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Research article

On the Waring–Goldbach problem for two squares and four cubes

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Abstract: Let $\mathcal{P}_r$ denote an almost–prime with at most r prime factors, counted according to multiplicity. In this paper, it is proved that for every sufficiently large even integer N, the following equation

$$N = p_1^2 + p_2^2 + x^3 + p_3^3 + p_4^3 + p_5^3$$

is solvable with x being an almost–prime $\mathcal{P}_7$ and the other variables primes. This result constitutes a deepening upon that of previous results.

Keywords: Waring–Goldbach problem; Hardy–Littlewood method; sieve method; almost–prime
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1. Introduction and main results

The famous Goldbach Conjecture states that every even integer $N \geqslant 6$ can be written as the sum of two odd primes, i.e.,

$$N = p_1 + p_2. \tag{1.1}$$

This conjecture still remains open. The recent developments on Goldbach Conjecture can be found in [21, 22, 30, 33, 34] and their references.

In view of Hua's theorem [13] on five squares of primes and Lagrange's theorem on four squares, it seems reasonable to conjecture that every sufficiently large integer satisfying some necessary congruence conditions can be written as the sum of four squares of primes, i.e.,

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2. \tag{1.2}$$

However, such a conjecture is out of reach at present. For the recent developments on conjecture (1.2), one can be found in [11, 12, 19] and their references.

Motivated by Hua's nine cubes of primes theorem [13], it seems reasonable to conjecture that every sufficiently large even integer is the sum of eight cubes of primes, i.e.,

$$N = p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + p_7^3 + p_8^3. \tag{1.3}$$

But unfortunately, such a conjecture (1.3) is still out of reach at present. For the recent developments on conjecture (1.3), one can see [17, 18] and its references.

Linnik [25, 26] proved that each sufficiently large odd integer N can be written as $N = p + n_1^2 + n_2^2$, which was firstly formulated by Hardy and Littlewood [9], where n_1 and n_2 are integers. In view of this result, it seems reasonable to conjecture that every sufficiently large integer satisfying some necessary congruence conditions is a sum of a prime and two squares of primes, i.e.,

$$N = p_1 + p_2^2 + p_3^2. \tag{1.4}$$

But current techniques lack the power to solve it. Many authors considered this problem and gave some approaches to approximate (1.4) (See [12, 13, 20, 23, 24, 27, 31, 40, 41, 43]). Meanwhile, we can regard this problem as the hybrid problem of (1.1) and (1.2).

In [29], Liu considered the hybrid problem of (1.1) and (1.3), i.e.,

$$N = p_1 + p_2^3 + p_3^3 + p_4^3 + p_5^3. \tag{1.5}$$

There are some approximations to (1.5). On one hand, as an approach to prove (1.5), Liu and Lü [28] proved that every sufficiently large odd integer can be written as the sum of a prime, four cubes of primes and bounded number of powers of 2, i.e.,

$$N = p_1 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_{K_1}},$$

and gave an acceptable value of K_1. On the other hand, Liu [29] gave another approximation to (1.5). He proved that every sufficiently large odd integer N can be written in the form $N = x + p_1^3 + p_2^3 + p_3^3 + p_4^3$, where p_1, p_2, p_3, p_4 are primes and x is an almost–prime $\mathcal{P}_2$. As usual, $\mathcal{P}_r$ always denotes an almost–prime with at most r prime factors, counted according to multiplicity. In [28], Liu and Lü also considered the hybrid problem of (1.2) and (1.3),

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3. \tag{1.6}$$

In their paper, they gave an approximation to (1.6) and proved that every sufficiently large even integer can be written as the sum of two squares of primes, four cubes of primes and 211 powers of 2, i.e.,

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + 2^{\nu_1} + 2^{\nu_2} + \cdots + 2^{\nu_{211}}. \tag{1.7}$$

Also, in 2016, Cai [6] gave another approximation to (1.6), and proved that any sufficiently large even integer N can be written in the form $N = x^2 + p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3$, where p_1, p_2, p_3, p_4, p_5 are primes and x is an almost–prime $\mathcal{P}_3$. Afterwards, Cai's studies in this direction were subsequently generalized by Zhang and Li [42].

In view of (1.7), the results of Cai [6] and Zhang and Li [42], in this paper, we shall continue giving some approximation to the conjecture (1.6).

Theorem 1.1. *Let $\mathscr{R}(N)$ denote the number of solutions of the following equation*

$$N = p_1^2 + p_2^2 + x^3 + p_3^3 + p_4^3 + p_5^3 \tag{1.8}$$

with x being an almost–prime $\mathcal{P}_7$ and the p_j's primes. Then, for every sufficiently large even integer N, there holds

$$\mathscr{R}(N) \gg N^{\frac{11}{9}} \log^{-6} N.$$

We approach Theorem 1.1 via the Hardy–Littlewood method, and in a certain sense by a unified approach. To be specific, we use the ideas, which were firstly created by Brüdern [1, 2] and developed by Brüdern and Kawada [3, 4], combining with Hardy–Littlewood method and Iwaniec's linear sieve method to give the proof of Theorem 1.1.

Notation. Throughout this paper, small italics denote integers when they do not obviously represent a function; p, p_1, p_2 . . . , with or without subscript, always stand for a prime number; ε always denotes an arbitrary small positive constant, which may not be the same at different occurrences; γ denotes Euler's constant; $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$; the constants in the O–term and $\ll$–symbol depend at most on ε; $\mathcal{P}_r$ always denotes an almost–prime with at most r prime factors, counted according to multiplicity. As usual, $\varphi(n)$ and $\tau_j(n)$ denote Euler's function and the j–dimensional divisor function respectively. Especially, we write $\tau(n) = \tau_2(n)$.We denote by $a(m)$ arithmetical function satisfying $|a(m)| \ll 1$; (s, t) denotes the greatest common divisor of s and t; $e(\alpha) = e^{2\pi i\alpha}$ for abbreviation.

2. Proof of Theorem 1.1: Preliminaries

In this section, we shall give some notations and preliminary lemmas. We always denote by χ a Dirichlet character (mod q), and by χ^0 the principal Dirichlet character (mod q). Let

$$A = 10^{300}, \qquad Q_0 = \log^{50A} N, \qquad X_j = \frac{1}{2}\Big(\frac{2N}{3}\Big)^{\frac{1}{j}}, \qquad X_j^* = \frac{1}{2}\Big(\frac{2N}{3}\Big)^{\frac{5}{6j}},$$

$$D = N^{\frac{1}{39}-60\varepsilon}, \qquad z = D^{\frac{1}{3}}, \qquad \mathscr{P} = \prod_{2<p<z} p, \qquad Q_1 = N^{\frac{1}{6}+60\varepsilon}, \qquad Q_2 = X_3^{\frac{27}{13}-60\varepsilon},$$

$$f_j(\alpha) = \sum_{X_j<p\leqslant 2X_j} (\log p)e(p^j\alpha), \qquad f_j^*(\alpha) = \sum_{X_j^*<p\leqslant 2X_j^*} (\log p)e(p^j\alpha),$$

$$G_j(\chi, a) = \sum_{m=1}^{q} \chi(m)e\Big(\frac{am^j}{q}\Big), \quad S_j^*(q, a) = G_j(\chi^0, a), \quad S_j(q, a) = \sum_{m=1}^{q} e\Big(\frac{am^j}{q}\Big),$$

$$v_j(\lambda) = \int_{X_j}^{2X_j} e(\lambda u^j)\mathrm{d}u, \qquad \mathscr{F}(\alpha) = \sum_{d\leqslant D} a(d) \sum_{\substack{X_3<m\leqslant 2X_3 \\ m\equiv 0 \ (\mathrm{mod}\ d)}} e(m^3\alpha),$$

$$v_j^*(\lambda) = \int_{X_j^*}^{2X_j^*} e(\lambda u^j)\mathrm{d}u, \qquad \mathcal{J}(N) = \int_{-\infty}^{+\infty} v_2^2(\lambda)v_3^2(\lambda)v_3^{*2}(\lambda)e(-N\lambda)\mathrm{d}\lambda,$$

$$B_d(q,N)=\sum_{\substack{a=1\\(a,q)=1}}^{q} S_2^{*2}(q,a)S_3(q,ad^3)S_3^{*3}(q,a)e\Big(-\frac{aN}{q}\Big),\qquad B(q,N)=B_1(q,N),$$

