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Research article

Some novel inequalities involving Atangana-Baleanu fractional integral operators and applications

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Abstract: As we know, Atangana and Baleanu developed great fractional integral operators which used the generalized Mittag-Leffler function as non-local and non-singular kernel. Inspired by these integral operators, we derive in this paper two new fractional integral identities involving Atangana-Baleanu fractional integrals. Using these identities as auxiliary results, we establish new fractional counterparts of classical inequalities essentially using first and second order differentiable higher order strongly n-polynomial convex functions. We also discuss several important special cases of the main results. In order to show the efficiency of our main results, we offer applications for special means and for differentiable functions of first and second order that are in absolute value bounded.

Keywords: Atangana-Baleanu fractional integrals; higher order strongly n-polynomial convex; Hölder's inequality; power mean inequality; special means; bounded functions
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1. Introduction and preliminaries

A set $K \subset \mathbb{R}$ is said to be convex, if $\forall\, \Theta, \Omega \in K$ and $\tau \in [0, 1]$, we have

$$(1-\tau)\Theta + \tau\Omega \in K.$$

A function $\Phi : K \to \mathbb{R}$ is said to be convex, if $\forall\, \Theta, \Omega \in K$ and $\tau \in [0, 1]$, we get

$$\Phi((1-\tau)\Theta + \tau\Omega) \le (1-\tau)\Phi(\Theta) + \tau\Phi(\Omega).$$

Theory of convexity has also played an important part in the development of theory of inequalities. Several results in theory of inequalities are direct consequences of the applications of convexity. Among these results one of the most extensively as well as intensively studied result is the Hermite-Hadamard inequality. This result reads as:
Let $\Phi : I = [a_1, a_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$\Phi\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \Phi(\Theta) d\Theta \leq \frac{\Phi(a_1) + \Phi(a_2)}{2}.$$

In recent years, different approaches have been used in obtaining new versions of Hermite-Hadamard inequality. Mohammed et al. [1] found new discrete inequalities of Hermite-Hadamard type for convex functions. Mohammed et al. [2] established generalized Hermite-Hadamard inequalities via the tempered fractional integrals. Mohammed et al. [3] obtained new fractional inequalities of Hermite-Hadamard type involving the incomplete gamma functions. Rahman et al. [4] derived certain fractional proportional integral inequalities via convex functions. Rahman et al. [5] give new bounds of generalized proportional fractional integrals in general form via convex functions and presented their applications. Sarikaya et al. [6] have used the approach of fractional calculus and obtained fractional analogues of Hermite-Hadamard inequality. Since then a variety of different approaches from fractional calculus have been used in obtaining fractional analogues of classical inequalities. For instance, Budak and Agarwal [7] obtained new generalized fractional midpoint type inequalities. Chu et al. [8] obtained new Simpson type of inequalities using Katugampola fractional integrals. Chu et al. [9] new generalized fractional Hermite-Hadamard inequality using χ_k-Hilfer fractional integrals. Kashuri et al. [10] obtained new generalized fractional integral identities and obtained new inequalities. Liu et al. [11] obtained new Hermite-Hadamard type of inequalities via ψ-fractional integrals. Onalan et al. [12] obtained fractional analogues of Hermite-Hadamard type integral inequalities via fractional integral operators with Mittag-Leffler kernel. Talib and Awan [13] obtained some new estimates of upper bounds for n-th order differentiable functions involving χ-Riemann-Liouville integrals via γ-preinvex functions. Wu et al. [14] obtained estimates of upper bound for a k-th order differentiable functions involving Riemann-Liouville integrals via higher order strongly h-preinvex functions. Wu et al. [15] established some integral inequalities for n-polynomial ζ-preinvex functions. Zhang et al. [16] obtained new k-fractional integral inequalities containing multi parameters via generalized (s, m)-preinvexity property of the functions. Huang et al. [17] derived some inequalities of the Hermite-Hadamard type via k-fractional conformable integrals. Rahman et al. [18] established certain inequalities via generalized proportional Hadamard fractional integral operators. Rahman et al. [19] obtained the Minkowski inequalities via generalized proportional fractional integral operators. For some more details, see [20–25].

In recent years, several new extensions and generalizations of classical convex functions have been defined in the literature. Chu et al. [8] introduced the notion of higher order strongly n-polynomial convex function as follows:

Definition 1.1. *[[8]] A function $\Phi : K \to \mathbb{R}$ is said to be higher order strongly n-polynomial convex, if $\forall\, \Theta, \Omega \in K, \tau \in [0, 1]$, and $u, \sigma > 0$, we have*

$$\Phi(\tau\Theta + (1 - \tau)\Omega)$$

$$\leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-\tau)^{s}]\Phi(\Theta)+\frac{1}{n}\sum_{s=1}^{n}[1-\tau^{s}]\Phi(\Omega)-u(\tau^{\sigma}(1-\tau)+\tau(1-\tau)^{\sigma})\|\Omega-\Theta\|^{\sigma}.$$

Remark 1.1. *Note that, if we take $u = 0$ in Definition 1.5, then we have the class of n-polynomial convex functions introduced and studied by Toplu et al. [26]. If we take $\sigma = 2$ in Definition 1.5, then we get the class of strongly n-polynomial convex functions. If we take $n = 1$ and $\sigma = 2$, then we obtain the class of strongly convex functions [27].*

Fractional calculus is an effective tool to explain physical phenomenas and also real world problems. The concept of fractional order derivative and integrals that will shed light on some unknown points about differential equations and solutions of some fractional order differential equations, which proved to be useless for their solution, is a novelty in applied sciences as well as in mathematics. New derivatives and integrals contribute to the solution of differential equations that are expressed and solved in classical analysis, as well as fractional order derivatives and integrals. In addition, it has increased its contribution to the literature with its applications in areas such as engineering, biostatistics and mathematical biology. Fractional derivative and integral operators not only differed from each other in terms of singularity, locality and kernels, but also brought innovations to fractional analysis in terms of their usage areas and spaces. Baleanu et al. [28], investigated existence results for solutions of a coupled system of hybrid boundary value problems with hybrid conditions. Khan et al. [29], analyzed positive solution and Hyers-Ulam stability for a class of singular fractional differential equations with p-Laplacian in Banach space. Khan et al. [30], give the existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel. For more details regarding fractional calculus and their applications, see [31–35].

Let us recall some basic concepts from fractional calculus which will be helpful in obtaining our main results.

Definition 1.2. *[36] Let $\Phi \in L_1[a_1, a_2]$. The Riemann-Liouville integrals $\mathrm{J}^{\nu}_{a_1^+}\Phi$ and $\mathrm{J}^{\nu}_{a_2^-}\Phi$ of order $\nu > 0$ are defined by*

$$\mathrm{J}^{\nu}_{a_1^+}\Phi(\Theta) = \frac{1}{\Gamma(\nu)}\int_{a_1}^{\Theta}(\Theta-\tau)^{\nu-1}\Phi(\tau)d\tau, \quad \Theta > a_1,$$

and

$$\mathrm{J}^{\nu}_{a_2^-}\Phi(\Theta) = \frac{1}{\Gamma(\nu)}\int_{\Theta}^{a_2}(\tau-\Theta)^{\nu-1}\Phi(\tau)d\tau, \quad \Theta < a_2.$$

In 2015 Caputo and Fabrizio suggested a new operator with fractional order, this derivative is based on the exponential kernel. Earlier this year 2016 Atangana and Baleanu developed another version which used the generalized Mittag-Leffler function as non-local and non-singular kernel. Both operators show some properties of filter. However, the Atangana and Baleanu version has in addition to this, all properties of fractional derivative. This shown effectiveness and advantages of the Atangana-Baleanu integral operators.

So, let see the following definitions about Atangana-Baleanu fractional derivatives and fractional integrals that are given in [37–39], respectively.

Definition 1.3. *Let $\Phi \in H^1(a_1, a_2)$ and not necessarily differentiable then, the definition of the new fractional derivative (Atangana-Baleanu fractional derivative in Riemann-Liouville sense) is given as follows:*

$$ {}_{a_1}^{\mathcal{ABR}}D_\tau^\alpha \Phi(\tau) = \frac{\mathcal{B}(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{a_1}^{\tau}\Phi(\Theta)E_\alpha\left[-\alpha\frac{(\tau-\Theta)^\alpha}{1-\alpha}\right]\mathrm{d}\Theta, \quad a_2 > a_1, \alpha \in [0,1], $$

where $E_\alpha(\Theta)$ is the well-known Mittag-Leffler function.

