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Research article

Hermite-Hadamard type inclusions via generalized Atangana-Baleanu fractional operator with application

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Abstract: Defining new fractional operators and employing them to establish well-known integral inequalities has been the recent trend in the theory of mathematical inequalities. To take a step forward, we present novel versions of Hermite-Hadamard type inequalities for a new fractional operator, which generalizes some well-known fractional integral operators. Moreover, a midpoint type fractional integral identity is derived for differentiable mappings, whose absolute value of the first-order derivatives are convex functions. Moreover, considering this identity as an auxiliary result, several improved inequalities are derived using some fundamental inequalities such as Hölder-İşcan, Jensen and Young inequality. Also, if we take the parameter $\rho = 1$ in most of the results, we derive new results for Atangana-Baleanu equivalence. One example related to matrices is also given as an application.

Keywords: convex functions; Hermite-Hadamard inequality; Atangana-Baleanu fractional integral operators; Young inequality; Jensen's inequality
Mathematics Subject Classification: 26A33, 26A51, 26D10

Abbreviations

H-H: Hermite-Hadamard; AB: Atangana-Baleanu; ABK: Atangana-Baleanu-Kashuri

1. Introduction

The concept of functions is one of the fundamental constructions of mathematics, and many researchers have concentrated on new functions and put forth attempts to order the space of functions. One such function characterized as a result of this extreme exertion is the convex function, which has a lot of applications in engineering, industrial optimization, probability theory, theory of mathematical inequalities (see [1–3]), etc. Starting now and into the foreseeable future various experts have inspected and applied different fractional operators for modelling of COVID-19 [4], modelling of Hepatitis-B epidemic [5], groundwater flow [6], RLC electric circuit [7], the hypothesis of viscoelasticity [8], fluid mechanics [9], etc.

The theory of convex function was preceded by Jensen in [10] to the literature for the first time and since then attracted consideration to the fact that it seems to be the basis of the concept of incremental function. Many researchers have concentrated on the connections between convexity and the Hermite-Hadamard inequality.

There are numerous inequalities in the literature for convexity theory. Hermite-Hadamard inequality may be the one that takes the most consideration of scientists and on which many investigations have been conducted. The concept of integral inequality is an intriguing topic of study in mathematical analysis. The fundamental integral inequalities can be used to cultivate convexity's subjective features. The convex function has a lovely representation based on an inequality shown when the functional value of a linear combination of two points in its domain does not exceed the linear combination of the functional values at those two locations.

Definition 1.1. *(see [11]) A real valued function $\Upsilon : \mathrm{K} \subseteq \mathcal{R} \to \mathcal{R}$ (set of real numbers) is said to be convex iff the following inequality satisfies*

$$\Upsilon(\mathfrak{u}\varrho_1 + (1-\mathfrak{u})\varrho_2) \leqq \mathfrak{u}\Upsilon(\varrho_1) + (1-\mathfrak{u})\Upsilon(\varrho_2),$$

for all $\varrho_1, \varrho_2 \in \mathrm{K}, \mathfrak{u} \in [0,1]$.

Let $\Upsilon : \mathrm{K} \subseteq \mathcal{R} \longrightarrow \mathcal{R}$ be a convex function with $\varrho_1 < \varrho_2$ and $\varrho_1, \varrho_2 \in \mathrm{K}$. Then the H-H inequality is expressed as follows (see [12]):

$$\Upsilon\left(\frac{\varrho_1+\varrho_2}{2}\right) \leqq \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} \Upsilon(\mathrm{x})d\mathrm{x} \leqq \frac{\Upsilon(\varrho_1)+\Upsilon(\varrho_2)}{2}. \tag{1.1}$$

Numerous mathematicians have recently generalized and extended the standard H-H inequality (1.1) under the premise of certain interesting new definitions as a generalization of a convex function.

It is known that fractional calculus aims at establishing mathematical models. As of late, it is seen that refining well-known integral inequalities utilizing fractional integral operators has become a surprising subject of exploration among mathematicians. Generalization of some existing integral inequalities such as Hermite-Hadamard, Simpson, Opial, Fejer and Ostrowski type inequalities using different types of fractional operators and innovative methodologies guided to a revolution in the inequality theory. Several researchers have included this concept in the premises of several types of convexities in recent decades. We refer the interested readers to check (see [13–19]) for collections of integral inequalities involving convexities.

Mathematicians have recently been fascinated by the prospect of refining well-known integral inequalities using fractional integral operators. Sarikaya et al. [20] suggested the idea of presenting the Hermite-Hadamard inequality using the Rieman-Liouville fractional integral operator. Liu et al. [21] generalized the Hermite-Hadamard inequality for the ψ-Riemann-Liouville fractional operator, which was inspired by Sarikaya's research. Similarly, several researchers, including Mumcu et al. [22] employed a generalized Proportional fractional integral. Gürbüz et al. [23] worked on the Caputo-Fabrizio operator, whereas Fernandez et al. [24] combined the Atangana-Baleanu fractional operator with the Mittag-Leffler kernel. Mohammed et al. [25] used Tempered fractional integrals, Sahoo et al. [26] used $k-$Riemann-Liouville fractional integrals and Khan et al. [27] worked on a generalised conformable operator.

This article is arranged as follows: In Section 2, some basic notions and properties in the frame of fractional calculus are presented. Section 3 deals with presenting the main results of our article, where we have established some new H-H type inequalities involving ABK fractional operator. In Section 4, application related to matrices is given. Finally, in Section 5, conclusions and plans of this paper are hinted at.

2. Preliminaries

Now, we will present some basic definitions of different types of fractional operators that are exceptionally appealing in many parts of mathematics as follows:

Definition 2.1. *(see [20, 28]) Let $\Upsilon \in \mathcal{L}[\varrho_1, \varrho_2]$. Then the left and right Riemann-Liouville fractional integrals $\mathrm{I}^{\alpha}_{\varrho_1^+}$ and $\mathrm{I}^{\alpha}_{\varrho_2^-}$ of order $\alpha > 0$ are defined by*

$$\mathrm{I}^{\alpha}_{\varrho_1^+}\Upsilon(\mathtt{x}) := \frac{1}{\Gamma(\alpha)}\int_{\varrho_1}^{\mathtt{x}} (\mathtt{x}-\mathrm{u})^{\alpha-1}\ \Upsilon(\mathrm{u})\,d\mathrm{u}, \qquad (0 \leqq \varrho_1 < \mathtt{x} < \varrho_2)$$

and

$$\mathrm{I}^{\alpha}_{\varrho_2^-}\Upsilon(\mathtt{x}) := \frac{1}{\Gamma(\alpha)}\int_{\mathtt{x}}^{\varrho_2} (\mathrm{u}-\mathtt{x})^{\alpha-1}\ \Upsilon(\mathrm{u})\,d\mathrm{u}, \qquad (0 \leqq \varrho_2 < \mathtt{x} < \varrho_2),$$

respectively.

Theorem 2.1. *(see [29]) Let $\Upsilon : [\varrho_1, \varrho_2] \longrightarrow \mathcal{R}$ be a convex function with $0 \leqq \varrho_1 \leqq \varrho_2$. If $\Upsilon \in \mathcal{L}[\varrho_1, \varrho_2]$, then the following inequality for Riemann-Liouville fractional integral operator holds true:*

$$\Upsilon\left(\frac{\varrho_1+\varrho_2}{2}\right) \leqq \frac{2^{\mu-1}\Gamma(\mu+1)}{(\varrho_2-\varrho_1)^{\mu}}\left[\mathrm{I}^{\mu}_{(\frac{\varrho_1+\varrho_2}{2})^+}\Upsilon(\varrho_2) + \mathrm{I}^{\mu}_{(\frac{\varrho_1+\varrho_2}{2})^-}\Upsilon(\varrho_1)\right] \leqq \frac{\Upsilon(\varrho_1)+\Upsilon(\varrho_2)}{2}.$$

Definition 2.2. *(see [30]) Let $[\varrho_1, \varrho_2] \subset \mathcal{R}$ be a finite interval. Then the left side and right side Katugampola fractional integral of order $\mu > 0$ of $\Upsilon \in X^{\rho}_{c}(\varrho_1, \varrho_2)$ are defined by:*

$${}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^+}\{\Upsilon(\mathtt{t})\} = \frac{\rho^{1-\mu}}{\Gamma(\mu)}\int_{\varrho_1}^{\mathtt{t}} \Upsilon(\mathtt{x})(\mathtt{t}^{\rho}-\mathtt{x}^{\rho})^{\mu-1}\mathtt{x}^{\rho-1}\mathtt{dx},$$

and

$${}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^-}\{\Upsilon(\mathtt{t})\} = \frac{\rho^{1-\mu}}{\Gamma(\mu)}\int_{\mathtt{t}}^{\varrho_2} \Upsilon(\mathtt{x})(\mathtt{x}^{\rho}-t^{\rho})^{\mu-1}\mathtt{x}^{\rho-1}\mathtt{dx}.$$

With the use of fractional integral operators, this study was able to grow and obtain a variety of well-known integral inequalities. Several mathematicians have recently described distinct forms of the Riemann-Liouville fractional operator in order to propose novel generalisations and refinements of integral inequalities involving differentiable functions. In various domains of applied sciences, studies pertaining to fractional calculus have provided a new perspective and direction. With the use of recently defined fractional operators, it has provided insight into a variety of real-world challenges. A few fundamental rules have separated all of these new fractional operators, and some favourable applications have been contrasted with others.