$$A_d(q,N)=\frac{B_d(q,N)}{q\varphi^5(q)},\ \ A(q,N)=A_1(q,N),\ \ \mathfrak{S}_d(N)=\sum_{q=1}^{\infty}A_d(q,N),\ \ \mathfrak{S}(N)=\mathfrak{S}_1(N),$$

$$\mathscr{M}_r=\{m:\ X_3<m\leqslant 2X_3,\ m=p_1p_2\cdots p_r,\ z\leqslant p_1\leqslant p_2\leqslant\cdots\leqslant p_r\},$$

$$\mathscr{N}_r=\{m:\ \ m=p_1p_2\cdots p_{r-1},\ z\leqslant p_1\leqslant p_2\leqslant\cdots\leqslant p_{r-1},\ p_1p_2\cdots p_{r-2}p_{r-1}^2\leqslant 2X_3\},$$

$$g_r(\alpha)=\sum_{\substack{X_3<\ell p\leqslant 2X_3\\ \ell\in\mathscr{N}_r}}\frac{\log p}{\log\frac{X_3}{\ell}}e\Big((\ell p)^3\alpha\Big),\qquad \log\boldsymbol{\Xi}=(\log 2X_2)^2(\log 2X_3)(\log 2X_3^*)^2,$$

$$\log\boldsymbol{\Theta}=(\log X_2)^2(\log X_3^*)^2.$$

Lemma 2.1. *For* $(a,q)=1$*, we have*

$$\text{(i)}\quad S_j(q,a)\ll q^{1-\frac{1}{j}};$$
$$\text{(ii)}\quad G_j(\chi,a)\ll q^{\frac{1}{2}+\varepsilon}.$$

In particular, for $(a,p)=1$*, we have*

$$\text{(iii)}\quad |S_j(p,a)|\leqslant((j,p-1)-1)\sqrt{p};$$
$$\text{(iv)}\quad |S_j^*(p,a)|\leqslant((j,p-1)-1)\sqrt{p}+1;$$
$$\text{(v)}\quad S_j^*(p^\ell,a)=0\ \textit{for}\ \ \ell\geqslant\gamma(p),\ \textit{where}$$
$$\gamma(p)=\begin{cases}\theta+2, & \textit{if}\ \ p^\theta\|j,\ \ p\neq 2\ \ \textit{or}\ \ p=2,\ \theta=0,\\ \theta+3, & \textit{if}\ \ p^\theta\|j,\ \ p=2,\ \ \theta>0.\end{cases}$$

Proof. For (i), (iii) and (iv), see Theorem 4.2 and Lemma 4.3 of Vaughan [38], respectively. For (ii), see Lemma 8.5 of Hua [14] or the Problem 14 of Chapter VI of Vinogradov [39]. For (v), see Lemma 8.3 of Hua [14]. □

Lemma 2.2. *We have*

$$\text{(i)}\ \int_0^1\Big|f_3(\alpha)f_3^{*2}(\alpha)\Big|^2\mathrm{d}\alpha\ll N^{\frac{8}{9}+\varepsilon},$$
$$\text{(ii)}\ \int_0^1\Big|f_2(\alpha)f_3^{*}(\alpha)\Big|^4\mathrm{d}\alpha\ll N^{\frac{19}{9}}(\log N)^2.$$

Proof. For (i), one can see the Theorem of Vaughan [37]. For (ii), in view of Satz 3 of Rieger [35], one has

$$\int_0^1\Big|f_2(\alpha)\Big|\mathrm{d}\alpha\ll X_2^2(\log N)^2,$$

which combined with the trivial estimate $f_3^*(\alpha)\ll X_3^*$ yields the desired result. □

Lemma 2.3. *Suppose that*

$$S(t) = \sum_{\nu} c(\nu)e(\nu t)$$

is an absolutely convergent exponential sum. Here the frequencies ν run over an arbitrary sequence of real numbers and the coefficients are complex. Let $\delta = \theta/T$ with $0 < \theta < 1$. Then

$$\int_{-T}^{T} |S(t)|^2 \mathrm{d}t \ll_\theta \int_{-\infty}^{+\infty} \left| \delta^{-1} \sum_{x<\nu\leqslant x+\delta} c(\nu) \right|^2 \mathrm{d}x.$$

Proof. See Lemma 1 of Gallagher [8]. □

Lemma 2.4. *For any $A > 0$ and non–principal Dirichlet character $\chi \pmod q$ with $q \ll (\log x)^A$, there holds*

$$\sum_{p\leqslant x} \chi(p) \ll x \exp\Big(-c(A)\sqrt{\log x}\Big),$$

where the implied constant depends only on A.

Proof. By partial summation and the arguments on p. 132 of Davenport [7], it is easy to derive the desired result. □

Lemma 2.5. *Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotonic, and $F'(x) \geqslant m > 0$, or $F'(x) \leqslant -m < 0$, throughout the interval $[a, b]$. Then we have*

$$\left| \int_a^b e^{iF(x)} \mathrm{d}x \right| \leqslant \frac{4}{m}.$$

Proof. See Lemma 4.2 of Titchmarsh [36]. □

Lemma 2.6. *Let $f(x)$ be a real differentiable function in the interval $[a, b]$. If $f'(x)$ is monotonic and satisfies $|f'(x)| \leqslant \theta < 1$. Then we have*

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)} \mathrm{d}x + O(1).$$

Proof. See Lemma 4.8 of Titchmarsh [36]. □

Lemma 2.7. *For $\alpha = \frac{a}{q} + \lambda$, define*

$$\mathfrak{N}(q, a) = \left[\frac{a}{q} - \frac{1}{qQ_0}, \frac{a}{q} + \frac{1}{qQ_0} \right], \tag{2.1}$$

$$\Delta_3(\alpha) = f_3(\alpha) - \frac{S_3^*(q, a)}{\varphi(q)} \sum_{X_3<n\leqslant 2X_3} e(n^3\lambda), \tag{2.2}$$

$$W(\alpha) = \sum_{d\leqslant D} \frac{a(d)}{dq} S_3(q, ad^3) v_3(\lambda). \tag{2.3}$$

Then we have

$$\sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{N}(q,a)} \left| W(\alpha)\Delta_3(\alpha) \right|^2 \mathrm{d}\alpha \ll N^{\frac{1}{3}} \exp\left(-\log^{1/4} N \right).$$

Proof. By Lemma 2.5, we know that

$$v_3(\lambda) \ll \frac{X_3}{1+N|\lambda|}. \tag{2.4}$$

It follows from (i) of Lemma 2.1 that

$$S_3(q, ad^3) \ll (q, d^3)\Big(\frac{q}{(q, d^3)}\Big)^{2/3} \ll q^{2/3}(q, d^3)^{1/3},$$

which combined with (2.4) and the trivial estimate $(q, d^3) \leqslant (q, d)^3$ yields

$$\begin{aligned} W(\alpha) &\ll \sum_{d\leqslant D} \frac{\tau(d)}{d}(q, d^3)^{1/3} q^{-1/3}|v_3(\lambda)| \\ &\ll \tau_3(q) q^{-1/3}|v_3(\lambda)| \log^2 N \ll \frac{\tau_3(q) X_3 \log^2 N}{q^{1/3}(1+N|\lambda|)}. \end{aligned} \tag{2.5}$$

For a Dirichlet character $\chi\,(\text{mod } q)$, define

$$\mathcal{W}_3(\chi, \lambda) = \sum_{X_3<p\leqslant 2X_3} (\log p)\chi(p) e(p^3\lambda) - \delta_\chi \sum_{X_3<m\leqslant 2X_3} e(m^3\lambda),$$

where $\delta_\chi = 1$ or 0 according to whether χ is principal or not. Then by the orthogonality of Dirichlet characters, (ii) of Lemma 2.1 and Cauchy's inequality, we derive that

$$\begin{aligned} \Delta_3(\alpha) &= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} G_3(\overline{\chi}, a)\mathcal{W}_3(\chi, \lambda) \\ &\ll q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \bmod q} \big|\mathcal{W}_3(\chi, \lambda)\big| \\ &\ll q^{\varepsilon}\Big(\sum_{\chi \bmod q} \big|\mathcal{W}_3(\chi, \lambda)\big|^2\Big)^{\frac{1}{2}}. \end{aligned} \tag{2.6}$$

