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Research article

A novel picture fuzzy Aczel-Alsina geometric aggregation information: Application to determining the factors affecting mango crops

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Abstract: Picture fuzzy (PF) sets are extremely reasonable to represent the uncertain, imprecise, and inconsistent information that exists in scientific and engineering fields. To meet decision makers' preference selection, the operational flexibility of aggregation operators shows its importance in dealing with the flexible decision-making problems in the PF environment. With assistance from Aczel-Alsina operations, we introduce the aggregation strategies of PFNs. We initially broaden the Aczel-Alsina norms to PF situations and present a few new operations of PFNs in view of which we build up a few new PF aggregation operators, for instance, the PF Aczel-Alsina weighted geometric, order weighted geometric, and hybrid weighted geometric operators. Furthermore, a decision support approach has been developed using the proposed aggregation operators under the PF environment. In this method, the aggregated results of each evaluated alternative are determined, and their score values are obtained. Then, all alternatives were ranked in decreasing order, and the best one was determined based on the highest score value. An illustrative example related to mango production is presented to investigate the most influential factor that resulted in mango production minimization. Finally, a comparison study was conducted on the proposed decision support method and the existing relative techniques. The result shows that the proposed method can overcome the insufficiency of lacking decision flexibility in the existing MAGDM method by the PF weighted geometric aggregation operators.

Keywords: Aczel-Alsina norms; picture fuzzy numbers; geometric operators; decision making
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1. Introduction

Multi-attribute decision making (MADM) is the process of analyzing, sorting, and selecting the best possibilities based on decision support data and a specific decision support model. Making a choice primarily involves the expert information and the selection of appropriate decision support techniques. Expertise and society's complexity have led to the need to expand and refine the MADM strategy in order to make better judgments in the decision support challenges. There is ambiguity and uncertainty in the decision-making process, and the theory of fuzzy sets by Zadeh [51] provides a highly effective way for dealing with these issues. Thereafter came the idea of intuitionistic fuzzy sets (IFS) [8], which employed positive and negative membership grades to represent uncertainty in decision support procedures. As time went on, decision-makers began to use intuitionistic fuzzy numbers to communicate their preferences for different options when faced with a decision-making challenge [36, 43, 46]. As a result, more and more researchers are becoming interested in intuitionistic fuzzy information.

We need to utilize some aggregation operators (AO) to accumulate the information obtained from experts. Xu [44] developed some AOs, such as the IF averaging operator. Wang and Liu [40] introduced some Einstein AOs, such as IFE averaging/geometric operators. Yu and Xu [48] established the list of prioritized AOs and discussed their applicability in decision making problems (DMP). Liu and Wang [28] developed some novel AOs under linguistic IF approach and generate an algorithm to address the complex uncertain DMP. Xu and Yager [45] introduced the decision-making approach (DMA) under IF Bonferroni means AO. Arora and Garg [3] established the Group DMA based on the prioritized IF AO under linguistic data set. Zhao et al. [52] developed the generalized IF AOs, such as the generalized IF averaging/geometric operators and explored their applications to tackle the uncertainty in DMP. Yu [49] presented the some confidence level based IF AOs and solved complex real life DMPs. Yu [50] developed the IF AO using Heronian mean and discussed the applicability in decision making. Jiang et al. [19] established the decision support approach based on IF power AO and entropy measure. Senapati et al. [35] introduced Aczel-Alsina norm based some IF Aczel-Alsina AOs and utilized them in the IF multi-attribute decision support process. Khan et al. [23] developed the novel generalized IF soft information based AOs and explored their applicability in decision-making.

Even though all these approaches are useful for describing incomplete information, they cannot handle indeterminate (neutral) information and inconsistent information in engineering practice. Cuong [10] established the picture fuzzy set (PFS) portrayed by the grades of positive membership, neutral membership and negative membership and the sum of such membership grades should not exceed one. Clearly, utilizing PFS to explain the dubious data tend to be more reasonable and exact than FSs and IFSs. After the invention of PFS, a huge number of researchers started working on PFS.

The aggregation of information is fundamental for obtaining the synthesis of the performance degree of criteria. Various aggregation operators [6, 7, 20, 21] have been developed by far, such as Ashraf et al. [4] invented the list of novel picture fuzzy (PF) algebraic AO and decision support approach to address the complex uncertain data in DMP. Garg [11] established the some AOs, such as PF geometric operators. Wei [41] developed the list of PF AOs and discussed their applicability in decision support problems. Khan et al. [24] developed the novel generalized PF soft information based AOs and explored their applicability in decision-making. Khan et al. [25] introduced some Einstein AOs, such as PF Einstein averaging/geometric operators. Qiyas et al. [34] developed the some PF averaging/geometric AO under algebraic norm and linguistic data set. Jana et al. [18] invented some Dombi AOs, such as PF Dombi averaging/ geometric operators under PF settings. Wei et al. [42] presented some PF AOs, such as PF Hamacher averaging/ geometric operators using Hamacher t-norm and s-norm. Ashraf et al. [5] invented the new distance measure based algebraic AOs under cubic PF environment. Khan et al. [26] developed the some logarithmic PF AOs and discussed their application in decision-making. Although these operators provide some inspirations for solving the MADM problems, the flexible decision-making corresponding to favorite priorities of alternatives were not considered comprehensively in the MADM process [38, 39].

It is widely accepted that the t-norms [31] and their associated t-conorms (e.g. Algebraic t-norm and t-conorm, Einstein t-norm and t-conorm, Hamacher t-norm and t-conorm) are crucial operations in fuzzy sets and other fuzzy systems. Aczel and Alsina (1982) presented new operations referred to as Aczel-Alsina t-norm and Aczel-Alsina t-conorm, which have the advantage of changeability by adjusting a parameter [1]. This study aims to propose the Aczel-Alsina t-norm and t-conorm operations and list of new aggregation operators under the picture fuzzy environment and to create a MAGDM approach using these operators for solving favorite priority of alternatives in multi-attribute DMP. An illustrative example related to Mango production is presented to investigate the most effecting factor that cause the minimization of Mango production. Base on the results, we can help the governments to take stance for better and much batter production of Mango crop. The comparison shows the proposed method has its advantage in flexible decision-making corresponding to favorite priorities of alternatives. The commitments of our technique are expressed in the following ways:

(1) We built up a some Aczel-Alsina operations for PFNs, that may triumph over the deficiency of algebraic, Einstein and Hamacher operations and capture the connection among diverse PFNs.

(2) We prolonged Aczel-Alsina operators to PF Aczel-Alsina operators: PF Aczel-Alsina weighted geometric (PFAWG) operator, PF Aczel-Alsina order weighted geometric (PFAOWG) operator, PF Aczel-Alsina hybrid weighted geometric (PFAHWG) operator in support of PF data, which can conquer the existing operator's disadvantages.

(3) We built up an algorithm to handle MAGDM issues utilizing PF data.

(4) To exhibit the adequacy and unwavering quality of the suggested PF Aczel-Alsina aggregation operators, we carried out the suggested operator to a MAGDM issue.

(5) The outcomes demonstrate that the suggested procedure is progressively powerful and gives an even more authentic output in comparison to current strategies.

The remaining portion of the paper is sorted out in the prescribed sequence: Some fundamental information associated with t-norms, t-conorms, Aczel-Alsina t-norms, PFSs and several working rules in terms of PFNs are characterized in Section 2. The Aczel-Alsina working rules and the features of PFNs are discussed in Section 3. In Section 4, we interpret some PF Aczel-Alsina aggregation operators and look at several of their desirable properties. In the next section, we tackle the MAGDM issue, utilizing PF Aczel-Alsina aggregation operators. In Section 6, we provide an illustrative example related to Mango production is presented to investigate the most effecting factor that cause the minimization of Mango production. In Section 7, we look at how a parameter affects decision-making outcomes. Section 8 presents a comparative evaluation of the considered aggregation operators with the prevailing aggregation operators. Section 9 concludes the paper and elaborates on future studies.

2. Preliminaries

In this section, we will look at some key concepts that will be important in the development of this paper.

2.1. Aczel-Alsina norm

Definition 1. *For any* $y, h, k \in [0,1]$, *a mapping* $\mathcal{W} : [0,1] \times [0,1] \to [0,1]$ *is a t-norm, if it fulfilled*
(1) $\mathcal{W}(y,h) = \mathcal{W}(h,y)$;
(2) $\mathcal{W}(y,h) \le \mathcal{W}(y,k)$ *if* $h \le k$;
(3) $\mathcal{W}(y, \mathcal{W}(h,k)) = \mathcal{W}(\mathcal{W}(y,h), k)$;
(4) $\mathcal{W}(y,1) = y$.

Definition 2. *For any* $y, h, k \in [0,1]$, *a mapping* $\mathcal{B} : [0,1] \times [0,1] \to [0,1]$ *is a s-norm, if it fulfilled*
(1) $\mathcal{B}(y,h) = \mathcal{B}(h,y)$;
(2) $\mathcal{B}(y,h) \le \mathcal{B}(y,k)$ *if* $h \le k$;
(3) $\mathcal{B}(y, \mathcal{B}(h,k)) = \mathcal{B}(\mathcal{B}(y,h), k)$;
(4) $\mathcal{B}(y,0) = y$.

Aczel-Alsina norms are two useful operations, which have advantages of changeability with the activity of parameters [18,19].

