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Research article

Study on a semilinear fractional stochastic system with multiple delays in control

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Abstract: This paper studies a semilinear fractional stochastic differential equation with multiple constant point delays in control. We transform the controllability problem into a fixed point problem. We obtain sufficient condition for the controllability by using Schauder's fixed point theorem. In addition, we discuss the optimal controllability of the problem. Some examples are given to illustrate the main result.

Keywords: Caputo fractional derivative; controllability; stochastic system; control; multiple delays
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1. Introduction

Consider the stochastic fractional differential equation with several delays in control:

$$\begin{cases} {}^{C}D_t^{\alpha} z(t) = Az(t) + \sum\limits_{q=0}^{r} E_q u(t-\eta_q) + \varrho_1\Big(t, z(t), \sum\limits_{q=0}^{r} u(t-\eta_q)\Big) + \varrho_2\Big(t, z(t), \sum\limits_{q=0}^{r} u(t-\eta_q)\Big)\dfrac{d\upsilon(t)}{dt}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,\ell], \\ z(0) = z_0, \quad z'(0) = z_1, \\ u(t) = 0, \quad t \in [-\eta_r, 0], \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$ and ${}^{C}D_t^{\alpha}$ is the standard Caputo fractional derivative of order α. Let $\mathcal{Z}$ and $\mathcal{W}$ be separable Hilbert spaces such that the state function $z(t) \in \mathcal{Z}$. The operator A generates a strongly continuous α-order fractional cosine family $\{C_\alpha(t) : t \geq 0\}$ in $\mathcal{Z}$. Let υ be a $\mathcal{W}$-valued Weiner process with a finite trace nuclear covariance operator $Q \geq 0$ on a complete probability space (Ω, Υ, P), where $\Upsilon_t \subset \Upsilon$, $t \in [0,\ell]$ is a normal filtration. Υ_t is a right continuous increasing family and Υ_0 contains all P-null sets. Also, let $L_2^0 = L_2(Q^{1/2}\mathcal{W}, \mathcal{Z})$, the space of all Hilbert-Schmidt operators from $Q^{1/2}\mathcal{W}$ to $\mathcal{Z}$

be a separable Hilbert space with the norm $\|\psi\|_Q^2 = tr[\psi Q\psi^*]$. Let the control function u takes value in a separable and reflexive Hilbert space $\mathcal{U}$ and $E_q \in \mathcal{L}(\mathcal{U}, \mathcal{Z}),\ q = 0, 1, 2, \ldots, r$ are linear continuous operators and $0 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_q < \cdots < \eta_{r-1} < \eta_r$ are delay points. For convenience, let $u(\cdot) \in \mathcal{U}_{ad} = \mathcal{L}^p_\Upsilon([0, \ell], \mathcal{U})$, where $\mathcal{U}_{ad}$ denotes the set of admissible control functions which is closed and convex. $\varrho_q,\ q = 1, 2$ are nonlinear functions that satisfy some suitable conditions which will be specified later. z_0 and z_1 denote Υ_0-measurable $\mathcal{Z}$-valued random variables.

Let $C([0, \ell], \mathcal{L}^p(\Upsilon, \mathcal{Z}))$ be the Banach space of continuous maps defined on $[0, \ell]$ into $\mathcal{L}^p(\Upsilon, \mathcal{Z})$ such that

$$\sup_{t\in[0,\ell]} \mathbb{E}\|z(t)\|^p_{\mathcal{Z}} < \infty,$$

where $\mathcal{L}^p(\Upsilon, \mathcal{Z})$ denotes the Banach space of all $\mathcal{Z}$ valued, p-integrable and Υ-measurable random variables and $\mathbb{E}$ is the expectation given by $\mathbb{E}(z) = \int_\Omega z(\upsilon)dP$. Take $C_2 = C_p([0, \ell], \mathcal{Z})$, then C_2 is a closed subspace of $C([0, \ell], \mathcal{L}^p(\Upsilon, \mathcal{Z}))$ endowed with the norm

$$\|z\|_{C_2} = \left(\sup_{t\in[0,\ell]} \mathbb{E}\|z(t)\|^p_{\mathcal{Z}}\right)^{\frac{1}{p}}.$$

Classical differential equations cannot adequately describe more and more phenomena as science and technology advance. Various physical processes, for example, have memory and heritability properties that the classical local differential operators cannot adequately represent. Many well-known mathematicians such as Euler, Liouville, Riemann, Caputo and Letnikov developed a new excellent tool to describe these nonlocal processes (fractional differential equations described by nonlocal operators) [13, 19]. In recent years, it has turned out that many phenomena in viscoelastic polymers, fluid mechanics, foams and animal tissues can be successfully modeled by fractional order derivatives.

The development of Fractional calculus theory is due to the significant contributions of many mathematicians such as Euler, Liouville, Riemann and Letnikov. The fractional theory deals with arbitrary order derivatives or integrals. Fractional calculus is an influential tool that plays an essential role in studying non-integer parametric models. Also, it emerges a significant role to specify bio-system, neuroscience, drug diffusion in the human body, fractional biological neurons, frequency-modulated systems, chemical technology and many real-life phenomena. Fractional integrals and derivatives also appear in control dynamical systems. Fractional differential equations describe many natural processes and phenomena studied in biotechnology, electric circuits, engineering science, optimal control, porous media, economics, etc. A comprehensive study of fractional calculus is essential and is now well-established. For the basic theory of fractional calculus and applications in control theory, refer to [2, 12, 33].

Noise and stochastic distress are so common in natural and man-made systems that they cannot be avoided. Furthermore, some randomness may appear. As a result, stochastic models are being considered for improved performance and are becoming more important tools for formulating and analyzing phenomena. In 1940, Kiyosi Ito, a Japanese mathematician, pioneered the mathematical theory of stochastic differential equations.

Many real dynamical systems have a fundamental feature of uncertainty. The theory of stochastic dynamical systems is now a well-established area of study that is still in active development and has many unresolved issues. Statistical physics, economic problems, decision problems, epidemiology, insurance mathematics, risk theory, reliability theory and other stochastic equation-based methods are important fields of applications. For more information on the fundamental theory of stochastic differential equations, see [20]. The control theory is used to study a wide range of stochastic systems. These papers [6, 24, 25, 31] show the fascinating property of such dynamical systems.