As of late, Atangana-Baleanu presented another fractional operator that involves a special function, i.e., the Mittag-Leffler function, which tackles the issue of recovering the original function. The Mittag-Leffler function is more reasonable than a power law in demonstrating the physical phenomenon around us. This made the AB fractional operator more powerful and accommodating. Thus, numerous researchers have shown a keen fascination for using this operator. Atangana-Baleanu presented the derivative both in Caputo and Reimann-Liouville sense.

Note. From now on we will use $\mathcal{B}(\mu) > 0$ as a normalization function satisfying $\mathcal{B}(0) = \mathcal{B}(1) = 1$.

Definition 2.3. *Let $p \in [1, \infty)$ and $[\varrho_1, \varrho_2]$ be an open subset of $\mathcal{R}$, the Sobolev space $\mathcal{H}^p(\varrho_1, \varrho_2)$ is defined by*

$$\mathcal{H}^p(\varrho_1, \varrho_2) = \left\{ \Upsilon \in L^2(\varrho_1, \varrho_2) \ : \ D^\mu \Upsilon \in L^2(\varrho_1, \varrho_2), \quad for \ \ all \ |\mu| \leq p \right\}.$$

Definition 2.4. *(see [31]) Let $\varrho_2 > \varrho_1$, $\mu \in [0, 1]$ and $\Upsilon \in \mathcal{H}^1(\varrho_1, \varrho_2)$. The new fractional derivative is given:*

$$ {}^{\mathtt{ABC}}_{\varrho_1}\mathrm{D}^{\mu}_{\mathtt{t}} \left[\Upsilon(\mathtt{t})\right] = \frac{\mathcal{B}(\mu)}{1-\mu} \int_{\varrho_1}^{\mathtt{t}} \Upsilon'(\mathtt{x}) \mathbb{E}_\mu \left[-\mu \frac{(\mathtt{t}-\mathtt{x})^\mu}{(1-\mu)} \right] \mathtt{dx}.$$

Definition 2.5. *(see [31]) Let $\Upsilon \in \mathcal{H}^1(\varrho_1, \varrho_2)$, $\varrho_1 < \varrho_2$, $\mu \in [0, 1]$. The new fractional derivative is given:*

$$ {}^{\mathtt{ABR}}_{\varrho_1}\mathrm{D}^{\mu}_{\mathtt{t}} \left[\Upsilon(\mathtt{t})\right] = \frac{\mathcal{B}(\mu)}{1-\mu} \frac{\mathtt{d}}{\mathtt{dt}} \int_{\varrho_1}^{\mathtt{t}} \Upsilon(\mathtt{x}) \mathbb{E}_\mu \left[-\mu \frac{(\mathtt{t}-\mathtt{x})^\mu}{(1-\mu)} \right] \mathtt{dx}.$$

However, in the same paper they have given the corresponding Atangana-Baleanu(AB)-fractional integral operator as:

Definition 2.6. *(see [31]) The fractional integral operator with non-local kernel of a function $\Upsilon \in \mathcal{H}^1(\varrho_1, \varrho_2)$ is defined as:*

$$ {}^{\mathtt{AB}}_{\ \varrho_1}\mathrm{I}^{\mu}_{\mathtt{t}} \left\{\Upsilon(\mathtt{t})\right\} = \frac{1-\mu}{\mathcal{B}(\mu)} \Upsilon(\mathtt{t}) + \frac{\mu}{\mathcal{B}(\mu)\Gamma(\mu)} \int_{\varrho_1}^{\mathtt{t}} \Upsilon(\mathtt{x})(\mathtt{t}-\mathtt{x})^{\mu-1} \mathtt{dx},$$

where $\varrho_2 > \varrho_1, \mu \in [0, 1]$.

In [32], the right hand side of Atangana-Baleanu fractional integral operator as following;

$$ {}^{\mathtt{AB}}_{\ \varrho_2}\mathrm{I}^{\mu}_{\mathtt{t}} \left\{\Upsilon(\mathtt{t})\right\} = \frac{1-\mu}{\mathcal{B}(\mu)} \Upsilon(\mathtt{t}) + \frac{\mu}{\mathcal{B}(\mu)\Gamma(\mu)} \int_{\mathtt{t}}^{\varrho_2} \Upsilon(\mathtt{x})(\mathtt{x}-\mathtt{t})^{\mu-1} \mathtt{dx},$$

where, $\Gamma(\mu)$ is the Gamma function defined by $\Gamma(\mu) = \int_0^\infty e^{-\mathtt{t}} \mathtt{t}^{\mu-1} d\mathtt{t}$.

Now, we present the definition of integral operator that generalizes Atangana-Baleanu integral operator and Katugampola integral operator.

Definition 2.7. *(see [33]) Let $[\varrho_1, \varrho_2] \subset \mathcal{R}$ be a finite interval. Then the left side and right side ABK-fractional integral of order $\mu > 0$ of $\Upsilon \in X_c^{\rho}(\varrho_1, \varrho_2)$ are defined as follows:*

$$ {}^{\mathrm{ABK}\rho}_{\varrho_1^+}\mathrm{I}^{\mu}_{\mathrm{t}}\{\Upsilon(\mathrm{t})\} = \frac{1-\mu}{\mathcal{B}(\mu)}\Upsilon(\mathrm{t}) + \frac{\rho^{1-\mu}\mu}{\mathcal{B}(\mu)\Gamma(\mu)}\int_{\varrho_1}^{\mathrm{t}} \Upsilon(\mathrm{x})(\mathrm{t}^{\rho} - \mathrm{x}^{\rho})^{\mu-1}\mathrm{x}^{\rho-1}\mathrm{dx}, $$

and

$$ {}^{\mathrm{ABK}\rho}_{\varrho_2^-}\mathrm{I}^{\mu}_{\mathrm{t}}\{\Upsilon(\mathrm{t})\} = \frac{1-\mu}{\mathcal{B}(\mu)}\Upsilon(\mathrm{t}) + \frac{\rho^{1-\mu}\mu}{\mathcal{B}(\mu)\Gamma(\mu)}\int_{\mathrm{t}}^{\varrho_2} \Upsilon(\mathrm{x})(\mathrm{x}^{\rho} - t^{\rho})^{\mu-1}\mathrm{x}^{\rho-1}\mathrm{dx}. $$

For other generalizations and detailed knowledge about fractional-calculus operators, the interested readers can see [34–45].

The purpose of this investigation is to get new estimations of Hermite-Hadamard type inequalities employing a generalized Atangana-Baleanu fractional integral operator, i.e., ABK for convex functions. The principal motivation for applying ABK fractional operators is that the consequences demonstrated by this operator permit us to generate unique and comprehensive inequalities of the Hermite-Hadamard type. A novel version of Hermite-Hadamard inequality and an integral identity including generalized Atangana-Baleanu fractional operator is established. Next, some refinements of H-H type inequalities in the setting of (ABK) fractional operator are discussed as well. It is apparent that if one selects ρ= 1, and $\mu = 1$ in the main results, multiple alternatives of the classical H-H inequality will be recaptured.