Definition 1.4. *Let $\Phi \in H^1(a_1, a_2)$ then, the definition of the new fractional derivative (Atangana-Baleanu derivative in Caputo sense) is given as:*

$$ {}_{a_1}^{\mathcal{ABC}}D_\tau^\alpha \Phi(\tau) = \frac{\mathcal{B}(\alpha)}{1-\alpha}\int_{a_1}^{\tau}\Phi'(\Theta)E_\alpha\left[-\alpha\frac{(\tau-\Theta)^\alpha}{1-\alpha}\right]\mathrm{d}\Theta, \quad a_2 > a_1, \alpha \in [0,1]. $$

Here, $\mathcal{B}(\alpha) > 0$ is the normalization function which satisfies the condition $\mathcal{B}(0) = \mathcal{B}(1) = 1$. They suggested that $\mathcal{B}(\alpha)$ has the same properties as in Caputo and Fabrizio case. The above definitions are very helpful to real world problem and also they have great advantages when Laplace transform is apply to solve some physical problems. Since the normalization function $\mathcal{B}(\alpha)$ is positive, it immediately follows that the fractional Atangana-Baleanu integral of a positive function is positive. It should be noted that, when the order $\alpha \to 1$, we recover the classical integral. Also, the initial function is recovered whenever the fractional order $\alpha \to 0$.

Definition 1.5. *The left hand side fractional integral related to the new fractional derivative with nonlocal kernel of a function $\Phi \in L_1[a_1, a_2]$ is defined as follows:*

$$ {}_{a_1}^{\mathcal{AB}}I_\tau^\alpha \Phi(\tau) = \frac{1-\alpha}{\mathcal{B}(\alpha)}\Phi(\tau) + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_{a_1}^{\tau}\Phi(\Theta)(\tau-\Theta)^{\alpha-1}\mathrm{d}\Theta, \quad a_2 > a_1, \alpha \in [0,1]. $$

The right hand side of Atangana-Baleanu fractional integral is given as:

$$ {}^{\mathcal{AB}}I_{a_2}^\alpha \Phi(\tau) = \frac{1-\alpha}{\mathcal{B}(\alpha)}\Phi(\tau) + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_{\tau}^{a_2}\Phi(\Theta)(\Theta-\tau)^{\alpha-1}\mathrm{d}\Theta. $$

Here, $\Gamma(\alpha)$ is the gamma function.

For more details about Atangana-Baleanu fractional integral operators and their applications, see [40–48].

Inspired by above results, the main motivation of this paper is to establish two new fractional integral identities involving Atangana-Baleanu fractional integrals. Using these identities as auxiliary results, we will derive new fractional counterparts of classical inequalities essentially using first and second order differentiable higher order strongly n-polynomial convex functions. Furthermore, in order to show the efficiency of our main results, we will offer applications for special means and for differentiable functions of first and second order that are in absolute value bounded. Finally, some conclusions and future research will be given.

2. Atangana-Baleanu fractional integral identities

In this section, we derive two new fractional integral identities using Atangana-Baleanu fractional integrals.

Lemma 2.1. *Let $\Phi : [a_1, a_2] \to \mathbb{R}$ be a differentiable function on (a_1, a_2) with $a_1 < a_2$. If $\Phi' \in L_1[a_1, a_2]$, then for all $\Theta \in [a_1, a_2]$, we have*

$$\begin{aligned}
&{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\} + {}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\tau}\{\Phi(a_1+a_2-\Theta)\} - \frac{\Phi(a_1+a_2-\Theta)}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha} + (a_2-\Theta)^{\alpha} + 2(1-\alpha)\Gamma(\alpha)] \\
&= \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\bigg[(\Theta-a_1)^{\alpha+1}\int_0^1 (1-\tau^{\alpha})\Phi'(\tau a_2 + (1-\tau)(a_1+a_2-\Theta))d\tau \\
&\quad + (a_2-\Theta)^{\alpha+1}\int_0^1 (\tau^{\alpha}-1)\Phi'(\tau a_1 + (1-\tau)(a_1+a_2-\Theta))d\tau\bigg].
\end{aligned}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\int_0^1 (1-\tau^{\alpha})\Phi'(\tau a_2 + (1-\tau)(a_1+a_2-\Theta))d\tau \\
&= -\frac{\Phi(a_1+a_2-\Theta)}{\Theta-a_1} + \frac{\alpha}{\Theta-a_1}\int_0^1 \tau^{\alpha-1}\Phi(\tau a_2 + (1-\tau)(a_1+a_2-\Theta))d\tau \\
&= -\frac{\Phi(a_1+a_2-\Theta)}{\Theta-a_1} + \frac{\alpha}{(\Theta-a_1)^{\alpha+1}}\int_{a_1+a_2-\Theta}^{a_2} (u-(a_1+a_2-\Theta))^{\alpha-1}\Phi(u)du.
\end{aligned}$$

Multiplying both sides of the last inequality by $\frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}$, and then adding the term $\frac{1-\alpha}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta)$, we get

$$\begin{aligned}
&\frac{(\Theta-a_1)^{\alpha+1}}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^1 (1-\tau^{\alpha})\Phi'(\tau a_2 + (1-\tau)(a_1+a_2-\Theta))d\tau + \frac{1-\alpha}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta) \\
&= -\frac{(\Theta-a_1)^{\alpha}}{\mathcal{B}(\alpha)\Gamma(\alpha)}\Phi(a_1+a_2-\Theta) + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_{a_1+a_2-\Theta}^{a_2}(u-(a_1+a_2-\Theta))^{\alpha-1}\Phi(u)du \\
&\quad + \frac{1-\alpha}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta) \\
&= -\frac{(\Theta-a_1)^{\alpha}}{\mathcal{B}(\alpha)\Gamma(\alpha)}\Phi(a_1+a_2-\Theta) + {}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}. \qquad (2.1)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{(a_2-\Theta)^{\alpha+1}}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^1 (\tau^{\alpha}-1)\Phi'(\tau a_1 + (1-\tau)(a_1+a_2-\Theta))d\tau + \frac{1-\alpha}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta) \\
&= -\frac{(a_2-\Theta)^{\alpha}}{\mathcal{B}(\alpha)\Gamma(\alpha)}\Phi(a_1+a_2-\Theta) + {}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1+a_2-\Theta)\}. \qquad (2.2)
\end{aligned}$$

By the identities (2.1) and (2.2), we obtain the required result. □

Remark 2.1. *Taking* $\alpha = 1$ *in Lemma 2.1, we get the following identity:*

$$\begin{aligned}
&\frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \Phi(\Theta)\mathrm{d}\Theta - \Phi(a_1 + a_2 - \Theta) \\
&= \frac{1}{a_2 - a_1}\bigg[(\Theta - a_1)^2 \int_0^1 (1-\tau)\Phi'(\tau a_2 + (1-\tau)(a_1 + a_2 - \Theta))\mathrm{d}\tau \\
&\quad + (a_2 - \Theta)^2 \int_0^1 (\tau - 1)\Phi'(\tau a_1 + (1-\tau)(a_1 + a_2 - \Theta))\mathrm{d}\tau\bigg].
\end{aligned}$$

Lemma 2.2. *Let* $\Phi : [a_1, a_2] \to \mathbb{R}$ *be a twice differentiable function on* (a_1, a_2) *with* $a_1 < a_2$*. If* $\Phi'' \in L_1[a_1, a_2]$*, then for all* $\Theta \in [a_1, a_2]$*, we have*

$$\begin{aligned}
&\frac{\Phi'(a_1 + a_2 - \Theta)}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}[(a_2 - \Theta)^{\alpha+1} - (\Theta - a_1)^{\alpha+1}] - [{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1 + a_2 - \Theta)\} + {}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1 + a_2 - \Theta)\}] \\
&\quad + \frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi(a_1 + a_2 - \Theta) + \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta - a_1)^{\alpha}\Phi(a_2) + (a_2 - \Theta)^{\alpha}\Phi(a_1)] \\
&= \frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\bigg[(\Theta - a_1)^{\alpha+2}\int_0^1 (1 - \tau^{\alpha+1})\Phi''(\tau a_2 + (1-\tau)(a_1 + a_2 - \Theta))\mathrm{d}\tau \\
&\quad - (a_2 - \Theta)^{\alpha+2}\int_0^1 (\tau^{\alpha+1} - 1)\Phi''(\tau a_1 + (1-\tau)(a_1 + a_2 - \Theta))\mathrm{d}\tau\bigg]. \qquad (2.3)
\end{aligned}$$