Many mathematicians, physicists and engineers have been drawn to control problems and significant contributions to theory and applications have been made. R. E. Kalman began systematically developing controllability theory in 1963. The fundamental concepts of controllability are described in Barnett (1975), Curtain and Zwart (1995) [26, 28]. The scientific community has grown increasingly interested in studying control problems described as abstract differential equations or inclusions in recent decades. The concept of controllability has been central to modern control theory throughout history. It plays an essential role in investigating and designing dynamical control systems and has applications in engineering and operations research. Stochastic systems can be used to model physical problems where some randomness appears. Many researchers have focused their efforts on determining the controllability of stochastic fractional semilinear systems. Most authors have looked into the controllability of autonomous systems [5, 8, 17]. Some authors, however, have investigated non-autonomous systems [31]. For more work, see [9, 18, 24, 30, 34].

The goal of controllability theory is to be able to control a specific system to the desired state by giving appropriate input functions in a finite time interval. Many authors demonstrated the control system with several delays [6, 14, 15, 17]. Optimal control theory extends the calculus of variations in which an optimized objective function is obtained. We minimize the cost functional due to optimization. It is important in a variety of scientific fields, including engineering, mathematics and biology. The papers [21, 25, 29] contain some works on controllability.

Fractional-order stochastic differential equations with multiple control delays have played a significant role in real-life problems. Furthermore, many practical problems have either constant point or time variable delay terms in their control. Differential equations with multiple delays have many applications in control, including population dynamics, electro-mechanical, control theory, biology, epidemiology, etc. Many researchers are now focusing on this theory and its applications. The controllability concept has numerous applications in control theory, electric bulk power systems, industrial and chemical process control, aerospace engineering and, more recently, quantum systems theory. For more information, refer to [4, 10, 32].

In 2006, P. Balasubramaniam and S. K. Ntouyas [23] provided the controllability result for partial stochastic functional differential inclusions with infinite delay. In 2017, R. Haloi [27] gave sufficient conditions for controllability of non-autonomous differential equations with a nonlocal finite delay with deviating arguments. In 2022, A. Afreen et al. [1] studied a semilinear stochastic system with constant delays in control.

More specifically, in 2012, K. Balachandran et al. [17] considered the following nonlinear fractional dynamical system with multiple delays in control

$$\begin{cases} {}^{C}D_t^q x(t) = Ax(t) + \sum_{i=0}^{M} B_i u(h_i(t)) + f(t, x(t), u(t)), \quad t \in [0, T] := J, \\ x(0) = x_0, \end{cases}$$

where $0 < q < 1,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^p$.

In 2019, A. Haq and N. Sukavanam [3] obtained sufficient conditions for the controllability of the following semilinear delay system

$$\begin{cases} \vartheta''(t) = A\vartheta(t) + \mathcal{B}_1 v(t) + \mathcal{B}_2 v(t-b) \\ \qquad\qquad +F(t, \vartheta_{a(t)}, v(t) + v(t-b)), \quad t \in (0, \beta], \\ \vartheta'(0) = \vartheta_1, \\ g(\vartheta) = \varphi, \quad v(t) = 0, \ \ t \in [-b, 0]. \end{cases}$$

In 2015, A. Shukla et al. [7] studied the approximate controllability of the following semilinear fractional control system of order $\alpha \in (1, 2]$

$$\begin{cases} {}^C D_t^\alpha y(t) = Ay(t) + Bv(t) + f(t, y_t, v), \quad 0 \le t \le T, \\ y_0(\theta) = \phi(\theta), \quad \theta \in [-h, 0], \\ y'(0) = y_0. \end{cases}$$

In [6,14,15], the authors have studied the controllability of a semilinear system with multiple delays in control. However, in 2017, A. Shukla et al. [9] examined fractional-order $\alpha \in (1, 2]$ stochastic system without control delay. Best of our knowledge, there are no papers concerned with the problem of nonlinear fractional stochastic systems with multiple delays in control in abstract spaces. To fill the gap, we have constructed the system (1.1), which is inspired by the works of [6, 7, 9, 14, 15]. Our aim is to examine the controllability of the considered system. To establish the results, first, we transform the controllability problem into a fixed-point problem.

The remaining part of the paper is designed as follows: Section 2 contains some basic definitions, lemmas and assumptions. In Section 3, the controllability problem is transformed into the existence of a fixed-point problem. Sections 4 and 5 contain the main results of the controllability. In Section 6, several examples are provided to show the effectuality of the result. In the end, a conclusion is added for further work.

2. Preliminaries and assumptions

Definition 1. *[13] The Caputo fractional derivative of order α for a function $g \in C^n([0, \ell], \mathbb{R})$ is defined by*

$${}^C D_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s) ds, \ \ n-1 < \alpha < n, n \in \mathbb{N}.$$

Consider the following linear fractional order system

$${}^C D_t^\alpha z(t) = Az(t), \quad z(0) = \xi, \quad z'(0) = 0, \tag{2.1}$$

where $\alpha \in (1, 2)$, $A : D(A) \subset \mathcal{Z} \to \mathcal{Z}$ is closed and densely defined operator in a Hilbert space $\mathcal{Z}$.

Definition 2. *[9] Let $\alpha \in (1, 2)$. A family $\{C_\alpha\}_{\alpha \ge 0} \subset \mathcal{L}(\mathcal{Z})$ (Banach space of all bounded linear operators on $\mathcal{Z}$) is called a solution operator (or strongly continuous α-order fractional cosine family) for (2.1) and A is called the infinitesimal generator of $C_\alpha(t)$, if the following conditions are satisfied*

(1) $C_\alpha(t)$ is strongly continuous for $t \ge 0$ and $C_\alpha(0) = I$;
(2) $C_\alpha(t)D(A) \subset D(A)$ and $AC_\alpha(t)\xi = C_\alpha(t)A\xi$ for all $\xi \in D(A)$, $t \ge 0$;
(3) $C_\alpha(t)\xi$ is a solution of $z(t) = \xi + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Az(s) ds$ for all $\xi \in D(A)$.

