
$$+\mathcal{F}(b)\mathcal{G}(b)\frac{p-q}{p^{s_1+s_2+1}-q^{s_1+s_2+1}}.$$

Also,

$$\int_0^1 \mathcal{F}(a+\tau\theta(b,a))\mathcal{G}(a+\tau\theta(b,a))\ {}_0\mathrm{d}_{p,q}\tau = \frac{1}{\theta(b,a)}\int_a^{a+\theta(b,a)} \mathcal{F}(x)\mathcal{G}(x)\ {}_a\mathrm{d}_{p,q}x,$$

which completes the proof. □

If we take $\theta(b,a) = b-a$ and $p = 1$ in Theorem 3, then we have Theorem 2.3 [14].

We will now discuss some more special cases of Theorem 3.

Theorem 4. *Let $\mathcal{F}:\mathcal{B}\to\mathbb{R}$ be s-preinvex functions. If $\mathrm{h}:\mathcal{B}\to\mathbb{R}$ is non-negative, integrable on $\mathcal{B}$ and symmetric about $\frac{2a+\theta(b,a)}{2}$ and if $\theta(.,.)$ satisfies the condition **C**, then the following (p,q)-inequality holds with $s\in(0,1]$:*

$$\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)\int_a^{a+\theta(b,a)} \mathrm{h}(x)\ {}_a\mathrm{d}_{p,q}x \leq 2^{1-s}\int_a^{a+\theta(b,a)} \mathcal{F}(x)\mathrm{h}(x)\ {}_a\mathrm{d}_{p,q}x. \tag{4.13}$$

Proof. By preinvexity of h, we have for every $x = a+\frac{1+\mu}{2}\theta(b,a), y = a+\frac{1-\mu}{2}\theta(b,a)\in[a,a+\theta(b,a)]$ with $\mu = [-1,1]$, we have

$$\mathcal{F}\left(x+\frac{\theta(y,x)}{2}\right) \leq 2^{-s}\mathcal{F}(x) + (1-2^{-1})^s\mathcal{F}(y).$$

Using the condition **C**, we have

$$\begin{aligned}
&\mathcal{F}\left(a+\frac{1+\mu}{2}(b,a)+\frac{\theta(a+\frac{1-\mu}{2}\theta(b,a),a+\frac{1+\mu}{2}\theta(b,a))}{2}\right)\\
&=\mathcal{F}\left(a+\frac{1+\mu}{2}\theta(b,a)-\frac{\mu\theta(b,a)}{2}\right)\\
&=\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)\\
&\leq 2^{-s}\mathcal{F}(a+\frac{1+\mu}{2}\theta(b,a))+(1-2^{-1})^{s}\mathcal{F}(a+\frac{1+\mu}{2}\theta(b,a)).
\end{aligned}$$

Multiplying the above inequality with $\mathrm{h}\left(a+\frac{1+\mu}{2}\theta(b,a)\right)$ on both sides and then integrating with respect to μ on $[-1,1]$, we have

$$\begin{aligned}
&\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)\int_{-1}^{1}\mathrm{h}\left(a+\frac{1+\mu}{2}\theta(b,a)\right)\,{}_{a}\mathrm{d}_{p,q}\mu\\
&\leq 2^{-s}\int_{-1}^{1}\mathcal{F}\left(a+\frac{1+\mu}{2}\theta(b,a)\right)\mathrm{h}\left(a+\frac{1+\mu}{2}\theta(b,a)\right)\,{}_{a}\mathrm{d}_{p,q}x\mu\\
&\quad+(2^{1-s}-2^{-s})\int_{-1}^{1}\mathcal{F}\left(a+\frac{1+\mu}{2}\theta(b,a)\right)\mathrm{h}\left(a+\frac{1+\mu}{2}\theta(b,a)\right)\,{}_{a}\mathrm{d}_{p,q}\mu.
\end{aligned}$$