Proof. Consider the right hand side of 2.3, we have

$$\begin{aligned}
I &:= \frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\bigg[(\Theta - a_1)^{\alpha+2}\int_0^1 (1 - \tau^{\alpha+1})\Phi''(\tau a_2 + (1-\tau)(a_1 + a_2 - \Theta))\mathrm{d}\tau \\
&\quad - (a_2 - \Theta)^{\alpha+2}\int_0^1 (\tau^{\alpha+1} - 1)\Phi''(\tau a_1 + (1-\tau)(a_1 + a_2 - \Theta))\mathrm{d}\tau\bigg] \\
&= \frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta - a_1)^{\alpha+2}I_1 - (a_2 - \Theta)^{\alpha+2}I_2], \qquad (2.4)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \int_0^1 (1 - \tau^{\alpha+1})\Phi''(\tau a_2 + (1-\tau)(a_1 + a_2 - \Theta))\mathrm{d}\tau \\
&= -\frac{\Phi'(a_1 + a_2 - \Theta)}{\Theta - a_1} + \frac{\alpha+1}{\Theta - a_1}\int_0^1 \tau^{\alpha}\Phi'(\tau a_2 + (1-\tau)(a_1 + a_2 - \Theta))\mathrm{d}\tau \\
&= -\frac{\Phi'(a_1 + a_2 - \Theta)}{\Theta - a_1} + \frac{\alpha+1}{(\Theta - a_1)}\bigg[\frac{\Phi(a_2)}{\Theta - a_1} - \frac{\alpha}{(\Theta - a_1)}\int_0^1 \tau^{\alpha-1}\Phi(\tau a_2 + (1-\tau)(a_1 + a_2 - \Theta))\mathrm{d}\tau\bigg] \\
&= -\frac{\Phi'(a_1 + a_2 - \Theta)}{\Theta - a_1} + \frac{(\alpha+1)\Phi(a_2)}{(\Theta - a_1)^2} - \frac{\alpha(\alpha+1)}{(\Theta - a_1)^2}\int_0^1 \tau^{\alpha-1}\Phi(\tau a_2 + (1-\tau)(a_1 + a_2 - \Theta))\mathrm{d}\tau \\
&= -\frac{\Phi'(a_1 + a_2 - \Theta)}{\Theta - a_1} + \frac{(\alpha+1)\Phi(a_2)}{(\Theta - a_1)^2} + \frac{(1+\alpha)\Gamma(\alpha)\Phi(a_1 + a_2 - \Theta)}{(\Theta - a_1)^{\alpha+2}} \\
&\quad - \frac{(1+\alpha)\mathcal{B}(\alpha)\Gamma(\alpha)}{(\Theta - a_1)^{\alpha+2}}{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1 + a_2 - \Theta)\}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
I_2 &:= \int_0^1 (\tau^{\alpha+1}-1)\Phi''(\tau a_1+(1-\tau)(a_1+a_2-\Theta))\mathrm{d}\tau \\
&= -\frac{\Phi'(a_1+a_2-\Theta)}{a_2-\Theta} - \frac{(\alpha+1)\Phi(a_1)}{(a_2-\Theta)^2} - \frac{(1+\alpha)\Gamma(\alpha)\Phi(a_1+a_2-\Theta)}{(a_2-\Theta)^{\alpha+2}} \\
&\quad + \frac{(1+\alpha)\mathcal{B}(\alpha)\Gamma(\alpha)}{(a_2-\Theta)^{\alpha+2}} {}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1+a_2-\Theta)\}.
\end{aligned}$$

By substituting the values of I_1 and I_2 in 2.4, we obtain our required result. □

Remark 2.2. *Taking $\alpha = 1$ in Lemma 2.2, we get the following identity:*

$$\begin{aligned}
&\Phi'(a_1+a_2-\Theta)[(a_2-\Theta)^2-(\Theta-a_1)^2] + 2[(\Theta-a_1)\Phi(a_2)+(a_2-\Theta)\Phi(a_1)] - 2\int_{a_1}^{a_2}\Phi(\Theta)\mathrm{d}\Theta \\
&= \Bigg[(\Theta-a_1)^3\int_0^1 (1-\tau^2)\Phi''(\tau a_2+(1-\tau)(a_1+a_2-\Theta))\mathrm{d}\tau \\
&\quad - (a_2-\Theta)^3\int_0^1(\tau^2-1)\Phi''(\tau a_1+(1-\tau)(a_1+a_2-\Theta))\mathrm{d}\tau\Bigg].
\end{aligned}$$

3. Inequalities involving Atangana-Baleanu fractional integrals

In this section, we discuss our main results.

Theorem 3.1. *Let $\Phi : [a_1, a_2] \to \mathbb{R}$ be a differentiable function on (a_1, a_2) with $0 \le a_1 < a_2$. If $|\Phi'|^q$ is higher order strongly n-polynomial convex on $[a_1, a_2]$ for $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, then for all $\Theta \in [a_1, a_2]$ the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\} + {}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\tau}\{\Phi(a_1+a_2-\Theta)\} - \frac{\Phi(a_1+a_2-\Theta)}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^\alpha+(a_2-\Theta)^\alpha+2(1-\alpha)\Gamma(\alpha)]\right| \\
&\le \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\Bigg[(\Theta-a_1)^{\alpha+1}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi'(a_2)|^q+|\Phi'(a_1+a_2-\Theta)|^q) - \frac{2u\|\Theta-a_1\|^\sigma}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}} \\
&\quad +(a_2-\Theta)^{\alpha+1}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi'(a_1)|^q+|\Phi'(a_1+a_2-\Theta)|^q) - \frac{2u\|a_2-\Theta\|^\sigma}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Proof. It is evident that

$$\int_0^1[1-(1-\tau)^s]\mathrm{d}\tau = \int_0^1[1-\tau^s]\mathrm{d}\tau = \frac{s}{s+1}.$$

Applying Lemma 2.1 and properties of modulus, we have

$$\begin{aligned}
&\left|{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\} + {}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\tau}\{\Phi(a_1+a_2-\Theta)\} - \frac{\Phi(a_1+a_2-\Theta)}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^\alpha+(a_2-\Theta)^\alpha+2(1-\alpha)\Gamma(\alpha)]\right| \\
&\le \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\Bigg[(\Theta-a_1)^{\alpha+1}\int_0^1|1-\tau^\alpha||\Phi'(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|\mathrm{d}\tau
\end{aligned}$$

$$+ (a_2-\Theta)^{\alpha+1}\int_0^1 |\tau^\alpha - 1||\Phi'(\tau a_1 + (1-\tau)(a_1+a_2-\Theta))|d\tau\Bigg].$$

Using Hölder's inequality and higher order strongly n-polynomial convexity of $|\Phi'|^q$, we get

$$\begin{aligned}
&\int_0^1 |1-\tau^\alpha||\Phi'(\tau a_2 + (1-\tau)(a_1+a_2-\Theta))|d\tau \\
&\le \left(\int_0^1 |1-\tau^\alpha|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1 |\Phi'(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|^q d\tau\right)^{\frac{1}{q}} \\
&\le \left(\int_0^1 (1-\tau^{\alpha p})d\tau\right)^{\frac{1}{p}}\Bigg(\int_0^1 \Bigg[\frac{1}{n}\sum_{s=1}^{n}[1-(1-\tau)^s]|\Phi'(a_2)|^q + \frac{1}{n}\sum_{s=1}^{n}[1-\tau^s]|\Phi'(a_1+a_2-\Theta)|^q \\
&-u(\tau^\sigma(1-\tau)+\tau(1-\tau)^\sigma)||\Theta-a_1||^\sigma\Bigg]d\tau\Bigg)^{\frac{1}{q}} \\
&= \left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi'(a_2)|^q + |\Phi'(a_1+a_2-\Theta)|^q) - \frac{2u||\Theta-a_1||^\sigma}{(\sigma+1)(\sigma+2)}\right]^{\frac{1}{q}}.
\end{aligned} \tag{3.1}$$

Here, we use

$$(A-\mathcal{B})^q \le A^q - \mathcal{B}^q$$

for any $A > \mathcal{B} \ge 0$ and $q \ge 1$.
Similarly, we have

$$\begin{aligned}
&\int_0^1 |\tau^\alpha - 1||\Phi'(\tau a_1 + (1-\tau)(a_1+a_2-\Theta))|d\tau \\
&\le \left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi'(a_1)|^q + |\Phi'(a_1+a_2-\Theta)|^q) - \frac{2u||a_2-\Theta||^\sigma}{(\sigma+1)(\sigma+2)}\right]^{\frac{1}{q}}.
\end{aligned} \tag{3.2}$$