Definition 3. *[9] The fractional sine family $S_\alpha : \mathbb{R}^+ \to \mathcal{L}(\mathcal{Z})$ associated with C_α is defined by*

$$S_\alpha(t) = \int_0^t C_\alpha(s)ds, \quad t \geq 0.$$

Definition 4. *[9] The fractional Riemann-Liouville family $P_\alpha : \mathbb{R}^+ \to \mathcal{L}(\mathcal{Z})$ associated with C_α is defined by*

$$P_\alpha(t) = I_t^{\alpha-1} C_\alpha(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} C_\alpha(s)ds, \quad t \geq 0.$$

Definition 5. *(Mild solution) A stochastic process $z \in C_2$ is said to be a mild solution of (1.1) if it satisfies*

$$\begin{aligned} z(t) &= C_\alpha(t)z_0 + S_\alpha(t)z_1 + \int_0^t P_\alpha(t-s) \sum_{q=0}^{r} E_q u(s-\eta_q)ds + \int_0^t P_\alpha(t-s)\varrho_1\Big(s, z(s), \sum_{q=0}^{r} u(s-\eta_q)\Big)ds \\ &\quad + \int_0^t P_\alpha(t-s)\varrho_2\Big(s, z(s), \sum_{q=0}^{r} u(s-\eta_q)\Big)d\upsilon(s). \end{aligned} \tag{2.2}$$

Definition 6. *The control system (1.1) is said to be controllable if the initial states of the system are changed to some other desired states by a controlled input in a finite duration of time. If the system is controllable for all z_0 at $t = 0$ and for all $z(\ell) = z_\ell$ at $t = \ell$, it will be called completely controllable on $[0, \ell]$.*

Lemma 1. *[9] For any $z_\ell \in \mathcal{L}^p(\Upsilon_\ell, \mathcal{Z})$, there exists $\varphi \in \mathcal{L}^p_\Upsilon([0, \ell], \mathcal{L}_2^0)$ such that*

$$z_\ell = \mathbb{E}z_\ell + \int_0^\ell \varphi(s)d\upsilon(s).$$

Lemma 2. *[31] Let $\mathcal{V} : [0, \ell] \times \Omega \to L_2^0$ be strongly measurable mapping such that $\int_0^\ell \mathbb{E}\big\|\mathcal{V}(s)\big\|^p_{L_2^0} ds < \infty$. Then*

$$\mathbb{E}\Big\|\int_0^t \mathcal{V}(s)d\upsilon(s)\Big\|^p \leq L_{\mathcal{V}} \int_0^t \mathbb{E}\big\|\mathcal{V}(s)\big\|^p ds,$$

for every $t \in [0, \ell]$ and $p \geq 2$, where $L_{\mathcal{V}}$ is the constant depending on p and ℓ.

Schauder's fixed-point theorem. Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $S \subset X$ is a non-empty closed, bounded and convex set. Any compact operator $A : S \to S$ has atleast one fixed point.

The following assumptions hold throughout the paper.

(C1) There exist constants $\mu_1 \geq 1$, $\mu_2 = \mu_1\ell$, $\mu_3 = \frac{\mu_1 \ell^{\alpha-1}}{\Gamma(\alpha)}$, $\mu_4 > 0$ such that $\|C_\alpha(t)\| \leq \mu_1$, $\|S_\alpha(t)\| \leq \mu_2$, $\|P_\alpha(t)\| \leq \mu_3$, $\mu_4 = \max\Big\{ \big\|E_q\big\| : q = 0, 1, 2, \ldots, r \Big\}$.

(C2) The nonlinear function $\varrho_1 : [0,\ell]\times\mathcal{Z}\times\mathcal{U}_{ad} \to \mathcal{Z}$ is continuous and there are real constants α_1, β_1 such that

$$\Big\|\varrho_1\Big(t, z(t), \sum_{q=0}^{r} u(t-\eta_q)\Big) - \varrho_1\Big(t, \tilde{z}(t), \sum_{q=0}^{r} \tilde{u}(t-\eta_q)\Big)\Big\|_{\mathcal{Z}}^p \leq \alpha_1\|z-\tilde{z}\|_{C_2}^p + \beta_1\|u-\tilde{u}\|_{\mathcal{U}_{ad}}^p,$$

where $\|u-\tilde{u}\|_{\mathcal{U}_{ad}}^p = \sum_{q=0}^{r}\Big\|(u-\tilde{u})(t-\eta_q)\Big\|^p$.

(C3) The nonlinear function $\varrho_2 : [0,\ell]\times\mathcal{Z}\times\mathcal{U}_{ad} \to L_2^0$ is continuous and there are real constants α_2, β_2 such that

$$\Big\|\varrho_2\Big(t, z(t), \sum_{q=0}^{r} u(t-\eta_q)\Big) - \varrho_2\Big(t, \tilde{z}(t), \sum_{q=0}^{r} \tilde{u}(t-\eta_q)\Big)\Big\|_{L_2^0}^p \leq \alpha_2\|z-\tilde{z}\|_{C_2}^p + \beta_2\|u-\tilde{u}\|_{\mathcal{U}_{ad}}^p,$$

where $\|u-\tilde{u}\|_{\mathcal{U}_{ad}}^p = \sum_{q=0}^{r}\Big\|(u-\tilde{u})(t-\eta_q)\Big\|^p$.

3. Transformation to a fixed-point problem

Our aim is to find a suitable control u which steers the stochastic solution of the dynamical system (1.1) from $z(0) = z_0$ to $z_\ell = \mathbb{E}z_\ell + \int_0^\ell \varphi(s)d\upsilon(s)$. Now for each $(y, w) \in \mathcal{M} = C([0,\ell], \mathcal{L}^p(\Upsilon, \mathcal{Z}))\times C([0,\ell], \mathcal{U}_{ad})$, consider the fractional linear system

$$\begin{cases} {}^C D_t^\alpha z(t) = Az(t) + \sum_{q=0}^{r} E_q u(t-\eta_q) + \varrho_1\Big(t, y(t), \sum_{q=0}^{r} w(t-\eta_q)\Big) + \varrho_2\Big(t, y(t), \sum_{q=0}^{r} w(t-\eta_q)\Big)\dfrac{d\upsilon(t)}{dt}, \\ \qquad t \in [0,\ell], \\ z(0) = z_0, \quad z'(0) = z_1, \\ u(t) = 0, \quad t \in [-\eta_r, 0], \end{cases} \tag{3.1}$$

where $1 < \alpha < 2$. $\mathcal{M} = C([0,\ell], \mathcal{L}^p(\Upsilon, \mathcal{Z})) \times C([0,\ell], \mathcal{U}_{ad})$ is the Banach space with the norm $\|(y, w)\|^p = \|y\|^p + \|w\|^p$, where

$$\|y\|^p = \sup_{t\in[0,\ell]} \mathbb{E}\|y(t)\|_{\mathcal{Z}}^p.$$

The solution of (3.1) is given by

$$\begin{aligned} z(t) = C_\alpha(t)z_0 + S_\alpha(t)z_1 + \int_0^t P_\alpha(t-s)\sum_{q=0}^{r} E_q u(s-\eta_q)ds + \int_0^t P_\alpha(t-s)\varrho_1\Big(s, y(s), \sum_{q=0}^{r} w(s-\eta_q)\Big)ds \\ + \int_0^t P_\alpha(t-s)\varrho_2\Big(s, y(s), \sum_{q=0}^{r} w(s-\eta_q)\Big)d\upsilon(s). \end{aligned} \tag{3.2}$$