Combining (2.4)–(2.6), we obtain

$$\begin{aligned} &\sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{R}(q,a)} \big|W(\alpha)\Delta_3(\alpha)\big|^2 \mathrm{d}\alpha \\ \ll &\sum_{1\leqslant q\leqslant Q_0} q^{-\frac{2}{3}+\varepsilon} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{|\lambda|\leqslant \frac{1}{qQ_0}} \frac{X_3^2 \log^4 N}{(1+N|\lambda|)^2} \sum_{\chi \bmod q} \big|\mathcal{W}_3(\chi, \lambda)\big|^2 \mathrm{d}\lambda \\ \ll &N^{\frac{2}{3}}(\log N)^4 \sum_{1\leqslant q\leqslant Q_0} q^{-\frac{2}{3}+\varepsilon} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \sum_{\chi \bmod q} \int_{|\lambda|\leqslant \frac{1}{qQ_0}} \frac{|\mathcal{W}_3(\chi, \lambda)|^2}{(1+N|\lambda|)^2} \mathrm{d}\lambda \\ \ll &N^{\frac{2}{3}}(\log N)^4 \sum_{1\leqslant q\leqslant Q_0} q^{-\frac{2}{3}+\varepsilon} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \sum_{\chi \bmod q} \bigg(\int_{|\lambda|\leqslant \frac{1}{N}} \big|\mathcal{W}_3(\chi, \lambda)\big|^2 \mathrm{d}\lambda \end{aligned}$$

$$+ (\log N) \times \max_{1 \leqslant Z \ll N/Q_0} \frac{1}{Z^2} \int_{\frac{Z}{N} \leqslant |\lambda| \leqslant \frac{2Z}{N}} \left|\mathcal{W}_3(\chi, \lambda)\right|^2 \mathrm{d}\lambda \Bigg). \tag{2.7}$$

By Lemma 2.3 and Lemma 2.4, we obtain

$$\begin{aligned}
&\int_{|\lambda| \leqslant \frac{1}{N}} \left|\mathcal{W}_3(\chi, \lambda)\right|^2 \mathrm{d}\lambda \\
&\ll \frac{1}{N^2} \int_{-\infty}^{+\infty} \Bigg| \sum_{\substack{X_3 < p \leqslant 2X_3 \\ y < p^3 \leqslant y + \frac{N}{20}}} (\log p)\chi(p) - \delta_\chi \sum_{\substack{X_3 < m \leqslant 2X_3 \\ y < m^3 \leqslant y + \frac{N}{20}}} 1 \Bigg|^2 \mathrm{d}y \\
&\ll \frac{1}{N^2} \int_{X_3^3 - \frac{N}{20}}^{(2X_3)^3} \Bigg| \sum_{\substack{X_3 < p \leqslant 2X_3 \\ y < p^3 \leqslant y + \frac{N}{20}}} (\log p)\chi(p) - \delta_\chi \sum_{\substack{X_3 < m \leqslant 2X_3 \\ y < m^3 \leqslant y + \frac{N}{20}}} 1 \Bigg|^2 \mathrm{d}y \\
&\ll \frac{1}{N^2} \cdot N \cdot X_3^2 \exp\left(- \log^{1/3} N\right) \ll N^{-\frac{1}{3}} \exp\left(- \log^{1/3} N\right).
\end{aligned} \tag{2.8}$$

Similarly, we derive that

$$\int_{\frac{Z}{N} \leqslant |\lambda| \leqslant \frac{2Z}{N}} \left|\mathcal{W}_3(\chi, \lambda)\right|^2 \mathrm{d}\lambda \ll Z^2 N^{-\frac{1}{3}} \exp\left(- \log^{1/3} N\right). \tag{2.9}$$

Combining (2.7)–(2.9), we obtain

$$\begin{aligned}
&\sum_{1 \leqslant q \leqslant Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{N}(q,a)} \left|W(\alpha)\Delta_3(\alpha)\right|^2 \mathrm{d}\alpha \\
&\ll N^{\frac{2}{3}} (\log N)^5 \sum_{1 \leqslant q \leqslant Q_0} q^{-\frac{2}{3}+\varepsilon} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \varphi(q) \cdot N^{-\frac{1}{3}} \exp\left(- \log^{1/3} N\right) \\
&\ll N^{\frac{1}{3}} (\log N)^5 \exp\left(- \log^{1/3} N\right) \sum_{1 \leqslant q \leqslant Q_0} q^{\frac{4}{3}+\varepsilon} \\
&\ll N^{\frac{1}{3}} Q_0^{\frac{7}{3}+\varepsilon} (\log N)^5 \exp\left(- \log^{1/3} N\right) \ll N^{\frac{1}{3}} \exp\left(- \log^{1/4} N\right).
\end{aligned}$$

This completes the proof of Lemma 2.7. □

Lemma 2.8. *For $\alpha = \frac{a}{q} + \lambda$, define*

$$V_j(\alpha) = \frac{S_j^*(q,a)}{\varphi(q)} v_j(\lambda). \tag{2.10}$$

Then we have

$$\sum_{1 \leqslant q \leqslant Q_0} \sum_{\substack{a=-q \\ (a,q)=1}}^{2q} \int_{\mathfrak{N}(q,a)} \left|V_3(\alpha)\right|^2 \mathrm{d}\alpha \ll N^{-\frac{1}{3}} \log^{51A} N, \tag{2.11}$$

and

$$\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{N}(q,a)}\left|W(\alpha)\right|^2\mathrm{d}\alpha \ll N^{-\frac{1}{3}+\varepsilon}. \tag{2.12}$$

where $\mathfrak{N}(q,a)$ *and* $W(\alpha)$ *are defined by (2.1) and (2.3), respectively.*

Proof. For (2.11), one can refer to the process of Lemma 2.6 of Cai [5]. For (2.12), by (2.5) we obtain

$$\begin{aligned}
&\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{N}(q,a)}\left|W(\alpha)\right|^2\mathrm{d}\alpha\\
\ll&\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{|\lambda|\leqslant\frac{1}{Q_0}}\frac{\tau_3^2(q)X_3^2\log^4 N}{q^{2/3}(1+N|\lambda|)^2}\mathrm{d}\lambda\\
\ll&N^{\frac{2}{3}}(\log N)^4\sum_{1\leqslant q\leqslant Q_0}q^{-\frac{2}{3}+\varepsilon}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\left(\int_{|\lambda|\leqslant\frac{1}{N}}\mathrm{d}\lambda+\int_{\frac{1}{N}<|\lambda|\leqslant\frac{1}{Q_0}}\frac{1}{N^2\lambda^2}\mathrm{d}\lambda\right)\\
\ll&N^{-\frac{1}{3}}(\log N)^4\sum_{1\leqslant q\leqslant Q_0}q^{\frac{1}{3}+\varepsilon}\ll N^{-\frac{1}{3}}Q_0^{\frac{4}{3}+\varepsilon}(\log N)^4\ll N^{-\frac{1}{3}+\varepsilon},
\end{aligned}$$

which completes the proof of Lemma 2.8. □

For $(a,q)=1,\ 1\leqslant a\leqslant q\leqslant Q_2$, define

$$\mathfrak{M}(q,a)=\left[\frac{a}{q}-\frac{1}{qQ_2},\frac{a}{q}+\frac{1}{qQ_2}\right],\qquad \mathfrak{M}=\bigcup_{1\leqslant q\leqslant Q_0^6}\bigcup_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}\mathfrak{M}(q,a),$$

$$\mathfrak{M}_0(q,a)=\left[\frac{a}{q}-\frac{Q_0}{N},\frac{a}{q}+\frac{Q_0}{N}\right],\qquad \mathfrak{M}_0=\bigcup_{1\leqslant q\leqslant Q_0^6}\bigcup_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}\mathfrak{M}_0(q,a),$$

$$\mathcal{I}_0=\left[-\frac{1}{Q_2},1-\frac{1}{Q_2}\right],\qquad \mathfrak{m}_0=\mathfrak{M}\setminus\mathfrak{M}_0,$$

$$\mathfrak{m}_1=\bigcup_{Q_0^6<q\leqslant Q_1}\bigcup_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}\mathfrak{M}(q,a),\qquad \mathfrak{m}_2=\mathcal{I}_0\setminus(\mathfrak{M}\cup\mathfrak{m}_1).$$

