Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(7): 12322–12341.
DOI: 10.3934/math.2022684
Received: 01 March 2022
Revised: 09 April 2022
Accepted: 19 April 2022
Published: 25 April 2022

Research article

A note on three different contractions in partially ordered complex valued G_b-metric spaces

Yiquan Li[1], **Chuanxi Zhu**[1,*], **Yingying Xiao**[1,2] **and Li Zhou**[1,2]

[1] School of Mathematics and Computer Sciences, Nanchang University, Nanchang, Jiangxi, 330031, China

[2] Department of Basic discipline, Nanchang JiaoTong Institute, Nanchang, Jiangxi, 330031, China

* **Correspondence:** Email: chuanxizhu@126.com.

Abstract: In this paper, we introduce the complex valued C^p-class function, a type of Geraghty contraction and a type of JS contraction in complete partially ordered complex valued G_b-metric spaces, prove three fixed point theorems in this space, and also we give some examples to support our results.

Keywords: complex valued G_b-metric space; complex valued C^p-class function; Geraghty contraction; JS contraction; partially ordered
Mathematics Subject Classification: 47H10, 54H25, 54D99, 54E99

1. Introduction

Fixed point theory in metric spaces occupies an extremely important position in modern mathematics, it has been generalized in various aspects. For example, G-metric spaces [1] were introduced and G_b-metric spaces were reported in [2], which successfully popularized the general metric and promoted the research of various types of fixed point theorems. These theorems are accompanied with different contractive conditions (see [3–12,21–27]), especially the new Geraghty contraction was given in [13] and the JS contraction was given in [14].

Recently, Shoaib et al. [15] introduced the ordered dislocated quasi G-metric spaces, and obtained some new fixed point results for a dominated mapping on a close ball in this space. On the other hand, Ege [16] also proposed the complex valued G_b-metric spaces as a new notion, the Banach contraction principle and Kannan's fixed point theorem were proved for this space. Moreover, there are also other interesting fixed point theorems in this space (see [17–20]).

In this work, we study some problems about the common solutions of the operator equations $F_n x = ux$ $(u \geq 1, n \in \mathbb{N}^*)$ in complete partially ordered complex valued G_b-metric spaces, introduce the complex valued C^p-class function and a type of Geraghty contraction to this space respectively, and we obtain the common solutions in a closed ball. Furthermore, we also introduce a type of JS contraction to this space and investigate a new theorem.

Firstly, we recall some basic concepts, which will be used later. For a real Banach space E, a nonempty closed subset $Q \subset E$ is called a cone, if
(a) for all $\zeta \in Q$ and $\tau \geq 0$, $\tau\zeta \in Q$;
(b) for all $\zeta_1, \zeta_2 \in Q$, $\zeta_1 + \zeta_2 \in Q$;
(c) $Q \cap (-Q) = 0$.

For $\xi_1, \xi_2 \in E$, given a cone Q, we define a partial order $\leq$ on E, which is induced by Q, i.e., $\xi_1 \leq \xi_2$ iff $\xi_2 - \xi_1 \in Q$. Furthermore, ξ_1, ξ_2 are said to be comparable if $\xi_1 \leq \xi_2$ or $\xi_2 \leq \xi_1$.

On the other hand, for all $\xi_1, \xi_2 \in \mathbb{C}$, the partial order $\precsim$ on $\mathbb{C}$ is defined as follows:

$$\xi_1 \precsim \xi_2 \Leftrightarrow Re(\xi_1) \leq Re(\xi_2) \text{ and } Im(\xi_1) \leq Im(\xi_2).$$

Therefore, $\xi_1 \precsim \xi_2$ if one of the following conditions holds:
(C_1) $Re(\xi_1) = Re(\xi_2)$ and $Im(\xi_1) = Im(\xi_2)$;
(C_2) $Re(\xi_1) = Re(\xi_2)$ and $Im(\xi_1) < Im(\xi_2)$;
(C_3) $Re(\xi_1) < Re(\xi_2)$ and $Im(\xi_1) = Im(\xi_2)$;
(C_4) $Re(\xi_1) < Re(\xi_2)$ and $Im(\xi_1) < Im(\xi_2)$.

Moreover, we denote $\xi_1 \prec \xi_2$ if only (C_4) holds. Obviously, $0 \precsim \xi_1 \precsim \xi_2 \Rightarrow |\xi_1| \leq |\xi_2|$, where $|\xi_i|$ is the magnitude of ξ_i, $i = 1, 2$. For more details, see [25].

Definition 1.1. ([16]) Let X be a nonempty set, for a real number $s \geq 1$, if the mapping $G_b : X \times X \times X \to \mathbb{C}$ satisfies:
(CG_b1) $G_b(\zeta_1, \zeta_2, \zeta_3) = 0$ if $\zeta_1 = \zeta_2 = \zeta_3$;
(CG_b2) $G_b(\zeta_1, \zeta_1, \zeta_2) \succ 0$ for all $\zeta_1, \zeta_2 \in X$ with $\zeta_1 \neq \zeta_2$;
(CG_b3) $G_b(\zeta_1, \zeta_1, \zeta_2) \precsim G_b(\zeta_1, \zeta_2, \zeta_3)$ for all $\zeta_1, \zeta_2, \zeta_3 \in X$ with $\zeta_3 \neq \zeta_2$;
(CG_b4) $G_b(\zeta_1, \zeta_2, \zeta_3) = G_b(R\{\zeta_1, \zeta_2, \zeta_3\})$, where R is an arbitrary permutation of $\{\zeta_1, \zeta_2, \zeta_3\}$;
(CG_b5) $G_b(\zeta_1, \zeta_2, \zeta_3) \precsim s[G_b(\zeta_1, \upsilon, \upsilon) + G_b(\upsilon, \zeta_2, \zeta_3)]$ for all $\zeta_1, \zeta_2, \zeta_3, \upsilon \in X$.

Then the function G_b is called a complex valued G_b-metric on X, the pair (X, G_b) is called a complex valued G_b-metric space.

Proposition 1.1. ([16]) For a complex valued G_b-metric space (X, G_b) and all $\zeta_1, \zeta_2, \zeta_3 \in X$, we have
(1) $G_b(\zeta_1, \zeta_2, \zeta_3) \precsim s[G_b(\zeta_1, \zeta_1, \zeta_2) + G_b(\zeta_1, \zeta_1, \zeta_3)]$;
(2) $G_b(\zeta_1, \zeta_2, \zeta_2) \precsim 2s[G_b(\zeta_1, \zeta_1, \zeta_2)]$.

Definition 1.2. ([16]) Let $\{x_n\}$ be a sequence in a complex valued G_b-metric space (X, G_b),
(1) $\{x_n\}$ is called complex valued G_b-convergent to $\zeta \in X$, if for any $\epsilon \in \mathbb{C}$ with $\epsilon \succ 0$, there exists $\xi \in \mathbb{N}$ such that $G_b(\zeta, x_n, x_m) \prec \epsilon$ for all $n, m \geq \xi$. We write $x_n \to \zeta$ as $n \to \infty$, or $\lim_{n\to\infty} x_n = \zeta$;
(2) $\{x_n\}$ is called complex valued G_b-Cauchy, if for any $\epsilon \in \mathbb{C}$ with $\epsilon \succ 0$, there exists $\xi \in \mathbb{N}$ such that $G_b(x_n, x_m, x_l) \prec \epsilon$ for all $n, m, l \geq \xi$;
(3) (X, G_b) is said to be complex valued G_b-complete, if any complex valued G_b-Cauchy sequence $\{x_n\}$ is complex valued G_b-convergent.

Theorem 1.1. ([16]) Let $\{x_n\}$ be a sequence in a complex valued G_b-metric space (X, G_b), and $\zeta \in X$, the following are equivalent:
(1) $\{x_n\}$ is complex valued G_b-convergent to ζ;
(2) $|G_b(x_n, x_m, \zeta)| \to 0$ as $n, m \to \infty$;

(3) $|G_b(x_n, \zeta, \zeta)| \to 0$ as $n \to \infty$;
(4) $|G_b(x_n, x_n, \zeta)| \to 0$ as $n \to \infty$.

Theorem 1.2. ([16]) A sequence $\{x_n\}$ is complex valued G_b-Cauchy sequence is equivalent to $|G_b(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$.

Definition 1.3. ([28]) Let $Q \subset \mathbb{R}^m$ be a cone, a mapping $S : Q \to \mathbb{R}^m$ is said to be dominated if $Sx \preceq x$ for all $x \in Q$.

Theorem 1.3. ([15]) Let $(X, \preceq, G)$ be an ordered complete dislocated quasi G-metric space, $S : X \to X$ be a mapping and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, \frac{1}{2})$ with

$$G(Sx, Sy, Sz) \leq k(G(x, Sx, Sx) + G(y, Sy, Sy) + G(z, Sz, Sz))$$

for all comparable elements $x, y, z \in \overline{B(x_0, r)}$, and

$$G(x_0, Sx_0, Sx_0) \leq (1 - \theta)r,$$

where $\theta = \frac{k}{1-2k}$. If for nonincreasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists a point $x^\star$ in $\overline{B(x_0, r)}$ such that $x^\star = Sx^\star$ and $G(x^\star, x^\star, x^\star) = 0$. Moreover, if for any three points $x, y, z \in \overline{B(x_0, r)}$, there exists a point v in $\overline{B(x_0, r)}$ such that $v \preceq x$ and $v \preceq y$, $v \preceq z$, where

$$G(x_0, Sx_0, Sx_0) + G(v, Sv, Sv) + G(v, Sv, Sv) \leq G(x_0, v, v) + G(Sx_0, Sv, Sv) + G(Sx_0, Sv, Sv),$$

then the point $x^\star$ is unique.

2. Results and examples

In this section, let $X = \mathbb{R}^N$, $\Omega_1 = \{Z_1 \in \mathbb{C} : 0 \precsim Z_1\}$, $\Omega_2 = \{Z_2 \in \mathbb{C} : 0 \prec Z_2\}$, $\Omega_3 = \{Z_3 \in \mathbb{C} : 1 \prec Z_3\}$. $(X, G_b, \preceq)$ is called a partially ordered complex valued G_b-metric space, which shows (X, G_b) is a complex valued G_b-metric space and $(X, \preceq)$ is a partially ordered set.