Using $u(t) = 0, \;\; t \in [-\eta_r, 0]$, we get

$$z(t) = C_\alpha(t)z_0 + S_\alpha(t)z_1 + \sum_{q=0}^{r}\int_0^{t-\eta_q} P_\alpha(t-s-\eta_q)E_q u(s)ds$$

$$+\int_0^t P_\alpha(t-s)\varrho_1\Big(s, y(s), \sum_{q=0}^{r} w(s-\eta_q)\Big)ds$$
$$+\int_0^t P_\alpha(t-s)\varrho_2\Big(s, y(s), \sum_{q=0}^{r} w(s-\eta_q)\Big)d\upsilon(s). \tag{3.3}$$

Putting $t = \ell$ in (3.3), we get

$$\begin{aligned} z(\ell) &= C_\alpha(\ell)z_0 + S_\alpha(\ell)z_1 + \sum_{q=0}^{r}\int_0^{\ell-\eta_q} P_\alpha(\ell-s-\eta_q)E_q u(s)ds \\ &\quad + \int_0^\ell P_\alpha(\ell-s)\varrho_1\Big(s, y(s), \sum_{q=0}^{r} w(s-\eta_q)\Big)ds \\ &\quad + \int_0^\ell P_\alpha(\ell-s)\varrho_2\Big(s, y(s), \sum_{q=0}^{r} w(s-\eta_q)\Big)d\upsilon(s). \end{aligned} \tag{3.4}$$

Now, let us introduce the following notation

$$\begin{aligned} \phi(z_0, z(\ell); y, w) &= \mathbb{E}z_\ell - C_\alpha(\ell)z_0 - S_\alpha(\ell)z_1 + \int_0^\ell \varphi(s)d\upsilon(s) - \int_0^\ell P_\alpha(\ell-s)\varrho_1\Big(s, y(s), \sum_{q=0}^{r} w(s-\eta_q)\Big)ds \\ &\quad - \int_0^\ell P_\alpha(\ell-s)\varrho_2\Big(s, y(s), \sum_{q=0}^{r} w(s-\eta_q)\Big)d\upsilon(s). \end{aligned} \tag{3.5}$$

Define the controllability Grammian operator

$$\zeta(0, \ell-\eta_q; y, w)\{\cdot\} = \sum_{q=0}^{r}\int_0^{\ell-\eta_q} P_\alpha(\ell-s-\eta_q)E_q\{P_\alpha(\ell-s-\eta_q)E_q\}^* \mathbb{E}\{\cdot|\Upsilon_s\}ds, \tag{3.6}$$

where $q = 0, 1, 2, \ldots, r$ and $*$ denotes the adjoint. System (3.1) is completely controllable if and only if the controllablity Grammian operator is nonsingular; or equivalently (see Theorem 1, [22])

$$\zeta(0, \ell-\eta_q; y, w) \geq bI, \tag{3.7}$$

where $b > 0$ and I stands for the identity operator.

If the system (3.1) satisfies the above condition, then one of the control that steers the state (3.3) to the desired state z_ℓ is given by

$$\begin{aligned} u(t) &\equiv (t, 0, z_0, \ell-\eta_q, z_\ell; y, w) \\ &= E_q^* P_\alpha^*(\ell-t-\eta_q)\zeta^{-1}\phi(z_0, z(\ell); y, w). \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.4) with (3.5) and (3.6), it is easy to verify that for each fixed $(y, w) \in \mathcal{M}$, the control $u(t)$ steers the initial state z_0 to the desired state $z_\ell = \mathbb{E}z_\ell + \int_0^\ell \varphi(s)d\upsilon(s)$.

If arbitrarily chosen vectors y, w agree with z, u that result from (3.3) and (3.8), respectively, then these vectors are also solutions of the semilinear system (1.1). Hence, the controllability problem for system (1.1) becomes an existence of a fixed point problem for (3.3) and (3.8).

4. Controllability result

Theorem 1. *Assume that for* $(z, \tilde{u}) \in \mathcal{M}$,

$$\lim_{\|(z,\tilde{u})\|^p \to \infty} \frac{\|\varrho_1(t, z, \tilde{u})\|^p_{\mathcal{Z}} + \|\varrho_2(t, z, \tilde{u})\|^p_{L^0_2}}{\|(z, \tilde{u})\|^p} = 0,$$

uniformly on $[0, \ell]$ *and (C1) holds. Further, if there exists a closed bounded convex subset* $\mathcal{H}$ *of* $\mathcal{M}$ *such that the operator* κ *defined by*

$$\kappa(y, \tilde{w}) = (z, \tilde{u}), \text{ for any } (y, \tilde{w}) \in \mathcal{H}, \tag{4.1}$$

where $\tilde{u} = \sum_{q=0}^{r} u(t - \eta_q)$, $\tilde{w} = \sum_{q=0}^{r} w(t - \eta_q)$, *has a fixed point in* $\mathcal{H}$, *then the semilinear fractional system (1.1) is completely controllable if it satisfies (3.7).*