Definition 3. *A mapping* $(\mathcal{W}_\alpha^\rho)_{\rho \in [0,\infty]}$ *is a Aczel-Alsina t-norm, if it fulfilled*

$$\mathcal{W}_\alpha^\rho(y,h) = \begin{cases} \mathcal{W}_\mathcal{D}(y,h), & \text{if} \quad \rho = 0 \\ \min(y,h), & \text{if} \quad \rho = \infty \\ e^{-((-\ln y)^\rho + (-\ln h)^\rho)^{\frac{1}{\rho}}}, & \text{otherwise} \end{cases}$$

where $y, h \in [0,1]$, ρ *is positive constant and* $\mathcal{W}_\mathcal{D}$ *is drastic t-norm, defined as*

$$\mathcal{W}_\mathcal{D}(y,h) = \begin{cases} y, & \text{if} \quad h = 1 \\ h, & \text{if} \quad y = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Definition 4. *A mapping* $(\mathcal{B}_\alpha^\rho)_{\rho \in [0,\infty]}$ *is a Aczel–Alsina s-norm, if it fulfilled*

$$\mathcal{B}_\alpha^\rho(y,h) = \begin{cases} \mathcal{B}_\mathcal{D}(y,h), & \text{if} \quad \rho = 0 \\ \max(y,h), & \text{if} \quad \rho = \infty \\ 1 - e^{-((-\ln(1-y))^\rho + (-\ln(1-h))^\rho)^{\frac{1}{\rho}}}, & \text{otherwise} \end{cases}$$

where $y, h \in [0,1]$, ρ *is positive constant and* $\mathcal{B}_\mathcal{D}$ *is drastic s-norm, defined as*

$$\mathcal{B}_\mathcal{D}(y,h) = \begin{cases} y, & \text{if} \quad h = 0 \\ h, & \text{if} \quad y = 0 \\ 0, & \text{otherwise} \end{cases}.$$

For every $\rho \in [0,\infty]$, the t-norm $\mathcal{W}_\alpha^\rho$ and s-norm $\mathcal{B}_\alpha^\rho$ are dual to each other.

2.2. Picture fuzzy sets

Definition 5. *[10] A PFS $\mathcal{Z}$ in $\mathcal{K}$ is defined as*

$$\mathcal{Z} = \{(O_{\mathcal{Z}}(\flat), \mathcal{Y}_{\mathcal{Z}}(\flat), \mathcal{G}_{\mathcal{Z}}(\flat)) | \flat \in \mathcal{K}\},$$

where positive grade $O_{\mathcal{Z}} \in [0,1]$, neutral grade $\mathcal{Y}_{\mathcal{Z}} \in [0,1]$ and negative grade $\mathcal{G}_{\mathcal{Z}} \in [0,1]$ of the element $\flat$ to picture fuzzy set $\mathcal{Z}$, fulfilled that $0 \leq O_{\mathcal{Z}}(\flat) + \mathcal{Y}_{\mathcal{Z}}(\flat) + \mathcal{G}_{\mathcal{Z}}(\flat) \leq 1$, for each $\flat \in \mathcal{K}$.

Definition 6. *[10] Let $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ be two picture fuzzy numbers (PFNs), where $(\mathcal{J} = 1,2)$.*
(1) $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$ iff $O_{\mathcal{Z}_1} \leq O_{\mathcal{Z}_2}, \mathcal{Y}_{\mathcal{Z}_1} \leq \mathcal{Y}_{\mathcal{Z}_2}$ and $\mathcal{G}_{\mathcal{Z}_1} \geq \mathcal{G}_{\mathcal{Z}_2}$ for all $\flat \in \mathcal{K}$;
(2) $\mathcal{Z}_1 = \mathcal{Z}_2$ if $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$ and $\mathcal{Z}_2 \subseteq \mathcal{Z}_1$;
(2) $\mathcal{Z}_1 \bigcap \mathcal{Z}_2 = \{\min(O_{\mathcal{Z}_1}, O_{\mathcal{Z}_2}), \min(\mathcal{Y}_{\mathcal{Z}_1}, \mathcal{Y}_{\mathcal{Z}_2}), \max(\mathcal{G}_{\mathcal{Z}_1}, \mathcal{G}_{\mathcal{Z}_2})\}$;
(3) $\mathcal{Z}_1 \bigcup \mathcal{Z}_2 = \{\max(O_{\mathcal{Z}_1}, O_{\mathcal{Z}_2}), \min(\mathcal{Y}_{\mathcal{Z}_1}, \mathcal{Y}_{\mathcal{Z}_2}), \min(\mathcal{G}_{\mathcal{Z}_1}, \mathcal{G}_{\mathcal{Z}_2})\}$;
(4) $(\mathcal{Z}_1)^c = \{\mathcal{G}_{\mathcal{Z}_1}, \mathcal{Y}_{\mathcal{Z}_1}, O_{\mathcal{Z}_1}\}$.

Definition 7. *[10] Let $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ be two PFNs, where $(\mathcal{J} = 1,2)$. The operations about any two PFNs are introduced as follows:*
(1) $\mathcal{Z}_1 \oplus \mathcal{Z}_2 = \{O_{\mathcal{Z}_1} + O_{\mathcal{Z}_2} - O_{\mathcal{Z}_1}O_{\mathcal{Z}_2}, \mathcal{Y}_{\mathcal{Z}_1}\mathcal{Y}_{\mathcal{Z}_2}, \mathcal{G}_{\mathcal{Z}_1}\mathcal{G}_{\mathcal{Z}_2}\}$;
(2) $\mathcal{Z}_1 \otimes \mathcal{Z}_2 = \{O_{\mathcal{Z}_1}O_{\mathcal{Z}_2}, \mathcal{Y}_{\mathcal{Z}_1}\mathcal{Y}_{\mathcal{Z}_2}, \mathcal{G}_{\mathcal{Z}_1} + \mathcal{G}_{\mathcal{Z}_2} - \mathcal{G}_{\mathcal{Z}_1}.\mathcal{G}_{\mathcal{Z}_2}\}$;
(3) $\eta \cdot \mathcal{Z}_1 = \{1 - (1 - O_{\mathcal{Z}_1})^\eta, (\mathcal{Y}_{\mathcal{Z}_1})^\eta, (\mathcal{G}_{\mathcal{Z}_1})^\eta\}, \eta > 0$;
(4) $(\mathcal{Z}_1)^\eta = \{(O_{\mathcal{Z}_1})^\eta, 1 - (1 - \mathcal{Y}_{\mathcal{Z}_1})^\eta, 1 - (1 - \mathcal{G}_{\mathcal{Z}_1})^\eta\}, \eta > 0$.

On the basis of Definition 7, Wei [41] derived following operations in the following ways:

Definition 8. *[41] Let $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ be collection of PFNs, where $(\mathcal{J} = 1,2,...,n)$ and $\eta_1, \eta_1 > 0$, then*
(1) $\mathcal{Z}_1 \oplus \mathcal{Z}_2 = \mathcal{Z}_2 \oplus \mathcal{Z}_1$;
(2) $\mathcal{Z}_1 \otimes \mathcal{Z}_2 = \mathcal{Z}_2 \otimes \mathcal{Z}_1$;
(3) $\eta_1(\mathcal{Z}_1 \oplus \mathcal{Z}_2) = \eta_1\mathcal{Z}_1 \oplus \eta_1\mathcal{Z}_2$;
(4) $(\mathcal{Z}_1 \otimes \mathcal{Z}_2)^{\eta_1} = \mathcal{Z}_1^{\eta_1} \otimes \mathcal{Z}_2^{\eta_1}$;
(5) $\eta_1\mathcal{Z}_1 \oplus \eta_2\mathcal{Z}_1 = (\eta_1 + \eta_2)\mathcal{Z}_1$;
(6) $\mathcal{Z}_1^{\eta_1} \otimes \mathcal{Z}_1^{\eta_2} = \mathcal{Z}_1^{(\eta_1+\eta_2)}$;
(7) $\left(\mathcal{Z}_1^{\eta_1}\right)^{\eta_2} = \mathcal{Z}_1^{\eta_1\eta_2}$.

Definition 9. *[4] Let $\mathcal{Z} = \{O_{\mathcal{Z}}, \mathcal{Y}_{\mathcal{Z}}, \mathcal{G}_{\mathcal{Z}}\}$ be a PFN. The score $\xi(\mathcal{Z})$ and accuracy $\alpha(\mathcal{Z})$ are given as follows:*
(1) $\xi(\mathcal{Z}) = \frac{1}{3}(O_{\mathcal{Z}} + 1 - \mathcal{Y}_{\mathcal{Z}} + 1 - \mathcal{G}_{\mathcal{Z}}) \in [0,1]$;
(2) $\alpha(\mathcal{Z}) = (O_{\mathcal{Z}} - \mathcal{Y}_{\mathcal{Z}} - \mathcal{G}_{\mathcal{Z}}) \in [-1,1]$.

Definition 10. *[4] Let $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ be two PFNs, where $(\mathcal{J} = 1,2)$. Then, the comparison technique of PFNs can be defined as:*
(1) $\xi(\mathcal{Z}_1) > \xi(\mathcal{Z}_2)$ implies that $\mathcal{Z}_1 > \mathcal{Z}_2$;
(2) $\xi(\mathcal{Z}_1) = \xi(\mathcal{Z}_2)$ and $\alpha(\mathcal{Z}_1) > \alpha(\mathcal{Z}_2)$ implies that $\mathcal{Z}_1 > \mathcal{Z}_2$;
(3) $\xi(\mathcal{Z}_1) = \xi(\mathcal{Z}_2)$ and $\alpha(\mathcal{Z}_1) = \alpha(\mathcal{Z}_2)$ implies that $\mathcal{Z}_1 = \mathcal{Z}_2$.

Ashraf et al. [4] prepared the algebraic aggregation operator under PFSs portrayed in the succeeding definition.

Definition 11. *[4] Let* $\mathcal{Z}_{\mathcal{J}} = \left\{\mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *be collection of PFNs, where* $(\mathcal{J} = 1, 2, ..., \ell)$. *A PF weighted averaging (PFWA) aggregation operator of dimension* ℓ *is a mapping* $\mathcal{P}^{\ell} \to \mathcal{P}$ *with weight vector* $\sigma = (\sigma_1, \sigma_1, ..., \sigma_\ell)^T$ *such that* $\sigma_{\mathcal{J}} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}} = 1$ *as*

$$\begin{aligned} PFWA(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) &= \sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}.\mathcal{Z}_{\mathcal{J}} \\ &= \left\{1 - \prod_{\mathcal{J}=1}^{\ell} \left(1 - \mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\sigma_{\mathcal{J}}}, \prod_{\mathcal{J}=1}^{\ell} \left(\mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\sigma_{\mathcal{J}}}, \prod_{\mathcal{J}=1}^{\ell} \left(\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\sigma_{\mathcal{J}}}\right\}. \end{aligned}$$

Definition 12. *[4] Let* $\mathcal{Z}_{\mathcal{J}} = \left\{\mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *be collection of PFNs, where* $(\mathcal{J} = 1, 2, ..., \ell)$. *A PF weighted geometric (PFWG) aggregation operator of dimension* ℓ *is a mapping* $\mathcal{P}^{\ell} \to \mathcal{P}$ *with weight vector* $\sigma = (\sigma_1, \sigma_1, ..., \sigma_\ell)^T$ *such that* $\sigma_{\mathcal{J}} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}} = 1$ *as*

$$\begin{aligned} PFWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) &= \prod_{\mathcal{J}=1}^{\ell} (\mathcal{Z}_{\mathcal{J}})^{\sigma_{\mathcal{J}}} \\ &= \left\{\begin{array}{c} \prod_{\mathcal{J}=1}^{\ell} \left(\mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\sigma_{\mathcal{J}}}, 1 - \prod_{\mathcal{J}=1}^{\ell} \left(1 - \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\sigma_{\mathcal{J}}}, \\ 1 - \prod_{\mathcal{J}=1}^{\ell} \left(1 - \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\sigma_{\mathcal{J}}} \end{array}\right\}. \end{aligned}$$

3. Aczel-Alsina operation for PFNs

In consideration of Aczel-Alsina t-norm and Aczel-Alsina t-conorm, we expounded Aczel-Alsina operations in connection with PFNs.