Definition 2.8. *(Hölder's inequality [46]) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If Υ and Ψ are real functions defined on $[\varrho_1, \varrho_2]$ and if $|\Upsilon|^p$ and $|\Psi|^q$ are integrable on $[\varrho_1, \varrho_2]$, then the following inequality holds true:*

$$ \int_{\varrho_1}^{\varrho_2} |\Upsilon(\mathrm{x})\Psi(\mathrm{x})| d\mathrm{x} \leqq \left(\int_{\varrho_1}^{\varrho_2} |\Upsilon(\mathrm{x})|^p d\mathrm{x}\right)^{\frac{1}{p}} \left(\int_{\varrho_1}^{\varrho_2} |\Psi(\mathrm{x})|^q d\mathrm{x}\right)^{\frac{1}{q}}. $$

Definition 2.9. *(Power-mean inequality [46]) Let $q \geqq 1$. If Υ and Ψ are real functions defined on $[\varrho_1, \varrho_2]$ and if $|\Upsilon|$, $|\Upsilon||\Psi|^q$ are integrable on $[\varrho_1, \varrho_2]$, then the following inequality holds true:*

$$ \int_{\varrho_1}^{\varrho_2} |\Upsilon(\mathrm{x})\Psi(\mathrm{x})| d\mathrm{x} \leqq \left(\int_{\varrho_1}^{\varrho_2} |\Upsilon(\mathrm{x})| d\mathrm{x}\right)^{1-\frac{1}{q}} \left(\int_{\varrho_1}^{\varrho_2} |\Upsilon(\mathrm{x})||\Psi(\mathrm{x})|^q d\mathrm{x}\right)^{\frac{1}{q}}. $$

Definition 2.10. *(Hölder-İşcan integral inequality [46]) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If Υ and Ψ are real functions defined on $[\varrho_1, \varrho_2]$ and if $|\Upsilon|^p$ and $|\Psi|^q$ are integrable on $[\varrho_1, \varrho_2]$, then the following inequality holds true:*

$$ \begin{aligned} \int_{\varrho_1}^{\varrho_2} |\Upsilon(\mathrm{x})\Psi(\mathrm{x})| d\mathrm{x} \leqq & \frac{1}{\varrho_2 - \varrho_1} \left(\int_{\varrho_1}^{\varrho_2} (\varrho_2 - \mathrm{x})|\Upsilon(\mathrm{x})|^p d\mathrm{x}\right)^{\frac{1}{p}} \left(\int_{\varrho_1}^{\varrho_2} (\varrho_2 - \mathrm{x})|\Psi(\mathrm{x})|^q d\mathrm{x}\right)^{\frac{1}{q}} \\ & + \frac{1}{\varrho_2 - \varrho_1} \left(\int_{\varrho_1}^{\varrho_2} (\mathrm{x} - \varrho_1)|\Upsilon(\mathrm{x})|^p d\mathrm{x}\right)^{\frac{1}{p}} \left(\int_{\varrho_1}^{\varrho_2} (\mathrm{x} - \varrho_1)|\Psi(\mathrm{x})|^q d\mathrm{x}\right)^{\frac{1}{q}}. \end{aligned} $$

3. H-H type inequalities of $\left(\frac{\varrho_1+\varrho_2}{2}\right)$ type

The main objective of this section is to establish several novel refinements of H-H type inequalities via ABK fractional operator.

Theorem 3.1. *Let $\Upsilon : X = [\varrho_1^\rho, \varrho_2^\rho] \longrightarrow \mathcal{R}$ be a function with $\varrho_2 > \varrho_1 \geqq 0$, $\rho > 0$ and $\Upsilon \in X_c^\rho(\varrho_1^\rho, \varrho_2^\rho)$. If Υ is a convex function on X, then the following ABK fractional integral inequalities hold true:*

$$\frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho)+\Upsilon(\varrho_2^\rho)\right]+\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)$$
$$\leqq\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho)+{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right]$$
$$\leqq\frac{\left[\Upsilon(\varrho_1^\rho+\Upsilon(\varrho_2^\rho))\right]}{2^\mu}\left[\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu\rho^{1-\mu}+2^\mu\Gamma(\mu)(1-\mu)}{\mathcal{B}(\mu)\Gamma(\mu)}\right],$$

where $\mu \in (0,1)$.

Proof. From the convexity theory, we have

$$\Upsilon\left(\frac{\mathrm{x}^\rho+\mathrm{y}^\rho}{2}\right)\leqq\frac{\Upsilon(\mathrm{x}^\rho)+\Upsilon(\mathrm{y}^\rho)}{2}.$$

Choosing

$$\mathrm{x}^\rho=\frac{\mathrm{u}^\rho}{2}\varrho_1^\rho+\frac{2-\mathrm{u}^\rho}{2}\varrho_2^\rho$$

and

$$\mathrm{y}^\rho=\frac{\mathrm{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathrm{u}^\rho}{2}\varrho_1^\rho.$$

Consequently, we get

$$2\Upsilon\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)\leqq\Upsilon\left(\frac{\mathrm{u}^\rho}{2}\varrho_1^\rho+\frac{2-\mathrm{u}^\rho}{2}\varrho_2^\rho\right)+\Upsilon\left(\frac{\mathrm{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathrm{u}^\rho}{2}\varrho_1^\rho\right).\tag{3.1}$$

Multiplying both sides of (3.1) by $\frac{\mu}{\mathcal{B}(\mu)\Gamma(\mu)}\mathrm{u}^{\rho\mu-1}$ $(\mu>0)$ and integrating over the closed inteval [0,1], we obtain

$$\frac{2}{\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)\leqq\frac{\mu2^\mu}{\mathcal{B}(\mu)\Gamma(\mu)}\int_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}}}^{\varrho_2}\left(\frac{\varrho_2^\rho-\mathrm{x}^\rho}{\varrho_2^\rho-\varrho_1^\rho}\right)^{\mu-1}\Upsilon(\mathrm{x}^\rho)\frac{\mathrm{x}^{\rho-1}}{\varrho_2^\rho-\varrho_1^\rho}\mathrm{dx}$$
$$+\frac{\mu2^\mu}{\mathcal{B}(\mu)\Gamma(\mu)}\int_{\varrho_1}^{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}}}\left(\frac{\mathrm{x}^\rho-\varrho_1^\rho}{\varrho_2^\rho-\varrho_1^\rho}\right)^{\mu-1}\Upsilon(\mathrm{x}^\rho)\frac{\mathrm{x}^{\rho-1}}{\varrho_2^\rho-\varrho_1^\rho}\mathrm{dx}.$$

Now, by using Definition 2.7 of ABK Fractional operator, we get

$$\frac{2\rho^{1-\mu}(\varrho_2^\rho-\varrho_1^\rho)^\mu}{2^\mu\rho\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)\leqq\frac{\mu\rho^{1-\mu}}{\mathcal{B}(\mu)\Gamma(\mu)}\int_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}}}^{\varrho_2}\left(\varrho_2^\rho-\mathrm{x}^\rho\right)^{\mu-1}\Upsilon(\mathrm{x}^\rho)\mathrm{x}^{\rho-1}\mathrm{dx}$$
$$+\frac{\mu\rho^{1-\mu}}{\mathcal{B}(\mu)\Gamma(\mu)}\int_{\varrho_1}^{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}}}\left(\mathrm{x}^\rho-\varrho_1^\rho\right)^{\mu-1}\Upsilon(\mathrm{x}^\rho)\mathrm{x}^{\rho-1}\mathrm{dx}.$$

This proves the first inequality asserted by Theorem 3.1.

$$\frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho)+\Upsilon(\varrho_2^\rho)\right]+\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)$$
$$\leqq\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho)+{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right].$$

For the proof of the second inequality in Theorem 3.1, we need the following results:

$$\Upsilon\left(\frac{\mathrm{u}^\rho}{2}\varrho_1^\rho+\frac{2-\mathrm{u}^\rho}{2}\varrho_2^\rho\right)+\Upsilon\left(\frac{\mathrm{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathrm{u}^\rho}{2}\varrho_1^\rho\right)$$
$$\leqq\frac{\mathrm{u}^\rho}{2}\Upsilon(\varrho_1^\rho)+\frac{2-\mathrm{u}^\rho}{2}\Upsilon(\varrho_2^\rho)+\frac{\mathrm{u}^\rho}{2}\Upsilon(\varrho_2^\rho)+\frac{2-\mathrm{u}^\rho}{2}\Upsilon(\varrho_1^\rho)$$
$$=\Upsilon(\varrho_1^\rho)+\Upsilon(\varrho_2^\rho). \tag{3.2}$$

Multiplying both sides of (3.2) by $\frac{\mu}{\mathcal{B}(\mu)\Gamma(\mu)}\mathrm{u}^{\rho\mu-1}$ $(\mu>0)$ and integrating over the closed inteval [0,1], we obtain

$$\frac{\mu}{\mathcal{B}(\mu)\Gamma(\mu)}\int_0^1\Upsilon\left(\frac{\mathrm{u}^\rho}{2}\varrho_1^\rho+\frac{2-\mathrm{u}^\rho}{2}\varrho_2^\rho\right)\mathrm{u}^{\rho\mu-1}d\mathrm{u}+\frac{\mu}{\mathcal{B}(\mu)\Gamma(\mu)}\int_0^1\Upsilon\left(\frac{\mathrm{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathrm{u}^\rho}{2}\varrho_1^\rho\right)\mathrm{u}^{\rho\mu-1}d\mathrm{u}$$
$$\leqq\frac{\mu\left[\Upsilon(\varrho_1^\rho)+\Upsilon(\varrho_2^\rho)\right]}{\mathcal{B}(\mu)\Gamma(\mu)}\int_0^1\mathrm{u}^{\rho\mu-1}d\mathrm{u}.$$