Proof. Define the operator $\kappa : \mathcal{H} \subset \mathcal{M} \to \mathcal{H}$ by

$$\kappa(y, \tilde{w}) = (z, \tilde{u}), \text{ for any } (y, \tilde{w}) \in \mathcal{H}, \tag{4.2}$$

where

$$\begin{aligned} u(t) &= E^*_q P^*_\alpha(\ell - t - \eta_q)\zeta^{-1}\phi(z_0, z(\ell); y, w) \\ &= E^*_q P^*_\alpha(\ell - t - \eta_q)\zeta^{-1} \times \Big[\mathbb{E}z_\ell - C_\alpha(\ell)z_0 - S_\alpha(\ell)z_1 + \int_0^\ell \varphi(s)d\upsilon(s) \\ &\quad - \int_0^\ell P_\alpha(\ell - s)\varrho_1\Big(s, y(s), \sum_{q=0}^{r} w(s - \eta_q)\Big)ds - \int_0^\ell P_\alpha(\ell - s)\varrho_2\Big(s, y(s), \sum_{q=0}^{r} w(s - \eta_q)\Big)d\upsilon(s)\Big], \end{aligned}$$

and

$$\begin{aligned} z(t) &= C_\alpha(t)z_0 + S_\alpha(t)z_1 + \sum_{q=0}^{r} \int_0^{t-\eta_q} P_\alpha(t - s - \eta_q)E_q \times E^*_q P^*_\alpha(\ell - s - \eta_q)\zeta^{-1}\phi(z_0, z(\ell); y, w)ds \\ &\quad + \int_0^t P_\alpha(t - s)\varrho_1\Big(s, y(s), \sum_{q=0}^{r} w(s - \eta_q)\Big)ds + \int_0^t P_\alpha(t - s)\varrho_2\Big(s, y(s), \sum_{q=0}^{r} w(s - \eta_q)\Big)d\upsilon(s). \end{aligned}$$

For simplicity, take
$\|\varrho_1\|^p = \sup_{s\in[0,\ell]} \mathbb{E}\Big\|\varrho_1\Big(s, y(s), \sum_{q=0}^{r} w(s - \eta_q)\Big)\Big\|^p$, $\|\varrho_2\|^p = \sup_{s\in[0,\ell]} \mathbb{E}\Big\|\varrho_2\Big(s, y(s), \sum_{q=0}^{r} w(s - \eta_q)\Big)\Big\|^p$,
$\lambda_1 = \int_0^\ell \mathbb{E}\big\|\varphi(s)\big\|^p ds$, $\Lambda = 6^{p-1}\mu_4^p\mu_3^p\big\|\zeta^{-1}\big\|^p$, $\dfrac{1}{p} + \dfrac{1}{\sigma} = 1$, $\delta_1 = \Lambda\ell^{1+p/\sigma}\mu_3^p$, $\delta_2 = \Lambda L_{\varrho_2}\ell\mu_3^p$,
$\delta_3 = 5^{p-1}\ell^{1+p/\sigma}\mu_3^p\Big\{(r + 1)\mu_4^p\delta_1 + 1\Big\}$, $\delta_4 = 5^{p-1}\ell^{1+p/\sigma}\mu_3^p\Big\{(r + 1)\mu_4^p\delta_2 + L_{\varrho_2}\Big\}$,
$N_1 = \Lambda\Big\{\mathbb{E}\big\|\mathbb{E}z_\ell\big\|^p + \mu_1^p\mathbb{E}\|z_0\|^p + \mu_2^p\mathbb{E}\|z_1\|^p + L_\varphi\lambda_1\Big\}$, $N_2 = 5^{p-1}\Big\{\mu_1^p\mathbb{E}\|z_0\|^p + \mu_2^p\mathbb{E}\|z_1\|^p + (r+1)\ell^{1+p/\sigma}\mu_3^p\mu_4^p N_1\Big\}$,
$N = \max\{N_1, N_2\}$, $\delta = \max\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

Using Lemma 2 and Holder's inequality with the assumption (C1), we have

$$\begin{aligned}
\mathbb{E}\|u(t)\|^p &\leq 6^{p-1}\Big\|E_q^*P_\alpha^*(\ell-t-\eta_q)\zeta^{-1}\Big\|^p\Big[\mathbb{E}\Big\|\mathbb{E}z_\ell\Big\|^p+\mathbb{E}\Big\|C_\alpha(\ell)z_0\Big\|^p+\mathbb{E}\Big\|S_\alpha(\ell)z_1\Big\|^p\\
&\quad +L_\varphi\int_0^\ell\mathbb{E}\Big\|\varphi(s)\Big\|^p ds+\ell^{p/\sigma}\int_0^\ell\mathbb{E}\Big\|P_\alpha(\ell-s)\varrho_1\Big(s,y(s),\sum_{q=0}^r w(s-\eta_q)\Big)\Big\|^p ds\\
&\quad +L_{\varrho_2}\int_0^\ell\mathbb{E}\Big\|P_\alpha(\ell-s)\varrho_2\Big(s,y(s),\sum_{q=0}^r w(s-\eta_q)\Big)\Big\|^p ds\Big]\\
&\leq 6^{p-1}\mu_4^p\mu_3^p\Big\|\zeta^{-1}\Big\|^p\Big[\mathbb{E}\Big\|\mathbb{E}z_\ell\Big\|^p+\mu_1^p\mathbb{E}\|z_0\|^p+\mu_2^p\mathbb{E}\|z_1\|^p+L_\varphi\lambda_1+\ell^{1+p/\sigma}\mu_3^p\|\varrho_1\|^p+L_{\varrho_2}\ell\mu_3^p\|\varrho_2\|^p\Big]\\
&= N_1+\delta_1\|\varrho_1\|^p+\delta_2\|\varrho_2\|^p\\
&\leq N+\delta\Big(\|\varrho_1\|^p+\|\varrho_2\|^p\Big),
\end{aligned}$$

$\Rightarrow \mathbb{E}\|\tilde{u}(t)\|^p \leq (r+1)\Big[N+\delta\Big(\|\varrho_1\|^p+\|\varrho_2\|^p\Big)\Big]$
and

$$\begin{aligned}
\mathbb{E}\|z(t)\|^p &\leq 5^{p-1}\Big[\mathbb{E}\Big\|C_\alpha(\ell)z_0\Big\|^p+\mathbb{E}\Big\|S_\alpha(\ell)z_1\Big\|^p+\ell^{p/\sigma}\sum_{q=0}^r\int_0^{\ell-\eta_q}\mathbb{E}\Big\|P_\alpha(\ell-s-\eta_q)E_qu(s)\Big\|^p ds\\
&\quad +\ell^{p/\sigma}\int_0^\ell\mathbb{E}\Big\|P_\alpha(\ell-s)\varrho_1\Big(s,y(s),\sum_{q=0}^r w(s-\eta_q)\Big)\Big\|^p ds\\
&\quad +L_{\varrho_2}\int_0^\ell\mathbb{E}\Big\|P_\alpha(\ell-s)\varrho_2\Big(s,y(s),\sum_{q=0}^r w(s-\eta_q)\Big)\Big\|^p ds\Big]\\
&\leq 5^{p-1}\Big[\mu_1^p\mathbb{E}\|z_0\|^p+\mu_2^p\mathbb{E}\|z_1\|^p+(r+1)\ell^{1+p/\sigma}\mu_3^p\mu_4^p\Big\{N_1+\delta_1\|\varrho_1\|^p+\delta_2\|\varrho_2\|^p\Big\}+\ell^{1+p/\sigma}\mu_3^p\|\varrho_1\|^p\\
&\quad +L_{\varrho_2}\ell\mu_3^p\|\varrho_2\|^p\Big]\\
&= N_2+\delta_3\|\varrho_1\|^p+\delta_4\|\varrho_2\|^p\\
&\leq N+\delta\Big(\|\varrho_1\|^p+\|\varrho_2\|^p\Big).
\end{aligned}$$