By the inequalities (3.1) and (3.2), we obtain required result. □

Corollary 3.1. *Taking $\alpha = 1$ in Theorem 3.1 and using Remark 2.1, we have*

$$\begin{aligned}
&\left|\Phi(a_1+a_2-\Theta) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\Phi(\Theta)d\Theta\right| \\
&\le \frac{1}{a_2-a_1}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\Bigg[(\Theta-a_1)^2\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi'(a_2)|^q + |\Phi'(a_1+a_2-\Theta)|^q) - \frac{2u||\Theta-a_1||^\sigma}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}} \\
&+(a_2-\Theta)^2\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi'(a_1)|^q + |\Phi'(a_1+a_2-\Theta)|^q) - \frac{2u||a_2-\Theta||^\sigma}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Corollary 3.2. *Taking $u \to 0^+$ in Theorem 3.1, we have*

$$\left|{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\} + {}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\tau}\{\Phi(a_1+a_2-\Theta)\} - \frac{\Phi(a_1+a_2-\Theta)}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^\alpha + (a_2-\Theta)^\alpha + 2(1-\alpha)\Gamma(\alpha)]\right|$$

$$\leq \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[(\Theta-a_1)^{\alpha+1}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi'(a_2)|^q+|\Phi'(a_1+a_2-\Theta)|^q)\right)^{\frac{1}{q}}\right.$$
$$\left.+(a_2-\Theta)^{\alpha+1}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi'(a_1)|^q+|\Phi'(a_1+a_2-\Theta)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 3.3. *Under assumptions of Theorem 3.1 with $\Theta=\frac{a_1+a_2}{2}$, we have the following inequality*

$$\left|{}^{\mathcal{AB}}I^{\alpha}_{a_2}\Phi\left(\frac{a_1+a_2}{2}\right)+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\left(\frac{a_1+a_2}{2}\right)}\Phi\left(\frac{a_1+a_2}{2}\right)-2\frac{\Phi\left(\frac{a_1+a_2}{2}\right)}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left[\left(\frac{a_2-a_1}{2}\right)^{\alpha}+(1-\alpha)\Gamma(\alpha)\right]\right|$$
$$\leq \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{a_2-a_1}{2}\right)^{\alpha+1}\left[\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}\left(|\Phi'(a_2)|^q+|\Phi'\left(\frac{a_1+a_2}{2}\right)|^q\right)-\frac{2u\|\left(\frac{a_2-a_1}{2}\right)\|^{\sigma}}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}\left(|\Phi'(a_1)|^q+|\Phi'\left(\frac{a_1+a_2}{2}\right)|^q\right)-\frac{2u\|\left(\frac{a_2-a_1}{2}\right)\|^{\sigma}}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}\right].$$

Theorem 3.2. *Let $\Phi:[a_1,a_2]\to\mathbb{R}$ be a differentiable function on (a_1,a_2) with $0\leq a_1<a_2$. If $|\Phi'|^q$ is higher order strongly n-polynomial convex on $[a_1,a_2]$ for $q\geq 1$, then for all $\Theta\in[a_1,a_2]$ the following inequality for fractional integrals holds:*

$$\left|{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\tau}\{\Phi(a_1+a_2-\Theta)\}-\frac{\Phi(a_1+a_2-\Theta)}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}+(a_2-\Theta)^{\alpha}+2(1-\alpha)\Gamma(\alpha)]\right|$$
$$\leq \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[(\Theta-a_1)^{\alpha+1}\left[\frac{|\Phi'(a_2)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)}\right)\right.\right.$$
$$+\frac{|\Phi'(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{1}{\alpha+s+1}\right)$$
$$\left.-u\|\Theta-a_1\|^{\sigma}\left(\frac{1}{(\sigma+1)(\sigma+2)}-\frac{1}{(1+\alpha+\sigma)(2+\alpha+\sigma)}-\frac{\Gamma(2+\alpha)\Gamma(1+\sigma)}{\Gamma(3+\alpha+\sigma)}\right)\right]^{\frac{1}{q}}$$
$$+(a_2-\Theta)^{\alpha+1}\left[\frac{|\Phi'(a_1)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)}\right)\right.$$
$$+\frac{|\Phi'(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{1}{\alpha+s+1}\right)$$
$$\left.\left.-u\|a_2-\Theta\|^{\sigma}\left(\frac{1}{(\sigma+1)(\sigma+2)}-\frac{1}{(1+\alpha+\sigma)(2+\alpha+\sigma)}-\frac{\Gamma(2+\alpha)\Gamma(1+\sigma)}{\Gamma(3+\alpha+\sigma)}\right)\right]^{\frac{1}{q}}\right].$$

Proof. From Lemma 2.1, properties of modulus and power mean inequality, we have

$$\left|{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\tau}\{\Phi(a_1+a_2-\Theta)\}-\frac{\Phi(a_1+a_2-\Theta)}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}+(a_2-\Theta)^{\alpha}+2(1-\alpha)\Gamma(\alpha)]\right|$$

$$\leq \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\Bigg[(\Theta-a_1)^{\alpha+1}\int_0^1 |1-\tau^{\alpha}||\Phi'(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|d\tau$$

$$+\ (a_2-\Theta)^{\alpha+1}\int_0^1 |\tau^{\alpha}-1||\Phi'(\tau a_1+(1-\tau)(a_1+a_2-\Theta))|d\tau\Bigg]$$

$$\leq \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\Bigg[(\Theta-a_1)^{\alpha+1}\left(\int_0^1 |1-\tau^{\alpha}|d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-\tau^{\alpha}||\Phi'(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|^q d\tau\right)^{\frac{1}{q}}$$

$$+(a_2-\Theta)^{\alpha+1}\left(\int_0^1 |\tau^{\alpha}-1|d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1 |\tau^{\alpha}-1||\Phi'(\tau a_1+(1-\tau)(a_1+a_2-\Theta))|^q d\tau\right)^{\frac{1}{q}}\Bigg].$$

Using the higher order strongly n-polynomial convexity of $|\Phi'|^q$, we get

$$\int_0^1 |1-\tau^{\alpha}||\Phi'(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|d\tau$$

$$\leq \int_0^1 (1-\tau^{\alpha})\Bigg[\frac{1}{n}\sum_{s=1}^{n}[1-(1-\tau)^s]|\Phi'(a_2)|^q$$

$$+\frac{1}{n}\sum_{s=1}^{n}[1-\tau^s]|\Phi'(a_1+a_2-\Theta)|^q-u(\tau^{\sigma}(1-\tau)+\tau(1-\tau)^{\sigma})\|\Theta-a_1\|^{\sigma}\Bigg]d\tau$$

$$=\frac{|\Phi'(a_2)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)}\right)$$

$$+\frac{|\Phi'(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{1}{\alpha+s+1}\right)$$

$$-u\|\Theta-a_1\|^{\sigma}\left(\frac{1}{(\sigma+1)(\sigma+2)}-\frac{1}{(1+\alpha+\sigma)(2+\alpha+\sigma)}-\frac{\Gamma(2+\alpha)\Gamma(1+\sigma)}{\Gamma(3+\alpha+\sigma)}\right). \tag{3.3}$$

Similarly, we have

$$\int_0^1 |\tau^{\alpha}-1||\Phi'(\tau a_1+(1-\tau)(a_1+a_2-\Theta))|d\tau$$

$$\leq \int_0^1 (\tau^{\alpha}-1)\Bigg[\frac{1}{n}\sum_{s=1}^{n}[1-(1-\tau)^s]|\Phi'(a_1)|^q$$

$$+\frac{1}{n}\sum_{s=1}^{n}[1-\tau^s]|\Phi'(a_1+a_2-\Theta)|^q-u(\tau^{\sigma}(1-\tau)+\tau(1-\tau)^{\sigma})\|a_2-\Theta\|^{\sigma}\Bigg]d\tau$$

$$=\frac{|\Phi'(a_1)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)}\right)$$

$$+\frac{|\Phi'(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{1}{\alpha+s+1}\right)$$

$$-u\|a_2-\Theta\|^{\sigma}\left(\frac{1}{(\sigma+1)(\sigma+2)}-\frac{1}{(1+\alpha+\sigma)(2+\alpha+\sigma)}-\frac{\Gamma(2+\alpha)\Gamma(1+\sigma)}{\Gamma(3+\alpha+\sigma)}\right). \tag{3.4}$$

By the inequalities (3.3) and (3.4), we obtain required result. □

Corollary 3.4. *Taking $\alpha = 1$ in Theorem 3.2 and using Remark 2.1, we have*

$$\begin{aligned}
&\left|\Phi(a_1+a_2-\Theta)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\Phi(\Theta)d\Theta\right| \\
&\leq \frac{1}{a_2-a_1}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[(\Theta-a_1)^2\left[\frac{|\Phi'(a_2)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{\Gamma(s+1)}{\Gamma(s+3)}\right)\right.\right. \\
&\quad+\frac{|\Phi'(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{1}{s+2}\right) \\
&\quad\left.-u\|\Theta-a_1\|^{\sigma}\left(\frac{1}{(\sigma+1)(\sigma+2)}-\frac{1}{(2+\sigma)(3+\sigma)}-\frac{2\Gamma(1+\sigma)}{\Gamma(4+\sigma)}\right)\right]^{\frac{1}{q}} \\
&\quad+(a_2-\Theta)^2\left[\frac{|\Phi'(a_1)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{\Gamma(s+1)}{\Gamma(s+3)}\right)\right. \\
&\quad+\frac{|\Phi'(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{1}{s+2}\right) \\
&\quad\left.\left.-u\|a_2-\Theta\|^{\sigma}\left(\frac{1}{(\sigma+1)(\sigma+2)}-\frac{1}{(2+\sigma)(3+\sigma)}-\frac{2\Gamma(1+\sigma)}{\Gamma(4+\sigma)}\right)\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 3.5. *Taking $u \to 0^+$ in Theorem 3.2, we have*