It follows

$$\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho)+{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right]\leqq\frac{\Upsilon(\varrho_1^\rho)+\Upsilon(\varrho_2^\rho)}{2^\mu}\left[\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu\rho^{1-\mu}+2^\mu\Gamma(\mu)(1-\mu)}{\mathcal{B}(\mu)\Gamma(\mu)}\right].$$

This leads us to the proof of the Theorem 3.1. □

Corollary 3.1. *When we choose* $\rho=1$, *Theorem* 3.1 *yields the following result for Atangana-Baleanu fractional operator.*

$$\frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1)+\Upsilon(\varrho_2)\right]+\frac{(\varrho_2-\varrho_1)^\mu}{2^{\mu-1}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1+\varrho_2}{2}\right)$$
$$\leqq\left[\mathrm{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\mathrm{I}^\mu_{\varrho_2}\Upsilon(\varrho_2)+\mathrm{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\mathrm{I}^\mu_{\varrho_1}\Upsilon(\varrho_1)\right]$$
$$\leqq\frac{\left[\Upsilon(\varrho_1+\Upsilon(\varrho_2))\right]}{2^\mu}\left[\frac{(\varrho_2-\varrho_1)^\mu+2^\mu\Gamma(\mu)(1-\mu)}{\mathcal{B}(\mu)\Gamma(\mu)}\right].$$

Lemma 3.1. *Let* $\Upsilon:X=[\varrho_1^\rho,\varrho_2^\rho]\longrightarrow\mathcal{R}$ *be a differentiable function on X, with* $0\leqq\varrho_1<\varrho_2$, $\rho>0$ *and* $\mu\in(0,1)$. *Then the following ABK fractional integral equality holds true:*

$$\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho)+{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right]-\frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho)+\Upsilon(\varrho_2^\rho)\right]+\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)$$

$$= \left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)^{\mu+1} \frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)} \left[\int_0^1 \Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_1^\rho + \frac{2-\mathfrak{u}^\rho}{2}\varrho_2^\rho\right)\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} + \int_0^1 \Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho + \frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right].$$

Proof. The proof directly follows by using integration by parts technique

$$\int_0^1 \Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_1^\rho + \frac{2-\mathfrak{u}^\rho}{2}\varrho_2^\rho\right)\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} + \int_0^1 \Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho + \frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} = I_1 + I_2.$$

$$\begin{aligned} I_1 &= \int_0^1 \Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_1^\rho + \frac{2-\mathfrak{u}^\rho}{2}\varrho_2^\rho\right)\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} \\ &= \int_0^1 \mathfrak{u}^{\rho\mu}\left[\frac{d}{d\mathfrak{u}}\left(\Upsilon\left(\frac{\mathfrak{u}^\rho}{2}\varrho_1^\rho + \frac{2-\mathfrak{u}^\rho}{2}\varrho_2^\rho\right)\right)\frac{2}{\rho(\varrho_1^\rho - \varrho_2^\rho)}\right]d\mathfrak{u}. \end{aligned}$$

Integrating by parts

$$\mathfrak{u}^{\rho\mu}\Upsilon\left(\frac{\mathfrak{u}^\rho}{2}\varrho_1^\rho + \frac{2-\mathfrak{u}^\rho}{2}\varrho_2^\rho\right)\frac{2}{\rho(\varrho_1^\rho - \varrho_2^\rho)}\Bigg|_0^1 - \int_0^1 \mu\mathfrak{u}^{\rho\mu-1}\Upsilon\left(\frac{\mathfrak{u}^\rho}{2}\varrho_1^\rho + \frac{2-\mathfrak{u}^\rho}{2}\varrho_2^\rho\right)\frac{2}{(\varrho_1^\rho - \varrho_2^\rho)}d\mathfrak{u}$$

$$= -\frac{2}{\rho(\varrho_2^\rho - \varrho_1^\rho)}\Upsilon\left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right) + \frac{2^{\mu+1}\mu}{(\varrho_2^\rho - \varrho_1^\rho)^{\mu+1}}\int_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}}}^{\varrho_2}(\varrho_2^\rho - x^\rho)^{\mu-1}\Upsilon(x^\rho)x^{\rho-1}dx.$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^1 \Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho + \frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} \\ &= \frac{2}{\rho(\varrho_2^\rho - \varrho_1^\rho)}\Upsilon\left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right) - \frac{2^{\mu+1}\mu}{(\varrho_2^\rho - \varrho_1^\rho)^{\mu+1}}\int_{\varrho_1}^{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}}}(x^\rho - \varrho_1^\rho)^{\mu-1}\Upsilon(x^\rho)x^{\rho-1}dx. \end{aligned}$$

Multiplying both I_1 *and* I_2 by $\frac{\rho^{1-\mu}}{\mathcal{B}(\mu)\Gamma(\mu)}\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\mu+1}$ and then subtracting the resultant I_2 from I_1, we have the desired proof. □

Theorem 3.2. *Let* $\Upsilon : X = [\varrho_1^\rho, \varrho_2^\rho] \longrightarrow \mathcal{R}$ *be a differentiable function on X, with* $0 \leqq \varrho_1 < \varrho_2$, $\rho > 0$ *and* $\mu \in (0,1)$. *If* $|\Upsilon'|$ *is a convex function, then the following ABK fractional integral inequality holds true:*

$$\begin{aligned} &\left|\left[{}^{ABK}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho) + {}^{ABK}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho) + \Upsilon(\varrho_2^\rho)\right] + \frac{\left(\varrho_2^\rho - \varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)\right| \\ &\leqq \left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\frac{|\Upsilon'(\varrho_1^\rho)| + |\Upsilon'(\varrho_2^\rho)|}{\rho(\mu+1)}\right]. \end{aligned}$$

Proof. Taking the equality given in Lemma 3.1 into consideration and convexity of $|\Upsilon'|$, we have

$$\left|\left[{}^{ABK}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho) + {}^{ABK}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho) + \Upsilon(\varrho_2^\rho)\right] + \frac{\left(\varrho_2^\rho - \varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)\right|$$

$$\leqq \left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)^{\mu+1} \frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)} \left[\int_0^1 \left(|\Upsilon'(\varrho_1^\rho)|\frac{\mathfrak{u}^\rho}{2} + \frac{2-\mathfrak{u}^\rho}{2}|\Upsilon'(\varrho_2^\rho)|\right)\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} + \int_0^1 \left(|\Upsilon'(\varrho_2^\rho)|\frac{\mathfrak{u}^\rho}{2} + \frac{2-\mathfrak{u}^\rho}{2}|\Upsilon'(\varrho_1^\rho)|\right)\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right]$$

$$\leqq \left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)^{\mu+1} \frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)} \left[|\Upsilon'(\varrho_1^\rho)| + |\Upsilon'(\varrho_2^\rho)\int_0^1 \mathfrak{u}^{\rho\mu+\rho-g}d\mathfrak{u}\right]$$

$$= \left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)^{\mu+1} \frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)} \left[\frac{|\Upsilon'(\varrho_1^\rho)| + |\Upsilon'(\varrho_2^\rho)}{\rho(\mu+1)}\right].$$

This completes the proof. □

Corollary 3.2. *Using the same notations as in Theorem* 3.2 *and choosing* $|\Upsilon'| \leqq \mathbb{M}$*, we have the following inequality:*

$$\left|\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho) + {}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho) + \Upsilon(\varrho_2^\rho)\right] + \frac{\left(\varrho_2^\rho - \varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)\right|$$
$$\leqq \left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)^{\mu+1} \frac{\mathbb{M}\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}.$$

Corollary 3.3. *Using the same notations as in Theorem* 3.2 *and choosing* $\rho = 1$*, we have the following inequality involving Atangana-Baleanu fractional operators:*

$$\left|\left[\mathrm{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\mathrm{I}^{\mu}_{\varrho_2}\Upsilon(\varrho_2) + \mathrm{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\mathrm{I}^{\mu}_{\varrho_1}\Upsilon(\varrho_1)\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1) + \Upsilon(\varrho_2)\right] + \frac{(\varrho_2 - \varrho_1)^\mu}{2^{\mu-1}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1 + \varrho_2}{2}\right)\right|$$
$$\leqq \left(\frac{\varrho_1 + \varrho_2}{2}\right)^{\mu+1} \frac{1}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\frac{|\Upsilon'(\varrho_1)| + |\Upsilon'(\varrho_2)}{(\mu+1)}\right].$$

Theorem 3.3. *Let* $\Upsilon : X = [\varrho_1^\rho, \varrho_2^\rho] \longrightarrow \mathcal{R}$ *be a differentiable function on X, with* $0 \leqq \varrho_1 < \varrho_2$*,* $\rho > 0$ *and* $\mu \in (0,1)$*. If* $|\Upsilon'|^q$ *is a convex function, then the following* ABK *fractional integral inequality holds true:*

$$\left|\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho) + {}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho) + \Upsilon(\varrho_2^\rho)\right] + \frac{\left(\varrho_2^\rho - \varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)\right|$$
$$\leqq \left(\frac{\varrho_1^\rho + \varrho_2^\rho}{2}\right)^{\mu+1} \frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\left(\frac{1}{\rho(\mu+1)}\right)^{1-\frac{1}{q}}\left\{\frac{1}{2\rho(\mu+2)}|\Upsilon'(\varrho_1^\rho)|^q + \frac{\mu+3}{2\rho(\mu+1)(\mu+2)}|\Upsilon'(\varrho_2^\rho)|^q\right\}^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{1}{\rho(\mu+1)}\right)^{1-\frac{1}{q}}\left\{\frac{1}{2\rho(\mu+2)}|\Upsilon'(\varrho_2^\rho)|^q + \frac{\mu+3}{2\rho(\mu+1)(\mu+2)}|\Upsilon'(\varrho_1^\rho)|^q\right\}^{\frac{1}{q}}\right],$$

where $q \geqq 1$.