Since the function ϱ_1 and ϱ_2 satisfy Proposition 1 of [16]. Therefore, for each pair of constants N and δ, there exists $\varepsilon > 0$ such that, if $\mathbb{E}\|y(t)\|^p \leq \frac{\varepsilon}{2}$ and $\mathbb{E}\|\tilde{w}(t)\|^p \leq \frac{\varepsilon}{2}$, i.e., $\|(y,\tilde{w})\|^p \leq \varepsilon$, then $N+\delta\Big(\|\varrho_1\|^p+\|\varrho_2\|^p\Big) \leq \varepsilon$. Therefore, $\mathbb{E}\|\tilde{u}(t)\|^p \leq (r+1)\varepsilon$ and $\mathbb{E}\|z(t)\|^p \leq \varepsilon$. Thus, we have proved that, if $\mathcal{H}(\varepsilon') = \Big\{(y,\tilde{w}) \in \mathcal{H} : \|y\|^p \leq \frac{\varepsilon}{2} \text{ and } \|\tilde{w}\|^p \leq \frac{\varepsilon}{2}\Big\}$, where $\varepsilon' = \max\{(r+2)\varepsilon, \varepsilon\}$ then κ maps $\mathcal{H}(\varepsilon')$ into iteslf. Since ϱ_1, ϱ_2 are continuous, therefore κ is continuous. The complete continuity of κ is followed by Arzela-Ascoli theorem. As $\mathcal{H}(\varepsilon')$ is closed, bounded and convex set, therefore by Schauder's fixed point theorem, κ has a fixed point $(y,\tilde{w}) \in \mathcal{H}(\varepsilon')$, i.e., $\kappa(y,\tilde{w}) = (y,\tilde{w}) \equiv (z,\tilde{u})$. Hence, we have

$$\begin{aligned}
z(t) &= C_\alpha(t)z_0+S_\alpha(t)z_1+\sum_{q=0}^r\int_0^{t-\eta_q}P_\alpha(t-s-\eta_q)E_qu(s)ds+\int_0^t P_\alpha(t-s)\varrho_1\Big(s,y(s),\sum_{q=0}^r w(s-\eta_q)\Big)ds\\
&\quad +\int_0^t P_\alpha(t-s)\varrho_2\Big(s,y(s),\sum_{q=0}^r w(s-\eta_q)\Big)d\upsilon(s).
\end{aligned}$$

It is easy to prove that the control $u(t)$ steers the system (1.1) from z_0 to z_ℓ. Hence, the system (1.1) is completely controllable. □

5. Optimal controllability

We consider the Lagrange problem to find an optimal state-control pair $(z^0, u^0) \in C_2 \times \mathcal{U}_{ad}$ satisfying [25]

$$\mathcal{I}(z^0, u^0) \leq \mathcal{I}(z, u), \quad \forall\, (z, u) \in C_2 \times \mathcal{U}_{ad},$$

where z denotes the mild solution of the stochastic system (1.1) corresponding to the control $u \in \mathcal{U}_{ad}$ and

$$\mathcal{I}(z, u) = \mathbb{E}\Big\{ \int_0^\ell \tilde{\mathcal{G}}\Big(t, z(t), \sum_{q=0}^{r} u(t-\eta_q)\Big)dt\Big\}. \tag{5.1}$$

To discuss the Lagrange problem, we assume the following conditions

(C4) The Borel measurable function $\tilde{\mathcal{G}} : [0, \ell] \times \mathcal{Z} \times \mathcal{U}_{ad} \to \mathbb{R} \cup \{\infty\}$ satisfies
(a) For almost all $t \in [0, \ell]$, $\tilde{\mathcal{G}}(t, z, \cdot)$ is convex on $\mathcal{U}_{ad}$ for each $z \in \mathcal{Z}$.
(b) For almost all $t \in [0, \ell]$, $\tilde{\mathcal{G}}(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $\mathcal{Z} \times \mathcal{U}_{ad}$.
(c) There exist constants $e_1 \geq 0$, $e_2 > 0$ and Φ is a non-negative function in $\mathcal{L}^1([0, \ell], \mathbb{R})$ such that

$$\tilde{\mathcal{G}}\Big(t, z(t), \sum_{q=0}^{r} u(t-\eta_q)\Big) \geq \Phi(t) + e_1\|z\|_{\mathcal{Z}} + e_2\|u\|^p_{\mathcal{U}_{ad}},$$

where $\|u\|^p_{\mathcal{U}_{ad}} = \sum_{q=0}^{r} \|u(t-\eta_q)\|^p$.

Balder's Theorem 2.1. [11]: The following three conditions
(a) $f(t, \cdot, \cdot)$ is sequentially l.s.c. on $X \times V$ μ-a.e.,
(b) $f(t, x, \cdot)$ is convex on V for every $x \in X$ μ-a.e.,
(c) there exist $M > 0$ and $\psi \in L^1_{\mathbb{R}}$ such that $f(t, x, v) \geq \psi(t) - M(\|x\| + |v|)$ for all $x \in X$, $v \in V$ μ-a.e.,
are sufficient for sequential strong-weak lower semicontinuity of I_f on $L^1_X \times L^1_V$. Moreover, they are also necessary, provided that $I_f(\bar{x}, \bar{v}) < +\infty$ for some $\bar{x} \in L^1_X$, $\bar{v} \in L^1_V$.

Theorem 2. *Let assumptions (C1)–(C4) hold. Further, if all the hypotheses of Theorem 1 are satisfied, then there exists an optimal pair of (1.1) if* $3^{p-1}\mu_3^p(\ell^{p/\sigma}\alpha_1 + \alpha_2 L_{\varrho_2}) < 1$.