Then we obtain the Farey dissection

$$\mathcal{I}_0=\mathfrak{M}_0\cup\mathfrak{m}_0\cup\mathfrak{m}_1\cup\mathfrak{m}_2. \tag{2.13}$$

Lemma 2.9. *For* $\alpha=\frac{a}{q}+\lambda$, *define*

$$V_j^*(\alpha)=\frac{S_j^*(q,a)}{\varphi(q)}v_j^*(\lambda).$$

Then for $\alpha = \frac{a}{q} + \lambda \in \mathfrak{M}_0$, *we have*

$$\text{(i) } f_j(\alpha) = V_j(\alpha) + O(X_j \exp(-\log^{1/3} N)),$$
$$\text{(ii) } f_j^*(\alpha) = V_j^*(\alpha) + O(X_j^* \exp(-\log^{1/3} N)),$$
$$\text{(iii) } g_r(\alpha) = \frac{c_r V_3(\alpha)}{\log X_3} + O(X_3 \exp(-\log^{1/3} N)),$$

where $V_j(\alpha)$ *is defined by (2.10), and*

$$c_r = (1 + O(\varepsilon)) \times \int_{r-1}^{38} \frac{\mathrm{d}t_1}{t_1} \int_{r-2}^{t_1-1} \frac{\mathrm{d}t_2}{t_2} \cdots \int_{3}^{t_{r-4}-1} \frac{\mathrm{d}t_{r-3}}{t_{r-3}} \int_{2}^{t_{r-3}-1} \frac{\log(t_{r-2}-1)}{t_{r-2}} \mathrm{d}t_{r-2}. \tag{2.14}$$

Proof. By some routine arguments and partial summation, (i)–(iii) follow from Siegel–Walfisz theorem and prime number theorem. □

Lemma 2.10. *Suppose that* $|\alpha - a/q| \leqslant q^{-1} X_3^{-27/13}$, *where* $(a, q) = 1$ *and* $q \leqslant X_3^{27/13}$. *Then we have*

$$\mathscr{F}(\alpha) \ll X_3^{\frac{3}{4}+\varepsilon} D^{\frac{1}{4}} + \frac{\tau(q) X_3 \log X_3}{(q + X_3^3 |q\alpha - a|)^{1/3}}.$$

Proof. We rewrite $\mathscr{F}(\alpha)$ as

$$\mathscr{F}(\alpha) = \sum_{d \leqslant D} a(d) \sum_{\frac{X_3}{d} < y \leqslant \frac{2X_3}{d}} e(y^3 d^3 \alpha).$$

By Dirichlet's theorem on Diophantine approximation, there exist coprime integers $b = b(d)$ and $r = r(d)$ with

$$r \leqslant \frac{10X_3^2}{d^2}, \qquad \left| d^3\alpha - \frac{b}{r} \right| \leqslant \frac{d^2}{10rX_3^2}.$$

By Weyl's inequality (see Lemma 2.4 of Vaughan [38]), we have

$$\sum_{\frac{X_3}{d} < y \leqslant \frac{2X_3}{d}} e(y^3 d^3 \alpha) \ll \left(\frac{X_3}{d} \right)^{1+\varepsilon} \left(\frac{1}{r} + \frac{d}{X_3} + \frac{rd^3}{X_3^3} \right)^{\frac{1}{4}}.$$

If $r > X_3 d^{-1}$, then we have

$$\sum_{\frac{X_3}{d} < y \leqslant \frac{2X_3}{d}} e(y^3 d^3 \alpha) \ll \left(\frac{X_3}{d} \right)^{\frac{3}{4}+\varepsilon},$$

and thus $\mathscr{F}(\alpha) \ll X_3^{\frac{3}{4}+\varepsilon} D^{\frac{1}{4}}$. For $r \leqslant X_3 d^{-1}$, by Theorem 4.1 and Lemma 2.8 of Vaughan [38], and (i) of Lemma 2.1, we obtain

$$\sum_{\frac{X_3}{d} < y \leqslant \frac{2X_3}{d}} e(y^3 d^3 \alpha) \ll r^{-1} \cdot S_3(r, b) \cdot v_3(d^3\alpha - b/r) + r^{\frac{1}{2}+\varepsilon}$$
$$\ll r^{-1} \cdot r^{\frac{2}{3}} \cdot \min\left(\frac{X_3}{d}, \left| d^3\alpha - \frac{b}{r} \right|^{-\frac{1}{3}} \right) + \left(\frac{X_3}{d} \right)^{\frac{1}{2}+\varepsilon}$$

$$\ll r^{-\frac{1}{3}} \cdot \frac{X_3}{d} \cdot \min\left(1, \left(\left(\frac{X_3}{d}\right)^3 \left|d^3\alpha - \frac{b}{r}\right|\right)^{-\frac{1}{3}}\right) + \left(\frac{X_3}{d}\right)^{\frac{1}{2}+\varepsilon}$$

$$\ll r^{-\frac{1}{3}} \cdot \frac{X_3}{d} \cdot \left(1 + \left(\frac{X_3}{d}\right)^3 \left|d^3\alpha - \frac{b}{r}\right|\right)^{-\frac{1}{3}} + \left(\frac{X_3}{d}\right)^{\frac{1}{2}+\varepsilon}.$$

If

$$r > \left(\frac{X_3}{d}\right)^{\frac{3}{4}} \qquad \text{or} \qquad \left|d^3\alpha - \frac{b}{r}\right| > \frac{1}{r}\left(\frac{d}{X_3}\right)^{\frac{9}{4}},$$

then we get

$$\sum_{\frac{X_3}{d} < y \leqslant \frac{2X_3}{d}} e(y^3 d^3 \alpha) \ll X_3^{\frac{3}{4}} d^{-\frac{3}{4}},$$

and hence $\mathscr{F}(\alpha) \ll X_3^{\frac{3}{4}+\varepsilon} D^{\frac{1}{4}}$. If there hold

$$r \leqslant \left(\frac{X_3}{d}\right)^{\frac{3}{4}} \qquad \text{and} \qquad \left|d^3\alpha - \frac{b}{r}\right| \leqslant \frac{1}{r}\left(\frac{d}{X_3}\right)^{\frac{9}{4}}, \tag{2.15}$$

then summing over d it follows that

$$\mathscr{F}(\alpha) \ll X_3^{\frac{3}{4}+\varepsilon} D^{\frac{1}{4}} + X_3 \sum_{d \in \mathfrak{D}} d^{-1} r^{-\frac{1}{3}} \left(1 + \left(\frac{X_3}{d}\right)^3 \left|d^3\alpha - \frac{b}{r}\right|\right)^{-\frac{1}{3}}, \tag{2.16}$$

where $\mathfrak{D}$ denotes the set of all $d \leqslant D$ for which the conditions (2.15) hold. For any $d \in \mathfrak{D}$, we compare (2.15) with the Diophantine approximation to α postulated in Lemma 2.10 and deduce that

$$\left|\frac{ad^3}{q} - \frac{b}{r}\right| \leqslant \left|\frac{ad^3}{q} - d^3\alpha\right| + \left|d^3\alpha - \frac{b}{r}\right| \leqslant \frac{d^3}{qX_3^{27/13}} + \frac{1}{r}\left(\frac{d}{X_3}\right)^{\frac{9}{4}},$$

which yields that

$$\left|ad^3 r - bq\right| \leqslant rd^3 X_3^{-\frac{27}{13}} + qd^{\frac{9}{4}} X_3^{-\frac{9}{4}} \leqslant \left(\frac{X_3}{d}\right)^{\frac{3}{4}} d^3 X_3^{-\frac{27}{13}} + X_3^{\frac{27}{13}} \cdot d^{\frac{9}{4}} X_3^{-\frac{9}{4}}$$
$$\leqslant D^{\frac{9}{4}} X_3^{-\frac{69}{52}} + D^{\frac{9}{4}} X_3^{-\frac{9}{52}} \leqslant 2D^{\frac{9}{4}} X_3^{-\frac{9}{52}} < 1.$$

This means $ad^3 r = bq$, and thus

$$\frac{ad^3}{q} = \frac{b}{r} \qquad \text{whence} \qquad r = \frac{q}{(q, d^3)}.$$