Proof. Taking the equality given in Lemma 3.1 into consideration and power-mean inequality, we have

$$\begin{aligned}
&\left|\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho)+{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right]-\frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho)+\Upsilon(\varrho_2^\rho)\right]+\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)\right|\\
&\leqq\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\left(\int_0^1 \mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_1^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_2^\rho\right)\right|^q\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 \mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\right|^q\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)^{\frac{1}{q}}\right]. \qquad (3.3)
\end{aligned}$$

Using the convexity of $|\Upsilon'|^q$ on X, we have

$$\begin{aligned}
&\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\right|^q\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\\
&=\int_0^1\left(\frac{\mathfrak{u}^\rho}{2}|\Upsilon'(\varrho_1^\rho)|+\frac{2-\mathfrak{u}^\rho}{2}|\Upsilon'(\varrho_2^\rho)|\right)\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\\
&=\frac{1}{2\rho(\mu+2)}|\Upsilon'(\varrho_1^\rho)|^q+\frac{\mu+3}{2\rho(\mu+1)(\mu+2)}|\Upsilon'(\varrho_2^\rho)|^q,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\right|^q\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\\
&=\int_0^1\left(\frac{\mathfrak{u}^\rho}{2}|\Upsilon'(\varrho_2^\rho)|+\frac{2-\mathfrak{u}^\rho}{2}|\Upsilon'(\varrho_1^\rho)|\right)\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\\
&=\frac{1}{2\rho(\mu+2)}|\Upsilon'(\varrho_2^\rho)|^q+\frac{\mu+3}{2\rho(\mu+1)(\mu+2)}|\Upsilon'(\varrho_1^\rho)|^q,
\end{aligned}$$

and also

$$\int_0^1 \mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}=\frac{1}{\rho(\mu+1)}.$$

Substitution of the above computations in (3.3), completes rest of the proof. □

Corollary 3.4. *Using the same notations as the above Theorem* 3.3 *and if we take* $|\Upsilon'|\leqq\mathbb{M}$, *then we get:*

$$\begin{aligned}
&\left|\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho)+{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right]-\frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho)+\Upsilon(\varrho_2^\rho)\right]+\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)\right|\\
&\leqq\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\mu+1}\frac{2\mathbb{M}\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left(\frac{1}{\rho(\mu+1)}\right)^{1-\frac{1}{q}}.
\end{aligned}$$

Corollary 3.5. *Using the same notations as in Theorem* 3.3 *and choosing* $\rho = 1$*, we have the following inequality involving Atangana-Baleanu fractional operators:*

$$\left|\left[\mathtt{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\mathrm{I}^{\mu}_{\varrho_2}\Upsilon(\varrho_2) + \mathtt{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\mathrm{I}^{\mu}_{\varrho_1}\Upsilon(\varrho_1)\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1)+\Upsilon(\varrho_2)\right] + \frac{(\varrho_2-\varrho_1)^{\mu}}{2^{\mu-1}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leqq \left(\frac{\varrho_1+\varrho_2}{2}\right)^{\mu+1}\frac{1}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\left(\frac{1}{(\mu+1)}\right)^{1-\frac{1}{q}}\left\{\frac{1}{2(\mu+2)}|\Upsilon'(\varrho_1)|^q + \frac{\mu+3}{2(\mu+1)(\mu+2)}|\Upsilon'(\varrho_2)|^q\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{(\mu+1)}\right)^{1-\frac{1}{q}}\left\{\frac{1}{2(\mu+2)}|\Upsilon'(\varrho_2)|^q + \frac{\mu+3}{2(\mu+1)(\mu+2)}|\Upsilon'(\varrho_1)|^q\right\}^{\frac{1}{q}}\right].$$

Theorem 3.4. *Let* $\Upsilon : X = [\varrho_1^{\rho}, \varrho_2^{\rho}] \longrightarrow \mathcal{R}$ *be a differentiable function on* X *with* $0 \leqq \varrho_1 < \varrho_2$, $\rho > 0$ *and* $\mu \in (0,1)$*. If* $|\Upsilon'|^n$ *is a convex function, then the following ABK fractional integral inequality holds true:*

$$\left|\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^{\rho}}\Upsilon(\varrho_2^{\rho}) + {}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^{\rho}}\Upsilon(\varrho_1^{\rho})\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^{\rho})+\Upsilon(\varrho_2^{\rho})\right] + \frac{\left(\varrho_2^{\rho}-\varrho_1^{\rho}\right)^{\mu}}{2^{\mu-1}\rho^{\mu}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)\right|$$
$$\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\frac{2}{m\rho(m\mu+1)} + \frac{|\Upsilon'(\varrho_1^{\rho})|^n + |\Upsilon'(\varrho_2^{\rho})|^n}{n\rho}\right],$$

where $\frac{1}{m}+\frac{1}{n} = 1$.

Proof. Taking the equality given in Lemma 3.1 into consideration, Young's inequality and convexity of $|\Upsilon'|^n$, we have

$$\left|\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^{\rho}}\Upsilon(\varrho_2^{\rho}) + {}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^{\rho}}\Upsilon(\varrho_1^{\rho})\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^{\rho})+\Upsilon(\varrho_2^{\rho})\right] + \frac{\left(\varrho_2^{\rho}-\varrho_1^{\rho}\right)^{\mu}}{2^{\mu-1}\rho^{\mu}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)\right|$$
$$\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^{\rho}}{2}\varrho_1^{\rho}+\frac{2-\mathfrak{u}^{\rho}}{2}\varrho_2^{\rho}\right)\right|\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} + \int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^{\rho}}{2}\varrho_2^{\rho}+\frac{2-\mathfrak{u}^{\rho}}{2}\varrho_1^{\rho}\right)\right|\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right].$$
$$\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\frac{1}{m}\int_0^1\mathfrak{u}^{m\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} + \frac{1}{n}\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^{\rho}}{2}\varrho_1^{\rho}+\frac{2-\mathfrak{u}^{\rho}}{2}\varrho_2^{\rho}\right)\right|^n\mathfrak{u}^{\rho-1}d\mathfrak{u}\right.$$
$$\left.+\frac{1}{m}\int_0^1\mathfrak{u}^{m\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} + \frac{1}{n}\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^{\rho}}{2}\varrho_2^{\rho}+\frac{2-\mathfrak{u}^{\rho}}{2}\varrho_1^{\rho}\right)\right|^n\mathfrak{u}^{\rho-1}d\mathfrak{u}\right].$$
$$\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\frac{1}{m}\int_0^1\mathfrak{u}^{m\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} + \int_0^1\left(|\Upsilon'(\varrho_1^{\rho})|^n\frac{\mathfrak{u}^{\rho}}{2}+\frac{2-\mathfrak{u}^{\rho}}{2}|\Upsilon'(\varrho_2^{\rho})|^n\right)\mathfrak{u}^{\rho-1}d\mathfrak{u}\right.$$
$$\left.\frac{1}{m}\int_0^1\mathfrak{u}^{m\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} + \int_0^1\left(|\Upsilon'(\varrho_2^{\rho})|^n\frac{\mathfrak{u}^{\rho}}{2}+\frac{2-\mathfrak{u}^{\rho}}{2}|\Upsilon'(\varrho_1^{\rho})|^n\right)\mathfrak{u}^{\rho-1}d\mathfrak{u}\right].$$
$$\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\frac{2}{m\rho(m\mu+1)} + \frac{|\Upsilon'(\varrho_1^{\rho})|^n + |\Upsilon'(\varrho_2^{\rho})|^n}{n\rho}\right],$$

where

$$\frac{1}{m}\int_0^1\mathfrak{u}^{m\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} = \frac{1}{m}\left[\frac{1}{\rho(m\mu+1)}\right],$$

$$\frac{1}{n}\int_0^1 \frac{\mathfrak{u}^{\rho}\mathfrak{u}^{\rho-1}}{2}d\mathfrak{u} = \frac{1}{4n\rho},$$

and

$$\frac{1}{n}\int_0^1 \frac{(2-\mathfrak{u}^{\rho})\mathfrak{u}^{\rho-1}}{2}d\mathfrak{u} = \frac{3}{4n\rho}.$$