Definition 13. *Let* $\mathcal{Z}_{\mathcal{J}} = \left\{\mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *be two PFNs, where* $(\mathcal{J} = 1, 2)$ *and* ρ *is positive constant. Then Aczel-Alsina norms based operations for PFNs are introduced as follows:*

(1) $\mathcal{Z}_1 \oplus \mathcal{Z}_2 = \left\{\begin{array}{c} 1 - e^{-\left(\left(-\ln\left(1-\mathcal{O}_{\mathcal{Z}_1}\right)\right)^{\rho} + \left(-\ln\left(1-\mathcal{O}_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\left(-\ln \mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho} + \left(-\ln \mathcal{Y}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\left(-\ln \mathcal{G}_{\mathcal{Z}_1}\right)^{\rho} + \left(-\ln \mathcal{G}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\}$;

(2) $\mathcal{Z}_1 \otimes \mathcal{Z}_2 = \left\{\begin{array}{c} e^{-\left(\left(-\ln \mathcal{O}_{\mathcal{Z}_1}\right)^{\rho} + \left(-\ln \mathcal{O}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\left(-\ln \mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho} + \left(-\ln \mathcal{Y}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_1}\right)\right)^{\rho} + \left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\}$;

(3) $\eta \cdot \mathcal{Z}_1 = \left\{1 - e^{-\left(\eta\left(-\ln\left(1-\mathcal{O}_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\eta\left(-\ln \mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\eta\left(-\ln \mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}\right\}, \eta > 0$;

(4) $(\mathcal{Z}_1)^{\eta} = \left\{e^{-\left(\eta\left(-\ln \mathcal{O}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\eta\left(-\ln \mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, 1 - e^{-\left(\eta\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}\right\}, \eta > 0$.

Theorem 1. *Let* $\mathcal{Z}_{\mathcal{J}} = \left\{\mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *be collection of PFNs, where* $(\mathcal{J} = 1, 2, ..., n)$ *and* $\eta_1, \eta_1 > 0$, *then*

(1) $\mathcal{Z}_1 \oplus \mathcal{Z}_2 = \mathcal{Z}_2 \oplus \mathcal{Z}_1$;
(2) $\mathcal{Z}_1 \otimes \mathcal{Z}_2 = \mathcal{Z}_2 \otimes \mathcal{Z}_1$;
(3) $\eta_1 (\mathcal{Z}_1 \oplus \mathcal{Z}_2) = \eta_1 \mathcal{Z}_1 \oplus \eta_1 \mathcal{Z}_2$;
(4) $(\mathcal{Z}_1 \otimes \mathcal{Z}_2)^{\eta_1} = \mathcal{Z}_1^{\eta_1} \otimes \mathcal{Z}_2^{\eta_1}$;

(5) $\eta_1\mathcal{Z}_1 \oplus \eta_2\mathcal{Z}_1 = (\eta_1 + \eta_2)\,\mathcal{Z}_1$;
(6) $\mathcal{Z}_1^{\eta_1} \otimes \mathcal{Z}_1^{\eta_2} = \mathcal{Z}_1^{(\eta_1+\eta_2)}$;
(7) $\left(\mathcal{Z}_1^{\eta_1}\right)^{\eta_2} = \mathcal{Z}_1^{\eta_1\eta_2}$.

Proof. (1) Since $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ be collection of PFNs, where ($\mathcal{J} = 1, 2, ..., n$) and $\eta_1, \eta_1 > 0$, then by the Definition 13, we have

$$\begin{aligned}
\mathcal{Z}_1 \oplus \mathcal{Z}_2 &= \left\{\begin{array}{c} 1 - e^{-\left(\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}+\left(-\ln\left(1-O_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\left(-\ln\mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}+\left(-\ln\mathcal{Y}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\left(-\ln\mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}+\left(-\ln\mathcal{G}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\} \\
&= \left\{\begin{array}{c} 1 - e^{-\left(\left(-\ln\left(1-O_{\mathcal{Z}_2}\right)\right)^{\rho}+\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\left(-\ln\mathcal{Y}_{\mathcal{Z}_2}\right)^{\rho}+\left(-\ln\mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\left(-\ln\mathcal{G}_{\mathcal{Z}_2}\right)^{\rho}+\left(-\ln\mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\} \\
&= \mathcal{Z}_2 \oplus \mathcal{Z}_1.
\end{aligned}$$

(2) By the Definition 13, we have

$$\begin{aligned}
\mathcal{Z}_1 \otimes \mathcal{Z}_2 &= \left\{\begin{array}{c} e^{-\left(\left(-\ln O_{\mathcal{Z}_1}\right)^{\rho}+\left(-\ln O_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\left(-\ln\mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}+\left(-\ln\mathcal{Y}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_1}\right)\right)^{\rho}+\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\} \\
&= \left\{\begin{array}{c} e^{-\left(\left(-\ln O_{\mathcal{Z}_2}\right)^{\rho}+\left(-\ln O_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\left(-\ln\mathcal{Y}_{\mathcal{Z}_2}\right)^{\rho}+\left(-\ln\mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_2}\right)\right)^{\rho}+\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\} \\
&= \mathcal{Z}_2 \otimes \mathcal{Z}_1.
\end{aligned}$$

(3) By the Definition 13, we have

$$\begin{aligned}
\eta_1\left(\mathcal{Z}_1 \oplus \mathcal{Z}_2\right) &= \eta_1\left\{\begin{array}{c} 1 - e^{-\left(\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}+\left(-\ln\left(1-O_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\left(-\ln\mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}+\left(-\ln\mathcal{Y}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\left(-\ln\mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}+\left(-\ln\mathcal{G}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\} \\
&= \left\{\begin{array}{c} 1 - e^{-\left(\eta_1\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}+\eta_1\left(-\ln\left(1-O_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\eta_1\left(-\ln\mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}+\eta_1\left(-\ln\mathcal{Y}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_1\left(-\ln\mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}+\eta_1\left(-\ln\mathcal{G}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\} \\
&= \left\{\begin{array}{c} 1 - e^{-\left(\eta_1\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_1\left(-\ln\mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_1\left(-\ln\mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\} \oplus \left\{\begin{array}{c} 1 - e^{-\left(\eta_1\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_1\left(-\ln\mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_1\left(-\ln\mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\} \\
&= \eta_1\mathcal{Z}_1 \oplus \eta_1\mathcal{Z}_2.
\end{aligned}$$

(4) It is obvious as (3).
(5) By the Definition 13, we have

$$\eta_1\mathcal{Z}_1 \oplus \eta_2\mathcal{Z}_1 = \left\{\begin{array}{c} 1 - e^{-\left(\eta_1\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_1\left(-\ln\mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_1\left(-\ln\mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\} \oplus \left\{\begin{array}{c} 1 - e^{-\left(\eta_2\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_2\left(-\ln\mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_2\left(-\ln\mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array}\right\}$$

$$\begin{aligned} &= \left\{ \begin{array}{c} 1 - e^{-\left(\eta_1\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}+\eta_2\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\eta_1\left(-\ln \mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}+\eta_2\left(-\ln \mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_1\left(-\ln \mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}+\eta_2\left(-\ln \mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\} \\ &= \left\{ \begin{array}{c} 1 - e^{-\left(\eta_1+\eta_2\left(-\ln\left(1-O_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\eta_1+\eta_2\left(-\ln \mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\eta_1+\eta_2\left(-\ln \mathcal{G}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\} \\ &= (\eta_1 + \eta_2)\, \mathcal{Z}_1. \end{aligned}$$

(6) & (7) are obvious as (5). □

4. Aczel-Alsina geometric aggregation operators for PFNs

Aczel-Alsina norms based list of novel aggregation operators under picture fuzzy settings are develop in this section.