$$\begin{aligned}
&\left|{}^{\mathcal{AB}}I_{a_2}^{\alpha}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I_{\tau}^{\alpha}\{\Phi(a_1+a_2-\Theta)\}-\frac{\Phi(a_1+a_2-\Theta)}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}+(a_2-\Theta)^{\alpha}+2(1-\alpha)\Gamma(\alpha)]\right| \\
&\leq \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[(\Theta-a_1)^{\alpha+1}\left[\frac{|\Phi'(a_2)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)}\right)\right.\right. \\
&\quad\left.+\frac{|\Phi'(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{1}{\alpha+s+1}\right)\right]^{\frac{1}{q}} \\
&\quad+(a_2-\Theta)^{\alpha+1}\left[\frac{|\Phi'(a_1)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)}\right)\right. \\
&\quad\left.\left.+\frac{|\Phi'(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{1}{\alpha+s+1}\right)\right]^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 3.6. *Under assumptions of Theorem 3.2 with $\Theta = \frac{a_1+a_2}{2}$, we have the following inequality*

$$\begin{aligned}
&\left|{}^{\mathcal{AB}}I_{a_2}^{\alpha}\Phi\left(\frac{a_1+a_2}{2}\right)+{}^{\mathcal{AB}}_{a_1}I_{\left(\frac{a_1+a_2}{2}\right)}^{\alpha}\Phi\left(\frac{a_1+a_2}{2}\right)-2\frac{\Phi\left(\frac{a_1+a_2}{2}\right)}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left[\left(\frac{a_2-a_1}{2}\right)^{\alpha}+(1-\alpha)\Gamma(\alpha)\right]\right| \\
&\leq \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{a_2-a_1}{2}\right)^{\alpha+1}\left[\left[\frac{|\Phi'(a_2)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)}\right)\right.\right. \\
&\quad+\frac{|\Phi'\left(\frac{a_1+a_2}{2}\right)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{1}{\alpha+s+1}\right)
\end{aligned}$$

$$-u\left\|\left(\frac{a_2-a_1}{2}\right)\right\|^{\sigma}\left(\frac{1}{(\sigma+1)(\sigma+2)}-\frac{1}{(1+\alpha+\sigma)(2+\alpha+\sigma)}-\frac{\Gamma(2+\alpha)\Gamma(1+\sigma)}{\Gamma(3+\alpha+\sigma)}\right)\Bigg]^{\frac{1}{q}}$$
$$+\Bigg[\frac{|\Phi'(a_1)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)}\right)$$
$$+\frac{|\Phi'\left(\frac{a_1+a_2}{2}\right)|^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{1}{\alpha+s+1}\right)$$
$$-u\left\|\left(\frac{a_2-a_1}{2}\right)\right\|^{\sigma}\left(\frac{1}{(\sigma+1)(\sigma+2)}-\frac{1}{(1+\alpha+\sigma)(2+\alpha+\sigma)}-\frac{\Gamma(2+\alpha)\Gamma(1+\sigma)}{\Gamma(3+\alpha+\sigma)}\right)\Bigg]^{\frac{1}{q}}\Bigg].$$

Theorem 3.3. *Let $\Phi : [a_1, a_2] \to \mathbb{R}$ be a twice differentiable function on (a_1, a_2) with $0 \le a_1 < a_2$. If $|\Phi''|^q$ is higher order strongly n-polynomial convex on $[a_1, a_2]$ for $q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$, then for all $\Theta \in [a_1, a_2]$ the following inequality for fractional integrals holds:*

$$\Bigg|\frac{\Phi'(a_1+a_2-\Theta)}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}[(a_2-\Theta)^{\alpha+1}-(\Theta-a_1)^{\alpha+1}]-[{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1+a_2-\Theta)\}]$$
$$+\frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta)+\frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}\Phi(a_2)+(a_2-\Theta)^{\alpha}\Phi(a_1)]\Bigg|$$
$$\le\frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{p(\alpha+1)}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\Bigg[(\Theta-a_1)^{\alpha+2}\Bigg(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi''(a_2)|^q+|\Phi''(a_1+a_2-\Theta)|^q)$$
$$-\frac{2u\|\Theta-a_1\|^{\sigma}}{(\sigma+1)(\sigma+2)}\Bigg)^{\frac{1}{q}}+(a_2-\Theta)^{\alpha+2}\Bigg(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi''(a_1)|^q+|\Phi''(a_1+a_2-\Theta)|^q)-\frac{2u\|a_2-\Theta\|^{\sigma}}{(\sigma+1)(\sigma+2)}\Bigg)^{\frac{1}{q}}\Bigg],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Applying the Lemma 2.2 and properties of modulus, we have

$$\Bigg|\frac{\Phi'(a_1+a_2-\Theta)}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}[(a_2-\Theta)^{\alpha+1}-(\Theta-a_1)^{\alpha+1}]-[{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1+a_2-\Theta)\}]$$
$$+\frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta)+\frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}\Phi(a_2)+(a_2-\Theta)^{\alpha}\Phi(a_1)]\Bigg|$$
$$\le\frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\Bigg[(\Theta-a_1)^{\alpha+2}\int_0^1|1-\tau^{\alpha+1}||\Phi''(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|d\tau$$
$$+(a_2-\Theta)^{\alpha+2}\int_0^1|\tau^{\alpha+1}-1||\Phi''(\tau a_1+(1-\tau)(a_1+a_2-\Theta))|d\tau\Bigg].$$

Using Hölder's inequality and higher order strongly n-polynomial convexity of $|\Phi''|^q$, we get

$$\int_0^1|1-\tau^{\alpha+1}||\Phi''(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|d\tau$$
$$\le\left(\int_0^1|1-\tau^{\alpha+1}|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1|\Phi''(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|^q d\tau\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 (1-\tau^{(\alpha+1)p})\mathrm{d}\tau\right)^{\frac{1}{p}} \left(\int_0^1 \left[\frac{1}{n}\sum_{s=1}^{n}[1-(1-\tau)^s]|\Phi''(a_2)|^q + \frac{1}{n}\sum_{s=1}^{n}[1-\tau^s]|\Phi''(a_1+a_2-\Theta)|^q\right.\right.$$

$$\left.\left. -u(\tau^\sigma(1-\tau)+\tau(1-\tau)^\sigma)\|\Theta-a_1\|^\sigma\right]\mathrm{d}\tau\right)^{\frac{1}{q}}$$

$$= \left(\frac{(\alpha+1)p}{(\alpha+1)p+1}\right)^{\frac{1}{p}} \left[\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi''(a_2)|^q + |\Phi''(a_1+a_2-\Theta)|^q) - \frac{2u\|\Theta-a_1\|^\sigma}{(\sigma+1)(\sigma+2)}\right]^{\frac{1}{q}}. \tag{3.5}$$

Here, we use

$$(A-\mathcal{B})^q \leq A^q - \mathcal{B}^q$$

for any $A > \mathcal{B} \geq 0$ and $q \geq 1$.
Similarly, we have

$$\int_0^1 |\tau^{\alpha+1}-1||\Phi''(\tau a_1 + (1-\tau)(a_1+a_2-\Theta))|\mathrm{d}\tau$$

$$\leq \left(\frac{(\alpha+1)p}{(\alpha+1)p+1}\right)^{\frac{1}{p}} \left[\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi''(a_1)|^q + |\Phi''(a_1+a_2-\Theta)|^q) - \frac{2u\|a_2-\Theta\|^\sigma}{(\sigma+1)(\sigma+2)}\right]^{\frac{1}{q}}. \tag{3.6}$$

By the inequalities (3.5) and (3.6), we obtain required result. □

Corollary 3.7. *Taking $\alpha = 1$ in Theorem 3.3 and using Remark 2.2, we have*

$$\left|\Phi'(a_1+a_2-\Theta)[(a_2-\Theta)^2-(\Theta-a_1)^2] + 2[(\Theta-a_1)\Phi(a_2)+(a_2-\Theta)\Phi(a_1)] - 2\int_{a_1}^{a_2}\Phi(\Theta)\mathrm{d}\Theta\right|$$

$$\leq \left(\frac{2p}{2p+1}\right)^{\frac{1}{p}} \left[(\Theta-a_1)^3\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi''(a_2)|^q + |\Phi''(a_1+a_2-\Theta)|^q)\right.\right.$$

$$\left.\left. -\frac{2u\|\Theta-a_1\|^\sigma}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}} + (a_2-\Theta)^3\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi''(a_1)|^q + |\Phi''(a_1+a_2-\Theta)|^q) - \frac{2u\|a_2-\Theta\|^\sigma}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}\right].$$