Let (X, G_b) be a complex valued G_b-metric space, for any $x_0 \in X$, $r \in \mathbb{C}$ and $r \succ 0$, the G_b-ball with ball center x_0 is $\overline{B(x_0, r)} = \{x \in X |\ G_b(x_0, x, x) \precsim r\}$. Moreover, for all $n \in \mathbb{N}^*$ and $x_1, x_2, ..., x_n \in \mathbb{C}$, the function $\max\{x_1, x_2, ..., x_n\} \succsim x_j$, $j = 1, 2, ..., n$.

Definition 2.1. A continuous mapping $P : \Omega_1^3 \to \mathbb{C}$ is called complex valued C^p-class function, if it satisfies $r \precsim P(r, s, t)$ for all $r, s, t \in \Omega_1$.

Example 2.1. Some examples of complex valued C^p-class function are given as follows:
(1) $P(r, s, t) = r + s + t$, where $r, s, t \in \Omega_1$;
(2) $P(r, s, t) = mr$, where $m \in [1, \infty)$ and $r, s, t \in \Omega_1$;
(3) $P(r, s, t) = \eta(r)r$, where $\eta : \Omega_1 \to [1, \infty)$ and $r, s, t \in \Omega_1$.

Theorem 2.1. Let $(X, G_b, \preceq)$ be a complete partially ordered complex valued G_b-metric space with $s \geq 1$, $Q \subset X$ be a cone, x_0 be an arbitrary element in Q, $\{S_n : X \to X,\ n \in \mathbb{N}^*\}$ be a dominated mapping sequence. If there exist $r \in \Omega_2$, and nonnegative numbers α, β, γ satisfy $\alpha - 2s\gamma \neq 0$, $\frac{\beta}{\alpha - 2\gamma} \in [0, \delta]$, $\delta < \frac{1}{s}$, such that

$$\begin{aligned} &P[\psi(\alpha G_b(S_i x, S_j y, S_j y)), \varphi(\alpha G_b(S_i x, S_j y, S_j y)), \varphi(\alpha G_b(S_i x, S_j y, S_j y))] \\ &\precsim \psi[\beta G_b(x, S_i x, S_i x) + \gamma G_b(y, S_j y, S_j z) + \gamma G_b(z, S_j z, S_j y)] \end{aligned} \tag{2.1}$$

for any comparable elements x, y, z in $\overline{B(x_0, r)}$, where $\overline{B(x_0, r)} \subset Q,\ i, j \in \mathbb{N}^*$, P is a complex valued C^p-class function, $\psi : \Omega_1 \to \Omega_1$ is a nondecreasing function, $\varphi : \Omega_1 \to \mathbb{C}$ is a continuous function. And

$$G_b(x_0, S_1 x_0, S_1 x_0) \precsim \frac{1 - s\delta}{s} r. \tag{2.2}$$

Define the operator equations $F_n x = ux$ by $F_n = uS_n,\ u \geq 1$. If a nonincreasing sequence $\{x_n\} \to \kappa$ such that $\kappa \preceq x_n$, then the operator equations have at least a common solution x^* in $\overline{B(x_0, r)}$. Moreover, if there exists an element v in $\overline{B(x_0, r)}$ such that $v \preceq x^*$, and

$$\beta G_b(x_0, S_1 x_0, S_1 x_0) + 2\gamma G_b(v, S_j v, S_j v) \precsim \beta G_b(x_0, v, v) + 2\gamma G_b(S_1 x_0, S_j v, S_j v), \tag{2.3}$$

then the operator equations have an unique solution.
Proof. By selecting the ball centre x_0 in $\overline{B(x_0, r)}$, we construct a sequence $\{x_n\}$, where $x_{n+1} = S_{n+1} x_n \preceq x_n,\ n \in \mathbb{N}$. From (2.2), we obtain $x_1 \in \overline{B(x_0, r)}$. Using (2.1), we have

$$\begin{aligned}
&\psi(\alpha G_b(S_1 x_0, S_2 x_1, S_2 x_1)) \\
&\precsim P[\psi(\alpha G_b(S_1 x_0, S_2 x_1, S_2 x_1)), \varphi(\alpha G_b(S_1 x_0, S_2 x_1, S_2 x_1)), \varphi(\alpha G_b(S_1 x_0, S_2 x_1, S_2 x_1))] \\
&\precsim \psi[\beta G_b(x_0, S_1 x_0, S_1 x_0) + 2\gamma G_b(x_1, S_2 x_1, S_2 x_1)].
\end{aligned}$$

Since the function ψ is nondecreasing, we can easily get

$$G_b(x_1, x_2, x_2) \precsim \frac{\beta}{\alpha - 2\gamma} G_b(x_0, x_1, x_1) \precsim \delta G_b(x_0, x_1, x_1).$$

Hence, $G_b(x_0, x_2, x_2) \precsim s[G_b(x_0, x_1, x_1) + G_b(x_1, x_2, x_2)] \precsim s(1 + \delta) G_b(x_0, x_1, x_1)$. Using (2.2), we get $G_b(x_0, x_2, x_2) \precsim (1 - \delta^2) r \prec r$, that is $x_2 \in \overline{B(x_0, r)}$.

Now we prove $\{x_n\} \subset \overline{B(x_0, r)}$. Suppose that $x_3, x_4, ..., x_k \in \overline{B(x_0, r)}$, according to (2.1), we have

$$\begin{aligned}
&\psi(\alpha G_b(S_k x_{k-1}, S_{k+1} x_k, S_{k+1} x_k)) \\
&\precsim P[\psi(\alpha G_b(S_k x_{k-1}, S_{k+1} x_k, S_{k+1} x_k)), \varphi(\alpha G_b(S_k x_{k-1}, S_{k+1} x_k, S_{k+1} x_k)), \varphi(\alpha G_b(S_k x_{k-1}, S_{k+1} x_k, S_{k+1} x_k))] \\
&\precsim \psi[\beta G_b(x_{k-1}, S_k x_{k-1}, S_k x_{k-1}) + 2\gamma G_b(x_k, S_{k+1} x_k, S_{k+1} x_k)].
\end{aligned}$$

Thus $G_b(x_k, x_{k+1}, x_{k+1}) \precsim \frac{\beta}{\alpha - 2\gamma} G_b(x_{k-1}, x_k, x_k) \precsim \delta G_b(x_{k-1}, x_k, x_k)$, it can easily get that

$$G_b(x_k, x_{k+1}, x_{k+1}) \precsim \delta^k G_b(x_0, x_1, x_1). \tag{2.4}$$

By using (CG_b5) and (2.4), it follows that

$$\begin{aligned}
G_b(x_0, x_{k+1}, x_{k+1}) &\precsim sG_b(x_0, x_1, x_1) + s^2 G_b(x_1, x_2, x_2) + ... + s^{k+1} G_b(x_k, x_{k+1}, x_{k+1}) \\
&\precsim (s + s^2\delta + ... + s^{k+1}\delta^k) G_b(x_0, x_1, x_1) \\
&\precsim s \cdot \frac{1}{1 - s\delta} \frac{1 - s\delta}{s} r \\
&= r,
\end{aligned}$$

i.e., $x_{k+1} \in \overline{B(x_0, r)}$, therefore, $\{x_n\} \subset \overline{B(x_0, r)}$.

Now we show that $\{x_n\}$ is a complex valued G_b-Cauchy sequence, from (2.4), we obtain

$$G_b(x_n, x_{n+1}, x_{n+1}) \precsim \delta^n G_b(x_0, x_1, x_1), \tag{2.5}$$

thus for all $n, m \in \mathbb{N}^*, n < m$, we have

$$\begin{aligned} G_b(x_n, x_m, x_m) &\precsim sG_b(x_n, x_{n+1}, x_{n+1}) + s^2 G_b(x_{n+1}, x_{n+2}, x_{n+2}) + ... + s^{m-n} G_b(x_{m-1}, x_m, x_m) \\ &\precsim (s\delta^n + s^2\delta^{n+1} + ... + s^{m-n}\delta^{m-1}) G_b(x_0, x_1, x_1) \\ &\precsim s\delta^n \cdot \frac{1}{1 - s\delta} G_b(x_0, x_1, x_1), \end{aligned}$$

which implies that

$$\lim_{n,m\to\infty} G_b(x_n, x_m, x_m) = 0.$$

Therefore, $\{x_n\}$ is a complex valued G_b-Cauchy sequence, and there exists an element x^* in $\overline{B(x_0, r)}$ such that $x_n \to x^*$.

Next we prove x^* is the common solution of the operator equations. For any $j \in \mathbb{N}^*$, we have

$$G_b(x^*, S_j x^*, S_j x^*) \precsim s[G_b(x^*, x^n, x^n) + G_b(x^n, S_j x^*, S_j x^*)].$$

Furthermore, since $S_j x^* \preceq x^* \preceq x_n \preceq x_{n-1}$, using (2.1), it can be easily get that

$$\alpha\, G_b(x^n, S_j x^*, S_j x^*) \precsim \beta\, G_b(x^{n-1}, x^n, x^n) + 2\gamma\, G_b(x^*, S_j x^*, S_j x^*).$$

Hence,

$$\begin{aligned} \alpha\, G_b(x^*, S_j x^*, S_j x^*) &\precsim s\alpha\, G_b(x^*, x^n, x^n) + s\alpha\, G_b(x^n, S_j x^*, S_j x^*) \\ &\precsim s\alpha\, G_b(x^*, x^n, x^n) + s\beta\, G_b(x^{n-1}, x^n, x^n) + 2s\gamma\, G_b(x^*, S_j x^*, S_j x^*). \end{aligned}$$

That is,

$$G_b(x^*, S_j x^*, S_j x^*) \precsim \frac{1}{\alpha - 2s\gamma}[s\alpha\, G_b(x^*, x^n, x^n) + s\beta\, G_b(x^{n-1}, x^n, x^n)].$$

Let $n \to \infty$ at both sides of the above inequality, we obtain $\lim_{n\to\infty} G_b(x^*, S_j x^*, S_j x^*) = 0$, i.e. $x^* = S_j x^*$. According to the arbitrariness of j, we get x^* is a common solution of the operator equations.

Uniqueness. Assume that y^* is another solution of the operator equations, $y^* \neq x^*$ and $y^* \in \overline{B(x_0, r)}$.