Since h is symmetric about $\frac{2a+\theta(b,a)}{2}$, we have

$$\begin{aligned}
&\mathcal{F}\left(\frac{2a+\theta(b,a)}{2}\right)\frac{2}{\theta(b,a)}\int_{a}^{a+\theta(b,a)}\mathrm{h}(x)\,{}_{a}\mathrm{d}_{p,q}x\\
&\leq 2^{-s}\frac{2}{\theta(b,a)}\int_{a}^{a+\theta(b,a)}\mathcal{F}(x)\mathrm{h}(x)\,{}_{a}\mathrm{d}_{p,q}x\\
&\quad+(2^{1-s}-2^{-s})\frac{2}{\theta(b,a)}\int_{a}^{a+\theta(b,a)}\mathcal{F}(x)\mathrm{h}\left(a+\theta(b,a)-x\right)\,{}_{a}\mathrm{d}_{p,q}x\\
&=2^{-s}\frac{2}{\theta(b,a)}\int_{a}^{a+\theta(b,a)}\mathcal{F}(x)\mathrm{h}(x)\,{}_{a}\mathrm{d}_{p,q}x\\
&\quad+(2^{1-s}-2^{-s})\frac{2}{\theta(b,a)}\int_{a}^{a+\theta(b,a)}\mathcal{F}(x)\mathrm{h}(x)\,{}_{a}\mathrm{d}_{p,q}x\\
&=2\frac{2^{1-s}}{\theta(b,a)}\int_{a}^{a+\theta(b,a)}\mathcal{F}(x)\mathrm{h}(x)\,{}_{a}\mathrm{d}_{p,q}x.
\end{aligned}$$

This completes the proof. □

If we take $\theta(b,a) = b - a$ and $p = 1$ in Theorem 4, then we have Theorem 2.4 [14].

5. Applications

We will now discuss some applications of the results discussed in the previous section.

Proposition 1. *Under the assumptions of Theorem 1, if we take* $\varepsilon + \delta = 1$, *then we have the following inequality*

$$\begin{aligned}&\left|\frac{2\varepsilon}{s+1}\mathcal{A}(a^{s+1}, b^{s+1}) + \frac{\delta - \varepsilon}{s+1}\mathcal{A}^{s+1}(a,b) - \frac{1}{(s+1)}\mathcal{D}(s,a,b;p,q)\right| \\ &\leq \theta(b,a)\left[a^s\left(2^{1-s}(L_1^\diamond(\varepsilon;p,q) + L_2^\diamond(\delta;p,q)) - (L_3^\diamond(s,\varepsilon;p,q) + L_4^\diamond(s,\delta;p,q))\right)\right. \\ &\quad\left.+\mathcal{L}_s^s\left(qb + (1-q)a, pb + (1-p)a\right)\left(L_3^\diamond(s,\varepsilon;p,q) + L_4^\diamond(s,\delta;p,q)\right)\right],\end{aligned}$$

where

$$\mathcal{D}(s,a,b;p,q) = (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\left(1 - \frac{q^n}{p^{n+1}}\right)a + \frac{q^n}{p^n}b\right)^{s+1}$$

and $L_1^\diamond, L_2^\diamond, L_3^\diamond$ *and* $L_4^\diamond$ *are given in Theorem 1.*

Proof. The proof directly follows by taking $\psi(x) = \frac{x^{s+1}}{s+1}$ and $\theta(b,a) = b - a$ in Theorem 1. □

Proposition 2. *Under the assumptions of Theorem 1, if we take* $\varepsilon + \delta = 1$, *then we have the following inequality*

$$\begin{aligned}&\left|2\varepsilon\mathcal{A}(a^{-s}, b^{-s}) + \delta - \varepsilon\mathcal{A}^{-s}(a,b) - \mathcal{E}(a,b;s,p,q)\right| \\ &\leq \theta(b,a)\left[\frac{s}{a^{s+1}}\left(2^{1-s}(L_1^\diamond(\varepsilon;p,q) + L_2^\diamond(\delta;p,q)) - (L_3^\diamond(s,\varepsilon;p,q) + L_4^\diamond(s,\delta;p,q))\right)\right. \\ &\quad\left.+s\mathcal{L}_{-s-1}^{-s-1}\left(qb + (1-q)a, pb + (1-p)a\right)\left(L_3^\diamond(s,\varepsilon;p,q) + L_4^\diamond(s,\delta;p,q)\right)\right],\end{aligned}$$

where

$$\mathcal{E}(a,b;s,p,q) = (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\left(1 - \frac{q^n}{p^{n+1}}\right)a + \frac{q^n}{p^n}b\right)^{-s}$$

and $L_1^\diamond, L_2^\diamond, L_3^\diamond$ *and* $L_4^\diamond$ *are given in Theorem 1.*