Corollary 3.8. *Taking $u \to 0^+$ in Theorem 3.3, we have*

$$\left|\frac{\Phi'(a_1+a_2-\Theta)}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}[(a_2-\Theta)^{\alpha+1}-(\Theta-a_1)^{\alpha+1}] - [{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\} + {}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1+a_2-\Theta)\}]\right.$$

$$\left. +\frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta) + \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^\alpha\Phi(a_2)+(a_2-\Theta)^\alpha\Phi(a_1)]\right|$$

$$\leq \frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{p(\alpha+1)}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \left[(\Theta-a_1)^{2\alpha+3}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi''(a_2)|^q + |\Phi''(a_1+a_2-\Theta)|^q)\right)^{\frac{1}{q}}\right.$$

$$\left. +(a_2-\Theta)^{2\alpha+3}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}(|\Phi''(a_1)|^q + |\Phi''(a_1+a_2-\Theta)|^q)\right)^{\frac{1}{q}}\right].$$

Corollary 3.9. *Under assumptions of Theorem 3.3 with* $\Theta = \frac{a_1+a_2}{2}$*, we have the following inequality*

$$\left|\frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi\left(\frac{a_1+a_2}{2}\right)-\left[{}^{\mathcal{AB}}I^{\alpha}_{a_2}\left\{\Phi\left(\frac{a_1+a_2}{2}\right)\right\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\left(\frac{a_1+a_2}{2}\right)}\left\{\Phi\left(\frac{a_1+a_2}{2}\right)\right\}\right]+\frac{(a_2-a_1)^{\alpha}}{2^{\alpha}\mathcal{B}(\alpha)\Gamma(\alpha)}[\Phi(a_1)+\Phi(a_2)]\right|$$
$$\leq \frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{p(\alpha+1)}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left[\left(\frac{a_2-a_1}{2}\right)^{\alpha+2}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}\left(|\Phi''(a_2)|^q+\left|\Phi''\left(\frac{a_1+a_2}{2}\right)\right|^q\right)\right.\right.$$
$$\left.\left.-\frac{2u\|a_2-a_1\|^{\sigma}}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}+\left(\frac{a_2-a_1}{2}\right)^{\alpha+2}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}\left(|\Phi''(a_1)|^q+\left|\Phi''\left(\frac{a_1+a_2}{2}\right)\right|^q\right)-\frac{2u\|a_2-\Theta\|^{\sigma}}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}\right].$$

Theorem 3.4. *Let* $\Phi : [a_1, a_2] \to \mathbb{R}$ *be a differentiable function on* (a_1, a_2) *with* $0 \leq a_1 < a_2$*. If* $|\Phi''|^q$ *is higher order strongly n-polynomial convex on* $[a_1, a_2]$ *for* $q \geq 1$*, then for all* $\Theta \in [a_1, a_2]$ *the following inequality for fractional integrals holds:*

$$\left|\frac{\Phi'(a_1+a_2-\Theta)}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}[(a_2-\Theta)^{\alpha+1}-(\Theta-a_1)^{\alpha+1}]-[{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1+a_2-\Theta)\}]\right.$$
$$\left.+\frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta)+\frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}\Phi(a_2)+(a_2-\Theta)^{\alpha}\Phi(a_1)]\right|$$
$$\leq \frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha+1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[(\Theta-a_1)^{\alpha+2}\left(\frac{|\Phi''(a_2)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{\Gamma(1+s)\Gamma(2+\alpha)}{\Gamma(3+s+\alpha)}\right)\right.\right.$$
$$+\frac{|\Phi''(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{1}{2+s+\alpha}\right)$$
$$\left.-u\|\Theta-a_1\|^{\sigma}\left(\frac{2}{(1+\sigma)(2+\sigma)}-\frac{1}{(2+\alpha+\sigma)(3+\alpha+\sigma)}-\frac{\Gamma(3+\alpha)\Gamma(1+\sigma)}{\Gamma(4+\alpha+\sigma)}\right)\right)^{\frac{1}{q}}$$
$$+(a_2-\Theta)^{\alpha+2}\left(\frac{|\Phi''(a_1)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{\Gamma(1+s)\Gamma(2+\alpha)}{\Gamma(3+s+\alpha)}\right)\right.$$
$$+\frac{|\Phi''(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{1}{2+s+\alpha}\right)$$
$$\left.\left.-u\|a_2-\Theta\|^{\sigma}\left(\frac{2}{(1+\sigma)(2+\sigma)}-\frac{1}{(2+\alpha+\sigma)(3+\alpha+\sigma)}-\frac{\Gamma(3+\alpha)\Gamma(1+\sigma)}{\Gamma(4+\alpha+\sigma)}\right)\right)^{\frac{1}{q}}\right].$$

Proof. Using Lemma 2.2 and the property of modulus, we have

$$\left|\frac{\Phi'(a_1+a_2-\Theta)}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}[(a_2-\Theta)^{\alpha+1}-(\Theta-a_1)^{\alpha+1}]-[{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1+a_2-\Theta)\}]\right.$$
$$\left.+\frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta)+\frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}\Phi(a_2)+(a_2-\Theta)^{\alpha}\Phi(a_1)]\right|$$
$$\leq \frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\left[(\Theta-a_1)^{\alpha+2}\int_0^1|1-\tau^{\alpha+1}||\Phi''(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|d\tau\right.$$
$$\left.+(a_2-\Theta)^{\alpha+2}\int_0^1|\tau^{\alpha+1}-1||\Phi''(\tau a_1+(1-\tau)(a_1+a_2-\Theta))|d\tau\right].$$

Using power mean inequality and the higher order strongly n-polynomial convexity of $|\Phi''|^q$, we get

$$\begin{aligned}
&\int_0^1 |1-\tau^{\alpha+1}||\Phi''(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|\mathrm{d}\tau\\
&\leq\left(\int_0^1|1-\tau^{\alpha+1}|\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^1(1-\tau^{\alpha+1})|\Phi''(\tau a_2+(1-\tau)(a_1+a_2-\Theta))|^q\mathrm{d}\tau\right)^{\frac{1}{q}}\\
&\leq\left(\frac{\alpha+1}{\alpha+2}\right)^{\frac{1}{p}}\left(\int_0^1(1-\tau^{\alpha+1})\left[\frac{1}{n}\sum_{s=1}^n[1-(1-\tau)^s]|\Phi''(a_2)|^q\right.\right.\\
&\quad\left.\left.+\frac{1}{n}\sum_{s=1}^n[1-\tau^s]|\Phi''(a_1+a_2-\Theta)|^q-u(\tau^\sigma(1-\tau)+\tau(1-\tau)^\sigma)||\Theta-a_1||^\sigma\right]\mathrm{d}\tau\right)^{\frac{1}{q}}\\
&=\left(\frac{\alpha+1}{\alpha+2}\right)^{\frac{1}{p}}\left(\frac{|\Phi''(a_2)|^q}{n}\sum_{s=1}^n\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{\Gamma(1+s)\Gamma(2+\alpha)}{\Gamma(3+s+\alpha)}\right)\right.\\
&\quad+\frac{|\Phi''(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^n\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{1}{2+s+\alpha}\right)\\
&\quad\left.-u||\Theta-a_1||^\sigma\left(\frac{2}{(1+\sigma)(2+\sigma)}-\frac{1}{(2+\alpha+\sigma)(3+\alpha+\sigma)}-\frac{\Gamma(3+\alpha)\Gamma(1+\sigma)}{\Gamma(4+\alpha+\sigma)}\right)\right)^{\frac{1}{q}}. \qquad (3.7)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 |\tau^{\alpha+1}-1||\Phi''(\tau a_1+(1-\tau)(a_1+a_2-\Theta))|\mathrm{d}\tau\\
&=\left(\frac{\alpha+1}{\alpha+2}\right)^{\frac{1}{p}}\left(\frac{|\Phi''(a_1)|^q}{n}\sum_{s=1}^n\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{\Gamma(1+s)\Gamma(2+\alpha)}{\Gamma(3+s+\alpha)}\right)\right.\\
&\quad+\frac{|\Phi''(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^n\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{1}{2+s+\alpha}\right)\\
&\quad\left.-u||a_2-\Theta||^\sigma\left(\frac{2}{(1+\sigma)(2+\sigma)}-\frac{1}{(2+\alpha+\sigma)(3+\alpha+\sigma)}-\frac{\Gamma(3+\alpha)\Gamma(1+\sigma)}{\Gamma(4+\alpha+\sigma)}\right)\right)^{\frac{1}{q}}. \qquad (3.8)
\end{aligned}$$