Definition 14. *Let $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ be collection of PFNs, where $(\mathcal{J} = 1, 2, ..., \ell)$. A PF Aczel-Alsina weighted geometric (PFAWG) aggregation operator of dimension ℓ is a mapping $\mathcal{P}^{\ell} \rightarrow \mathcal{P}$ with weight vector $\sigma = (\sigma_1, \sigma_1, ..., \sigma_{\ell})^T$ such that $\sigma_{\mathcal{J}} > 0$ and $\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}} = 1$ as*

$$PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_{\ell}) = \prod_{\mathcal{J}=1}^{\ell} (\mathcal{Z}_{\mathcal{J}})^{\sigma_{\mathcal{J}}}.$$

Theorem 2. *Suppose $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ be collection of PFNs, where $(\mathcal{J} = 1, 2, ..., \ell)$. A PF Aczel-Alsina weighted geometric (PFAWG) aggregation operator of dimension ℓ is a mapping $\mathcal{P}^{\ell} \rightarrow \mathcal{P}$ with weight vector $\sigma = (\sigma_1, \sigma_1, ..., \sigma_{\ell})^T$ such that $\sigma_{\mathcal{J}} > 0$ and $\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}} = 1$ is defined as:*

$$\begin{aligned} PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_{\ell}) &= \prod_{\mathcal{J}=1}^{\ell} (\mathcal{Z}_{\mathcal{J}})^{\sigma_{\mathcal{J}}} \\ &= \left\{ \begin{array}{c} e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln O_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\}. \end{aligned}$$

Proof. The proof of Theorem 2 is derived by implementation of mathematical induction as follows:

For $\ell = 2$, we get

$$PFAWG(\mathcal{Z}_1, \mathcal{Z}_2) = (\mathcal{Z}_1)^{\sigma_1} \otimes (\mathcal{Z}_2)^{\sigma_2}.$$

By the Definition 13, we have

$$(\mathcal{Z}_1)^{\sigma_1} = \left\{ e^{-\left(\sigma_1\left(-\ln O_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sigma_1\left(-\ln\left(\mathcal{Y}_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, 1 - e^{-\left(\sigma_1\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \right\},$$

and

$$(\mathcal{Z}_2)^{\sigma_2} = \left\{ e^{-\left(\sigma_2\left(-\ln O_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sigma_2\left(-\ln\left(\mathcal{Y}_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, 1 - e^{-\left(\sigma_2\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \right\},$$

therefore

$$\begin{aligned}
PFAWG(\mathcal{Z}_1, \mathcal{Z}_2) &= \left\{ \begin{array}{c} e^{-\left(\sigma_1\left(-\ln \mathcal{O}_{\mathcal{Z}_1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\sigma_1\left(-\ln\left(\mathcal{Y}_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sigma_1\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\} \otimes \left\{ \begin{array}{c} e^{-\left(\sigma_2\left(-\ln \mathcal{O}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\sigma_2\left(-\ln\left(\mathcal{Y}_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sigma_2\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\} \\
&= \left\{ \begin{array}{c} e^{-\left(\sigma_1\left(-\ln \mathcal{O}_{\mathcal{Z}_1}\right)^{\rho}+\sigma_2\left(-\ln \mathcal{O}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sigma_1\left(-\ln \mathcal{Y}_{\mathcal{Z}_1}\right)^{\rho}+\sigma_2\left(-\ln \mathcal{Y}_{\mathcal{Z}_2}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sigma_1\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_1}\right)\right)^{\rho}+\sigma_2\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_2}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\} \\
&= \left\{ \begin{array}{c} e^{-\left(\sum_{\mathcal{J}=1}^{2}\sigma_{\mathcal{J}}\left(-\ln \mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sum_{\mathcal{J}=1}^{2}\sigma_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sum_{\mathcal{J}=1}^{2}\sigma_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\}.
\end{aligned}$$

Thus Theorem 2 is true $\ell = 2$.
Now, we suppose that Theorem 2 is true $\ell = d$, we have

$$PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_d) = \left\{ \begin{array}{c} e^{-\left(\sum_{\mathcal{J}=1}^{d}\sigma_{\mathcal{J}}\left(-\ln \mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sum_{\mathcal{J}=1}^{d}\sigma_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sum_{\mathcal{J}=1}^{d}\sigma_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\}.$$

We have to show that Theorem 2 is true for $\ell = d + 1$.

$$PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_d, \mathcal{Z}_{d+1}) = \prod_{\mathcal{J}=1}^{\ell}(\mathcal{Z}_{\mathcal{J}})^{\sigma_{\mathcal{J}}} \otimes (\mathcal{Z}_{d+1})^{\sigma_{d+1}}$$

$$\begin{aligned}
\prod_{\mathcal{J}=1}^{\ell}(\mathcal{Z}_{\mathcal{J}})^{\sigma_{\mathcal{J}}} \otimes (\mathcal{Z}_{d+1})^{\sigma_{d+1}} &= \left\{ \begin{array}{c} e^{-\left(\sum_{\mathcal{J}=1}^{d}\sigma_{\mathcal{J}}\left(-\ln \mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\sum_{\mathcal{J}=1}^{d}\sigma_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sum_{\mathcal{J}=1}^{d}\sigma_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\} \otimes \left\{ \begin{array}{c} e^{-\left(\sigma_{d+1}\left(-\ln \mathcal{O}_{\mathcal{Z}_{d+1}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\sigma_{d+1}\left(-\ln\left(\mathcal{Y}_{\mathcal{Z}_{d+1}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sigma_{d+1}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_{d+1}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\} \\
&= \left\{ \begin{array}{c} e^{-\left(\sum_{\mathcal{J}=1}^{d}\sigma_{\mathcal{J}}\left(-\ln \mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}+\sigma_{d+1}\left(-\ln \mathcal{O}_{\mathcal{Z}d+1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ e^{-\left(\sum_{\mathcal{J}=1}^{d}\sigma_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}+\sigma_{d+1}\left(-\ln \mathcal{Y}_{\mathcal{Z}d+1}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sum_{\mathcal{J}=1}^{d}\sigma_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}+\sigma_{d+1}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}d+1}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\} \\
&= \left\{ \begin{array}{c} e^{-\left(\sum_{\mathcal{J}=1}^{d+1}\sigma_{\mathcal{J}}\left(-\ln \mathcal{O}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sum_{\mathcal{J}=1}^{d+1}\sigma_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sum_{\mathcal{J}=1}^{d+1}\sigma_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\}.
\end{aligned}$$

Hence, Theorem 2 is valid for all ℓ. □

We may demonstrate the accompanying properties effectively by utilizing the operator PFAWG.

Theorem 3. *(Idempotent) Let* $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\} (\mathcal{J} = 1, 2, ..., \ell)$ *be collection of equivalent PFNs, i.e.,* $\mathcal{Z}_{\mathcal{J}} = \mathcal{Z}$ *for each* $(\mathcal{J} = 1, 2, ..., \ell)$. *Then*

$$PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) = \mathcal{Z}.$$

Proof. Since

$$PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) = \left\{ \begin{array}{c} e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln O_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\}.$$

Put $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\} = \mathcal{Z} \ (\mathcal{J} = 1, 2, ..., \ell)$, we have

$$\begin{aligned} PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) &= \left\{ \begin{array}{c} e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln O_{\mathcal{Z}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\} \\ &= \left\{ \begin{array}{c} e^{-\left(\left(-\ln O_{\mathcal{Z}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\left(-\ln \mathcal{Y}_{\mathcal{Z}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\} \\ &= \left(O_{\mathcal{Z}}, \mathcal{Y}_{\mathcal{Z}}, \mathcal{G}_{\mathcal{Z}}\right) = \mathcal{Z}. \end{aligned}$$

Thus, $PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) = \mathcal{Z}$ holds. □

Theorem 4. *(Boundedness) Let* $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\} (\mathcal{J} = 1, 2, ..., \ell)$ *be collection of PFNs. Let* $\mathcal{Z}_{\mathcal{J}}^{-} = \left(\min_{\mathcal{J}}\left\{O_{\mathcal{Z}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right\}, \max_{\mathcal{J}}\left\{\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}\right)$ *and* $\mathcal{Z}_{\mathcal{J}}^{+} = \left(\max_{\mathcal{J}}\left\{O_{\mathcal{Z}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}\right)$ $(\mathcal{J} = 1, 2, ..., \ell)$. *Then,*

$$\mathcal{Z}_{\mathcal{J}}^{-} \leq PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) \leq \mathcal{Z}_{\mathcal{J}}^{+}.$$

Proof. We have $\min_{\mathcal{J}}\left\{O_{\mathcal{Z}_{\mathcal{J}}}\right\} \leq O_{\mathcal{Z}_{\mathcal{J}}} \leq \max_{\mathcal{J}}\left\{O_{\mathcal{Z}_{\mathcal{J}}}\right\}$, i.e.,

$$e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln\left(\min O_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \leq e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln O_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}} \leq e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln\left(\max O_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}},$$

similarly, we have

$$e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln\left(\min \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \leq e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)^{\rho}\right)^{\frac{1}{\rho}}} \leq e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln\left(\min \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}.$$

Now, also we have

$$\begin{aligned} 1 - e^{-\left(\sum_{\mathcal{J}=1}^{d} \sigma_{\mathcal{J}}\left(-\ln\left(\max\left(1-\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} &\leq 1 - e^{-\left(\sum_{\mathcal{J}=1}^{d} \sigma_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \\ &\leq 1 - e^{-\left(\sum_{\mathcal{J}=1}^{d} \sigma_{\mathcal{J}}\left(-\ln\left(\min\left(1-\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right)\right)\right)^{\rho}\right)^{\frac{1}{\rho}}}. \end{aligned}$$

Therefore,

$$\mathcal{Z}_{\mathcal{J}}^{-} \leq PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) \leq \mathcal{Z}_{\mathcal{J}}^{+}.$$

□

Theorem 5. *Let* $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *and* $\mathcal{Z}^*_{\mathcal{J}} = \left\{O^*_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}^*_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}^*_{\mathcal{Z}_{\mathcal{J}}}\right\}$ $(\mathcal{J} = 1, 2, ..., \ell)$ *be two collection of PFNs, if* $\mathcal{Z}_{\mathcal{J}} \leq \mathcal{Z}^*_{\mathcal{J}}$ *for* $(\mathcal{J} = 1, 2, ..., \ell)$ *. Then,*

$$PFAWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) \leq PFAWG(\mathcal{Z}^*_1, \mathcal{Z}^*_2, ..., \mathcal{Z}^*_\ell).$$

Proof. Obviously. □

Definition 15. *Let* $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *be collection of PFNs, where* $(\mathcal{J} = 1, 2, ..., \ell)$ *. A PF Aczel–Alsina ordered weighted geometric (PFAOWG) aggregation operator of dimension* ℓ *is a mapping* $\mathcal{P}^\ell \to \mathcal{P}$ *with weight vector* $\sigma = (\sigma_1, \sigma_1, ..., \sigma_\ell)^T$ *such that* $\sigma_{\mathcal{J}} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}} = 1$ *as*

$$PFAOWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) = \prod_{\mathcal{J}=1}^{\ell} (\mathcal{Z}_{\tau(\mathcal{J})})^{\sigma_{\mathcal{J}}},$$

where $(\tau(1), \tau(2), ..., \tau(\ell))$ *are the permutation in such a way as* $\mathcal{Z}_{\tau(\mathcal{J})} \leq \mathcal{Z}_{\tau(\mathcal{J}-1)}$.