This completes the proof. □

Corollary 3.6. *Using the same notations as in Theorem* 3.4 *and choosing* $\rho = 1$*, we have the following inequality involving Atangana-Baleanu fractional operators:*

$$\begin{aligned}&\left|\left[\mathtt{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\mathrm{I}^{\mu}_{\varrho_2}\Upsilon(\varrho_2) + \mathtt{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\mathrm{I}^{\mu}_{\varrho_1}\Upsilon(\varrho_1)\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1)+\Upsilon(\varrho_2)\right] + \frac{(\varrho_2-\varrho_1)^{\mu}}{2^{\mu-1}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1+\varrho_2}{2}\right)\right| \\ &\leqq \left(\frac{\varrho_1+\varrho_2}{2}\right)^{\mu+1}\frac{1}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\frac{2}{m(m\mu+1)} + \frac{|\Upsilon'(\varrho_1)|^n + |\Upsilon'(\varrho_2)|^n}{n}\right].\end{aligned}$$

Theorem 3.5. *Let* $\Upsilon : X = [\varrho_1^{\rho}, \varrho_2^{\rho}] \longrightarrow \mathcal{R}$ *be a differentiable function on X with* $0 \leqq \varrho_1 < \varrho_2$*,* $\rho > 0$ *and* $\mu \in (0,1)$*. If* $|\Upsilon'|$ *is a concave function on X, then the following ABK fractional integral inequality holds true:*

$$\begin{aligned}&\left|\left[{}^{\mathtt{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^{\rho}}\Upsilon(\varrho_2^{\rho}) + {}^{\mathtt{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^{\rho}}\Upsilon(\varrho_1^{\rho})\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^{\rho})+\Upsilon(\varrho_2^{\rho})\right] + \frac{\left(\varrho_2^{\rho}-\varrho_1^{\rho}\right)^{\mu}}{2^{\mu-1}\rho^{\mu}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)\right| \\ &\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left(\frac{1}{\rho(\mu+1)}\right)\left[\left|\Upsilon'\left(\frac{(\mu+1)\varrho_2^{\rho}+(\mu+3)\varrho_1^{\rho}}{2(\mu+2)}\right)\right| + \left|\Upsilon'\left(\frac{(\mu+1)\varrho_1^{\rho}+(\mu+3)\varrho_2^{\rho}}{2(\mu+2)}\right)\right|\right].\end{aligned}$$

Proof. Taking the equality given in Lemma 3.1 into consideration and Jensen's inequality, we have

$$\begin{aligned}&\left|\left[{}^{\mathtt{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^{\rho}}\Upsilon(\varrho_2^{\rho}) + {}^{\mathtt{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^{\rho}}\Upsilon(\varrho_1^{\rho})\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^{\rho})+\Upsilon(\varrho_2^{\rho})\right] + \frac{\left(\varrho_2^{\rho}-\varrho_1^{\rho}\right)^{\mu}}{2^{\mu-1}\rho^{\mu}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)\right| \\ &\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^{\rho}}{2}\varrho_1^{\rho}+\frac{2-\mathfrak{u}^{\rho}}{2}\varrho_2^{\rho}\right)\right|\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u} + \int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^{\rho}}{2}\varrho_2^{\rho}+\frac{2-\mathfrak{u}^{\rho}}{2}\varrho_1^{\rho}\right)\right|\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right]. \\ &\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\left(\int_0^1\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)\left|\Upsilon'\left(\frac{\int_0^1\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}\left(\frac{\mathfrak{u}^{\rho}}{2}\varrho_1^{\rho}+\frac{2-\mathfrak{u}^{\rho}}{2}\varrho_2^{\rho}\right)d\mathfrak{u}}{\int_0^1\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}}\right)\right|\right. \\ &\qquad\left. + \left(\int_0^1\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)\left|\Upsilon'\left(\frac{\int_0^1\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}\left(\frac{\mathfrak{u}^{\rho}}{2}\varrho_2^{\rho}+\frac{2-\mathfrak{u}^{\rho}}{2}\varrho_1^{\rho}\right)d\mathfrak{u}}{\int_0^1\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}}\right)\right|\right] \\ &\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\frac{1}{\rho(\mu+1)}\left|\Upsilon'\left(\frac{\frac{\varrho_1^{\rho}}{2\rho(\mu+2)}+\frac{(\mu+3)\varrho_2^{\rho}}{2\rho(\mu+1)(\mu+2)}}{\frac{1}{\rho(\mu+1)}}\right)\right| + \frac{1}{\rho(\mu+1)}\left|\Upsilon'\left(\frac{\frac{\varrho_2^{\rho}}{2\rho(\mu+2)}+\frac{(\mu+3)\varrho_1^{\rho}}{2\rho(\mu+1)(\mu+2)}}{\frac{1}{\rho(\mu+1)}}\right)\right|\right] \\ &\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left(\frac{1}{\rho(\mu+1)}\right)\left[\left|\Upsilon'\left(\frac{(\mu+1)\varrho_2^{\rho}+(\mu+3)\varrho_1^{\rho}}{2(\mu+2)}\right)\right| + \left|\Upsilon'\left(\frac{(\mu+1)\varrho_1^{\rho}+(\mu+3)\varrho_2^{\rho}}{2(\mu+2)}\right)\right|\right].\end{aligned}$$

This led us to the proof of the desired Theorem 3.5. □

Corollary 3.7. *Using the same notations as in Theorem 3.5 and choosing $\rho = 1$, we have the following inequality involving Atangana-Baleanu fractional operators:*

$$\left|\left[\mathtt{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\mathrm{I}^{\mu}_{\varrho_2}\Upsilon(\varrho_2) + \mathtt{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\mathrm{I}^{\mu}_{\varrho_1}\Upsilon(\varrho_1)\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1)+\Upsilon(\varrho_2)\right] + \frac{(\varrho_2-\varrho_1)^{\mu}}{2^{\mu-1}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$
$$\leqq \left(\frac{\varrho_1+\varrho_2}{2}\right)^{\mu+1}\frac{1}{\mathcal{B}(\mu)\Gamma(\mu)}\left(\frac{1}{(\mu+1)}\right)\left[\left|\Upsilon'\left(\frac{(\mu+1)\varrho_2+(\mu+3)\varrho_1}{2(\mu+2)}\right)\right| + \left|\Upsilon'\left(\frac{(\mu+1)\varrho_1+(\mu+3)\varrho_2}{2(\mu+2)}\right)\right|\right].$$

Theorem 3.6. *Let $\Upsilon : X = [\varrho_1^{\rho}, \varrho_2^{\rho}] \longrightarrow \mathcal{R}$ be a differentiable function on $[\varrho_1^{\rho}, \varrho_2^{\rho}]$ with $0 \leqq \varrho_1 < \varrho_2$, $\rho > 0$ and $\mu \in (0,1)$. If $|\Upsilon'|^q$ is a convex function on X, then the following inequality ABK fractional integral inequality holds true:*

$$\left|\left[{}^{\mathtt{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^{\rho}}\Upsilon(\varrho_2^{\rho}) + {}^{\mathtt{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^{\rho}}\Upsilon(\varrho_1^{\rho})\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^{\rho})+\Upsilon(\varrho_2^{\rho})\right] + \frac{\left(\varrho_2^{\rho}-\varrho_1^{\rho}\right)^{\mu}}{2^{\mu-1}\rho^{\mu}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)\right|$$
$$\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}$$
$$\times\left[\left(\frac{1}{\rho\mu s+\rho}-\frac{1}{\rho\mu s+\rho+1}\right)^{\frac{1}{s}}\left(\frac{|\Upsilon'(\varrho_1^{\rho})|^q}{2}\left[\frac{1}{2\rho(2\rho+1)}\right] + \frac{|\Upsilon'(\varrho_2^{\rho})|^q}{2}\left[\frac{3}{2\rho}+\frac{1}{2\rho+1}-\frac{2}{\rho+1}\right]\right)\right.$$
$$+\left(\frac{1}{\rho\mu s+\rho+1}\right)^{\frac{1}{s}}\left(\frac{|\Upsilon'(\varrho_1^{\rho})|^q}{2(2\rho+1)}+\frac{|\Upsilon'(\varrho_2^{\rho})|^q(3\rho+1)}{2(\rho+1)(2\rho+1)}\right)^{\frac{1}{q}} + \left(\frac{1}{\rho\mu s+\rho+1}\right)^{\frac{1}{s}}\left(\frac{|\Upsilon'(\varrho_2^{\rho})|^q}{2(2\rho+1)}+\frac{|\Upsilon'(\varrho_1^{\rho})|^q(3\rho+1)}{2(\rho+1)(2\rho+1)}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{\rho\mu s+\rho}-\frac{1}{\rho\mu s+\rho+1}\right)^{\frac{1}{s}}\left(\frac{|\Upsilon'(\varrho_2^{\rho})|^q}{2}\left[\frac{1}{2\rho(2\rho+1)}\right] + \frac{|\Upsilon'(\varrho_1^{\rho})|^q}{2}\left[\frac{3}{2\rho}+\frac{1}{2\rho+1}-\frac{2}{\rho+1}\right]\right)\right],$$

where $\frac{1}{s}+\frac{1}{q}=1$.