By using the trivial estimate $(q, d^3) \leqslant (q, d)^3$, we deduce that

$$\sum_{d \in \mathfrak{D}} d^{-1} r^{-\frac{1}{3}} \left(1 + \left(\frac{X_3}{d}\right)^3 \left|d^3\alpha - \frac{b}{r}\right|\right)^{-\frac{1}{3}}$$
$$= \sum_{d \in \mathfrak{D}} d^{-1} \frac{(q, d^3)^{1/3}}{q^{1/3}} \left(1 + \left(\frac{X_3}{d}\right)^3 d^3 \left|\alpha - \frac{a}{q}\right|\right)^{-\frac{1}{3}}$$

$$\ll q^{-\frac{1}{3}}\Big(1+X_3^3\Big|\alpha-\frac{a}{q}\Big|\Big)^{-\frac{1}{3}}\sum_{d\leqslant D}\frac{(q,d)}{d}$$
$$\ll \Big(q+X_3^3\big|q\alpha-a\big|\Big)^{-\frac{1}{3}}\sum_{k|q}\sum_{d\leqslant D/k}\frac{1}{d}\ll\frac{\tau(q)\log X_3}{(q+X_3^3|q\alpha-a|)^{1/3}},$$

which combined with (2.16) yields the desired result of Lemma 2.10. □

3. Mean value theorems

In this section, we shall give the mean value theorems for the proof of Theorem 1.1.

Proposition 3.1. *Let*

$$J(N,d)=\sum_{\substack{p_1^2+p_2^2+m^3+p_3^3+p_4^3+p_5^3=N\\ X_2<p_1,p_2\leqslant 2X_2\\ X_3<m,p_3\leqslant 2X_3\\ X_3^*<p_4,p_5\leqslant 2X_3^*\\ m\equiv 0\,(\mathrm{mod}\ d)}}\prod_{j=1}^{5}\log p_j.$$

Then we have

$$\sum_{d\leqslant D}a(d)\Big(J(N,d)-\frac{\mathfrak{S}_d(N)}{d}\mathcal{J}(N)\Big)\ll N^{\frac{11}{9}}\log^{-A}N.$$

Proof. Let

$$\mathcal{K}(\alpha)=\mathscr{F}(\alpha)f_2^2(\alpha)f_3(\alpha)f_3^{*2}(\alpha)e(-N\alpha).$$

By the Farey dissection (2.13), we have

$$\sum_{d\leqslant D}a(d)J(N,d)=\int_{\mathcal{I}_0}\mathcal{K}(\alpha)\mathrm{d}\alpha=\Big(\int_{\mathfrak{M}_0}+\int_{\mathfrak{m}_0}+\int_{\mathfrak{m}_1}+\int_{\mathfrak{m}_2}\Big)\mathcal{K}(\alpha)\mathrm{d}\alpha. \tag{3.1}$$

By Lemma 2.10, for $\alpha\in\mathfrak{m}_2$, there holds

$$\mathscr{F}(\alpha)\ll X_3^{\frac{3}{4}+\varepsilon}D^{\frac{1}{4}}+\frac{\tau(q)X_3\log X_3}{(q+X_3^3|q\alpha-a|)^{1/3}}$$
$$\ll N^{\frac{1}{4}+\varepsilon}D^{\frac{1}{4}}+N^{\frac{1}{3}+\varepsilon}Q_1^{-\frac{1}{3}}\ll N^{\frac{5}{18}-14\varepsilon}. \tag{3.2}$$

From (i) of Lemma 2.2, Hua's lemma (see Lemma 2.5 of Vaughan [38]), and Cauchy's inequality, we obtain

$$\int_0^1\big|f_2^2(\alpha)f_3(\alpha)f_3^{*2}(\alpha)\big|\mathrm{d}\alpha\ll\Big(\int_0^1\big|f_2(\alpha)\big|^4\mathrm{d}\alpha\Big)^{\frac{1}{2}}\Big(\int_0^1\big|f_3(\alpha)f_3^{*2}(\alpha)\big|^2\mathrm{d}\alpha\Big)^{\frac{1}{2}}$$
$$\ll(X_2^{2+\varepsilon})^{\frac{1}{2}}(N^{\frac{8}{9}+\varepsilon})^{\frac{1}{2}}\ll N^{\frac{17}{18}+\varepsilon}, \tag{3.3}$$

which combined with (3.2) gives

$$\int_{\mathfrak{m}_2}\mathcal{K}(\alpha)\mathrm{d}\alpha\ll\sup_{\alpha\in\mathfrak{m}_2}|\mathscr{F}(\alpha)|\times\int_0^1\big|f_2^2(\alpha)f_3(\alpha)f_3^{*2}(\alpha)\big|\mathrm{d}\alpha$$

$$\ll N^{\frac{5}{18}-14\varepsilon}\cdot N^{\frac{17}{18}+\varepsilon}\ll N^{\frac{11}{9}-13\varepsilon}. \tag{3.4}$$

For $\alpha\in\mathfrak{m}_1$, it follows from Theorem 4.1 of Vaughan [38] that

$$\mathscr{F}(\alpha)=W(\alpha)+O(DQ_1^{\frac{1}{2}+\varepsilon})=W(\alpha)+O(N^{\frac{17}{156}-28\varepsilon}), \tag{3.5}$$

where $W(\alpha)$ is defined by (2.3). Define

$$\mathcal{K}_1(\alpha)=W(\alpha)f_2^2(\alpha)f_3(\alpha)f_3^{*2}(\alpha)e(-N\alpha).$$

Then by (3.3) and (3.5) we have

$$\int_{\mathfrak{m}_1}\mathcal{K}(\alpha)\mathrm{d}\alpha=\int_{\mathfrak{m}_1}\mathcal{K}_1(\alpha)\mathrm{d}\alpha+O(N^{\frac{493}{468}-27\varepsilon}). \tag{3.6}$$

Let

$$\mathfrak{N}_0(q,a)=\Big[\frac{a}{q}-\frac{1}{N^{11/12}},\frac{a}{q}+\frac{1}{N^{11/12}}\Big],\qquad \mathfrak{N}_0=\bigcup_{1\leqslant q\leqslant Q_0}\bigcup_{\substack{-q\leqslant a\leqslant 2q\\(a,q)=1}}\mathfrak{N}_0(q,a),$$

$$\mathfrak{N}_1(q,a)=\mathfrak{N}(q,a)\setminus\mathfrak{N}_0(q,a),\qquad \mathfrak{N}_1=\bigcup_{1\leqslant q\leqslant Q_0}\bigcup_{\substack{-q\leqslant a\leqslant 2q\\(a,q)=1}}\mathfrak{N}_1(q,a),$$

$$\mathfrak{N}=\bigcup_{1\leqslant q\leqslant Q_0}\bigcup_{\substack{-q\leqslant a\leqslant 2q\\(a,q)=1}}\mathfrak{N}(q,a),$$

where $\mathfrak{N}(q,a)$ is defined by (2.1). Then we have $\mathfrak{m}_1\subseteq\mathcal{I}_0\subseteq\mathfrak{N}$. By Dirichlet's theorem on Diophantine rational approximation, we derive that

$$\int_{\mathfrak{m}_1}\mathcal{K}_1(\alpha)\mathrm{d}\alpha\ll\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_1\cap\mathfrak{N}_0(q,a)}|\mathcal{K}_1(\alpha)|\mathrm{d}\alpha$$
$$+\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_1\cap\mathfrak{N}_1(q,a)}|\mathcal{K}_1(\alpha)|\mathrm{d}\alpha. \tag{3.7}$$

By (2.5) it is easy to see that, for $\alpha\in\mathfrak{N}_1(q,a)$, there holds

$$W(\alpha)\ll N^{\frac{1}{4}}\log^2 N,$$

which combined with (3.3) yields

$$\sum_{1\leqslant q\leqslant Q_0}\sum_{\substack{a=-q\\(a,q)=1}}^{2q}\int_{\mathfrak{m}_1\cap\mathfrak{N}_1(q,a)}|\mathcal{K}_1(\alpha)|\mathrm{d}\alpha$$
$$\ll N^{\frac{1}{4}}\log^2 N\times\int_0^1\Big|f_2^2(\alpha)f_3(\alpha)f_3^{*2}(\alpha)\Big|\mathrm{d}\alpha\ll N^{\frac{11}{9}-\varepsilon}. \tag{3.8}$$