Case 1. If x^* and y^* are comparable, using (2.1), it follows that

$$\begin{aligned} \alpha\, G_b(x^*, y^*, y^*) &= \alpha\, G_b(S_i x^*, S_j y^*, S_j y^*) \\ &\precsim \beta\, G_b(x^*, S_i x^*, S_i x^*) + 2\gamma\, G_b(y^*, S_j y^*, S_j y^*) \\ &= \beta\, G_b(x^*, x^*, x^*) + 2\gamma\, G_b(y^*, y^*, y^*) \\ &= 0, \end{aligned}$$

as a result, $x^* = y^*$.

Case 2. If x^* and y^* are not comparable, then there exists an element $v \in \overline{B(x_0, r)}$ such that $v \preceq x^*$ and $v \preceq y^*$, for any $j \in \mathbb{N}^*$, we will show $\{S_j^n x_n\} \subset \overline{B(x_0, r)}$. Owing to (2.1) and (2.3), we have

$$\begin{aligned} \alpha\, G_b(S_1 x_0, S_j v, S_j v) &\precsim \beta\, G_b(x_0, S_1 x_0, S_1 x_0) + 2\gamma\, G_b(v, S_j v, S_j v) \\ &\precsim \beta\, G_b(x_0, v, v) + 2\gamma\, G_b(S_1 x_0, S_j v, S_j v), \end{aligned}$$

i.e.,

$$G_b(S_1x_0, S_jv, S_jv) \precsim \frac{\beta}{\alpha-2\gamma}G_b(x_0, v, v) \precsim \delta\, r.$$

Hence,

$$\begin{aligned}G_b(x_0, S_jv, S_jv) &\precsim s[G_b(x_0, x_1, x_1) + G_b(x_1, S_jv, S_jv)]\\ &\precsim s(\frac{1-s\delta}{s}r + \delta\, r)\\ &= r,\end{aligned}$$

that is $S_jv \in \overline{B(x_0, r)}$. Suppose that $S_j^2v, S_j^3v, ..., S_j^kv \in \overline{B(x_0, r)}$, obviously, $S_j^kv \preceq S_j^{k-1}v \preceq ... \preceq S_j^2v \preceq S_jv \preceq v \preceq x^* \preceq x_n \preceq ... \preceq x_0$. From (2.1), we can immediately obtain

$$\alpha\, G_b(S_j^kv, S_j^{k+1}v, S_j^{k+1}v) \precsim \beta\, G_b(S_j^{k-1}v, S_j^kv, S_j^kv) + 2\gamma\, G_b(S_j^kv, S_j^{k+1}v, S_j^{k+1}v),$$

so we have

$$G_b(S_j^kv, S_j^{k+1}v, S_j^{k+1}v) \precsim \frac{\beta}{\alpha-2\gamma}G_b(S_j^{k-1}v, S_j^kv, S_j^kv) \precsim \delta\, G_b(S_j^{k-1}v, S_j^kv, S_j^kv),$$

as a result,

$$\begin{aligned}G_b(S_j^kv, S_j^{k+1}v, S_j^{k+1}v) &\precsim \delta\, G_b(S_j^{k-1}v, S_j^kv, S_j^kv)\\ &\precsim ...\\ &\precsim \delta^k\, G_b(v, S_jv, S_jv).\end{aligned} \tag{2.6}$$

In addition, using (2.1), (2.3), (2.5) and (2.6), we can also immediately obtain

$$\begin{aligned}\alpha\, G_b(x_{k+1}, S_j^{k+1}v, S_j^{k+1}v) &\precsim \beta\, G_b(x_k, x_{k+1}, x_{k+1}) + 2\gamma\, G_b(S_j^kv, S_j^{k+1}v, S_j^{k+1}v)\\ &\precsim \beta\delta^k G_b(x_0, x_1, x_1) + 2\gamma\delta^k G_b(v, S_jv, S_jv)\\ &\precsim \beta\delta^k G_b(x_0, v, v) + 2\gamma\delta^k G_b(S_1x_0, S_jv, S_jv)\\ &\precsim \beta\delta^k G_b(x_0, v, v) + 2\gamma\delta^k\frac{\beta}{\alpha-2\gamma}G_b(x_0, v, v)\\ &\precsim (\beta\delta^k + 2\gamma\delta^{k+1})G_b(x_0, v, v),\end{aligned}$$

i.e.,

$$\begin{aligned}G_b(x_{k+1}, S_j^{k+1}v, S_j^{k+1}v) &\precsim \frac{(\beta\delta^k + 2\gamma\delta^{k+1})}{\alpha}G_b(x_0, v, v)\\ &\precsim \frac{(\alpha-2\gamma)\delta^{k+1} + 2\gamma\delta^{k+1}}{\alpha}G_b(x_0, v, v)\\ &= \delta^{k+1}G_b(x_0, v, v).\end{aligned}$$

Thus,

$$\begin{aligned}G_b(x_0, S_j^{k+1}v, S_j^{k+1}v) &\precsim sG_b(x_0, x_1, x_1) + ... + s^{k+1}G_b(x_k, x_{k+1}, x_{k+1}) + s^{k+1}G_b(x_{k+1}, S_j^{k+1}v, S_j^{k+1}v)\\ &\precsim (s + s^2\delta + ... + s^{k+1}\delta^k)G_b(x_0, x_1, x_1) + s^{k+1}\delta^{k+1}G_b(x_0, v, v)\\ &\precsim s\cdot\frac{1-(s\delta)^{k+1}}{1-s\delta}\cdot\frac{1-s\delta}{s}r + (s\delta)^{k+1}\cdot r\\ &= [1-(s\delta)^{k+1} + (s\delta)^{k+1}]r\\ &= r,\end{aligned}$$

which implies $S_j^{k+1}v \in \overline{B(x_0, r)}$, so $\{S_j^n x_n\} \subset \overline{B(x_0, r)}$. From (2.6), we obtain

$$G_b(S_j^n v, S_j^{n+1} v, S_j^{n+1} v) \precsim \delta^n G_b(v, S_j v, S_j v),$$

and

$$\lim_{n\to\infty} G_b(S_j^n v, S_j^{n+1} v, S_j^{n+1} v) = 0. \tag{2.7}$$

From (2.1), we can easily get

$$\begin{aligned}\alpha G_b(x^*, S_j^n v, S_j^n v) &= \alpha G_b(S_i x^*, S_j^n v, S_j^n v)\\ &\precsim \beta G_b(x^*, S_i x^*, S_i x^*) + 2\gamma G_b(S_j^{n-1} v, S_j^n v, S_j^n v)\\ &= 2\gamma G_b(S_j^{n-1} v, S_j^n v, S_j^n v).\end{aligned}$$

Owing to (2.7), we have

$$\lim_{n\to\infty} G_b(x^*, S_j^n v, S_j^n v) = 0. \tag{2.8}$$

Similarly,

$$\begin{aligned}\alpha G_b(S_j^n v, y^*, y^*) &= \alpha G_b(S_j^n v, S_i y^*, S_i y^*)\\ &\precsim \beta G_b(S_j^{n-1} v, S_j^n v, S_j^n v) + 2\gamma G_b(y^*, S_i y^*, S_i y^*)\\ &= \beta G_b(S_j^{n-1} v, S_j^n v, S_j^n v).\end{aligned}$$

According to (2.7), we also have

$$\lim_{n\to\infty} G_b(S_j^n v, y^*, y^*) = 0. \tag{2.9}$$

Since $G_b(x^*, y^*, y^*) \precsim s[G_b(x^*, S_j^n v, S_j^n v) + G_b(S_j^n v, y^*, y^*)]$, using (2.8) and (2.9), we obtain

$$G_b(x^*, y^*, y^*) = \lim_{n\to\infty} G_b(x^*, y^*, y^*) \precsim 0.$$

Therefore, $x^* = y^*$, the proof is completed.

Following the proof process of Theorem 2.1, we can obtain the following corollary.

Corollary 2.1. Let $(X, G_b, \preceq)$ be a complete partially ordered complex valued G_b-metric space with $s \geq 1$, $Q \subset X$ be a cone, $\{S_n : X \to Q,\ n \in \mathbb{N}^*\}$ be a dominated mapping sequence. If there exist nonnegative numbers α, β, γ satisfy $\alpha - 2s\gamma \neq 0$, $\frac{\beta}{\alpha - 2\gamma} \in [0, \frac{1}{s})$, such that

$$\begin{aligned}&\eta(\psi(\alpha G_b(S_i x, S_j y, S_j y)))\psi(\alpha G_b(S_i x, S_j y, S_j y))\\ &\precsim \psi[\beta G_b(x, S_i x, S_i x) + \gamma G_b(y, S_j y, S_j z) + \gamma G_b(z, S_j z, S_j y)]\end{aligned}$$

for any comparable elements x, y, z in Q, where $i, j \in \mathbb{N}^*$, $\eta : \Omega_1 \to [1, \infty)$, $\psi : \Omega_1 \to \Omega_1$ is a nondecreasing function.

Define the operator equations $F_n x = ux$ by $F_n = uS_n$, $u \geq 1$. If a nonincreasing sequence $\{x_n\} \to \kappa$ such that $\kappa \preceq x_n$, then the operator equations have at least a common solution x^* in Q. Moreover, if there exists an element v in Q such that $v \preceq x^*$, then the operator equations have an unique solution.

Example 2.2. Let $X = R$, $Q = [0, \infty)$, $\alpha = 5, \beta = \gamma = 1, \delta = \frac{1}{3}$, $G_b : X \times X \times X \to \mathbb{C}$ be defined by $G_b(\xi_1, \xi_2, \xi_3) = \max\{|\xi_1 - \xi_2|^2, |\xi_2 - \xi_3|^2, |\xi_1 - \xi_3|^2\} + \max\{|\xi_1 - \xi_2|^2, |\xi_2 - \xi_3|^2, |\xi_1 - \xi_3|^2\}i$ with $s = 2$, and $\psi(r) = \eta(r)r = r$ for any r in Ω_1.