Proof. Taking the equality given in Lemma 3.1 into consideration and using Hölder-İşcan inequality

$$\left|\left[{}^{\mathtt{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^{\rho}}\Upsilon(\varrho_2^{\rho}) + {}^{\mathtt{ABK}}_{\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^{\rho}}\Upsilon(\varrho_1^{\rho})\right] - \frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^{\rho})+\Upsilon(\varrho_2^{\rho})\right] + \frac{\left(\varrho_2^{\rho}-\varrho_1^{\rho}\right)^{\mu}}{2^{\mu-1}\rho^{\mu}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)\right|$$
$$\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}$$
$$\times\left[\int_0^1\left|\Upsilon'\left(\frac{u^{\rho}}{2}\varrho_1^{\rho}+\frac{2-u^{\rho}}{2}\varrho_2^{\rho}\right)\right|u^{\rho\mu}u^{\rho-1}du + \int_0^1\left|\Upsilon'\left(\frac{u^{\rho}}{2}\varrho_2^{\rho}+\frac{2-u^{\rho}}{2}\varrho_1^{\rho}\right)\right|u^{\rho\mu}u^{\rho-1}du\right].$$
$$\leqq \left(\frac{\varrho_1^{\rho}+\varrho_2^{\rho}}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}$$
$$\times\left[\left(\int_0^1(1-u)u^{\rho\mu s}u^{\rho-1}du\right)^{\frac{1}{s}}\left(\int_0^1(1-u)u^{\rho-1}\left|\Upsilon'\left(\frac{u^{\rho}}{2}\varrho_1^{\rho}+\frac{2-u^{\rho}}{2}\varrho_2^{\rho}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1 u u^{\rho\mu s}u^{\rho-1}du\right)^{\frac{1}{s}}\left(\int_0^1 u u^{\rho-1}\left|\Upsilon'\left(\frac{u^{\rho}}{2}\varrho_1^{\rho}+\frac{2-u^{\rho}}{2}\varrho_2^{\rho}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 (1-\mathfrak{u})\mathfrak{u}^{\rho\mu s}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)^{\frac{1}{s}}\left(\int_0^1 (1-\mathfrak{u})\mathfrak{u}^{\rho-1}\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \mathfrak{u}\mathfrak{u}^{\rho\mu s}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)^{\frac{1}{s}}\left(\int_0^1 \mathfrak{u}\mathfrak{u}^{\rho-1}\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\right|^q\right)^{\frac{1}{q}}\Bigg]. \tag{3.4}$$

By the convexity of $|\Upsilon'|^q$, we have

$$\int_0^1 (\mathfrak{u})\mathfrak{u}^{\rho-1}\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_1^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_2^\rho\right)\right|^q d\mathfrak{u}=\frac{|\Upsilon'(\varrho_1^\rho)|^q}{2(2\rho+1)}+\frac{|\Upsilon'(\varrho_2^\rho)|^q}{2}\left[\frac{2}{\rho+1}-\frac{1}{2\rho+1}\right],$$

$$\int_0^1 (1-\mathfrak{u})\mathfrak{u}^{\rho-1}\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_1^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_2^\rho\right)\right|^q d\mathfrak{u}=\frac{|\Upsilon'(\varrho_1^\rho)|^q}{2}\left[\frac{1}{2\rho}-\frac{1}{2\rho+1}\right]+\frac{|\Upsilon'(\varrho_2^\rho)|^q}{2}\left[\frac{3}{2\rho}-\frac{2}{\rho+1}+\frac{1}{2\rho+1}\right],$$

$$\int_0^1 (\mathfrak{u})\mathfrak{u}^{\rho-1}\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\right|^q d\mathfrak{u}=\frac{|\Upsilon'(\varrho_2^\rho)|^q}{2(2\rho+1)}+\frac{|\Upsilon'(\varrho_1^\rho)|^q}{2}\left[\frac{2}{\rho+1}-\frac{1}{2\rho+1}\right],$$

$$\int_0^1 (1-\mathfrak{u})\mathfrak{u}^{\rho-1}\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\right|^q d\mathfrak{u}=\frac{|\Upsilon'(\varrho_2^\rho)|^q}{2}\left[\frac{1}{2\rho}-\frac{1}{2\rho+1}\right]+\frac{|\Upsilon'(\varrho_1^\rho)|^q}{2}\left[\frac{3}{2\rho}-\frac{2}{\rho+1}+\frac{1}{2\rho+1}\right].$$

Also,

$$\int_0^1 (1-\mathfrak{u})\mathfrak{u}^{\rho\mu s}\mathfrak{u}^{\rho-1}d\mathfrak{u}=\frac{1}{\rho\mu s+\rho}-\frac{1}{\rho\mu s+\rho+1},$$

$$\int_0^1 (\mathfrak{u})\mathfrak{u}^{\rho\mu s}\mathfrak{u}^{\rho-1}d\mathfrak{u}=\frac{1}{\rho\mu s+\rho+1}.$$

Upon, substituting the above computations in (3.4), we have the required result. □

Corollary 3.8. *Using the same notations as in Theorem* 3.6 *and choosing* $\rho=1$*, we have the following inequality involving Atangana-Baleanu fractional operators:*

$$\left|\left[\mathtt{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\mathrm{I}^\mu_{\varrho_2}\Upsilon(\varrho_2)+\mathtt{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\mathrm{I}^\mu_{\varrho_1}\Upsilon(\varrho_1)\right]-\frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1)+\Upsilon(\varrho_2)\right]+\frac{(\varrho_2-\varrho_1)^\mu}{2^{\mu-1}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|$$

$$\leqq\left(\frac{\varrho_1+\varrho_2}{2}\right)^{\mu+1}\frac{1}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\left(\frac{1}{\mu s+1}-\frac{1}{\mu s+2}\right)^{\frac{1}{s}}\left(\frac{|\Upsilon'(\varrho_1)|^q}{12}+\frac{5|\Upsilon'(\varrho_2)|^q}{12}\right)\right.$$

$$+\left(\frac{1}{\mu s+2}\right)^{\frac{1}{s}}\left(\frac{|\Upsilon'(\varrho_1)|^q}{6}+\frac{|\Upsilon'(\varrho_2)|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{1}{\mu s+2}\right)^{\frac{1}{s}}\left(\frac{|\Upsilon'(\varrho_2)|^q}{6}+\frac{7|\Upsilon'(\varrho_1)|^q}{3}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{1}{\mu s+1}-\frac{1}{\mu s+2}\right)^{\frac{1}{s}}\left(\frac{|\Upsilon'(\varrho_2)|^q}{12}+\frac{5|\Upsilon'(\varrho_1)|^q}{12}\right)\right].$$

Theorem 3.7. *Let* $\Upsilon : X=[\varrho_1^\rho,\varrho_2^\rho]\longrightarrow\mathcal{R}$ *be a differentiable function with* $0\leqq\varrho_1<\varrho_2$ $\rho>0$*,* $\mu\in(0,1)$*. If* $|\Upsilon'|^q$ *is a convex function on X, for* $q>1$ *and* $\frac{1}{s}+\frac{1}{q}=1$*, then the following ABK fractional integral inequality holds true:*

$$\left|\left[{}^{\mathtt{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^\rho\mathrm{I}^\mu_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho)+{}^{\mathtt{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^\rho\mathrm{I}^\mu_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right]-\frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho)+\Upsilon(\varrho_2^\rho)\right]+\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)\right|$$

$$\leqq \left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left(\frac{1}{s(\rho\mu)+\rho}\right)^{\frac{1}{s}}\left\{\left(\frac{|\Upsilon'(\varrho_1^\rho)|^q}{4\rho}+\frac{3|\Upsilon'(\varrho_2^\rho)|^q}{4\rho}\right)^{\frac{1}{q}}+\left(\frac{|\Upsilon'(\varrho_2^\rho)|^q}{4\rho}+\frac{3|\Upsilon'(\varrho_1^\rho)|^q}{4\rho}\right)^{\frac{1}{q}}\right\}.$$