Theorem 6. *Suppose* $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *be collection of PFNs, where* $(\mathcal{J} = 1, 2, ..., \ell)$*. A PF Aczel-Alsina ordered weighted geometric (PFAOWG) aggregation operator of dimension* ℓ *is a mapping* $\mathcal{P}^\ell \to \mathcal{P}$ *with weight vector* $\sigma = (\sigma_1, \sigma_1, ..., \sigma_\ell)^T$ *such that* $\sigma_{\mathcal{J}} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}} = 1$ *is defined as:*

$$\begin{aligned} PFAOWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) &= \prod_{\mathcal{J}=1}^{\ell} (\mathcal{Z}_{\tau(\mathcal{J})})^{\sigma_{\mathcal{J}}} \\ &= \left\{ \begin{array}{c} e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln O_{\mathcal{Z}_{\tau(\mathcal{J})}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}_{\tau(\mathcal{J})}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}_{\tau(\mathcal{J})}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\}, \end{aligned}$$

where $(\tau(1), \tau(2), ..., \tau(\ell))$ *are the permutation in such a way as* $\mathcal{Z}_{\tau(\mathcal{J})} \leq \mathcal{Z}_{\tau(\mathcal{J}-1)}$.

We may demonstrate the accompanying properties effectively by utilizing the operator PFAOWG.

Theorem 7. *(1) (Idempotent) Let* $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ $(\mathcal{J} = 1, 2, ..., \ell)$ *be collection of equivalent PFNs, i.e.,* $\mathcal{Z}_{\mathcal{J}} = \mathcal{Z}$ *for each* $(\mathcal{J} = 1, 2, ..., \ell)$ *. Then*

$$PFAOWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) = \mathcal{Z}.$$

(2) (Boundedness) Let $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ $(\mathcal{J} = 1, 2, ..., \ell)$ *be collection of PFNs. Let* $\mathcal{Z}^-_{\mathcal{J}} = \left(\min_{\mathcal{J}}\left\{O_{\mathcal{Z}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right\}, \max_{\mathcal{J}}\left\{\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}\right)$ *and*

$$\mathcal{Z}^+_{\mathcal{J}} = (\max_{\mathcal{J}}\left\{O_{\mathcal{Z}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\})$$

$(\mathcal{J} = 1, 2, ..., \ell)$ *. Then,*

$$\mathcal{Z}^-_{\mathcal{J}} \leq PFAOWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) \leq \mathcal{Z}^+_{\mathcal{J}}.$$

(3) Let $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *and* $\mathcal{Z}^*_{\mathcal{J}} = \left\{O^*_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}^*_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}^*_{\mathcal{Z}_{\mathcal{J}}}\right\}$ $(\mathcal{J} = 1, 2, ..., \ell)$ *be two collection of PFNs, if* $\mathcal{Z}_{\mathcal{J}} \leq \mathcal{Z}^*_{\mathcal{J}}$ *for* $(\mathcal{J} = 1, 2, ..., \ell)$ *. Then,*

$$PFAOWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) \leq PFAOWG(\mathcal{Z}^*_1, \mathcal{Z}^*_2, ..., \mathcal{Z}^*_\ell).$$

Proof. Prove of this theorem is similarly done by using Theorems 3–5. □

Definition 16. *Let* $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *be collection of PFNs, where* $(\mathcal{J} = 1, 2, ..., \ell)$. *A PF Aczel-Alsina hybrid weighted geometric (PFAHWG) aggregation operator of dimension* ℓ *is a mapping* $\mathcal{P}^{\ell} \to \mathcal{P}$ *with weight vector* $\sigma = (\sigma_1, \sigma_2, ..., \sigma_{\ell})^T$ *such that* $\sigma_{\mathcal{J}} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}} = 1$ *as*

$$PFAHWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_{\ell}) = \prod_{\mathcal{J}=1}^{\ell} \left(\mathcal{Z}^*_{\tau(\mathcal{J})}\right)^{\Psi_{\mathcal{J}}},$$

where $\Psi = (\Psi_1, \Psi_2, ..., \Psi_{\ell})^T$ *are the associated weights such that* $\Psi_{\mathcal{J}} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \Psi_{\mathcal{J}} = 1$, *also* $\mathcal{Z}^*_{\tau(\mathcal{J})} = \left(\mathcal{Z}^*_{\tau(\mathcal{J})} = n\sigma_{\mathcal{J}}\mathcal{Z}_{\tau(\mathcal{J})}\right)$ $(\mathcal{J} = 1, 2, ..., \ell)$ *and* $(\tau(1), \tau(2), ..., \tau(\ell))$ *are the permutation in such a way as* $\mathcal{Z}^*_{\tau(\mathcal{J})} \leq \mathcal{Z}^*_{\tau(\mathcal{J}-1)}$.

Theorem 8. *Suppose* $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *be collection of PFNs, where* $(\mathcal{J} = 1, 2, ..., \ell)$. *A PF Aczel-Alsina hybrid weighted geometric (PFAHWG) aggregation operator of dimension* ℓ *is a mapping* $\mathcal{P}^{\ell} \to \mathcal{P}$ *with weight vector* $\sigma = (\sigma_1, \sigma_1, ..., \sigma_{\ell})^T$ *such that* $\sigma_{\mathcal{J}} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \sigma_{\mathcal{J}} = 1$ *is defined as:*

$$\begin{aligned} PFAHWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_{\ell}) &= \prod_{\mathcal{J}=1}^{\ell} \left(\mathcal{Z}^*_{\tau(\mathcal{J})}\right)^{\Psi_{\mathcal{J}}} \\ &= \left\{ \begin{array}{c} e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \Psi_{\mathcal{J}}\left(-\ln O_{\mathcal{Z}^*_{\tau(\mathcal{J})}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \Psi_{\mathcal{J}}\left(-\ln \mathcal{Y}_{\mathcal{Z}^*_{\tau(\mathcal{J})}}\right)^{\rho}\right)^{\frac{1}{\rho}}}, \\ 1 - e^{-\left(\sum_{\mathcal{J}=1}^{\ell} \Psi_{\mathcal{J}}\left(-\ln\left(1-\mathcal{G}_{\mathcal{Z}^*_{\tau(\mathcal{J})}}\right)\right)^{\rho}\right)^{\frac{1}{\rho}}} \end{array} \right\}, \end{aligned}$$

where $\Psi = (\Psi_1, \Psi_2, ..., \Psi_{\ell})^T$ *are the associated weights such that* $\Psi_{\mathcal{J}} > 0$ *and* $\sum_{\mathcal{J}=1}^{\ell} \Psi_{\mathcal{J}} = 1$, *also* $\mathcal{Z}^*_{\tau(\mathcal{J})} = \left(\mathcal{Z}^*_{\tau(\mathcal{J})} = n\sigma_{\mathcal{J}}\mathcal{Z}_{\tau(\mathcal{J})}\right)$ $(\mathcal{J} = 1, 2, ..., \ell)$ *and* $(\tau(1), \tau(2), ..., \tau(\ell))$ *are the permutation in such a way as* $\mathcal{Z}^*_{\tau(\mathcal{J})} \leq \mathcal{Z}^*_{\tau(\mathcal{J}-1)}$.

We may demonstrate the accompanying properties effectively by utilizing the operator PFAHWG.

Theorem 9. *(1) (Idempotent) Let* $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ $(\mathcal{J} = 1, 2, ..., \ell)$ *be collection of equivalent PFNs, i.e.,* $\mathcal{Z}_{\mathcal{J}} = \mathcal{Z}$ *for each* $(\mathcal{J} = 1, 2, ..., \ell)$. *Then*

$$PFAHWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_{\ell}) = \mathcal{Z}.$$

(2) (Boundedness) Let $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ $(\mathcal{J} = 1, 2, ..., \ell)$ *be collection of PFNs. Let* $\mathcal{Z}^-_{\mathcal{J}} = \left(\min_{\mathcal{J}}\left\{O_{\mathcal{Z}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right\}, \max_{\mathcal{J}}\left\{\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}\right)$ *and*

$$\mathcal{Z}^+_{\mathcal{J}} = \left(\max_{\mathcal{J}}\left\{O_{\mathcal{Z}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}\right\}, \min_{\mathcal{J}}\left\{\mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}\right)$$

$(\mathcal{J} = 1, 2, ..., \ell)$. *Then,*

$$\mathcal{Z}^-_{\mathcal{J}} \leq PFAHWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_{\ell}) \leq \mathcal{Z}^+_{\mathcal{J}}.$$

(3) Let $\mathcal{Z}_{\mathcal{J}} = \left\{O_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}}}\right\}$ *and* $\mathcal{Z}^*_{\mathcal{J}} = \left\{O^*_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{Y}^*_{\mathcal{Z}_{\mathcal{J}}}, \mathcal{G}^*_{\mathcal{Z}_{\mathcal{J}}}\right\}$ $(\mathcal{J} = 1, 2, ..., \ell)$ *be two collection of PFNs, if* $\mathcal{Z}_{\mathcal{J}} \leq \mathcal{Z}^*_{\mathcal{J}}$ *for* $(\mathcal{J} = 1, 2, ..., \ell)$. *Then,*

$$PFAHWG(\mathcal{Z}_1, \mathcal{Z}_2, ..., \mathcal{Z}_\ell) \leq PFAHWG(\mathcal{Z}^*_1, \mathcal{Z}^*_2, ..., \mathcal{Z}^*_\ell).$$

Proof. Prove of this theorem is similarly done by using Theorems 3–5. □

5. Decision support algorithm

In order to verify the effectiveness of the PF Aczel-Alsina geometric aggregation operators in this paper, a new MCGDM approach is established to tackle the complex uncertain data in real life decision support problems. The specific steps are as follows.

Assume that there is a set of ℓ alternatives $\{£_1, £_2, ..., £_\ell\}$, and satisfactorily assessed by a set of attributes $\{\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_m\}$. Then, the impotence of various attributes $\mathcal{R}_\imath\,(\imath = 1, 2, ..., m)$ is specified by a weight vector $\sigma = (\sigma_1, \sigma_1, ..., \sigma_m)^T$ such that $\sigma_\imath > 0$ and $\sum_{\imath=1}^m \sigma_\imath = 1$.