For any ξ in X, $0 < \nu^n \leq \frac{1}{4}$ and $n \in \mathbb{N}^*$, take $S_n\xi = \nu^n\xi$ and $F_n = uS_n$, where $u \geq 1$. The partial order $\preceq$ on X is the usual order $\leq$ of R, for any ξ_1, ξ_2, ξ_3 in Q, we have

$$\alpha G_b(S_n\xi_1, S_n\xi_2, S_n\xi_2) = 5\nu^{2n}(\xi_1 - \xi_2)^2 + 5\nu^{2n}(\xi_1 - \xi_2)^2 i,$$

and

$$\beta|\xi_1 - \nu^n\xi_1|^2 + \gamma|\xi_2 - \nu^n\xi_2|^2 + \gamma|\xi_3 - \nu^n\xi_3|^2 = (1 - \nu^n)^2(\xi_1^2 + \xi_2^2 + \xi_3^2).$$

Hence,

$$\begin{aligned}
&\alpha G_b(S_n\xi_1, S_n\xi_2, S_n\xi_2)\\
&\precsim \beta|\xi_1 - \nu^n\xi_1|^2 + \gamma|\xi_2 - \nu^n\xi_2|^2 + \gamma|\xi_3 - \nu^n\xi_3|^2 + [\beta|\xi_1 - \nu^n\xi_1|^2 + \gamma|\xi_2 - \nu^n\xi_2|^2 + \gamma|\xi_3 - \nu^n\xi_3|^2]i\\
&\precsim \beta G_b(\xi_1, S_n\xi_1, S_n\xi_1) + \gamma G_b(\xi_2, S_n\xi_2, S_n\xi_3) + \gamma G_b(\xi_3, S_n\xi_3, S_n\xi_2).
\end{aligned}$$

It follows that the operator equations $F_n\xi = u\xi$ have a common solution $\xi^* = 0$ in Q, and there exists an element $\nu = 0$ in Q such that $\nu \preceq \xi^*$. Therefore, all conditions of Corollary 2.1 are satisfied, the operator equations $F_n\xi = u\xi$ have an unique solution $\xi^* = 0$.

Let $\mathscr{B}$ be the set of functions $\beta : \Omega_1 \to [0, \frac{1}{s})$, which satisfies if $\lim_{n\to\infty} \beta(x_n) = \frac{1}{s}$, then $\lim_{n\to\infty} x_n = 0$.

Theorem 2.2. Let $(X, G_b, \preceq)$ be a complete partially ordered complex valued G_b-metric space with $s \geq 1$, $Q \subset X$ be a cone, x_0 be an arbitrary element in Q, $\{S_n : X \to X, n \in \mathbb{N}^*\}$ be a dominated mapping sequence. Suppose that there exist $\beta \in \mathscr{B}$, $i, j \in \mathbb{N}^*$ and $r \in \Omega_2$, such that

$$G_b(S_ix, S_jy, S_jz) \precsim \beta(M(x, y, z))M(x, y, z) \tag{2.10}$$

for any comparable elements x, y, z in $\overline{B(x_0, r)}$, where $\overline{B(x_0, r)} \subset Q$,

$$M(x, y, z) = \max\{G_b(x, y, z), \frac{G_b(x, S_ix, S_ix)G_b(y, S_jy, S_jz)}{1 + G_b(x, y, z)}, \frac{G_b(x, S_ix, S_ix)G_b(x, S_jy, S_jz)}{1 + s[G_b(x, y, z) + G_b(S_ix, S_jy, S_jz)]}\}, \tag{2.11}$$

and

$$G_b(x_0, S_1x_0, S_1x_0) \precsim \frac{1 - s\delta}{s}r, \tag{2.12}$$

where $\delta \in (0, \frac{1}{s})$.

Define the operator equations $F_nx = ux$ by $F_n = uS_n$, $u \geq 1$. If a nonincreasing sequence $\{x_n\} \to \kappa$ such that $\kappa \preceq x_n$, then the operator equations have at least a common solution x^* in $\overline{B(x_0, r)}$.

Proof. By selecting the ball centre x_0 in $\overline{B(x_0, r)}$, we construct a sequence $\{x_n\}$, where $x_{n+1} = S_{n+1}x_n \preceq x_n$, $n \in \mathbb{N}$. From (2.12), we know $x_1 \in \overline{B(x_0, r)}$. Using (2.10), we have

$$G_b(x_1, x_2, x_2) = G_b(S_1x_0, S_2x_1, S_2x_1) \precsim \beta(M(x_0, x_1, x_1))M(x_0, x_1, x_1), \tag{2.13}$$

where

$$\begin{aligned}
M(x_0, x_1, x_1) = \max\{&G_b(x_0, x_1, x_1),\\
&\frac{G_b(x_0, S_1x_0, S_1x_0)G_b(x_1, S_2x_1, S_2x_1)}{1 + G_b(x_0, x_1, x_1)}, \frac{G_b(x_0, S_1x_0, S_1x_0)G_b(x_0, S_2x_1, S_2x_1)}{1 + s[G_b(x_0, x_1, x_1) + G_b(S_1x_0, S_2x_1, S_2x_1)]}\}.
\end{aligned}$$

Since

$$\frac{G_b(x_0,S_1x_0,S_1x_0)G_b(x_1,S_2x_1,S_2x_1)}{1+G_b(x_0,x_1,x_1)} \precsim G_b(x_1,S_2x_1,S_2x_1) = G_b(x_1,x_2,x_2),$$

and

$$\begin{aligned}\frac{G_b(x_0,S_1x_0,S_1x_0)G_b(x_0,S_2x_1,S_2x_1)}{1+s[G_b(x_0,x_1,x_1)+G_b(S_1x_0,S_2x_1,S_2x_1)]} &\precsim \frac{s[G_b(x_0,x_1,x_1)+G_b(x_1,S_2x_1,S_2x_1)]G_b(x_0,S_1x_0,S_1x_0)}{1+s[G_b(x_0,x_1,x_1)+G_b(S_1x_0,S_2x_1,S_2x_1)]}\\ &\precsim G_b(x_0,S_1x_0,S_1x_0)\\ &= G_b(x_0,x_1,x_1),\end{aligned}$$

thus $M(x_0,x_1,x_1) \precsim \max\{G_b(x_0,x_1,x_1),\ G_b(x_1,x_2,x_2)\}$.

If $\max\{G_b(x_0,x_1,x_1),\ G_b(x_1,x_2,x_2)\} = G_b(x_1,x_2,x_2)$, then we have

$$G_b(x_1,x_2,x_2) \precsim \beta(M(x_0,x_1,x_1))M(x_0,x_1,x_1) \prec \frac{1}{s}G_b(x_1,x_2,x_2),$$

which is a contradiction, thus

$$\max\{G_b(x_0,x_1,x_1),\ G_b(x_1,x_2,x_2)\} = G_b(x_0,x_1,x_1),$$

and

$$G_b(x_1,x_2,x_2) \precsim \beta(M(x_0,x_1,x_1))M(x_0,x_1,x_1) \precsim \delta\, G_b(x_0,x_1,x_1).$$

So we have

$$\begin{aligned}G_b(x_0,x_2,x_2) &\precsim s[G_b(x_0,x_1,x_1)+G_b(x_1,x_2,x_2)]\\ &\precsim s(1+\delta)\cdot\frac{1-s\delta}{s}r\\ &\precsim (1-\delta^2)r\\ &\prec r,\end{aligned}$$

as a result, $x_2 \in \overline{B(x_0,r)}$.

Now we will show $\{x_n\} \subset \overline{B(x_0,r)}$. Assume that $x_3, x_4, ..., x_k \in \overline{B(x_0,r)}$, owing to (2.10), we get

$$G_b(x_k,x_{k+1},x_{k+1}) = G_b(S_kx_{k-1},S_{k+1}x_k,S_{k+1}x_k) \precsim \beta(M(x_{k-1},x_k,x_k))M(x_{k-1},x_k,x_k).$$

Following the above proof process, we can obtain

$$M(x_{k-1},x_k,x_k) \precsim \max\{G_b(x_{k-1},x_k,x_k), G_b(x_k,x_{k+1},x_{k+1})\} = G_b(x_{k-1},x_k,x_k). \tag{2.14}$$

Thus,

$$\begin{aligned}G_b(x_k,x_{k+1},x_{k+1}) &\precsim \delta\, G_b(x_{k-1},x_k,x_k)\\ &\precsim \delta^2\, G_b(x_{k-2},x_{k-1},x_{k-1})\\ &\precsim ...\\ &\precsim \delta^k\, G_b(x_0,x_1,x_1).\end{aligned} \tag{2.15}$$

By using (CG_b5) and (2.15), it follows that

$$\begin{aligned}G_b(x_0, x_{k+1}, x_{k+1}) &\precsim sG_b(x_0, x_1, x_1) + s^2 G_b(x_1, x_2, x_2) + ... + s^{k+1} G_b(x_k, x_{k+1}, x_{k+1})\\ &\precsim (s + s^2\delta + ... + s^{k+1}\delta^k) G_b(x_0, x_1, x_1)\\ &\precsim s \cdot \frac{1-(s\delta)^{k+1}}{1-s\delta} \cdot \frac{1-s\delta}{s} r\\ &\prec r.\end{aligned}$$

Hence, $x_{k+1} \in \overline{B(x_0, r)}$, so $\{x_n\} \subset \overline{B(x_0, r)}$. As a result, for all $n \in \mathbb{N}^*$,

$$G_b(x_n, x_{n+1}, x_{n+1}) = G_b(S_n x_{n-1}, S_{n+1} x_n, S_{n+1} x_n) \precsim \beta(M(x_{n-1}, x_n, x_n)) M(x_{n-1}, x_n, x_n), \tag{2.16}$$

thus we have $G_b(x_n, x_{n+1}, x_{n+1}) \prec \frac{1}{s} G_b(x_{n-1}, x_n, x_n)$.

If $s > 1$, then $G_b(x_n, x_{n+1}, x_{n+1}) \prec (\frac{1}{s})^n G_b(x_0, x_1, x_1) \to 0$ as $n \to \infty$.

If $s = 1$, then $G_b(x_n, x_{n+1}, x_{n+1}) \prec G_b(x_{n-1}, x_n, x_n)$, which implies that $\{G_b(x_n, x_{n+1}, x_{n+1})\}$ is a decreasing sequence.