Proof. Taking the equality given in Lemma 3.1 into consideration and Hölder inequality, we have

$$\begin{aligned}&\left|\left[{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}+}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\Upsilon(\varrho_2^\rho)+{}^{\mathrm{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}-}}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\Upsilon(\varrho_1^\rho)\right]-\frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1^\rho)+\Upsilon(\varrho_2^\rho)\right]+\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)\right|\\&\leqq\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\mu+1}\frac{\rho^{\mu-1}}{\mathcal{B}(\mu)\Gamma(\mu)}\left[\left(\int_0^1 \mathfrak{u}^{\rho\mu s}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)^{\frac{1}{s}}\left(\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_1^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_2^\rho\right)\right|^q\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_0^1 \mathfrak{u}^{\rho\mu s}\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)^{\frac{1}{s}}\left(\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\right|^q\mathfrak{u}^{\rho-1}d\mathfrak{u}\right)^{\frac{1}{q}}\right].\end{aligned}\tag{3.5}$$

Using the convexity of $|\Upsilon'|^q$ on $[\varrho_1^\rho,\varrho_2^\rho]$, we have

$$\begin{aligned}\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\right|^q\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}&=\int_0^1\left(\frac{\mathfrak{u}^\rho}{2}|\Upsilon'(\varrho_1^\rho)|+\frac{2-\mathfrak{u}^\rho}{2}|\Upsilon'(\varrho_2^\rho)|\right)\mathfrak{u}^{\rho-1}d\mathfrak{u}\\&=\frac{|\Upsilon'(\varrho_1^\rho)|^q}{4\rho}+\frac{3|\Upsilon'(\varrho_2^\rho)|^q}{4\rho},\end{aligned}$$

and

$$\begin{aligned}\int_0^1\left|\Upsilon'\left(\frac{\mathfrak{u}^\rho}{2}\varrho_2^\rho+\frac{2-\mathfrak{u}^\rho}{2}\varrho_1^\rho\right)\right|^q\mathfrak{u}^{\rho\mu}\mathfrak{u}^{\rho-1}d\mathfrak{u}&=\int_0^1\left(\frac{\mathfrak{u}^\rho}{2}|\Upsilon'(\varrho_2^\rho)|+\frac{2-\mathfrak{u}^\rho}{2}|\Upsilon'(\varrho_1^\rho)|\right)\mathfrak{u}^{\rho-1}d\mathfrak{u}\\&=\frac{|\Upsilon'(\varrho_2^\rho)|^q}{4\rho}+\frac{3|\Upsilon'(\varrho_1^\rho)|^q}{4\rho}.\end{aligned}$$

Substituting the above computations in (3.5), completes rest of the proof. □

Corollary 3.9. *Using the same notations as in Theorem* 3.7 *and choosing* $\rho = 1$*, we have the following inequality involving Atangana-Baleanu fractional operators:*

$$\begin{aligned}&\left|\left[\mathrm{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^+}\mathrm{I}^\mu_{\varrho_2}\Upsilon(\varrho_2)+\mathrm{AB}_{\left(\frac{\varrho_1+\varrho_2}{2}\right)^-}\mathrm{I}^\mu_{\varrho_1}\Upsilon(\varrho_1)\right]-\frac{1-\mu}{\mathcal{B}(\mu)}\left[\Upsilon(\varrho_1)+\Upsilon(\varrho_2)\right]+\frac{(\varrho_2-\varrho_1)^\mu}{2^{\mu-1}\mathcal{B}(\mu)\Gamma(\mu)}\Upsilon\left(\frac{\varrho_1+\varrho_2}{2}\right)\right|\\&\leqq\left(\frac{\varrho_1+\varrho_2}{2}\right)^{\mu+1}\frac{1}{\mathcal{B}(\mu)\Gamma(\mu)}\left(\frac{1}{s\mu}\right)^{\frac{1}{s}}\left\{\left(\frac{|\Upsilon'(\varrho_1)|^q}{4}+\frac{3|\Upsilon'(\varrho_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\Upsilon'(\varrho_2)|^q}{4}+\frac{3|\Upsilon'(\varrho_1)|^q}{4}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

4. Application

We represent $\mathbb{C}^n$ as the set of $n\times n$ complex matrices and $\mathbb{M}_n$ as the algebra of $n\times n$ complex matrices and $\mathbb{M}_n^+$ represents as the strictly positive matrices in $\mathbb{M}$. That is, $\mathrm{A}\in\mathbb{M}_n^+$, if $\langle \mathrm{A}u,u\rangle>0$ for all nonzero $u\in\mathbb{C}^n$.

Sababheh [47], proved that $\Upsilon(\mathfrak{u}) = \| \mathrm{A}^{\mathfrak{u}}\mathrm{X}\mathrm{B}^{1-\mathfrak{u}}+\mathrm{A}^{1-\mathfrak{u}}\mathrm{X}\mathrm{B}^{\mathfrak{u}}\|$, $\mathrm{A},\mathrm{B}\in\mathbb{M}_n^+,\mathrm{X}\in\mathbb{M}_n$ is convex for all $\mathfrak{u}\in[0,1]$. Then, by using Theorem 3.1, we have

$$
\begin{aligned}
&\frac{1-\mu}{\mathcal{B}(\mu)}\left[\left\|A^{\varrho_1^\rho}XB^{1-\varrho_1^\rho}+A^{1-\varrho_1^\rho}XB^{\varrho_1^\rho}\right\|+\left\|A^{\varrho_2^\rho}XB^{1-\varrho_2^\rho}+A^{1-\varrho_2^\rho}XB^{\varrho_2^\rho}\right\|\right]\\
&\quad+\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu}{2^{\mu-1}\rho^\mu\mathcal{B}(\mu)\Gamma(\mu)}\left\|A^{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)}XB^{1-\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)}+A^{1-\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)}XB^{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)}\right\|\\
&\leqq\frac{2^{\alpha-s}\Gamma(\alpha+1)}{(\varrho_2^\rho-\varrho_1^\rho)^\alpha}\left[{}^{\text{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}}+}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_2^\rho}\left\|A^{\varrho_2^\rho}XB^{1-\varrho_2^\rho}+A^{1-\varrho_2^\rho}XB^{\varrho_2^\rho}\right\|+{}^{\text{ABK}}_{\left(\frac{\varrho_1^\rho+\varrho_2^\rho}{2}\right)^{\frac{1}{\rho}}-}{}^{\rho}\mathrm{I}^{\mu}_{\varrho_1^\rho}\left\|A^{\varrho_1^\rho}XB^{1-\varrho_1^\rho}+A^{1-\varrho_1^\rho}XB^{\varrho_1^\rho}\right\|\right]\\
&\leqq\frac{\left\|A^{\varrho_1^\rho}XB^{1-\varrho_1^\rho}+A^{1-\varrho_1^\rho}XB^{\varrho_1^\rho}\right\|+\left\|A^{\varrho_2^\rho}XB^{1-\varrho_2^\rho}+A^{1-\varrho_2^\rho}XB^{\varrho_2^\rho}\right\|}{2^\mu}\left[\frac{\left(\varrho_2^\rho-\varrho_1^\rho\right)^\mu\rho^{1-\mu}+2^\mu\Gamma(\mu)(1-\mu)}{\mathcal{B}(\mu)\Gamma(\mu)}\right].
\end{aligned}
$$

5. Conclusions

Recently, fractional calculus has been one of the often utilized concepts to acquire new variants of some well-known integral inequalities. This improvement in the field of mathematical inequality dealing with fractional calculus has prompted another direction in different areas of mathematics and applied sciences. In this article, we have investigated ABK fractional integrals and, by using these fractional integrals, we have derived H-H type inequality (see Theorem 3.1) and then we presented an identity (see Lemma 3.1). Using the identity, some refinements of H-H type inequalities (see Theorems 3.2–3.7) are discussed for convex functions. If we choose $\rho = 1$, our presented results give some new inequalities of $\left(\frac{\varrho_1+\varrho_2}{2}\right)$ type for AB fractional operator. The ABK fractional operator being a new and unified operator, it will be quite interesting to check whether we can apply this integral operator to establish Ostrowski type inequality, Simpson type inequality. For future work, we will apply the ABK operator using Minkowski and Markov inequalities. Also, we will check its applicability on interval valued analysis and on coordinates for integral inequalities.

Conflict of interest

The authors declare no conflict of interest.

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