For $\alpha \in \mathfrak{N}_0(q,a)$, it follows from Lemma 2.6 that

$$f_3(\alpha) = \Delta_3(\alpha) + V_3(\alpha) + O(1).$$

Hence, one has

$$\begin{aligned}
&\sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{m}_1\cap\mathfrak{N}_0(q,a)} |\mathcal{K}_1(\alpha)|\mathrm{d}\alpha \\
\ll\ & \sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{m}_1\cap\mathfrak{N}_0(q,a)} \Big|W(\alpha)\Delta_3(\alpha)f_2^2(\alpha)f_3^{*2}(\alpha)\Big|\mathrm{d}\alpha \\
&+ \sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{m}_1\cap\mathfrak{N}_0(q,a)} \Big|W(\alpha)V_3(\alpha)f_2^2(\alpha)f_3^{*2}(\alpha)\Big|\mathrm{d}\alpha \\
&+ \sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{m}_1\cap\mathfrak{N}_0(q,a)} \Big|W(\alpha)f_2^2(\alpha)f_3^{*2}(\alpha)\Big|\mathrm{d}\alpha \\
=:\ & \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3,
\end{aligned} \tag{3.9}$$

where $\Delta_3(\alpha)$ and $V_3(\alpha)$ are defined by (2.2) and (2.10), respectively.

It follows from Cauchy's inequality, (ii) of Lemma 2.2, and Lemma 2.7 that

$$\begin{aligned}
\mathfrak{I}_1 &\ll \Bigg(\sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{N}(q,a)} \Big|W(\alpha)\Delta_3(\alpha)\Big|^2\mathrm{d}\alpha\Bigg)^{\frac12} \Bigg(\int_0^1 \Big|f_2(\alpha)f_3^*(\alpha)\Big|^4\mathrm{d}\alpha\Bigg)^{\frac12} \\
&\ll \Big(N^{\frac13}\exp\big(-\log^{1/4}N\big)\Big)^{\frac12}\Big(N^{\frac{19}{9}}\log^2 N\Big)^{\frac12} \ll N^{\frac{11}{9}}\exp\big(-\log^{1/5}N\big).
\end{aligned} \tag{3.10}$$

By (2.5), it is easy to see that, for $\alpha \in \mathfrak{m}_1$, there holds

$$\sup_{\alpha\in\mathfrak{m}_1} |W(\alpha)| \ll N^{\frac13}\log^{-50A} N. \tag{3.11}$$

Therefore, by (2.11), (ii) of Lemma 2.2, (3.11) and Cauchy's inequality, we derive that

$$\begin{aligned}
\mathfrak{I}_2 &\ll \sup_{\alpha\in\mathfrak{m}_1}|W(\alpha)| \cdot \Bigg(\sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{N}(q,a)} \Big|V_3(\alpha)\Big|^2\mathrm{d}\alpha\Bigg)^{\frac12} \Bigg(\int_0^1 \Big|f_2(\alpha)f_3^*(\alpha)\Big|^4\mathrm{d}\alpha\Bigg)^{\frac12} \\
&\ll (N^{\frac13}\log^{-50A}N)\cdot(N^{-\frac13}\log^{51A}N)^{\frac12}\cdot(N^{\frac{19}{9}}\log^2 N)^{\frac12} \\
&\ll N^{\frac{11}{9}}\log^{-20A}N.
\end{aligned} \tag{3.12}$$

It follows from (ii) of Lemma 2.2, (2.12) and Cauchy's inequality that

$$\mathfrak{I}_3 \ll \Bigg(\sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{N}(q,a)} \Big|W(\alpha)\Big|^2\mathrm{d}\alpha\Bigg)^{\frac12} \Bigg(\int_0^1 \Big|f_2(\alpha)f_3^*(\alpha)\Big|^4\mathrm{d}\alpha\Bigg)^{\frac12}$$

$$\ll (N^{-\frac{1}{3}+\varepsilon})^{\frac{1}{2}} \cdot (N^{\frac{19}{9}} \log^2 N)^{\frac{1}{2}} \ll N^{\frac{8}{9}+\varepsilon} \ll N^{\frac{11}{9}-\varepsilon}. \tag{3.13}$$

Combining (3.9), (3.10), (3.12) and (3.13), we can deduce that

$$\sum_{1\leqslant q\leqslant Q_0} \sum_{\substack{a=-q\\(a,q)=1}}^{2q} \int_{\mathfrak{m}_1\cap\mathfrak{N}_0(q,a)} |\mathcal{K}_1(\alpha)|\mathrm{d}\alpha \ll N^{\frac{11}{9}} \log^{-20A} N. \tag{3.14}$$

From (3.6), (3.7), (3.8) and (3.14), we deduce that

$$\int_{\mathfrak{m}_1} \mathcal{K}(\alpha)\mathrm{d}\alpha \ll N^{\frac{11}{9}} \log^{-20A} N. \tag{3.15}$$

Similarly, we obtain

$$\int_{\mathfrak{m}_0} \mathcal{K}(\alpha)\mathrm{d}\alpha \ll N^{\frac{11}{9}} \log^{-20A} N. \tag{3.16}$$

For $\alpha \in \mathfrak{M}_0$, define

$$\mathcal{K}_0(\alpha) = W(\alpha)V_2^2(\alpha)V_3(\alpha)V_3^{*2}(\alpha)e(-N\alpha).$$

By noticing that (3.5) still holds for $\alpha \in \mathfrak{M}_0$, it follows from Lemma 2.9 and (3.5) that

$$\mathcal{K}(\alpha) - \mathcal{K}_0(\alpha) \ll N^{\frac{20}{9}} \exp(-\log^{1/4} N),$$

which implies that

$$\int_{\mathfrak{M}_0} \mathcal{K}(\alpha)\mathrm{d}\alpha = \int_{\mathfrak{M}_0} \mathcal{K}_0(\alpha)\mathrm{d}\alpha + O(N^{\frac{11}{9}} \log^{-A} N). \tag{3.17}$$

By the well–known standard technique in the Hardy–Littlewood method, we deduce that

$$\int_{\mathfrak{M}_0} \mathcal{K}_0(\alpha)\mathrm{d}\alpha = \sum_{d\leqslant D} \frac{a(d)}{d} \mathfrak{S}_d(N)\mathcal{J}(N) + O(N^{\frac{11}{9}} \log^{-A} N), \tag{3.18}$$

and

$$\mathcal{J}(N) \asymp N^{\frac{11}{9}}. \tag{3.19}$$

Finally, Proposition 3.1 follows from (3.1), (3.4) and (3.15)–(3.19). This completes the proof of Proposition 3.1. □

By the same method, we have the following Proposition.

Proposition 3.2. *Define*

$$J_r(N,d) = \sum_{\substack{p_1^2+p_2^2+m^3+(\ell p)^3+p_4^3+p_5^3=N\\ X_3<\ell p\leqslant 2X_3,\ \ell\in\mathcal{N}_r,\ m\equiv 0 \ (\mathrm{mod}\ d)\\ X_2<p_1,\,p_2\leqslant 2X_2\\ X_3^*<p_4,\,p_5\leqslant 2X_3^*}} \left(\frac{\log p}{\log \dfrac{X_3}{\ell}} \prod_{j=1,2,4,5} \log p_j \right).$$

Then we have

$$\sum_{d\leqslant D} a(d)\Big(J_r(N,d) - \frac{c_r\mathfrak{S}_d(N)}{d\log X_3}\mathcal{J}(N)\Big) \ll N^{\frac{11}{9}} \log^{-A} N,$$

where c_r is defined by (2.14).

4. On the function $\omega(d)$

In this section, we shall investigate the function $\omega(d)$ which is defined in (4.10) and required in the proof of Theorem 1.1.