Suppose that

$$\lim_{n\to\infty} G_b(x_n, x_{n+1}, x_{n+1}) = r \succ 0,$$

owing to (2.14) and (2.16), we obtain

$$\begin{aligned}r &= \lim_{n\to\infty} G_b(x_n, x_{n+1}, x_{n+1})\\ &\precsim \lim_{n\to\infty} \beta(M(x_{n-1}, x_n, x_n)) M(x_{n-1}, x_n, x_n)\\ &\precsim \lim_{n\to\infty} \frac{1}{s} G_b(x_{n-1}, x_n, x_n)\\ &\precsim r,\end{aligned}$$

thus $\lim_{n\to\infty} \beta(M(x_{n-1}, x_n, x_n)) = 1$, which implies $\lim_{n\to\infty} G_b(x_{n-1}, x_n, x_n) = 0$, contradiction. As a result, $\lim_{n\to\infty} G_b(x_n, x_{n+1}, x_{n+1}) = 0$.

Now we prove $\{x_n\}$ is a complex valued G_b-Cauchy sequence. Suppose that contrary, then there exist $\epsilon \succ 0$ and two subsequences x_{m_k} and x_{n_k} of x_n, such that

$$G_b(x_{m_k}, x_{n_k}, x_{n_k}) \succsim \epsilon \text{ and } G_b(x_{m_k}, x_{n_k-1}, x_{n_k-1}) \prec \epsilon.$$

So we have

$$\epsilon \precsim G_b(x_{m_k}, x_{n_k}, x_{n_k}) \precsim s[G_b(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}}) + G_b(x_{m_{k+1}}, x_{n_k}, x_{n_k})].$$

Let $k \to \infty$, we get

$$\epsilon \precsim \liminf_{k\to\infty} G_b(x_{m_k}, x_{n_k}, x_{n_k}) \precsim s \liminf_{k\to\infty} G_b(x_{m_{k+1}}, x_{n_k}, x_{n_k}).$$

Furthermore, using (2.10) and (2.14),

$$\begin{aligned}\liminf_{k\to\infty} G_b(x_{m_k+1}, x_{n_k}, x_{n_k}) &\precsim \liminf_{k\to\infty} \beta(M(x_{m_k}, x_{n_k-1}, x_{n_k-1})) M(x_{m_k}, x_{n_k-1}, x_{n_k-1})\\ &\precsim \liminf_{k\to\infty} \beta(M(x_{m_k}, x_{n_k-1}, x_{n_k-1})) G_b(x_{m_k}, x_{n_k-1}, x_{n_k-1})\\ &\precsim \liminf_{k\to\infty} \beta(M(x_{m_k}, x_{n_k-1}, x_{n_k-1}))\epsilon,\end{aligned}$$

thus we have

$$\begin{aligned}\frac{\epsilon}{s} &\precsim \liminf_{k\to\infty} G_b(x_{m_{k+1}}, x_{n_k}, x_{n_k})\\ &\precsim \liminf_{k\to\infty} \beta(M(x_{m_k}, x_{n_k-1}, x_{n_k-1}))\epsilon\\ &\precsim \limsup_{k\to\infty} \beta(M(x_{m_k}, x_{n_k-1}, x_{n_k-1}))\epsilon\\ &\precsim \frac{\epsilon}{s}.\end{aligned}$$

Therefore, $\lim_{k\to\infty}\beta(M(x_{m_k}, x_{n_k-1}, x_{n_k-1})) = \frac{1}{s}$, thus $\lim_{k\to\infty} G_b(x_{m_k}, x_{n_k-1}, x_{n_k-1}) = 0$. As a result,

$$\epsilon \precsim G_b(x_{m_k}, x_{n_k}, x_{n_k}) \precsim s[G_b(x_{m_k}, x_{n_k-1}, x_{n_k-1}) + G_b(x_{n_k-1}, x_{n_k}, x_{n_k})] \to 0 \text{ as } k \to \infty,$$

which is a contradiction. Therefore, $\{x_n\}$ is a complex valued G_b-Cauchy sequence, and there exists an element x^* in $\overline{B(x_0, r)}$ such that $x_n \to x^*$.

Finally, we show that x^* is a common solution of the operator equations. Let $x = x_{i-1}, y = z = x^*$ in (2.10), we have

$$\lim_{i\to\infty} G_b(S_i x_{i-1}, S_j x^*, S_j x^*) \precsim \lim_{i\to\infty}\beta(M(x_{i-1}, x^*, x^*))M(x_{i-1}, x^*, x^*) \precsim \lim_{i\to\infty}\frac{1}{s}M(x_{i-1}, x^*, x^*),$$

where

$$\begin{aligned}M(x_{i-1}, x^*, x^*) = \max\{&G_b(x_{i-1}, x^*, x^*), \frac{G_b(x_{i-1}, S_i x_{i-1}, S_i x_{i-1})G_b(x^*, S_j x^*, S_j x^*)}{1 + G_b(x_{i-1}, x^*, x^*)},\\ &\frac{G_b(x_{i-1}, S_i x_{i-1}, S_i x_{i-1})G_b(x_{i-1}, S_j x^*, S_j x^*)}{1 + s[G_b(x_{i-1}, x^*, x^*) + G_b(S_i x_{i-1}, S_j x^*, S_j x^*)]}\}.\end{aligned}$$

It can be easily deduced that $\lim_{i\to\infty} M(x_{i-1}, x^*, x^*) = 0$ and $\lim_{i\to\infty} G_b(S_i x_{i-1}, S_j x^*, S_j x^*) = 0$, thus

$$G_b(x^*, S_j x^*, S_j x^*) \precsim s[G_b(x^*, S_i x_{i-1}, S_i x_{i-1}) + G_b(S_i x_{i-1}, S_j x^*, S_j x^*)] \to 0 \text{ as } i \to \infty.$$

As a result, $x^* = S_j x^*$, owing to the arbitrariness of j, we obtain that x^* is a common solution of the operator equations, the proof is completed.

Similarly, following the proof process of Theorem 2.2, the following corollary will be established.

Corollary 2.2. Let $(X, G_b, \preceq)$ be a complete partially ordered complex valued G_b-metric space with $s \geq 1$, $Q \subset X$ be a cone, $\{S_n : X \to Q, n \in \mathbb{N}^*\}$ be a dominated mapping sequence. Suppose that there exist $i, j \in \mathbb{N}^*$ such that

$$G_b(S_i x, S_j y, S_j z) \precsim \lambda M(x, y, z)$$

for any comparable elements x, y, z in Q, where $\lambda \in [0, \frac{1}{s})$, and

$$M(x, y, z) = \max\{G_b(x, y, z), \frac{G_b(x, S_i x, S_i x)G_b(y, S_j y, S_j z)}{1 + G_b(x, y, z)}, \frac{G_b(x, S_i x, S_i x)G_b(x, S_j y, S_j z)}{1 + s[G_b(x, y, z) + G_b(S_i x, S_j y, S_j z)]}\}.$$

Define the operator equations $F_n x = ux$ by $F_n = uS_n$, $u \geq 1$. If a nonincreasing sequence $\{x_n\} \to \kappa$ such that $\kappa \preceq x_n$, then the operator equations have at least a common solution x^* in Q.

Example 2.3. Let $X = R$, $Q = [0,\infty)$, $G_b : X \times X \times X \to \mathbb{C}$ be defined by $G_b(\xi_1,\xi_2,\xi_3) = (|\xi_1 - \xi_2| + |\xi_2 - \xi_3| + |\xi_1 - \xi_3|)^2 + (|\xi_1 - \xi_2| + |\xi_2 - \xi_3| + |\xi_1 - \xi_3|)^2 i$ with $s = 2$, $\delta = \frac{1}{5}$, $x_0 = 1$, $r = 4 + 4i$. For all $t \in \Omega_1$, take

$$\beta(t) = \begin{cases} \dfrac{1}{3}, & t = 0; \\ \dfrac{1}{2 + \frac{|t|}{2}}, & 0 < |t| \leq 1; \\ \dfrac{1}{\frac{5}{2} + \frac{1}{2+e^{|t|}}}, & |t| > 1. \end{cases}$$

Obviously, $\frac{1}{3} \leq \beta(t) < \frac{1}{2}$, and

$$\overline{B(1, 4+4i)} = \{x | G_b(1, x, x) \precsim 4 + 4i\} = \{x | 4|1 - x|^2 + 4|1 - x|^2 i \precsim 4 + 4i\} = [0, 2].$$

Moreover, for any ξ in X, let $S_n\xi = \frac{|\xi|}{\sqrt{3}n}, n \in \mathbb{N}^*$ and $F_n = uS_n$, where $u \geq 1$. The partial order $\leq$ on X is the usual order $\leq$ of R, for any ξ_1, ξ_2, ξ_3 in $\overline{B(1, 4+4i)}$, we have

$$G_b(S_n\xi_1, S_n\xi_2, S_n\xi_3) = \frac{1}{3n^2}[(|\xi_1 - \xi_2| + |\xi_2 - \xi_3| + |\xi_1 - \xi_3|)^2 + (|\xi_1 - \xi_2| + |\xi_2 - \xi_3| + |\xi_1 - \xi_3|)^2 i],$$

and

$$G_b(\xi_1,\xi_2,\xi_3) = (|\xi_1 - \xi_2| + |\xi_2 - \xi_3| + |\xi_1 - \xi_3|)^2 + (|\xi_1 - \xi_2| + |\xi_2 - \xi_3| + |\xi_1 - \xi_3|)^2 i.$$

It follows that

$$\begin{aligned} G_b(S_n\xi_1, S_n\xi_2, S_n\xi_3) &\precsim \frac{1}{3} G_b(\xi_1,\xi_2,\xi_3) \\ &\precsim \frac{1}{3} M(\xi_1,\xi_2,\xi_3) \\ &\precsim \beta(M(\xi_1,\xi_2,\xi_3)) M(\xi_1,\xi_2,\xi_3), \end{aligned}$$

and

$$G_b(1, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}) = \frac{16 - 8\sqrt{3}}{3} + \frac{16 - 8\sqrt{3}}{3} i \prec \frac{3}{10}(4 + 4i).$$

It is clearly that all conditions of Theorem 2.2 are satisfied, as a result, the operator equations $F_n\xi = u\xi$ have a common solution $\xi^* = 0$ in $\overline{B(1, 4+4i)}$.