Proof. The proof directly follows by taking $\psi(x) = x^{-s}$ and $\theta(b,a) = b - a$ in Theorem 1. □

Proposition 3. *Under the assumptions of Theorem 2, if we take* $\varepsilon + \delta = 1$, *then we have the following inequality*

$$\begin{aligned}&\left|\frac{2\varepsilon}{s+1}\mathcal{A}(a^{s+1}, b^{s+1}) + \frac{\delta - \varepsilon}{s+1}\mathcal{A}^{s+1}(a,b) - \frac{1}{(s+1)}\mathcal{D}(s,a,b;p,q)\right| \\ &\leq \theta(b,a)\left(A_1^{\frac{1}{m}}(\varepsilon,p,q;m)\left(|a^s|^r(2^{-s} - A_2(s;p,q)) + |\mathcal{L}_s^s\left(qb + (1-q)a, pb + (1-p)a\right)|^r A_2(s;p,q)\right)^{\frac{1}{r}}\right. \\ &\quad\left.+A_3^{\frac{1}{m}}(\delta,p,q;m)\left(|a^s|^r(2^{-s} - A_4(s;p,q)) + |\mathcal{L}_s^s\left(qb + (1-q)a, pb + (1-p)a\right)|^r A_4(s;p,q)\right)^{\frac{1}{r}}\right),\end{aligned}$$

Proof. The proof directly follows by taking $\psi(x) = \frac{x^{s+1}}{s+1}$ and $\theta(b, a) = b - a$ in Theorem 2. □

Proposition 4. *Under the assumptions of Theorem 1, if we take* $\varepsilon + \delta = 1$, *then we have the following inequality*

$$\begin{aligned}&\left|2\varepsilon\mathcal{A}(a^{-s}, b^{-s}) + \delta - \varepsilon\mathcal{A}^{-s}(a, b) - \mathcal{E}(a, b; s, p, q)\right|\\ &\le \theta(b, a)\left(A_1^{\frac{1}{m}}(\varepsilon, p, q; m)\left(\frac{s}{a^{s+1}}|^r(2^{-s} - A_2(s; p, q)) + |s\mathcal{L}_{-s-1}^{-s-1}(qb + (1-q)a, pb + (1-p)a)|^r A_2(s; p, q)\right)^{\frac{1}{r}}\right.\\ &\quad\left.+A_3^{\frac{1}{m}}(\delta, p, q; m)\left(|\frac{s}{a^{s+1}}|^r(2^{-s} - A_4(s; p, q)) + |s\mathcal{L}_{-s-1}^{-s-1}(qb + (1-q)a, pb + (1-p)a)|^r A_4(s; p, q)\right)^{\frac{1}{r}}\right).\end{aligned}$$

Proof. The proof directly follows by taking $\psi(x) = x^{-s}$ and $\theta(b, a) = b - a$ in Theorem 2. □

6. Conclusions

We have derived a new parametric generalized (p, q)-integral identity. By taking suitable choices for the parameters involved in the identity we obtained specialized versions of this generalized identity. Using this identity as an auxiliary result, we then obtained some of Simpson's type of integral inequalities in the setting of (p, q)-calculus. We have also discussed several special cases of the main results which proved that our results are quite unifying as it relates to several other unrelated results. Finally, we have presented some applications to means that described the significance of our theoretical results. It has been discussed in detail that integral inequalities have a wide range of applications in various fields of pure and applied sciences; thus we hope that our results will inspire interested readers working in this field.

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Conflict of interest

The authors declare that they have no competing interests.

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