Proof. It is enough to show that there exists $(z^0, u^0) \in C_2 \times \mathcal{U}_{ad}$ which minimize $\mathcal{I}(z, u)$.

If $\inf\{\mathcal{I}(z, u) : (z, u) \in \mathcal{Z} \times \mathcal{U}_{ad}\} = \infty$, then result holds trivially.

If $\inf\{\mathcal{I}(z, u) : (z, u) \in \mathcal{Z} \times \mathcal{U}_{ad}\} = \epsilon_0 < \infty$, then there exists a minimizing sequence $\{(z^n, u^n)\}$ such that $\mathcal{I}(z^n, u^n) \to \epsilon_0$ as $n \to \infty$. Since $\mathcal{U}_{ad}$ is closed and convex, therefore, sequence $\{u^n\}$ has a weakly convergent subsequence $u^m \to u^0 \in \mathcal{U}_{ad}$ by Marzur Lemma.
Using Theorem 1, for each $u^m \in \mathcal{U}_{ad}$, there exists a mild solution z^m of (1.1),

$$z^m(t) \;=\; C_\alpha(t)z_0 + S_\alpha(t)z_1 + \int_0^t P_\alpha(t-s)\sum_{q=0}^{r} E_q u^m(s-\eta_q)ds$$

$$+\int_0^t P_\alpha(t-s)\varrho_1\Big(s, z^m(s), \sum_{q=0}^r u^m(s-\eta_q)\Big)ds + \int_0^t P_\alpha(t-s)\varrho_2\Big(s, z^m(s), \sum_{q=0}^r u^m(s-\eta_q)\Big)d\upsilon(s).$$

Similarly, corresponding to u^0, we have

$$\begin{aligned} z^0(t) &= C_\alpha(t)z_0 + S_\alpha(t)z_1 + \int_0^t P_\alpha(t-s)\sum_{q=0}^r E_q u^0(s-\eta_q)ds \\ &\quad + \int_0^t P_\alpha(t-s)\varrho_1\Big(s, z^0(s), \sum_{q=0}^r u^0(s-\eta_q)\Big)ds + \int_0^t P_\alpha(t-s)\varrho_2\Big(s, z^0(s), \sum_{q=0}^r u^0(s-\eta_q)\Big)d\upsilon(s). \end{aligned}$$

We have,

$$\begin{aligned} &\mathbb{E}\|z^m(t) - z^0(t)\|^p \\ &\leq 3^{p-1}\mathbb{E}\Big\|\int_0^t P_\alpha(t-s)\sum_{q=0}^r \Big\{E_q u^m(s) - E_q u^0(s)\Big\}ds\Big\|^p \\ &\quad +3^{p-1}\mathbb{E}\Big\|\int_0^t P_\alpha(t-s)\Big\{\varrho_1\Big(s, z^m(s), \sum_{q=0}^r u^m(s-\eta_q)\Big) - \varrho_1\Big(s, z^0(s), \sum_{q=0}^r u^0(s-\eta_q)\Big)\Big\}ds\Big\|^p \\ &\quad +3^{p-1}\mathbb{E}\Big\|\int_0^t P_\alpha(t-s)\Big\{\varrho_2\Big(s, z^m(s), \sum_{q=0}^r u^m(s-\eta_q)\Big) - \varrho_2\Big(s, z^0(s), \sum_{q=0}^r u^0(s-\eta_q)\Big)\Big\}d\upsilon(s)\Big\|^p \\ &\leq 3^{p-1}\ell^{p/\sigma}\mu_3^p\mu_4^p(r+1)\int_0^t \Big\|u^m - u^0\Big\|^p ds + 3^{p-1}\ell^{p/\sigma}\mu_3^p\int_0^t \Big\{\alpha_1\Big\|z^m - z^0\Big\|^p + \beta_1\Big\|u^m - u^0\Big\|^p\Big\}ds \\ &\quad +3^{p-1}\mu_3^p L_{\varrho_2}\int_0^t \Big\{\alpha_2\Big\|z^m - z^0\Big\|^p + \beta_2\Big\|u^m - u^0\Big\|^p\Big\}ds \\ &\leq 3^{p-1}\mu_3^p\Big(\ell^{p/\sigma}\mu_4^p(r+1) + \ell^{p/\sigma}\beta_1 + \beta_2 L_{\varrho_2}\Big)\int_0^t \Big\|u^m - u^0\Big\|^p ds + 3^{p-1}\mu_3^p(\ell^{p/\sigma}\alpha_1 + \alpha_2 L_{\varrho_2})\int_0^t \Big\|z^m - z^0\Big\|^p ds. \end{aligned}$$

Since $3^{p-1}\mu_3^p(\ell^{p/\sigma}\alpha_1 + \alpha_2 L_{\varrho_2}) < 1$ and $\Big\|u^m - u^0\Big\|^p \to 0$, we conclude that $z^m \to z^0$.
Applying Balder's theorem (see Theorem 2.1, [11]), we obtain

$$\begin{aligned} \epsilon_0 &= \lim_{m\to\infty}\mathbb{E}\Big\{\int_0^\ell \tilde{\mathcal{G}}\Big(t, z^m(t), \sum_{q=0}^r u^m(t-\eta_q)\Big)dt\Big\} \\ &\geq \mathbb{E}\Big\{\int_0^\ell \tilde{\mathcal{G}}\Big(t, z^0(t), \sum_{q=0}^r u^0(t-\eta_q)\Big)dt\Big\} \\ &= \mathcal{I}(z^0, u^0) \geq \epsilon_0. \end{aligned}$$

This shows that $\mathcal{I}(z^0, u^0) = \epsilon_0$. □

6. Applications

Example 6.1. Consider the following fractional system with single point delay in control

$$\begin{cases} {}^{C}D_t^{1.7} z(t,x) = z_{xx}(t,x) + E_0 u(t,x) + 5u(t-2\pi, x) \\ \qquad\qquad + t^3 \sin(3\pi t)\Big(z(t,x) + u(t,x) + u(t-2\pi,x)\Big) \\ \qquad\qquad + t\cos(t)\Big(3\dfrac{\partial z_t(x)}{\partial x} + u(t,x) + u(t-2\pi,x)\Big)\dfrac{d\upsilon(t)}{dt}, \quad x \in [0,\pi], \quad t \in [0,7], \\ z(0,x) = z_0(x), \quad \dfrac{\partial z(t,x)}{\partial t}\Big|_{t=0} = z_1(x), \quad u(t) = 0, \quad t \in [-2\pi, 0], \\ z(t,0) = z(t,\pi) = 0, \quad t \in [0,7]. \end{cases} \tag{6.1}$$