Lemma 4.1. *Let $\mathscr{K}(q,N)$ and $\mathscr{L}(q,N)$ denote the number of solutions of the congruences*

$$x_1^2+x_2^2+y_2^3+y_3^3+y_4^3\equiv N \pmod q,\quad 1\leqslant x_i,y_j\leqslant q,\quad (x_iy_j,q)=1,$$

and

$$x_1^2+x_2^2+m^3+y_2^3+y_3^3+y_4^3\equiv N \pmod q,\quad 1\leqslant x_i,y_j,m\leqslant q,\quad (x_iy_j,q)=1,$$

respectively. Then, for all $N\equiv 0 \pmod 2$, we have $\mathscr{L}(p,N)>\mathscr{K}(p,N)$ for all primes. Moreover, there hold

$$\mathscr{L}(p,n)=p^5+O(p^4),$$

and

$$\mathscr{K}(p,n)=p^4+O(p^3).$$

Proof. Let $\mathscr{L}^*(q,N)$ denote the number of solutions of the congruence

$$x_1^2+x_2^2+m^3+y_2^3+y_3^3+y_4^3\equiv N \pmod q,\quad 1\leqslant x_i,y_j,m\leqslant q,\quad (mx_iy_j,q)=1.$$

Then by the orthogonality of Dirichlet characters, we have

$$\begin{aligned} p\cdot\mathscr{L}^*(p,N) &= \sum_{a=1}^{p} S_2^{*2}(p,a)S_3^{*4}(p,a)e\Big(-\frac{aN}{p}\Big)\\ &=(p-1)^6+E_p, \end{aligned}\tag{4.1}$$

where

$$E_p=\sum_{a=1}^{p-1} S_2^{*2}(p,a)S_3^{*4}(p,a)e\Big(-\frac{aN}{p}\Big).$$

By (iv) of Lemma 2.1, we have

$$|E_p|\leqslant (p-1)(\sqrt{p}+1)^2(2\sqrt{p}+1)^4.\tag{4.2}$$

It is easy to check that $|E_p|<(p-1)^6$ for $p\geqslant 11$. Hence we get $\mathscr{L}^*(p,N)>0$ for all $p\geqslant 11$. On the other hand, for $p=2,3,5,7$, we can check $\mathscr{L}^*(p,N)>0$ directly by hand. Therefore, we obtain $\mathscr{L}^*(p,N)>0$ for all primes with $N\equiv 0 \pmod 2$, and

$$\mathscr{L}(p,N)=\mathscr{L}^*(p,N)+\mathscr{K}(p,N)>\mathscr{K}(p,N).\tag{4.3}$$

From (4.1) and (4.2), we derive that

$$\mathscr{L}^*(p,N)=p^5+O(p^4).\tag{4.4}$$

By a similar argument of (4.1) and (4.2), we have

$$\mathscr{K}(p,N)=p^4+O(p^3).\tag{4.5}$$

Combining (4.3)–(4.5), we obtain the desired results. □

Lemma 4.2. *The series $\mathfrak{S}(N)$ is absolutely convergent. Moreover, there exists an absolute positive constant $c^* > 0$ such that for $N \equiv 0 \pmod 2$, there holds*

$$\mathfrak{S}(N) \geqslant c^* > 0.$$

Proof. From (i) and (ii) of Lemma 2.1, we obtain

$$|A(q,N)| \ll \frac{|B(q,N)|}{q\varphi^5(q)} \ll \frac{q^{\frac{13}{6}+5\varepsilon}}{\varphi^4(q)} \ll \frac{q^{\frac{13}{6}+5\varepsilon}(\log\log q)^4}{q^4} \ll \frac{1}{q^{5/3}}.$$

Thus, the series

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} A(q,N)$$

converges absolutely. Noting the fact that $A(q,N)$ is multiplicative in q and by (v) of Lemma 2.1, we get

$$\mathfrak{S}(N) = \prod_p \left(1 + \sum_{t=1}^{\infty} A(p^t,N)\right) = \prod_p (1 + A(p,N)). \tag{4.6}$$

From (iii) and (iv) of Lemma 2.1, we know that, for $p \geqslant 17$, there holds

$$|A(p,N)| \leqslant \frac{(p-1)\cdot 2\sqrt{p}(\sqrt{p}+1)^2(2\sqrt{p}+1)^3}{p(p-1)^5} \leqslant \frac{50}{p^2}.$$

Therefore, we get

$$\prod_{p\geqslant 17} (1 + A(p,N)) \geqslant \prod_{p\geqslant 17} \left(1 - \frac{50}{p^2}\right) \geqslant c_1 > 0. \tag{4.7}$$

On the other hand, it is easy to see that

$$1 + A(p,N) = \frac{\mathscr{L}(p,N)}{(p-1)^5}. \tag{4.8}$$

By Lemma 4.1, we have $\mathscr{L}(p,N) > 0$ for all primes p with $N \equiv 0 \pmod 2$, and thus $1 + A(p,N) > 0$. Consequently, we obtain

$$\prod_{p<17} (1 + A(p,N)) \geqslant c_2 > 0. \tag{4.9}$$

Combining (4.6), (4.7), (4.9), and taking $c^* = c_1 c_2 > 0$, we conclude that $\mathfrak{S}(N) \geqslant c^* > 0$, which completes the proof of Lemma 4.2. □

In view of Lemma 4.2, we define

$$\omega(d) = \frac{\mathfrak{S}_d(N)}{\mathfrak{S}(N)}. \tag{4.10}$$

Similar to (4.6), we have

$$\mathfrak{S}_d(N) = \prod_p (1 + A_d(p,N)). \tag{4.11}$$

If $(d, q) = 1$, then we have $S_k(q, ad^k) = S_k(q, a)$. Moreover, if $p|d$, then we get $A_d(p, N) = A_p(p, N)$. Therefore, we derive that

$$\omega(p) = \frac{1 + A_p(p, N)}{1 + A(p, N)}, \qquad \omega(d) = \prod_{p|d} \omega(p). \tag{4.12}$$

Also, it is easy to see that

$$1 + A_p(p, N) = \frac{p}{(p-1)^5} \mathscr{K}(p, N). \tag{4.13}$$

Using (4.8), (4.12) and (4.13), we derive

$$\omega(p) = \frac{p \cdot \mathscr{K}(p, N)}{\mathscr{L}(p, N)},$$

which combined with Lemma 4.1 yields the following lemma.

Lemma 4.3. *The function $\omega(d)$ is multiplicative and satisfies*

$$0 \leqslant \omega(p) < p, \qquad \omega(p) = 1 + O(p^{-1}). \tag{4.14}$$

5. Proof of Theorem 1.1

In this section, let $f(s)$ and $F(s)$ denote the classical functions in the linear sieve theory. Then it follows from (2.8) and (2.9) of Chapter 8 in [10] that

$$F(s) = \frac{2e^\gamma}{s}, \quad 1 \leqslant s \leqslant 3; \qquad f(s) = \frac{2e^\gamma \log(s-1)}{s}, \quad 2 \leqslant s \leqslant 4.$$

In the proof of Theorem 1.1, let $\lambda^{\pm}(d)$ be the lower and upper bounds for Rosser's weights of level D, hence for any positive integer d we have

$$|\lambda^{\pm}(d)| \leqslant 1, \quad \lambda^{\pm}(d) = 0 \quad \text{if} \quad d > D \quad \text{or} \quad \mu(d) = 0.$$

For further properties of Rosser's weights we refer to Iwaniec [15]. Define

$$\mathscr{W}(z) = \prod_{2<p<z} \Big(1 - \frac{\omega(p)}{p}\Big).$$

Then from Lemma 4.3 and Mertens' prime number theorem (See [32]) we obtain

$$\mathscr{W}(z) \asymp \frac{1}{\log N}. \tag{5.1}$$

In order to prove Theorem 1.1, we need the following lemma.