On the other hand, let Θ be the set of functions $\theta : \Omega_2 \to \Omega_3$, which satisfies the following conditions:
Θ_1 : θ is continuous;
Θ_2 : θ is nondecreasing, i.e. $\theta(x_1) \succsim \theta(x_2)$ if $x_1 \succsim x_2$;
Θ_3 : $\lim_{n\to\infty} \theta(x_n) = 1 \Leftrightarrow \lim_{n\to\infty} x_n = 0^+$, where $\{x_n\} \subset \Omega_2$.

Theorem 2.3. Let $(X, G_b, \leq)$ be a complete partially ordered complex valued G_b-metric space with $s \geq 1$, $Q \subset X$ be a cone, $\{S_n : X \to Q, \ n \in \mathbb{N}^*\}$ be a dominated mapping sequence. Suppose that there exist $\theta \in \Theta$, $i, j \in \mathbb{N}^*$, $k \in (0,1)$, $\alpha \geq 0$ such that

$$|\theta(G_b(S_i x, S_j y, S_j z))| \leq |\theta(\frac{1}{s} M(x, y, z) - \alpha)|^k \tag{2.17}$$

for any comparable elements x, y, z in Q, where $G_b(S_i x, S_j y, S_j z) \neq 0$, and

$$M(x,y,z) = \max\{G_b(x, S_i x, S_i x), G_b(y, S_j y, S_j z), G_b(z, S_j z, S_j y), G_b(x,y,z)\}. \tag{2.18}$$

Define the operator equations $F_n x = ux$ by $F_n = uS_n$, $u \geq 1$. If a nonincreasing sequence $\{x_n\} \to \kappa$ such that $\kappa \preceq x_n$, then the operator equations have at least a common solution x^* in Q. Moreover, if there exists an element v in Q such that $v \preceq x^*$, and

$$G_b(S_j^{n-1} v, S_j^n v, S_j^n v) \precsim G_b(x^*, S_j^{n-1} v, S_j^{n-1} v), \tag{2.19}$$

then the operator equations have an unique solution.
Proof. By selecting a point x_0 in Q, we construct a sequence $\{x_n\}$, where $x_{n+1} = S_{n+1} x_n \preceq x_n$, $n \in \mathbb{N}$. Let $x = x_{n-1}, y = z = x_n$ in (2.17), we have

$$\begin{aligned}
|\theta(\frac{1}{s} G_b(S_n x_{n-1}, S_{n+1} x_n, S_{n+1} x_n))| &\leq |\theta(G_b(S_n x_{n-1}, S_{n+1} x_n, S_{n+1} x_n))| \\
&\leq |\theta(\frac{1}{s} M(x_{n-1}, x_n, x_n) - \alpha)|^k \\
&\leq |\theta(\frac{1}{s} M(x_{n-1}, x_n, x_n))|^k,
\end{aligned}$$

where

$$\begin{aligned}
M(x_{n-1}, x_n, x_n) &= \max\{G_b(x_{n-1}, S_n x_{n-1}, S_n x_{n-1}), G_b(x_n, S_{n+1} x_n, S_{n+1} x_n), G_b(x_{n-1}, x_n, x_n)\} \\
&= \max\{G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1})\},
\end{aligned}$$

thus we get

$$|\theta(\frac{1}{s} G_b(x_n, x_{n+1}, x_{n+1}))| \leq |\theta(\frac{1}{s} \max\{G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1})\})|^k.$$

If $\max\{G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1})\} = G_b(x_n, x_{n+1}, x_{n+1})$, then

$$|\theta(\frac{1}{s} G_b(x_n, x_{n+1}, x_{n+1}))| \leq |\theta(\frac{1}{s} G_b(x_n, x_{n+1}, x_{n+1}))|^k, \text{ which is contradiction with } k \in (0,1),$$

hence,

$$|\theta(\frac{1}{s} G_b(x_n, x_{n+1}, x_{n+1}))| \leq |\theta(G_b(x_n, x_{n+1}, x_{n+1}))| \leq |\theta(\frac{1}{s} G_b(x_{n-1}, x_n, x_n))|^k.$$

It follows that

$$|\theta(\frac{1}{s} G_b(x_n, x_{n+1}, x_{n+1}))| \leq |\theta(\frac{1}{s} G_b(x_{n-1}, x_n, x_n))|^k \leq ... \leq |\theta(\frac{1}{s} G_b(x_0, x_1, x_1))|^{k^n},$$

and

$$\lim_{n\to\infty} |\theta(\frac{1}{s} G_b(x_n, x_{n+1}, x_{n+1}))| \leq \lim_{n\to\infty} |\theta(\frac{1}{s} G_b(x_0, x_1, x_1))|^{k^n} = 1,$$

therefore,

$$\lim_{n\to\infty} G_b(x_n, x_{n+1}, x_{n+1}) = 0 \text{ and } \lim_{n\to\infty} G_b(x_n, x_n, x_{n+1}) = 0.$$

Now we show $\{x_n\}$ is a complex valued G_b-Cauchy sequence. If not, then there exist $\epsilon \succ 0$ and two subsequences x_{m_i} and x_{n_i} of x_n, where $i \leq n_i \leq m_i$, such that

$$G_b(x_{n_i}, x_{n_i}, x_{m_i}) \succsim \epsilon \text{ and } G_b(x_{n_i}, x_{n_i}, x_{m_i-1}) \prec \epsilon.$$

Using (CG_b5), we have

$$\epsilon \precsim G_b(x_{n_i}, x_{n_i}, x_{m_i}) \precsim s[G_b(x_{n_i}, x_{n_i}, x_{n_i+1}) + G_b(x_{n_i+1}, x_{n_i+1}, x_{m_i})],$$

let $i \to \infty$ at the above inequality, we get

$$\frac{\epsilon}{s} \precsim \lim_{i\to\infty} G_b(x_{m_i}, x_{n_i+1}, x_{n_i+1}). \tag{2.20}$$

In addition, owing to (2.17), we obtain

$$|\theta(G_b(S_{m_i}x_{m_{i-1}}, S_{n_i+1}x_{n_i}, S_{n_i+1}x_{n_i}))| \leq |\theta(\frac{1}{s}M(x_{m_{i-1}}, x_{n_i}, x_{n_i}) - \alpha)|^k,$$

i.e.,

$$\begin{aligned} |\theta(\frac{1}{s}G_b(x_{m_i}, x_{n_i+1}, x_{n_i+1}))| &\leq |\theta(G_b(x_{m_i}, x_{n_i+1}, x_{n_i+1}))| \\ &\leq |\theta(\frac{1}{s}M(x_{m_{i-1}}, x_{n_i}, x_{n_i}) - \alpha)|^k \\ &\leq |\theta(\frac{1}{s}M(x_{m_{i-1}}, x_{n_i}, x_{n_i}))|^k, \end{aligned}$$

where

$$M(x_{m_{i-1}}, x_{n_i}, x_{n_i}) = \max\{G_b(x_{m_{i-1}}, x_{m_i}, x_{m_i}), G_b(x_{n_i}, x_{n_{i+1}}, x_{n_{i+1}}), G_b(x_{n_i}, x_{n_i}, x_{m_{i-1}})\}.$$

Since

$$\lim_{i\to\infty} G_b(x_{m_{i-1}}, x_{m_i}, x_{m_i}) = \lim_{i\to\infty} G_b(x_{n_i}, x_{n_{i+1}}, x_{n_{i+1}}) = 0,$$

obviously, $M(x_{m_{i-1}}, x_{n_i}, x_{n_i}) = G_b(x_{n_i}, x_{n_i}, x_{m_{i-1}})$, it follows that

$$|\theta(G_b(x_{m_i}, x_{n_i+1}, x_{n_i+1}))| \leq |\theta(\frac{1}{s}G_b(x_{n_i}, x_{n_i}, x_{m_{i-1}}))|^k. \tag{2.21}$$

Using (2.20) and (2.21), we have

$$|\theta(\frac{\epsilon}{s})| \leq \lim_{i\to\infty} |\theta(G_b(x_{m_i}, x_{n_i+1}, x_{n_i+1}))| \leq \lim_{i\to\infty} |\theta(\frac{1}{s}G_b(x_{n_i}, x_{n_i}, x_{m_{i-1}}))|^k < |\theta(\frac{\epsilon}{s})|^k,$$

which is a contradiction with $k \in (0, 1)$. As a result, $\{x_n\}$ is a complex valued G_b-Cauchy sequence, and there exists an element x^* in Q such that $x_n \to x^*$.

Now we prove that x^* is a common solution of the operator equations. For all $i, j \in \mathbb{N}^*$, we have

$$G_b(x^*, S_jx^*, S_jx^*) \precsim s[G_b(x^*, x_i, x_i) + G_b(x_i, S_jx^*, S_jx^*)],$$

and let $i \to \infty$ at the above inequality, we get

$$G_b(x^*, S_jx^*, S_jx^*) \precsim \lim_{i\to\infty} sG_b(x_i, S_jx^*, S_jx^*). \tag{2.22}$$

In addition, since $x^* \preceq x_{i-1}$, according to (2.17), we obtain

$$|\theta(G_b(S_i x_{i-1}, S_j x^*, S_j x^*))| \leq |\theta(\frac{1}{s}M(x_{i-1}, x^*, x^*) - \alpha)|^k \leq |\theta(\frac{1}{s}M(x_{i-1}, x^*, x^*))|^k,$$

where

$$M(x_{i-1}, x^*, x^*) = \max\{G_b(x_{i-1}, x_i, x_i), G_b(x^*, S_j x^*, S_j x^*), G_b(x_{i-1}, x^*, x^*)\}.$$

If $M(x_{i-1}, x^*, x^*) = G_b(x^*, S_j x^*, S_j x^*)$, using (2.22), it follows that

$$\lim_{i\to\infty} |\theta(G_b(x_i, S_j x^*, S_j x^*))| \leq \lim_{i\to\infty} |\theta(\frac{1}{s}G_b(x^*, S_j x^*, S_j x^*))|^k \leq \lim_{i\to\infty} |\theta(G_b(x_i, S_j x^*, S_j x^*))|^k,$$

contradiction, thus we can easily get

$$|\theta(G_b(x_i, S_j x^*, S_j x^*))| \leq |\theta(\frac{1}{s}G_b(x_{i-1}, x_i, x_i))|^k \to 1 \quad \text{as } i \to \infty,$$

or

$$|\theta(G_b(x_i, S_j x^*, S_j x^*))| \leq |\theta(\frac{1}{s}G_b(x_{i-1}, x^*, x^*))|^k \to 1 \quad \text{as } i \to \infty,$$

hence,

$$\lim_{i\to\infty} G_b(x_i, S_j x^*, S_j x^*) = 0.$$

From (2.22), we have

$$G_b(x^*, S_j x^*, S_j x^*) \precsim \lim_{i\to\infty} sG_b(x_i, S_j x^*, S_j x^*) = 0.$$

As a result, $x^* = S_j x^*$, owing to the arbitrariness of j, we obtain x^* is a common solution of the operator equations.