By the inequalities (3.7) and (3.8), we obtain required result. □

Corollary 3.10. *Taking $\alpha=1$ in Theorem 3.4 and using Remark 2.2, we have*

$$\begin{aligned}
&\left|\Phi'(a_1+a_2-\Theta)[(a_2-\Theta)^2-(\Theta-a_1)^2]+2[(\Theta-a_1)\Phi(a_2)+(a_2-\Theta)\Phi(a_1)]-2\int_{a_1}^{a_2}\Phi(\Theta)\mathrm{d}\Theta\right|\\
&\leq\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[(\Theta-a_1)^3\left(\frac{|\Phi''(a_2)|^q}{n}\sum_{s=1}^n\left(\frac{2}{3}-\frac{1}{1+s}+\frac{2\Gamma(1+s)}{\Gamma(4+s)}\right)\right.\right.\\
&\quad+\frac{|\Phi''(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^n\left(\frac{2}{3}-\frac{1}{1+s}+\frac{1}{3+s}\right)
\end{aligned}$$

$$-u\|\Theta-a_1\|^{\sigma}\left(\frac{2}{(1+\sigma)(2+\sigma)}-\frac{1}{(3+\sigma)(4+\sigma)}-\frac{6\Gamma(1+\sigma)}{\Gamma(5+\sigma)}\right)\Bigg)^{\frac{1}{q}}$$
$$+(a_2-\Theta)^3\Bigg(\frac{|\Phi''(a_1)|^q}{n}\sum_{s=1}^{n}\left(\frac{2}{3}-\frac{1}{1+s}+\frac{2\Gamma(1+s)}{\Gamma(4+s)}\right)$$
$$+\frac{|\Phi''(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(\frac{2}{3}-\frac{1}{1+s}+\frac{1}{3+s}\right)$$
$$-u\|a_2-\Theta\|^{\sigma}\left(\frac{2}{(1+\sigma)(2+\sigma)}-\frac{1}{(3+\sigma)(4+\sigma)}-\frac{6\Gamma(1+\sigma)}{\Gamma(5+\sigma)}\right)\Bigg)^{\frac{1}{q}}\Bigg].$$

Corollary 3.11. *Taking $u\to 0^+$ in Theorem 3.4, we have*

$$\Bigg|\frac{\Phi'(a_1+a_2-\Theta)}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}[(a_2-\Theta)^{\alpha+1}-(\Theta-a_1)^{\alpha+1}]-[{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1+a_2-\Theta)\}]$$
$$+\frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta)+\frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}\Phi(a_2)+(a_2-\Theta)^{\alpha}\Phi(a_1)]\Bigg|$$
$$\leq\frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha+1}{\alpha+2}\right)^{1-\frac{1}{q}}\Bigg[(\Theta-a_1)^{2\alpha+3}\Bigg(\frac{|\Phi''(a_2)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{\Gamma(1+s)\Gamma(2+\alpha)}{\Gamma(3+s+\alpha)}\right)$$
$$+\frac{|\Phi''(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{1}{2+s+\alpha}\right)\Bigg)^{\frac{1}{q}}$$
$$+(a_2-\Theta)^{2\alpha+3}\Bigg(\frac{|\Phi''(a_1)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{\Gamma(1+s)\Gamma(2+\alpha)}{\Gamma(3+s+\alpha)}\right)$$
$$+\frac{|\Phi''(a_1+a_2-\Theta)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{1}{2+s+\alpha}\right)\Bigg)^{\frac{1}{q}}\Bigg].$$

Corollary 3.12. *Under assumptions of Theorem 3.4 with $\Theta=\frac{a_1+a_2}{2}$, we have the following inequality*

$$\Bigg|\frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi\left(\frac{a_1+a_2}{2}\right)+\frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{a_2-a_1}{2}\right)^{\alpha}[\Phi(a_1)+\Phi(a_2)]$$
$$-\left[{}^{\mathcal{AB}}I^{\alpha}_{a_2}\left\{\Phi\left(\frac{a_1+a_2}{2}\right)\right\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\left(\frac{a_1+a_2}{2}\right)}\left\{\Phi\left(\frac{a_1+a_2}{2}\right)\right\}\right]\Bigg|$$
$$\leq\frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha+1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{a_2-a_1}{2}\right)^{\alpha+2}\Bigg[\Bigg(\frac{|\Phi''(a_2)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{\Gamma(1+s)\Gamma(2+\alpha)}{\Gamma(3+s+\alpha)}\right)$$
$$+\frac{|\Phi''\left(\frac{a_1+a_2}{2}\right)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{1}{2+s+\alpha}\right)$$
$$-u\|\frac{a_2-a_1}{2}\|^{\sigma}\left(\frac{2}{(1+\sigma)(2+\sigma)}-\frac{1}{(2+\alpha+\sigma)(3+\alpha+\sigma)}-\frac{\Gamma(3+\alpha)\Gamma(1+\sigma)}{\Gamma(4+\alpha+\sigma)}\right)\Bigg)^{\frac{1}{q}}$$
$$+\Bigg(\frac{|\Phi''(a_1)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{\Gamma(1+s)\Gamma(2+\alpha)}{\Gamma(3+s+\alpha)}\right)$$

$$+\frac{|\Phi''\left(\frac{a_1+a_2}{2}\right)|^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{1}{2+s+\alpha}\right)$$
$$\left.-u\|\frac{a_2-a_1}{2}\|^{\sigma}\left(\frac{2}{(1+\sigma)(2+\sigma)}-\frac{1}{(2+\alpha+\sigma)(3+\alpha+\sigma)}-\frac{\Gamma(3+\alpha)\Gamma(1+\sigma)}{\Gamma(4+\alpha+\sigma)}\right)\right)^{\frac{1}{q}}\Bigg].$$

Remark 3.1. *Taking $\Theta = a_1$ or $\Theta = a_2$ in our main results, we can obtain several important special cases. We omit here their proofs and the details are left to the interested readers.*

4. Applications

In this section, we discuss applications of our main results.

4.1. Applications to special means

In this section, we discuss some applications of our main results to special means of positive real numbers. First of all, we recall some previously known concepts. For $a_1 \neq a_2$, we have

(1) The arithmetic mean: $A(a_1, a_2) = \frac{a_1+a_2}{2}$.

(2) The logarithmic mean: $L(a_1, a_2) = \frac{a_2-a_1}{\ln(a_2)-\ln(a_1)}$.

(3) The generalized logarithmic mean: $L_n^n(a_1, a_2) = \left[\frac{{a_2}^{n+1}-{a_1}^{n+1}}{(a_2-a_1)(n+1)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z}\setminus\{-1, 0\}$.

Proposition 4.1. *Suppose all the assumptions of Theorem 3.1 are satisfied, then*

(1)

$$\left|A^m(a_1,a_2)-L_m^m(a_1,a_2)\right| \leq \frac{a_2-a_1}{4}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left[\left\{\sum_{s=1}^{n}\frac{s}{s+1}\left(m{a_2}^{(m-1)q}+mA^{(n-1)q}(a_1,a_2)\right)\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\sum_{s=1}^{n}\frac{s}{s+1}\left(m{a_1}^{(m-1)q}+mA^{(m-1)q}(a_1,a_2)\right)\right\}^{\frac{1}{q}}\right],$$

(2)

$$\left|L^{-1}(a_1,a_2)-A^{-1}(a_1,a_2)\right| \leq \frac{a_2-a_1}{4}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left[\left\{\sum_{s=1}^{n}\frac{s}{s+1}\left({a_2}^{-2q}+A^{-2q}(a_1,a_2)\right)\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left\{\sum_{s=1}^{n}\frac{s}{s+1}\left({a_1}^{-2q}+A^{-2q}(a_1,a_2)\right)\right\}^{\frac{1}{q}}\right].$$

Proof. The proof is direct consequence of Theorem 3.1, by setting $x = \frac{a_1+a_2}{2}, \alpha = 1, \mu = 0$ and $\Phi(x) = x^m$, and $\Phi(x) = \frac{1}{x}$, respectively. □

Proposition 4.2. *Suppose all the assumptions of Theorem 3.2 are satisfied, then*

(1)

$$
\begin{aligned}
&\left|A^m(a_1,a_2)-L_m^m(a_1,a_2)\right|\\
&\leq \frac{a_2-a_1}{4}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\frac{m{a_2}^{(m-1)q}}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{1}{(s+1)(s+2)}\right)\right.\\
&\quad\left.+\frac{mA^{(m-1)q}(a_1,a_2)}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{1}{s+2}\right)\right]^{\frac{1}{q}}\\
&\quad+\left[\frac{m{a_1}^{(m-1)q}}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{1}{(s+1)(s+2)}\right)+\frac{mA^{(m-1)q}(a_1,a_2)}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{1}{s+2}\right)\right]^{\frac{1}{q}},
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\left|L^{-1}(a_1,a_2)-A^{-1}(a_1,a_2)\right|\\
&\leq \frac{a_2-a_1}{4}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\frac{{a_2}^{-2q}}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{1}{(s+1)(s+2)}\right)\right.\\
&\quad\left.+\frac{A^{-2q}(a_1,a_2)}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{1}{s+2}\right)\right]^{\frac{1}{q}}\\
&\quad+\left[\frac{{a_1}^{-2q}}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{1}{(s+1)(s+2)}\right)+\frac{A^{-2q}(a_1,a_2)}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{2}+\frac{1}{s+2}\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. The proof is direct consequence of Theorem 3.2, by taking $x=\frac{a_1+a_2}{2}, \alpha=1, \mu=0$ and $\Phi(x)=x^m$, and $\Phi(x)=\frac{1}{x}$, respectively. □

4.2. Applications to bounded functions

In this last section, we discuss applications regarding bounded functions in absolute value of the results obtained from our main results. We suppose that the following conditions are satisfied:

$$
|\Phi'|\leq F_1 \quad\text{and}\quad |\Phi''|\leq F_2.
$$

Applying the above conditions, we have the following results.