Let $\mathcal{Z}_{\mathcal{J}\imath} = \left\{O_{\mathcal{Z}_{\mathcal{J}\imath}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}\imath}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}\imath}}\right\}$ for $O_{\mathcal{Z}_{\mathcal{J}\imath}}$, $\mathcal{Y}_{\mathcal{Z}_{\mathcal{J}\imath}}$, $\mathcal{G}_{\mathcal{Z}_{\mathcal{J}\imath}} \in [0, 1]$ be the satisfactory assessment of each attribute for each alternative, where $O_{\mathcal{Z}_{\ell m}}$ indicates the positive grade function that the alternative $£_{\mathcal{J}}$ $(\mathcal{J} = 1, 2, \ldots, \ell)$ satisfies $\mathcal{R}_\imath\,(\imath = 1, 2, ..., m)$. $\mathcal{Y}_{\mathcal{Z}_{\ell m}}$ and $\mathcal{G}_{\mathcal{Z}_{\ell m}}$ indicate the neutral grade function and the negative grade function, respectively. According to all assessment values, we can yield the decision matrix of PFNs: $\mathcal{Z} = (\mathcal{Z}_{\mathcal{J}\imath})_{\ell m}$.

In this study, the developed PF Aczel-Alsina geometric operators were applied to solve the MCDDM problem, and the procedure for determining the best alternative is provided as the following steps:

Step-1. Determine a collection of attributes that are appropriate for the assessment problem under consideration possibilities:
A literature study is used to gather prospective evaluation attributes, and then an expert committee is assembled to filter the attributes and come up with a suitable set of assessment attributes

$\mathcal{R}_\imath\,(\imath = 1, 2, ..., m)$.

$$D_{\mathcal{J}\times\imath} = \begin{matrix} \\ £_1 \\ £_2 \\ \vdots \\ £_\ell \end{matrix} \begin{pmatrix} \mathcal{R}_1 & \mathcal{R}_2 & & \mathcal{R}_m \\ (O_{\mathcal{Z}_{11}}, \mathcal{Y}_{\mathcal{Z}_{11}}, \mathcal{G}_{\mathcal{Z}_{11}}) & (O_{\mathcal{Z}_{12}}, \mathcal{Y}_{\mathcal{Z}_{12}}, \mathcal{G}_{\mathcal{Z}_{12}}) & \cdots & (O_{\mathcal{Z}_{1m}}, \mathcal{Y}_{\mathcal{Z}_{1m}}, \mathcal{G}_{\mathcal{Z}_{1m}}) \\ (O_{\mathcal{Z}_{21}}, \mathcal{Y}_{\mathcal{Z}_{21}}, \mathcal{G}_{\mathcal{Z}_{21}}) & (O_{\mathcal{Z}_{22}}, \mathcal{Y}_{\mathcal{Z}_{22}}, \mathcal{G}_{\mathcal{Z}_{22}}) & \cdots & (O_{\mathcal{Z}_{2m}}, \mathcal{Y}_{\mathcal{Z}_{2m}}, \mathcal{G}_{\mathcal{Z}_{2m}}) \\ \vdots & \vdots & \ddots & \vdots \\ (O_{\mathcal{Z}_{\ell 1}}, \mathcal{Y}_{\mathcal{Z}_{\ell 1}}, \mathcal{G}_{\mathcal{Z}_{\ell 1}}) & (O_{\mathcal{Z}_{\ell 2}}, \mathcal{Y}_{\mathcal{Z}_{\ell 2}}, \mathcal{G}_{\mathcal{Z}_{\ell 2}}) & \cdots & (O_{\mathcal{Z}_{\ell m}}, \mathcal{Y}_{\mathcal{Z}_{\ell m}}, \mathcal{G}_{\mathcal{Z}_{\ell m}}) \end{pmatrix}$$

Step-2. Obtain the normalized decision matrix through normalization as follow:

$$\mathcal{N}_{\mathcal{J}\times\imath} = \begin{cases} \left(O_{\mathcal{Z}_{\mathcal{J}\imath}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}\imath}}, \mathcal{G}_{\mathcal{Z}_{\mathcal{J}\imath}}\right) & if \quad C_I \\ \left(\mathcal{G}_{\mathcal{Z}_{\mathcal{J}\imath}}, \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}\imath}}, O_{\mathcal{Z}_{\mathcal{J}\imath}}\right) & if \quad C_{II} \end{cases} \tag{5.1}$$

where C_I refers to "if $\mathcal{R}_\imath\,(\imath = 1, 2, ..., m)$ is a benefit criterion" and C_{II} refers to "if $\mathcal{R}_\imath\,(\imath = 1, 2, ..., m)$ is a cost criterion".

Step-3. Collected expert uncertain data: PFWA/PFWG aggregation operators are utilized to aggregate the expert uncertain data of decision support problems.

Step-4. Identify the importance of consider attributes $\{\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_m\}$ using PF entropy measure as follows:

$$\kappa_\iota = \frac{1 + \frac{1}{\ell}\sum_{\mathcal{J}=1}^{\ell}\left(O_{\mathcal{Z}_{\mathcal{J}_\iota}} \log\left(O_{\mathcal{Z}_{\mathcal{J}_\iota}}\right) + \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}_\iota}} \log\left(\mathcal{Y}_{\mathcal{Z}_{\mathcal{J}_\iota}}\right) + \mathcal{G}_{\mathcal{Z}_{\mathcal{J}_\iota}} \log\left(\mathcal{G}_{\mathcal{Z}_{\mathcal{J}_\iota}}\right)\right)}{\sum_{\iota=1}^{m}\left(1 + \frac{1}{\ell}\sum_{\mathcal{J}=1}^{\ell} O_{\mathcal{Z}_{\mathcal{J}_\iota}} \log\left(O_{\mathcal{Z}_{\mathcal{J}_\iota}}\right) + \mathcal{Y}_{\mathcal{Z}_{\mathcal{J}_\iota}} \log\left(\mathcal{Y}_{\mathcal{Z}_{\mathcal{J}_\iota}}\right) + \mathcal{G}_{\mathcal{Z}_{\mathcal{J}_\iota}} \log\left(\mathcal{G}_{\mathcal{Z}_{\mathcal{J}_\iota}}\right)\right)}$$

Step-5. Aggregated data: Developed PF Aczel-Alsina geometric operators are utilized to aggregate the expert uncertain data of decision support problems.

Step-5(a). Utilized PFAWG operator to integrate aggregated data.

Step-5(b). Utilized PFAOWG operator to integrate aggregated data.

Step-5(c). Utilized PFAHWG operator to integrate aggregated data.

Step-6. According to the score function in Definition 9, the score values of $£_{\mathcal{J}}$ ($\mathcal{J} = 1, 2, \ldots, \ell$) are obtained.

Step-7. Based on the score values, all alternatives are ranked in a decreasing order, and select the best one concerning the biggest score value.

6. Numerical illustration of developed technique

In order to validate the effectiveness and applicability of the developed technique to the MADM problem, we address an uncertain real-life problem of detecting the factors effecting mango productivity in its probable solutions below.

Case study

Mango is a prominent fruit crop in Pakistan, and the country is the fourth largest producer and exporter of mangoes in the world [15]. Mango (Mangifera Indica L.) plays a leading role among the diverse range of horticultural crops grown in Pakistan, owing to its robust production base, high domestic demand (95%), and export potential, all of which contribute to the country's socioeconomic development. Mango is frequently dubbed the "Fruit King". It is the second most important fruit crop in the region in terms of cultivation and production, trailing only citrus in Pakistan's development. Internationally, Pakistani mangoes are renowned for their sweetness, juiciness, nutrition, and unusual flavors. Throughout the summer season, domestic demand for mangoes is fairly high. They are consumed both fresh and processed in the form of jams, pickles, juices, squash, and jellies. Pakistani mangoes are considered to be among the best on the world market [12]. Although the role of other sectors such as services in the Pakistani economy is rapidly increasing (59.9 percent), agriculture is still considered one of the most significant sectors of the economy, accounting for 19.8 percent of the country's GDP. Thus, agriculture is critical to Pakistan's economy since it employs a greater share of the worker force. According to a Pakistani economic assessment, agriculture employs 42.3% of the total labour force directly or indirectly [13]. Agriculture is divided into numerous subsectors, with horticulture being one of the most significant. Pakistan is the world's sixth largest country [29]. As a result, the horticulture sector is critical in meeting the basic requirement of food for a big population. Horticulture includes fruit growing, and the subcontinent is a mango habitat [15].

Now, we enlist the four factors (alternatives) that effect on the production of the Mango crop.
(1) Incidence of Pests and Diseases ($£_1$)**:** The mango is subject to several situations during its development, from seedling plants to fruits stored or transported. If the diseases are virulent, they can cause crop failure. The main pests and diseases in mango crop are mango borer, mango hoper, mites, anthracnose, and powdery mildew [47]. Water deficiency, extreme summer and winter temperatures, hard soil, and limited nutrient consumption in mango plantations are the key abiotic variables [17]. Researchers found infested arbours, incorrect pruning and irrigation, improper intercropping with inadequate crops, lack of macro and micro-nutrients, and deep ploughing, all of which resulted in root injuries and disease infestation. The most common mango bacterial and fungal illnesses are as following in Pakistan; Mangiferaes, bacterial black spot (Xanthomonas campestris pv. Mangiferae), bacterial black spot (Xanthomonas campestris pv.), powdery mash (Oidium mangiferae), fruit red (Aspergillus niger), anthracnose (Colletotrichum gloeosporioides), trunk blight or die (Ceratic Fimbriata, Lasiodiplodea, theobromae), raw root (CR and Fusarium species) as well [2]. A tree that looks outwardly healthy, but dies in a matter of days is called a "fast fall." The dieback tip is part of the reduction in mango over the wide period. Symptoms initial were bark gummy and branch death, as well as other symptoms, including a vascular discolouration under rubber. Tree death often took place within 6 months of the first appearance of the symptom [30]. Moreover, inadequate pest management hinders export growth, posing another enormous difficulty for farmers. Farmers appear to lack comprehension of pest management strategies and their impact on fruit export potential. Fruit fly infestations in mango orchards continue to reduce mango exportability. The majority of farmers continue to let their trees grow until they reach 40 feet tall (Low density plantation). This height not only hinders spraying and harvesting but also reduces fruit quality. Modern tree management techniques like pruning and canopy control were rare. Even those who chose these procedures used axes to prune. Traditional orchard management in Pakistan perpetuates fruit diseases. All mango cultivars in Pakistan are affected by powdery mildew (Oidium mangifera), anthracnose (Glomerella cingulata-Colletotrichum gloeosporioides) and stem blight (Diplodia spp) [9].
(2) Lack of Technical Knowledge and Traditional Varieties of Mango ($£_2$)**:** Sixty-five percent of mango growers were unaware of the technical knowledge required to properly manage their crops. Improper harvesting, post-harvest technological handling, and low yields due to lack of knowledge about effective and timely use of farm inputs resulted in massive loss. Unbalanced fertilisers, inadequate irrigation, and inappropriate pesticide use all impact productivity [22]. Numerous studies have discovered the factors that influence mango production in Pakistan. The same survey found that most mango growers do not use recommended pesticide, irrigation, or fertilizer applications. Many responsible factors have been studied, however the efficiency of mango farmers has been underestimated. Low mango crop yields may be due to farmers' technical efficiency. Enlarging mango growers' technical efficiency and output requires understanding technical inefficiencies. Singh [37] investigated the possibility for exporting non-traditional African crops. He noticed that most non-traditional crops in Africa were planted for export, notably to Europe. In the 1990s, the European