Uniqueness. If y^* is another solution of the operator equations, $y^* \neq x^*$, then $G_b(x^*, y^*, y^*) \neq 0$.
Case 1. x^* and y^* are comparable, using (2.17), it follows that

$$|\theta(G_b(x^*, y^*, y^*))| = |\theta(G_b(S_i x^*, S_j y^*, S_j y^*))| \leq |\theta(\frac{1}{s}M(x^*, y^*, y^*) - \alpha)|^k \leq |\theta(\frac{1}{s}M(x^*, y^*, y^*))|^k.$$

Obviously, $M(x^*, y^*, y^*) = G_b(x^*, y^*, y^*)$, so we have

$$|\theta(G_b(x^*, y^*, y^*))| \leq |\theta(\frac{1}{s}G_b(x^*, y^*, y^*))|^k,$$

which is a contradiction. As a result, $y^* = x^*$.
Case 2. x^* and y^* are not comparable, then there exists an element $v \in Q$ such that $v \preceq x^*$ and $v \preceq y^*$, for any $i, j \in \mathbb{N}^*$, we have

$$x^* = S_i x^* = S_i^2 x^* = ... = S_i^n x^*, \ \ y^* = S_j y^* = S_j^2 y^* = ... = S_j^n y^*,$$

and

$$S_j^n v \preceq ... \preceq S_j v \preceq v \preceq x^*, \ \ S_j^n v \preceq ... \preceq S_j v \preceq v \preceq y^*.$$

From (2.17), we get

$$|\theta(G_b(S_i^n x^*, S_j^n v, S_j^n v))| \le |\theta(\frac{1}{s}M(S_i^{n-1}x^*, S_j^{n-1}v, S_j^{n-1}v) - \alpha)|^k \le |\theta(\frac{1}{s}M(S_i^{n-1}x^*, S_j^{n-1}v, S_j^{n-1}v))|^k,$$

where

$$M(S_i^{n-1}x^*, S_j^{n-1}v, S_j^{n-1}v) = \max\{G_b(S_i^{n-1}x^*, S_i^n x^*, S_i^n x^*), G_b(S_j^{n-1}v, S_j^n v, S_j^n v), G_b(S_i^{n-1}x^*, S_j^{n-1}v, S_j^{n-1}v)\}.$$

According to (2.19), we obtain $M(S_i^{n-1}x^*, S_j^{n-1}v, S_j^{n-1}v) = G_b(S_i^{n-1}x^*, S_j^{n-1}v, S_j^{n-1}v)$, and

$$|\theta(G_b(S_i^n x^*, S_j^n v, S_j^n v))| \le |\theta(\frac{1}{s}G_b(S_i^{n-1}x^*, S_j^{n-1}v, S_j^{n-1}v))|^k \le |\theta(G_b(S_i^{n-1}x^*, S_j^{n-1}v, S_j^{n-1}v))|^k,$$

so that we have

$$|\theta(G_b(S_i^n x^*, S_j^n v, S_j^n v))| \le |\theta(G_b(S_i^{n-1}x^*, S_j^{n-1}v, S_j^{n-1}v))|^k \le ... \le |\theta(G_b(x^*, v, v))|^{k^n}.$$

It follows that

$$\lim_{n\to\infty} |\theta(G_b(S_i^n x^*, S_j^n v, S_j^n v))| \le \lim_{n\to\infty} |\theta(G_b(x^*, v, v))|^{k^n} = 1,$$

hence,

$$\lim_{n\to\infty} G_b(S_i^n x^*, S_j^n v, S_j^n v) = 0. \tag{2.23}$$

Similarly, using (2.17) and (2.19), we get

$$\begin{aligned}|\theta(G_b(S_j^n y^*, S_j^n v, S_j^n v))| &\le |\theta(\frac{1}{s}M(S_j^{n-1}y^*, S_j^{n-1}v, S_j^{n-1}v) - \alpha)|^k \\ &\le |\theta(M(S_j^{n-1}y^*, S_j^{n-1}v, S_j^{n-1}v))|^k,\end{aligned}$$

where

$$\begin{aligned}M(S_j^{n-1}y^*, S_j^{n-1}v, S_j^{n-1}v)) &= \max\{G_b(S_j^{n-1}y^*, S_j^n y^*, S_j^n y^*), G_b(S_j^{n-1}v, S_j^n v, S_j^n v), G_b(S_j^{n-1}y^*, S_j^{n-1}v, S_j^{n-1}v)\} \\ &= \max\{0, G_b(S_j^{n-1}v, S_j^n v, S_j^n v), G_b(S_j^{n-1}y^*, S_j^{n-1}v, S_j^{n-1}v)\} \\ &= G_b(S_j^{n-1}y^*, S_j^{n-1}v, S_j^{n-1}v).\end{aligned}$$

Therefore,

$$|\theta(G_b(S_j^n y^*, S_j^n v, S_j^n v))| \le |\theta(G_b(S_j^{n-1}y^*, S_j^{n-1}v, S_j^{n-1}v))|^k \le ... \le |\theta(G_b(y^*, v, v))|^{k^n},$$

let $n \to \infty$, we have

$$\lim_{n\to\infty} |\theta(G_b(S_j^n y^*, S_j^n v, S_j^n v))| \le \lim_{n\to\infty} |\theta(G_b(y^*, v, v))|^{k^n} = 1,$$

so we obtain

$$\lim_{n\to\infty} G_b(S_j^n y^*, S_j^n v, S_j^n v) = 0,$$

and also

$$\lim_{n\to\infty} G_b(S_j^n v, S_j^n y^*, S_j^n y^*) = 0. \tag{2.24}$$

Using (2.23) and (2.24), we also have

$$G_b(S_i^n x^*, S_j^n y^*, S_j^n y^*) \precsim s[G_b(S_i^n x^*, S_j^n v, S_j^n v) + G_b(S_j^n v, S_j^n y^*, S_j^n y^*)] \to 0 \text{ as } n \to \infty.$$

Owing to $G_b(x^*, y^*, y^*) = G_b(S_i^n x^*, S_j^n y^*, S_j^n y^*)$, as a result, $x^* = y^*$, the proof is completed.

Example 2.4. Let $X = R$, $Q = [0, \infty)$, $\theta(t) = 1 + t$, $\alpha = 0$, $k = \frac{1}{2}$, $G_b : X \times X \times X \to \mathbb{C}$ be defined by $G_b(\xi_1, \xi_2, \xi_3) = \max\{|\xi_1 - \xi_2|^2, |\xi_2 - \xi_3|^2, |\xi_1 - \xi_3|^2\} + \max\{|\xi_1 - \xi_2|^2, |\xi_2 - \xi_3|^2, |\xi_1 - \xi_3|^2\}i$ with $s = 2$. For any ξ in X, take $S_n\xi = \frac{|\xi|}{4n}$ and $F_n = uS_n$, where $u \geq 1, n \in \mathbb{N}^*$, the partial order $\preceq$ on X is the usual order $\leq$ of R.

Suppose that $\xi_1 \geq \xi_2 \geq \xi_3$, if $\xi_1 - \xi_3 \leq 1$ for any ξ_1, ξ_2, ξ_3 in Q, or $\xi_1, \xi_2, \xi_3 \in [0, 1]$, we can easily obtain

$$1 + G_b(S_n\xi_1, S_n\xi_2, S_n\xi_3) = 1 + \left(\frac{\xi_1 - \xi_3}{4n}\right)^2 + \left(\frac{\xi_1 - \xi_3}{4n}\right)^2 i,$$

and

$$1 + \frac{1}{2}G_b(\xi_1, \xi_2, \xi_3) = 1 + \frac{1}{2}(\xi_1 - \xi_3)^2 + \frac{1}{2}(\xi_1 - \xi_3)^2 i.$$

Hence,

$$\begin{aligned}
|1 + G_b(S_n\xi_1, S_n\xi_2, S_n\xi_3)|^4 &= \left[\sqrt{\left(1 + \left(\frac{\xi_1 - \xi_3}{4n}\right)^2\right)^2 + \left(\frac{\xi_1 - \xi_3}{4n}\right)^4}\,\right]^4 \\
&= 1 + 4\left(\frac{\xi_1 - \xi_3}{4n}\right)^2 + 8\left(\frac{\xi_1 - \xi_3}{4n}\right)^4 + 8\left(\frac{\xi_1 - \xi_3}{4n}\right)^6 + 4\left(\frac{\xi_1 - \xi_3}{4n}\right)^8 \\
&\leq 1 + 12\left(\frac{\xi_1 - \xi_3}{4n}\right)^2 + 12\left(\frac{\xi_1 - \xi_3}{4n}\right)^4 \\
&\leq 1 + (\xi_1 - \xi_3)^2 + \frac{1}{2}(\xi_1 - \xi_3)^4,
\end{aligned}$$

and

$$|1 + \frac{1}{2}G_b(\xi_1, \xi_2, \xi_3)|^2 = 1 + (\xi_1 - \xi_3)^2 + \frac{1}{2}(\xi_1 - \xi_3)^4.$$

Thus we obtain

$$|1 + G_b(S_n\xi_1, S_n\xi_2, S_n\xi_3)| \leq |1 + \frac{1}{2}G_b(\xi_1, \xi_2, \xi_3)|^{\frac{1}{2}} \leq |1 + \frac{1}{2}M(\xi_1, \xi_2, \xi_3)|^{\frac{1}{2}}.$$

It follows that the operator equations $F_n\xi = u\xi$ have a common solution $\xi^* = 0$ in Q and (2.19) is established with $v = 0$. Therefore, all conditions of Theorem 2.3 are satisfied, the operator equations $F_n\xi = u\xi$ have an unique solution $\xi^* = 0$.