Corollary 4.1. *Under the assumptions of Theorem 3.1, the following inequality holds:*

$$
\begin{aligned}
&\left|{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\tau}\{\Phi(a_1+a_2-\Theta)\}-\frac{\Phi(a_1+a_2-\Theta)}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}+(a_2-\Theta)^{\alpha}+2(1-\alpha)\Gamma(\alpha)]\right|\\
&\leq \frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha p}{\alpha p+1}\right)^{\frac{1}{p}}\left[(\Theta-a_1)^{\alpha+1}\left(\frac{2F_1^q}{n}\sum_{s=1}^{n}\frac{s}{s+1}-\frac{2u\|\Theta-a_1\|^{\sigma}}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+(a_2-\Theta)^{\alpha+1}\left(\frac{2F_1^q}{n}\sum_{s=1}^{n}\frac{s}{s+1}-\frac{2u\|a_2-\Theta\|^{\sigma}}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Corollary 4.2. *Under the assumptions of Theorem 3.2, the following inequality holds:*

$$\left|{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{\tau}\{\Phi(a_1+a_2-\Theta)\}-\frac{\Phi(a_1+a_2-\Theta)}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}+(a_2-\Theta)^{\alpha}+2(1-\alpha)\Gamma(\alpha)]\right|$$
$$\leq\frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[(\Theta-a_1)^{\alpha+1}\left[\frac{F_1^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)}\right)\right.\right.$$
$$+\frac{F_1^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{1}{\alpha+s+1}\right)$$
$$\left.-u\|\Theta-a_1\|^{\sigma}\left(\frac{1}{(\sigma+1)(\sigma+2)}-\frac{1}{(1+\alpha+\sigma)(2+\alpha+\sigma)}-\frac{\Gamma(2+\alpha)\Gamma(1+\sigma)}{\Gamma(3+\alpha+\sigma)}\right)\right]^{\frac{1}{q}}$$
$$+(a_2-\Theta)^{\alpha+1}\left[\frac{F_1^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{\Gamma(1+s)\Gamma(1+\alpha)}{\Gamma(2+s+\alpha)}\right)\right.$$
$$+\frac{F_1^q}{n}\sum_{s=1}^{n}\left(\frac{s}{s+1}-\frac{1}{\alpha+1}+\frac{1}{\alpha+s+1}\right)$$
$$\left.\left.-u\|a_2-\Theta\|^{\sigma}\left(\frac{1}{(\sigma+1)(\sigma+2)}-\frac{1}{(1+\alpha+\sigma)(2+\alpha+\sigma)}-\frac{\Gamma(2+\alpha)\Gamma(1+\sigma)}{\Gamma(3+\alpha+\sigma)}\right)\right]^{\frac{1}{q}}\right].$$

Corollary 4.3. *Under the assumptions of Theorem 3.3, the following inequality holds:*

$$\left|\frac{\Phi'(a_1+a_2-\Theta)}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}[(a_2-\Theta)^{\alpha+1}-(\Theta-a_1)^{\alpha+1}]-[{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1+a_2-\Theta)\}]\right.$$
$$\left.+\frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta)+\frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}\Phi(a_2)+(a_2-\Theta)^{\alpha}\Phi(a_1)]\right|$$
$$\leq\frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{p(\alpha+1)}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left[(\Theta-a_1)^{2\alpha+3}\left(\frac{2F_2^q}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right.\right.$$
$$\left.\left.-\frac{2u\|\Theta-a_1\|^{\sigma}}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}+(a_2-\Theta)^{2\alpha+3}\left(\frac{2F_2^q}{n}\sum_{s=1}^{n}\frac{s}{s+1}-\frac{2u\|a_2-\Theta\|^{\sigma}}{(\sigma+1)(\sigma+2)}\right)^{\frac{1}{q}}\right].$$

Corollary 4.4. *Under the assumptions of Theorem 3.4, the following inequality holds:*

$$\left|\frac{\Phi'(a_1+a_2-\Theta)}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}[(a_2-\Theta)^{\alpha+1}-(\Theta-a_1)^{\alpha+1}]-[{}^{\mathcal{AB}}I^{\alpha}_{a_2}\{\Phi(a_1+a_2-\Theta)\}+{}^{\mathcal{AB}}_{a_1}I^{\alpha}_{a_1+a_2-\Theta}\{\Phi(a_1+a_2-\Theta)\}]\right.$$
$$\left.+\frac{2(1-\alpha)}{\mathcal{B}(\alpha)}\Phi(a_1+a_2-\Theta)+\frac{1}{\mathcal{B}(\alpha)\Gamma(\alpha)}[(\Theta-a_1)^{\alpha}\Phi(a_2)+(a_2-\Theta)^{\alpha}\Phi(a_1)]\right|$$
$$\leq\frac{1}{(\alpha+1)\mathcal{B}(\alpha)\Gamma(\alpha)}\left(\frac{\alpha+1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[(\Theta-a_1)^{2\alpha+3}\left(\frac{F_2^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{\Gamma(1+s)\Gamma(2+\alpha)}{\Gamma(3+s+\alpha)}\right)\right.\right.$$
$$+\frac{F_2^q}{n}\sum_{s=1}^{n}\left(1-\frac{1}{1+s}-\frac{1}{2+\alpha}+\frac{1}{2+s+\alpha}\right)$$
$$\left.-u\|\Theta-a_1\|^{\sigma}\left(\frac{2}{(1+\sigma)(2+\sigma)}-\frac{1}{(2+\alpha+\sigma)(3+\alpha+\sigma)}-\frac{\Gamma(3+\alpha)\Gamma(1+\sigma)}{\Gamma(4+\alpha+\sigma)}\right)\right)^{\frac{1}{q}}$$

$$+ (a_2 - \Theta)^{2\alpha+3} \left(\frac{F_2^q}{n} \sum_{s=1}^{n} \left(1 - \frac{1}{1+s} - \frac{1}{2+\alpha} + \frac{\Gamma(1+s)\Gamma(2+\alpha)}{\Gamma(3+s+\alpha)} \right) \right.$$

$$+ \frac{F_2^q}{n} \sum_{s=1}^{n} \left(1 - \frac{1}{1+s} - \frac{1}{2+\alpha} + \frac{1}{2+s+\alpha} \right)$$

$$\left. \left. -u\|a_2 - \Theta\|^{\sigma} \left(\frac{2}{(1+\sigma)(2+\sigma)} - \frac{1}{(2+\alpha+\sigma)(3+\alpha+\sigma)} - \frac{\Gamma(3+\alpha)\Gamma(1+\sigma)}{\Gamma(4+\alpha+\sigma)} \right) \right)^{\frac{1}{q}} \right].$$

5. Conclusions

In 2015 Caputo and Fabrizio suggested a new operator with fractional order, this derivative is based on the exponential kernel. Earlier this year 2016 Atangana and Baleanu developed another version which used the generalized Mittag-Leffler function as non-local and non-singular kernel. Both operators show some properties of filter. However, the Atangana and Baleanu version has in addition to this, all properties of fractional derivative. This shown effectiveness and advantages of the Atangana-Baleanu integral operators. Inspired by this great fact that own Atangana-Baleanu integral operators, we found two new fractional integral identities involving Atangana-Baleanu fractional integrals. Applying these identities as auxiliary results, we derived new fractional counterparts of classical inequalities essentially using first and second order differentiable higher order strongly n-polynomial convex functions. We have discussed several important special cases from our main results. The efficiency of our main results is demonstrated via special means and differentiable functions of first and second order that are in absolute value bounded. We will derive as future works several new fractional integral inequalities using Chebyshev, Markov, Young and Minkowski inequalities. Since the class of higher order strongly n-polynomial convex functions have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, and mathematical inequalities and may stimulate further research in different areas of pure and applied sciences. Studies relating convexity, partial convexity, and preinvex functions (as contractive operators) may have useful applications in complex interdisciplinary studies, such as maximizing the likelihood from multiple linear regressions involving Gauss-Laplace distribution. For more details, see [49–56]. We hope that our ideas and techniques of this paper will inspire interested readers working in this field.

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Conflicts of interest

The authors declare that they have no competing interests.

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