Union (EU) imported more fresh fruits and vegetables than any other agricultural product. In response, many African countries expanded their agricultural production to include EU-favored commodities. A number of factors have been suggested for the increase in African horticulture exports, including trade agreements like the Lome Convention, which gave African exporters preferential treatment on the European market. As a result, African countries privatized state-owned companies, liberalized commercial restrictions, and subsidized exports to tap the booming European market. International corporations helped several African counterparts by transferring technology, providing logistical support, and branding African products in importing markets. So that they might compete more effectively on the global market, several African countries organized regional economic groups.

(3) Climate Change ($£_3$)**:** Agriculture is very climate-dependent. Increases in temperature and carbon dioxide (CO2) levels have been shown to boost agricultural yields in some regions. However, in order to reap these benefits, nutrient levels, soil moisture, and water availability must all be met. Drought and flood recurrence and intensity changes could create difficulties for farmers and ranchers and jeopardies food safety [14]. Meanwhile, rising water temperatures are projected to change the habitat ranges of numerous fish and shellfish species, resulting in ecosystem disruption. Climate change, in general, may make it more difficult to grow crops, rear animals, and fish in the same manner and locations as in the past. Climate change effects must also be taken into account alongside other emerging elements affecting agricultural production, such as changes in farming practises and technology [33]. In the same way, as crop water use increases and drought conditions deteriorate, water required for food production may become scarcer. When climate change makes certain areas unusable for agricultural, competition for land could heat up. On the other hand, extreme weather events connected to climate change could reduce agricultural productivity, causing a spike in agricultural commodity prices. Because of this, key producing areas such as Russia, Ukraine, and Kazakhstan saw output losses in the summer of 2010, resulting in significant price increases for staple products. Climate change can cause food insecurity because of rising food prices, as evidenced by the fact that an increasing proportion of local inhabitants now live in poverty.

(4) Salinity ($£_4$)**:** Pakistan is an agricultural country that feeds its 207.77 million people with food and jobs. However, salt threatens the economy, leading to increased environmental deterioration and, in particular, mango yield. Moreover, Pakistani society consists of just 22,05 million hectares of cultivated land out of a total of 79,6 million hectales [12], with a salt or salt space of 6,28 million hectares [13]. The bigger the higher. The effects of salinity on agricultural production in any country are varied and serious, including rural-urban migration, living standards, health difficulties and transport delays [16]. In short, the government should compensate for these impacts by building a strong adaptation model for crop development, macro- and micro-economic policies, taking into consideration the detrimental effects of the salinity in the country's economy and budget [32].

Probable solutions: Following literature suggested the probable remedies to overcome on the mentioned problems.

(1) Mango Disease Management ($\mathcal{R}_1$);
(2) Conservation of horticulture ($\mathcal{R}_2$);
(3) Age and educational factor of farmers ($\mathcal{R}_3$);
(4) Adapt mango cultivation to climate change ($\mathcal{R}_4$).

There are three agriculture experts, that provided their observational data using PF information. The weight of experts is $(0.34, 0.33, 0.33)^T$.

Step-1. The expert matrices of PFNs is enclosed in Tables 1–3:

Table 1. Expert Matrix-1 of PFNs.

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
£$_1$	(0.3, 0.2, 0.4)	(0.4, 0.2, 0.3)	(0.3, 0.2, 0.3)	(0.5, 0.1, 0.3)
£$_2$	(0.2, 0.3, 0.5)	(0.3, 0.3, 0.3)	(0.4, 0.2, 0.3)	(0.3, 0.3, 0.3)
£$_3$	(0.4, 0.2, 0.2)	(0.2, 0.2, 0.5)	(0.3, 0.2, 0.4)	(0.2, 0.3, 0.4)
£$_4$	(0.4, 0.3, 0.1)	(0.3, 0.2, 0.4)	(0.2, 0.3, 0.4)	(0.2, 0.3, 0.3)

Table 2. Expert Matrix-2 of PFNs.

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
£$_1$	(0.3, 0.3, 0.2)	(0.3, 0.2, 0.4)	(0.2, 0.4, 0.3)	(0.4, 0.2, 0.3)
£$_2$	(0.3, 0.2, 0.5)	(0.4, 0.2, 0.3)	(0.4, 0.2, 0.3)	(0.3, 0.3, 0.2)
£$_3$	(0.3, 0.2, 0.4)	(0.4, 0.3, 0.1)	(0.5, 0.3, 0.1)	(0.4, 0.3, 0.2)
£$_4$	(0.5, 0.2, 0.1)	(0.2, 0.4, 0.3)	(0.2, 0.3, 0.3)	(0.3, 0.1, 0.4)

Table 3. Expert Matrix-3 of PFNs.

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
£$_1$	(0.3, 0.2, 0.3)	(0.2, 0.2, 0.5)	(0.2, 0.2, 0.4)	(0.2, 0.3, 0.4)
£$_2$	(0.4, 0.3, 0.1)	(0.2, 0.2, 0.5)	(0.2, 0.3, 0.4)	(0.1, 0.3, 0.4)
£$_3$	(0.3, 0.2, 0.4)	(0.3, 0.2, 0.3)	(0.4, 0.2, 0.3)	(0.4, 0.3, 0.2)
£$_4$	(0.1, 0.2, 0.5)	(0.4, 0.2, 0.3)	(0.3, 0.2, 0.3)	(0.3, 0.3, 0.2)

Step-2. The normalized decision matrices through normalization are evaluated in Tables 4–6:

Table 4. Normalized expert Matrix-1 of PFNs.

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
£$_1$	(0.4, 0.2, 0.3)	(0.4, 0.2, 0.3)	(0.3, 0.2, 0.3)	(0.5, 0.1, 0.3)
£$_2$	(0.5, 0.3, 0.2)	(0.3, 0.3, 0.3)	(0.4, 0.2, 0.3)	(0.3, 0.3, 0.3)
£$_3$	(0.2, 0.2, 0.4)	(0.2, 0.2, 0.5)	(0.3, 0.2, 0.4)	(0.2, 0.3, 0.4)
£$_4$	(0.1, 0.3, 0.4)	(0.3, 0.2, 0.4)	(0.2, 0.3, 0.4)	(0.2, 0.3, 0.3)

Table 5. Normalized expert Matrix-2 of PFNs.

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
£$_1$	(0.2, 0.3, 0.3)	(0.3, 0.2, 0.4)	(0.2, 0.4, 0.3)	(0.4, 0.2, 0.3)
£$_2$	(0.5, 0.2, 0.3)	(0.4, 0.2, 0.3)	(0.4, 0.2, 0.3)	(0.3, 0.3, 0.2)
£$_3$	(0.4, 0.2, 0.3)	(0.4, 0.3, 0.1)	(0.5, 0.3, 0.1)	(0.4, 0.3, 0.2)
£$_4$	(0.1, 0.2, 0.5)	(0.2, 0.4, 0.3)	(0.2, 0.3, 0.3)	(0.3, 0.1, 0.4)

Table 6. Normalized expert Matrix-3 of PFNs.

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
£$_1$	(0.3, 0.2, 0.3)	(0.2, 0.2, 0.5)	(0.2, 0.2, 0.4)	(0.2, 0.3, 0.4)
£$_2$	(0.1, 0.3, 0.4)	(0.2, 0.2, 0.5)	(0.2, 0.3, 0.4)	(0.1, 0.3, 0.4)
£$_3$	(0.4, 0.2, 0.3)	(0.3, 0.2, 0.3)	(0.4, 0.2, 0.3)	(0.4, 0.3, 0.2)
£$_4$	(0.5, 0.2, 0.1)	(0.4, 0.2, 0.3)	(0.3, 0.2, 0.3)	(0.3, 0.3, 0.2)

Step-3. Collected expert uncertain data is evaluated in Table 7:

Table 7. Collected expert uncertain data under PFNs.

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
£$_1$	$(0.305, 0.228, 0.300)$	$(0.305, 0.200, 0.390)$	$(0.235, 0.251, 0.329)$	$(0.379, 0.180, 0.329)$
£$_2$	$(0.392, 0.262, 0.287)$	$(0.304, 0.229, 0.355)$	$(0.340, 0.228, 0.329)$	$(0.239, 0.300, 0.288)$
£$_3$	$(0.338, 0.200, 0.330)$	$(0.303, 0.228, 0.248)$	$(0.404, 0.228, 0.230)$	$(0.338, 0.300, 0.253)$
£$_4$	$(0.258, 0.229, 0.272)$	$(0.304, 0.251, 0.330)$	$(0.234, 0.262, 0.330)$	$(0.267, 0.208, 0.288)$

Step-4. Evaluated the importance of consider attributes $\{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4\}$ using PF entropy measure as follows:

$$\sigma = (\sigma_1 = 256408, \sigma_2 = 263857, \sigma_3 = 243739, \sigma_4 = 235996)^T$$

Step-5(a). Utilized PFAWG operator to integrate aggregated data enclosed in Table 8:

Table 8. Picture fuzzy aggregated data (PFAWG).