The following two corollaries can be easily obtained, if we let $\theta(t) = e^{|t|} + t$ and $\theta(t) = 2 - \frac{2}{\pi}\arctan(\frac{1}{|t|^\gamma})$ in Theorem 2.3 respectively.

Corollary 2.3. Let $(X, G_b, \preceq)$ be a complete partially ordered complex valued G_b-metric space with $s \geq 1$, $Q \subset X$ be a cone, $\{S_n : X \to Q,\ n \in \mathbb{N}^*\}$ be a dominated mapping sequence. Suppose that there exist $i, j \in \mathbb{N}^*, k \in (0, 1), \alpha \geq 0$ such that

$$|e^{|G_b(S_ix, S_jy, S_jz)|} + G_b(S_ix, S_jy, S_jz)| \leq |e^{|\frac{1}{s}M(x,y,z) - \alpha|} + \frac{1}{s}M(x, y, z) - \alpha|^k$$

for any comparable elements x, y, z in Q, where $G_b(S_ix, S_jy, S_jz) \neq 0$, and

$$M(x,y,z) = \max\{G_b(x, S_ix, S_ix), G_b(y, S_jy, S_jz), G_b(z, S_jz, S_jy), G_b(x,y,z)\}.$$

Define the operator equations $F_nx = ux$ by $F_n = uS_n$, $u \geq 1$. If a nonincreasing sequence $\{x_n\} \to \kappa$ such that $\kappa \preceq x_n$, then the operator equations have at least a common solution x^* in Q. Moreover, if there exists an element v in Q such that $v \preceq x^*$, and

$$G_b(S_j^{n-1}v, S_j^nv, S_j^nv) \precsim G_b(x^*, S_j^{n-1}v, S_j^{n-1}v),$$

then the operator equations have an unique solution.

Corollary 2.4. Let $(X, G_b, \preceq)$ be a complete partially ordered complex valued G_b-metric space with $s \geq 1$, $Q \subset X$ be a cone, $\{S_n : X \to Q,\ n \in \mathbb{N}^*\}$ be a dominated mapping sequence. Suppose that there exist $i, j \in \mathbb{N}^*, \gamma, k \in (0,1), \alpha \geq 0$ such that

$$2 - \frac{2}{\pi}\arctan\left(\frac{1}{|G_b(S_ix, S_jy, S_jz)|^{\gamma}}\right) \leq |2 - \frac{2}{\pi}\arctan\left(\frac{1}{|\frac{1}{s}M(x,y,z) - \alpha|^{\gamma}}\right)|^{k}$$

for any comparable elements x, y, z in Q, where $G_b(S_ix, S_jy, S_jz) \neq 0$, and

$$M(x,y,z) = \max\{G_b(x, S_ix, S_ix), G_b(y, S_jy, S_jz), G_b(z, S_jz, S_jy), G_b(x,y,z)\}.$$

Define the operator equations $F_nx = ux$ by $F_n = uS_n$, $u \geq 1$. If a nonincreasing sequence $\{x_n\} \to \kappa$ such that $\kappa \preceq x_n$, then the operator equations have at least a common solution x^* in Q. Moreover, if there exists an element v in Q such that $v \preceq x^*$, and

$$G_b(S_j^{n-1}v, S_j^nv, S_j^nv) \precsim G_b(x^*, S_j^{n-1}v, S_j^{n-1}v),$$

then the operator equations have an unique solution.

3. Conclusions

In this paper, we have obtained some new theorems for the common solutions of the operator equations $F_nx = ux$ ($u \geq 1, n \in \mathbb{N}^*$) via complex valued C^p-class function, a type of Geraghty contraction and a type of JS contraction in complete partially ordered complex valued G_b-metric spaces, and some of which are established in a closed ball. These new results generalize many known results in complex valued G_b-metric spaces and G_b-metric spaces, in addition, it would be interesting and worthwhile to further investigate some similar problems in other types of spaces.

Acknowledgments

This work was supported by National Natural Science Foundation of China (Grant No. 11771198).

Conflict of interest

The authors declare no conflict of interest.

References

1. Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex A.*, **7** (2006), 289–297.
2. A. Aghajani, M. Abbas, J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces, *Filomat*, **28** (2014), 1087–1101. http://dx.doi.org/10.2298/FIL1406087A
3. L. Zhu, C. X. Zhu, C. F. Chen, Common fixed point theorems for fuzzy mappings in G-metric spaces, *Fixed Point Theory A.*, **2012** (2012), 159. http://dx.doi.org/10.1186/1687-1812-2012-159
4. M. Asadi, E. Karapınar, P. Salimi, A new approach to G-metric and related fixed point theorems, *J. Inequal. Appl.*, **2013** (2013), 454. http://dx.doi.org/10.1186/1029-242X-2013-454
5. J. R. Roshan, N. Shobkolaei, S. Sedghi, V. Parvaneh, S. Radenović, Common fixed point theorems for three maps in discontinuous G_b-metric spaces, *Acta Math. Sci.*, **34** (2014), 1643–1654. http://dx.doi.org/10.1016/S0252-9602(14)60110-7
6. J. Chen, C. X. Zhu, L. Zhu, A note on some fixed point theorems on G-metric spaces, *J. Appl. Anal. Comput.*, **11** (2021), 101–112. http://dx.doi.org/10.11948/20190125
7. Y. U. Gaba, Fixed point theorems in G-metric spaces, *J. Math. Anal. Appl.*, **455** (2017), 528–537. http://dx.doi.org/10.1016/j.jmaa.2017.05.062
8. M. Liang, C. X. Zhu, C. F. Chen, Z. Q. Wu, Some new theorems for cyclic contractions in G_b-metric spaces and some applications, *Appl. Math. Comput.*, **346** (2019), 545–558. http://dx.doi.org/10.1016/j.amc.2018.10.028
9. C. X. Zhu, J. Chen, C. F. Chen, J. H. Chen, H. P. Huang, A new generalization of $\mathcal{F}$-metric spaces and some fixed point theorems and an application, *J. Appl. Anal. Comput.*, **11** (2021), 2649–2663. http://dx.doi.org/10.11948/20210244
10. C. X. Zhu, J. Chen, X. J. Huang, J. H. Chen, Fixed point theorems in modular spaces with simulation functions and altering distance functions with applications, *J. Nonlinear Convex A.*, 2020, 1403–1424.
11. Y. X. Wang, C. F. Chen, Two new Geraghty type contractions in G_b-metric spaces, *J. Funct. Space.*, **2019** (2019). http://dx.doi.org/10.1155/2019/7916486
12. M. Jleli, E. Karapınar, B. Samet, Further generalizations of the Banach contraction principle, *J. Inequal. Appl.*, **2014** (2014), 439. http://dx.doi.org/10.1186/1029-242X-2014-439
13. H. Aydi, A. Felhi, H. Afshari, New Geraghty type contractions on metric-like spaces, *J. Nonlinear Sci. Appl.*, **10** (2017), 780–788. http://dx.doi.org/10.22436/jnsa.010.02.38
14. M. Jleli, B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, **2014** (2014), 38. http://dx.doi.org/10.1186/1029-242X-2014-38
15. A. Shoaib, M. Arshad, T. Rasham, M. Abbas, Unique fixed point results on closed ball for dislocated quasi G-metric spaces, *T. A. Razmadze Math. In.*, **171** (2017), 221–230. http://dx.doi.org/10.1016/j.trmi.2017.01.002
16. O. Ege, Complex valued G_b-metric spaces, *J. Comput. Anal. Appl.*, **21** (2016), 363–368.

17. O. Ege, Some fixed point theorems in complex valued G_b-metric spaces, *J. Nonlinear Convex A.*, **18** (2017), 1997–2005.

18. O. Ege, C. Park, A. H. Ansari, A different approach to complex valued G_b-metric spaces, *Adv. Differ. Equ.*, **2020** (2020), 152. http://dx.doi.org/10.1186/s13662-020-02605-0

19. O. Ege, I. Karaca, Common fixed point results on complex valued G_b-metric spaces, *Thai J. Math.*, **16** (2018), 775–787.

20. A. H. Ansari, O. Ege, S. Radenović, Some fixed point results on complex valued G_b-metric spaces, *RACSAM Rev. R. Acad. A*, **112** (2018), 463–472. http://dx.doi.org/10.1007/s13398-017-0391-x

21. H. Afshari, Solution of fractional differential equations in quasi-b-metric and b-metric-like spaces, *Adv. Differ. Equ.*, **2019** (2019), 285. http://dx.doi.org/10.1186/s13662-019-2227-9

22. H. Afshari, M. Atapour, E. Karapınar, A discussion on a generalized Geraghty multi-valued mappings and applications, *Adv. Differ. Equ.*, **2020** (2020). http://dx.doi.org/10.1186/s13662-020-02819-2

23. H. Afshari1, S. Kalantari, D. Baleanu, Solution of fractional differential equations via α-ψ-Geraghty type mappings, *Adv. Differ. Equ.*, **2018** (2018), 347. https://doi.org/10.1186/s13662-018-1807-4

24. M. Jleli, B. Samet, Remarks on G-metric spaces and fixed point theorems, *Fixed Point Theory A.*, **2012** (2012), 210. http://dx.doi.org/10.1186/1687-1812-2012-210

25. R. P. Agarwal, H. H. Alsulami, E. Karapınar, F. Khojasteh, Remarks on some recent fixed point results on quaternion-valued metric spaces, *Abstr. Appl. Anal.*, **2014** (2014). http://dx.doi.org/10.1155/2014/171624

26. A. Shoaib, S. Mustafa, A. Shahzad, Common fixed point of multivalued mappings in ordered dislocated quasi G-metric spaces, *Punjab Univ. J. Math.*, **52** (2020).

27. A. E. Al-Mazrooei, A. Shoaib, J. Ahmad, Unique fixed-point results for β-admissible mapping under (β-$\breve{\psi}$)-contraction in complete dislocated G_d-metric space, *Mathematics*, **8** (2020). http://dx.doi.org/10.3390/math8091584

28. M. Abbas, S. Z. Németh, Finding solutions of implict complementarity problems by isotonicty of the metric projection, *Nonlinear Anal.*, **75** (2012), 2349–2361. http://dx.doi.org/10.1016/j.na.2011.10.033

AIMS Press