Here, $\eta_0 = 0, \eta_1 = 2\pi$.
Let $\mathcal{Z} = \mathcal{L}^2[0,\pi]$ and the operator $A : \mathcal{Z} \to \mathcal{Z}$ is defined as $Az = z''$ with

$$D(A) = \{z \in \mathcal{Z} : z,\ z' \text{ are absolutely continuous, } z'' \in \mathcal{Z} \text{ and } z(0) = z(\pi) = 0\}.$$

The operator A has discrete spectrum with normalized eigenvectors $e_q(x) = \sqrt{\frac{2}{\pi}}\sin(qx)$ corresponding to the eigenvalues $\lambda_q = -q^2,\ q \in \mathbb{N}$. The set $\{e_q : q \in \mathbb{N}\}$ forms an orthogonal basis for $\mathcal{Z}$. Thus, we have

$$Az = \sum_{q\in\mathbb{N}} -q^2 \langle z, e_q\rangle e_q,\ z \in D(A).$$

A generates strongly continuous cosine and sine family given by

$$C(t)z = \sum_{q\in\mathbb{N}} \cos(qt)\langle z, e_q\rangle e_q,$$

and

$$S(t)z = \sum_{q\in\mathbb{N}} \frac{\sin(qt)}{q}\langle z, e_q\rangle e_q,\ t \in \mathbb{R},$$

respectively. For $\alpha \in (1,2)$ [12]

$$C_\alpha(t) = \int_0^\infty t^{-\alpha/2}\psi_{\alpha/2}(st^{-\alpha/2})C(s)ds,$$

where

$$\psi_\mu(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma(-\mu n + 1 - \mu)},\ \ 0 < \mu < 1,\ \ t > 0.$$

Define

$$\mathcal{U} = \Big\{u : u = \sum_{q=2}^{\infty} u_q e_q(x) \ \Big|\ \sum_{q=2}^{\infty} u_q^2 < \infty\Big\},$$

with the norm $\|u\| = \Big(\sum_{q=2}^{\infty} u_q^2\Big)^{1/2}$.
Define the operator $E_0 : \mathcal{U} \to \mathcal{Z}$ by $E_0 u = (Eu)(t)$, where $E \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ such that

$$Eu(t) = 2u_2(t)e_1(x) + \sum_{q=2}^{\infty} u_q(t)e_q(x).$$

If we define $z(t)$ and $u(t)$ as

$$z(t)(\cdot) = z(t, \cdot), \quad u(t)(\cdot) = u(t, \cdot).$$

Then

$$\varrho_1\Big(t, z(t), \sum_{q=0}^{1} u(t-\eta_q)\Big)(x) = t^3 \sin(3\pi t)\Big(z(t,x) + u(t,x) + u(t-2\pi, x)\Big),$$

and

$$\varrho_2\Big(t, z(t), \sum_{q=0}^{1} u(t-\eta_q)\Big)(x) = t\cos(t)\Big(3\frac{\partial z_t(x)}{\partial x} + u(t,x) + u(t-2\pi, x)\Big).$$

Now the system (6.1) can be written as in the abstract form (1.1). Thus, (6.1) has a solution. Clearly, all the requirements of Theorem 1 are satisfied, therefore, the system (6.1) is completely controllable on $[0, 7]$.

Example 6.2. Consider the following system

$$\begin{cases} {}^C D_t^{1.85} z(t,x) = z_{xx}(t,x) + E_0 u(t,x) + \sum\limits_{q=1}^{4} u(t-3q, x) + \dfrac{5}{1+\sin t} z(t,x) + 2\pi u(t,x) \\ \qquad\qquad + e^2 u(t-3,x) + u(t-6,x) + \dfrac{1}{7e} u(t-9,x) + u(t-12,x) \\ \qquad\qquad + \Big(\pi e^{-5t} z(t,x) + \dfrac{3}{1+e^{2t}} \sum\limits_{q=0}^{4} u(t-3q,x)\Big)\dfrac{d\upsilon(t)}{dt}, \quad x \in [0,\pi], \quad t \in [0,14], \\ z(0,x) = z_0(x), \quad \dfrac{\partial z(t,x)}{\partial t}\Big|_{t=0} = z_1(x), \quad u(t) = 0, \quad t \in [-12, 0], \\ z(t,0) = z(t,\pi) = 0, \quad t \in [0,14]. \end{cases} \tag{6.2}$$

Collecting the above definitions and following Example 6.1, we can easily conclude the result.

Example 6.3. Consider the following system having multiple delays

$$\begin{cases} {}^C D_t^{1.5} z(t,x) = z_{xx}(t,x) + \dfrac{7}{9}\sum\limits_{q=0}^{n} u(t-2q,x) + e^{-3t} z(t,x) + 8e^{-4t} \sum\limits_{q=0}^{n} u(t-2q,x) \\ \qquad\qquad + \Big(5z(t,x) + \dfrac{2}{1+\pi}\sum\limits_{q=0}^{n} u(t-2q,x)\Big)\dfrac{d\upsilon(t)}{dt}, \quad x \in [0,\pi], \quad t \in [0,3n], \\ z(0,x) = z_0(x), \quad \dfrac{\partial z(t,x)}{\partial t}\Big|_{t=0} = z_1(x), \quad u(t) = 0, \quad t \in [-2n, 0], \\ z(t,0) = z(t,\pi) = 0, \quad t \in [0,3n]. \end{cases} \tag{6.3}$$

Following Example 6.1, we can conclude that the system (6.3) is completely controllable on $[0, 3n]$.

7. Conclusions

In the present paper, we have established sufficient conditions for the controllability of a semilinear fractional stochastic system with multiple delays in control. The controllability problem has been transformed into a fixed point problem. The existence of a subset on which the operator is invariant is shown to be a sufficient condition for controllability using Schauder's fixed point theorem. It is also shown that the problem admits at least one optimal pair of state-control under some natural assumptions. Several examples are provided to demonstrate the efficacy of the results. In the future, the above work could be extended to a multi-term time-fractional impulsive system using the same technique or the Picard iterative technique.

Conflict of interest

The authors declare no conflicts of interest.

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