£$_1$	$(0.29785, 0.21218, 0.34062)$
£$_2$	$(0.31075, 0.25123, 0.31766)$
£$_3$	$(0.34127, 0.23349, 0.27030)$
£$_4$	$(0.26432, 0.23636, 0.30698)$

Step-5(b). Utilized PFAOWG operator to integrate aggregated data enclosed in Table 9:

Table 9. Picture fuzzy aggregated data (PFAOWG).

£$_1$	$(0.29971, 0.21152, 0.33908)$
£$_2$	$(0.31143, 0.25121, 0.31706)$
£$_3$	$(0.34200, 0.23391, 0.26892)$
£$_4$	$(0.26389, 0.23520, 0.30569)$

Step-5(c). Utilized PFAHWG operator (under the associated weights $(0.2, 0.3, 0.1, 0.4)^T$) to integrate aggregated data enclosed in Table 10:

Table 10. Picture fuzzy aggregated data (PFAHWG).

£$_1$	$(0.28325, 0.22186, 0.32809)$
£$_2$	$(0.28247, 0.23961, 0.32170)$
£$_3$	$(0.31747, 0.22245, 0.26950)$
£$_4$	$(0.21320, 0.19100, 0.34393)$

Step-6. According to the score function in Definition 9, the score values of £$_{\mathcal{J}}$ ($\mathcal{J} = 1, 2, 3, 4$) are enclosed in Table 11:

Table 11. Score and ranking of PFNs.

	Score				
Operators	ξ(£$_1$)	ξ(£$_2$)	ξ(£$_3$)	ξ(£$_4$)	Ranking
PFAWG	0.5816	0.5806	0.6124	0.5736	£$_3$ > £$_1$ > £$_2$ > £$_4$
PFAOWG	0.5830	0.5810	0.6130	0.5743	£$_3$ > £$_1$ > £$_2$ > £$_4$
PFAHWG	0.5777	0.5737	0.6085	0.5594	£$_3$ > £$_1$ > £$_2$ > £$_4$

Step-7. Under all the proposed Aczel-Alsina operators, £$_3$ has the highest score value, therefore £$_3$ (**Climate Change**) is our best alternative with respect to offer attributes listed in factor that effecting Mango crop.

7. Sensitivity analysis

In this section, we modify the parameter ρ value from 0 to 30 to study the distinct patterns of the scores and ranking of the alternatives using the provided picture fuzzy Aczel-Alsina aggregation techniques. Table 12 contains the findings received from proposed PFAWG, PFAOWG, and PFAHWG operators. The acquired findings demonstrate to the decision makers that they can get the optimal alternative based on their preferences.

Table 12. Sensitivity analysis of parameter ρ.

Υ	Operators	Score ξ(£$_1$)	ξ(£$_2$)	ξ(£$_3$)	ξ(£$_4$)	Ranking Best Alternative
$\rightarrow 0.2$	*PFAWG*	0.5843	0.5834	0.6144	0.5746	£$_3$
	PFAOWG	0.5857	0.5839	0.6150	0.5752	£$_3$
$\rightarrow 1$	*PFAWG*	0.5831	0.5822	0.6136	0.5741	£$_3$
	PFAOWG	0.5845	0.5826	0.6141	0.5748	£$_3$
$\rightarrow 2$	*PFAWG*	0.5816	0.5806	0.6124	0.5736	£$_3$
	PFAOWG	0.5830	0.5810	0.6130	0.5743	£$_3$
$\rightarrow 5$	*PFAWG*	0.5772	0.5759	0.6087	0.5722	£$_3$
	PFAOWG	0.5785	0.5763	0.6092	0.5729	£$_3$
$\rightarrow 10$	*PFAWG*	0.5709	0.5693	0.6033	0.5703	£$_3$
	PFAOWG	0.5720	0.5697	0.6036	0.5710	£$_3$
$\rightarrow 15$	*PFAWG*	0.5664	0.5649	0.5998	0.5691	£$_3$
	PFAOWG	0.5673	0.5653	0.6000	0.5698	£$_3$
$\rightarrow 30$	*PFAWG*	0.5603	0.5588	0.5955	0.5674	£$_3$
	PFAOWG	0.5609	0.5591	0.5955	0.5679	£$_3$

The graphical representation of the parametric values is given in Figure 1.

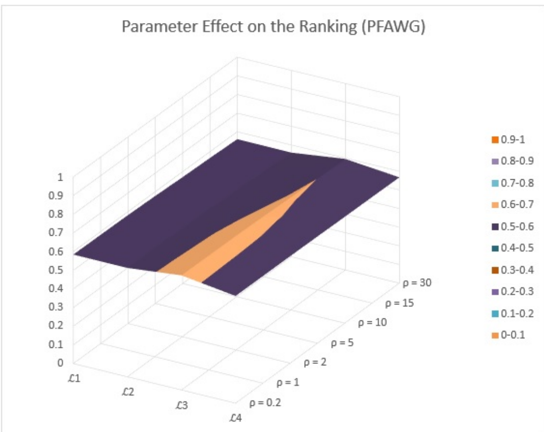

Figure 1. Effect of parametric values

8. Comparison analysis

Compare the features of developed picture fuzzy Aczel-Alsina aggregation operators with the existing decision support approaches is illustrated here to demonstrate the applicability and validity of the established picture fuzzy Aczel-Alsina operators based methodology.

Comparison with Li et al. [27]:

Li et al. [27] developed the list of novel picture fuzzy weighted interaction aggregation operators to sort out the best alternative. Tables 14 and 15 give the comparison findings and shows that the PFWG operator is a special instance of our developed PFAWG operator, and it acquires when $\rho = 1$.

For this reason, our recommended techniques are likely to become more comprehensive and more adaptable than a few existing techniques to control picture fuzzy MADM challenges.

Collected expert data [27] is presented in Table 13:

Table 13. Collected expert data under PFNs.

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
£$_1$	$(0.30, 0.23, 0.31)$	$(0.31, 0.20, 0.39)$	$(0.24, 0.27, 0.33)$	$(0.38, 0.23, 0.39)$
£$_2$	$(0.39, 0.28, 0.33)$	$(0.31, 0.24, 0.36)$	$(0.34, 0.23, 0.33)$	$(0.24, 0.27, 0.31)$
£$_3$	$(0.34, 0.20, 0.33)$	$(0.30, 0.25, 0.29)$	$(0.41, 0.25, 0.24)$	$(0.34, 0.27, 0.27)$
£$_4$	$(0.26, 0.24, 0.30)$	$(0.26, 0.27, 0.33)$	$(0.23, 0.27, 0.34)$	$(0.27, 0.34, 0.31)$

Comparative studies with collected expert data by [27] is enclosed in Table 14(a):

Table-14(a). Score and ranking of SFNs.

Li et al. [27]	Score				Ranking
	ξ(£$_1$)	ξ(£$_2$)	ξ(£$_3$)	ξ(£$_4$)	
PFWIA	0.2678	0.5767	1.000	0.002	£$_3$ > £$_1$ > £$_2$ > £$_4$
Developed Method	**Score**				**Ranking**
	ξ(£$_1$)	ξ(£$_2$)	ξ(£$_3$)	ξ(£$_4$)	
PFAWG	0.5816	0.5806	0.6124	0.5736	£$_3$ > £$_1$ > £$_2$ > £$_4$
PFAOWG	0.5830	0.5810	0.6130	0.5743	£$_3$ > £$_1$ > £$_2$ > £$_4$
PFAHWG	0.5777	0.5737	0.6085	0.5594	£$_3$ > £$_1$ > £$_2$ > £$_4$

(2) **Comparison with Garg [11] algebraic operators**

Comparative studies with collected expert data by [11] is enclosed in Table 14(b):

Table 14(b). Alternative ranking.

Garg [11]	Score				Ranking
	ξ(£$_1$)	ξ(£$_2$)	ξ(£$_3$)	ξ(£$_4$)	
PFWA	0.5776	0.5886	0.6099	0.5615	£$_3$ > £$_2$ > £$_1$ > £$_4$
Developed Method	**Score**				**Ranking**
	ξ(£$_1$)	ξ(£$_2$)	ξ(£$_3$)	ξ(£$_4$)	
PFAWG	0.5816	0.5806	0.6124	0.5736	£$_3$ > £$_1$ > £$_2$ > £$_4$
PFAOWG	0.5830	0.5810	0.6130	0.5743	£$_3$ > £$_1$ > £$_2$ > £$_4$
PFAHWG	0.5777	0.5737	0.6085	0.5594	£$_3$ > £$_1$ > £$_2$ > £$_4$

Comparison of attributes within a few currently exist techniques in Table 15:

Table 15. Comparison.

Techniques	Wether utilized fuzzy data	Wether make a data aggregation through parameter
Li et al. [27]	Yes	No
Garg [11]	Yes	No
Developed Method	Yes	Yes

9. Conclusions

In the present study, we extended the Aczel-Alsina t-norm and t-conorm to PF scenarios, proposed a few innovative working rules for PFNs, and investigated their features and interconnections. At that moment, centered on such novel working rules, a few new aggregation operators, in particular, the PFAWG operator, PFAOWG operator and PFAHWG operator have now been introduced to meet the scenarios where the given conflicts are in PFNs. Different alluring features and some particular instances of those operators have now been examined in further detail, as well as the linkages between those operators. The suggested operators, along with PF data, were placed on MAGDM problems, and a mathematical formulation was presented to show the decision-making mechanism. The effect of parameter F on decision-making outcomes has been examined. The most favorable alternative can be acquired with proposed operators by appropriately setting the parameter F. As a result, the suggested aggregation operators provide decision-makers with a new flexible method for reducing PF MAGDM difficulties. In other words, by providing a parameter, we can simply represent fuzzy information and make the information aggregation system more transparent than certain other current techniques. The existing aggregation operators [11,27], on the other hand, do not make data aggregation more flexible. As a result, our proposed aggregation operators are more sophisticated and trustworthy in PF data decision-making.

In future work, we shall further study the applications of Aczel-Alsina weighted aggregation operators of PFNs in other domains, such as intelligent manufacturing, machine learning, and data mining.

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Conflict of interest

The authors declare no conflict of interest.

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