Lemma 5.1. *Under the condition (4.14), then if $z \leqslant D$, there holds*

$$\sum_{d|\mathscr{P}} \frac{\lambda^-(d)\omega(d)}{d} \geqslant \mathscr{W}(z)\Big(f\Big(\frac{\log D}{\log z}\Big) + O(\log^{-1/3} D)\Big), \tag{5.2}$$

and if $z \leqslant D^{1/2}$, there holds

$$\sum_{d|\mathscr{P}} \frac{\lambda^+(d)\omega(d)}{d} \leqslant \mathscr{W}(z)\Big(F\Big(\frac{\log D}{\log z}\Big) + O(\log^{-1/3} D)\Big). \tag{5.3}$$

Proof. See Iwaniec [16], (12) and (13) of Lemma 3. □

From the definition of $\mathscr{M}_r$, we know that $r \leqslant 39$. Hence we obtain

$$\begin{aligned}\mathscr{R}(N) \geqslant & \sum_{\substack{p_1^2+p_2^2+m^3+p_3^3+p_4^3+p_5^3=N\\ X_2<p_1,\,p_2\leqslant 2X_2\\ X_3<m,\,p_3\leqslant 2X_3\\ X_3^*<p_4,\,p_5\leqslant 2X_3^*\\ (m,\mathscr{P})=1}} 1 - \sum_{r=8}^{39} \sum_{\substack{p_1^2+p_2^2+m^3+p_3^3+p_4^3+p_5^3=N\\ m\in\mathscr{M}_r,\ X_3<p_3\leqslant 2X_3\\ X_2<p_1,\,p_2\leqslant 2X_2\\ X_3^*<p_4,\,p_5\leqslant 2X_3^*}} 1\\ =: & \ \Upsilon_0 - \sum_{r=8}^{39} \Upsilon_r. \end{aligned} \tag{5.4}$$

By the property (5.2) of Rosser's weight $\lambda^-(d)$ and Proposition 3.1, we get

$$\begin{aligned}\Upsilon_0 \geqslant & \frac{1}{\log \Xi} \sum_{\substack{p_1^2+p_2^2+m^3+p_3^3+p_4^3+p_5^3=N\\ X_2<p_1,\,p_2\leqslant 2X_2\\ X_3<m,\,p_3\leqslant 2X_3\\ X_3^*<p_4,\,p_5\leqslant 2X_3^*\\ (m,\mathscr{P})=1}} \prod_{j=1}^{5} \log p_j\\ = & \frac{1}{\log \Xi} \sum_{\substack{p_1^2+p_2^2+m^3+p_3^3+p_4^3+p_5^3=N\\ X_2<p_1,\,p_2\leqslant 2X_2\\ X_3<m,\,p_3\leqslant 2X_3\\ X_3^*<p_4,\,p_5\leqslant 2X_3^*}} \Bigg(\prod_{j=1}^{5} \log p_j\Bigg) \sum_{d|(m,\mathscr{P})} \mu(d)\\ \geqslant & \frac{1}{\log \Xi} \sum_{\substack{p_1^2+p_2^2+m^3+p_3^3+p_4^3+p_5^3=N\\ X_2<p_1,\,p_2\leqslant 2X_2\\ X_3<m,\,p_3\leqslant 2X_3\\ X_3^*<p_4,\,p_5\leqslant 2X_3^*}} \Bigg(\prod_{j=1}^{5} \log p_j\Bigg) \sum_{d|(m,\mathscr{P})} \lambda^-(d)\\ = & \frac{1}{\log \Xi} \sum_{d|\mathscr{P}} \lambda^-(d) J(N,d)\\ = & \frac{1}{\log \Xi} \sum_{d|\mathscr{P}} \frac{\lambda^-(d)\mathfrak{S}_d(N)}{d} \mathcal{J}(N) + O(N^{\frac{11}{9}} \log^{-A} N)\\ = & \frac{1}{\log \Xi} \Bigg(\sum_{d|\mathscr{P}} \frac{\lambda^-(d)\omega(d)}{d}\Bigg) \mathfrak{S}(N)\mathcal{J}(N) + O(N^{\frac{11}{9}} \log^{-A} N)\\ \geqslant & \frac{\mathfrak{S}(N)\mathcal{J}(N)\mathscr{W}(z)}{\log \Xi} f(3)\Big(1 + O\big(\log^{-1/3} D\big)\Big) + O(N^{\frac{11}{9}} \log^{-A} N). \end{aligned} \tag{5.5}$$

By the property (5.3) of Rosser's weight $\lambda^+(d)$ and Proposition 3.2, we have

$$\Upsilon_r \leqslant \sum_{\substack{p_1^2+p_2^2+m^3+(\ell p)^3+p_4^3+p_5^3=N\\ \ell\in\mathscr{N}_r,\ X_3<\ell p\leqslant 2X_3,\ (m,\mathscr{P})=1\\ X_2<p_1,\,p_2\leqslant 2X_2\\ X_3^*<p_4,\,p_5\leqslant 2X_3^*}} 1$$

$$
\begin{aligned}
&\leqslant \frac{1}{\log \boldsymbol{\Theta}} \sum_{\substack{p_1^2+p_2^2+m^3+(\ell p)^3+p_4^3+p_5^3=N \\ \ell\in\mathscr{N}_r,\ X_3<\ell p\leqslant 2X_3,\ (m,\mathscr{P})=1 \\ X_2<p_1,\,p_2\leqslant 2X_2 \\ X_3^*<p_4,\,p_5\leqslant 2X_3^*}} \frac{\log p}{\log \frac{X_3}{\ell}} \prod_{j=1,2,4,5} \log p_j \\
&= \frac{1}{\log \boldsymbol{\Theta}} \sum_{\substack{p_1^2+p_2^2+m^3+(\ell p)^3+p_4^3+p_5^3=N \\ \ell\in\mathscr{N}_r,\ X_3<\ell p\leqslant 2X_3 \\ X_2<p_1,\,p_2\leqslant 2X_2 \\ X_3^*<p_4,\,p_5\leqslant 2X_3^*}} \left(\frac{\log p}{\log \frac{X_3}{\ell}} \prod_{j=1,2,4,5} \log p_j\right) \sum_{d|(m,\mathscr{P})} \mu(d) \\
&\leqslant \frac{1}{\log \boldsymbol{\Theta}} \sum_{\substack{p_1^2+p_2^2+m^3+(\ell p)^3+p_4^3+p_5^3=N \\ \ell\in\mathscr{N}_r,\ X_3<\ell p\leqslant 2X_3 \\ X_2<p_1,\,p_2\leqslant 2X_2 \\ X_3^*<p_4,\,p_5\leqslant 2X_3^*}} \left(\frac{\log p}{\log \frac{X_3}{\ell}} \prod_{j=1,2,4,5} \log p_j\right) \sum_{d|(m,\mathscr{P})} \lambda^+(d) \\
&= \frac{1}{\log \boldsymbol{\Theta}} \sum_{d|\mathscr{P}} \lambda^+(d) J_r(N,d) \\
&= \frac{1}{\log \boldsymbol{\Theta}} \sum_{d|\mathscr{P}} \frac{\lambda^+(d) c_r \mathfrak{S}_d(N)}{d \log X_3} \mathcal{J}(N) + O(N^{\frac{11}{9}} \log^{-A} N) \\
&= \frac{c_r \mathfrak{S}(N)\mathcal{J}(N)}{(\log X_3)\log \boldsymbol{\Theta}} \sum_{d|\mathscr{P}} \frac{\lambda^+(d)\omega(d)}{d} + O(N^{\frac{11}{9}} \log^{-A} N) \\
&\leqslant \frac{c_r \mathfrak{S}(N)\mathcal{J}(N)\mathscr{W}(z)}{\log \boldsymbol{\Xi}} F(3)\Big(1 + O\big(\log^{-1/3} D\big)\Big) + O(N^{\frac{11}{9}} \log^{-A} N). \qquad (5.6)
\end{aligned}
$$

According to simple numerical calculations, we obtain

$$
c_8 \leqslant 0.1519148, \quad c_9 \leqslant 0.0324018, \quad c_j \leqslant 0.00555945 \quad \text{with } 10 \leqslant j \leqslant 39. \qquad (5.7)
$$

From (5.1) and (5.4)–(5.7), we deduce that

$$
\begin{aligned}
\mathscr{R}(N) &\geqslant \left(f(3) - F(3)\sum_{r=8}^{39} c_r\right)\Big(1 + O\big(\log^{-1/3} D\big)\Big) \\
&\quad \times \frac{\mathfrak{S}(N)\mathcal{J}(N)\mathscr{W}(z)}{\log \boldsymbol{\Xi}} + O(N^{\frac{11}{9}} \log^{-A} N) \\
&\geqslant \frac{2e^\gamma}{3}(\log 2 - 0.1519148 - 0.0324018 - 0.00555945 \times 30)\Big(1 + O\big(\log^{-1/3} D\big)\Big) \\
&\quad \times \frac{\mathfrak{S}(N)\mathcal{J}(N)\mathscr{W}(z)}{\log \boldsymbol{\Xi}} + O(N^{\frac{11}{9}} \log^{-A} N) \\
&\gg N^{\frac{11}{9}} \log^{-6} N,
\end{aligned}
$$

which completes the proof of Theorem 1.1.

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Conflict of interest

The authors declare that no conflict of interest exists in this manuscript.

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