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Research article

Certain dynamic iterative scheme families and multi-valued fixed point results

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Abstract: The article presents a systematic investigation of an extension of the developments concerning F-contraction mappings which were proposed in 2012 by Wardowski. We develop the notion of F-contractions to the case of non-linear $(F,\ F_H)$-dynamic-iterative scheme for Branciari Ćirić type-contractions and prove multi-valued fixed point results in controlled-metric spaces. An approximation of the dynamic-iterative scheme instead of the conventional Picard sequence is determined. The paper also includes a tangible example and a graphical interpretation that displays the motivation for such investigations. The work is illustrated by providing an application of the proposed non-linear $(F,\ F_H)$-dynamic-iterative scheme to the Liouville-Caputo fractional derivatives and fractional differential equations.

Keywords: fixed points; controlled-metric space; $(F,\ F_H)$-dynamic-iterative scheme; Liouville-Caputo fractional differential equation
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1. Introduction

In the frame of functional analysis, the study of metric fixed point theory (MFPT) untied a portal to a new area of pure and applied mathematics. It states sufficient conditions for the existence of a (unique) fixed point and also provides an iterative system by which we can make approximations to the fixed point and error bounds. This idea was explored and furthered by a good number of researchers (see [2, 3, 29, 31]). It is widely held that MFPT originated in the year 1922 through the work of S. Banach [7] when he established the famous contraction mapping principle (CMP). Iteration systems are used in each branch of applied mathematics, and the criteria for convergence proofs and error estimates are very often produced by an application of the CMP and its variants. Moreover, E. Sperner (1928) proved the combinatorial geometric well-known lemma on the decomposition of a triangle which displays an important rule in the theory of CMP. These are the most important tool for proving tlre existence and uniqueiess of solutions to different mathematical models (differential, integral, ordinary and partial differential quations, variational inequalities). Some other fields are steady-state tempreture ditribution, chemical reactions, neutron transport theoty, economic, flow of fluids, optimal control theory, fractals, etc.

The classical fixed point theorems of Banach and Brouwer marked the development of the two most prominent and complementary facets of the theory, namely, the metric fixed point theory and the topological fixed point theory. The metric theory encompasses results and methods that involve properties of an essentially isometric nature. It originates with the concept of Picard successive approximations for establishing existence and uniqueness of solutions to nonlinear initial value problems of the 1st order and goes back as far as Cauchy, Liouville, Lipschitz, Peano, Fredholm, and most particularly, Emile Picard. The concept was investigated by extending metric spaces into its extensions. Kamran et al. [19] initiated the idea of an extended b-metric space, which is one of the most highlight extension of a b-metric space. After, in 2018, Mlaiki et al. [23] generalized the notion of an extended b-metric space to a controlled metric space. Many recent developments on metric structures and fixed point theory are investigated in [10, 12, 13] and also in the references therein. Later on, Nadler [24] used the idea of the Pompeiu-Hausdorff metric and gave the contraction theorem for set-valued maps instead of single-valued maps. In 2002, Branciari [9] introduced a well known contraction, known as the Branciari contraction. In 2012, Wardowski [32] initiated a new class of contractions, known as an F-contraction mapping and investigated the existence and uniqueness of fixed point results (see more [14, 15, 22, 26]). Recently, many developments on fractional calculus and fixed point results based on (generalized) F-contraction mappings are investigated in [8, 11, 20, 25, 28, 33] and also in the references therein in the associated approach.

2. Preliminaries

Let $N(\Psi)$ represent the family of all non-empty subsets of a non-empty set Ψ, and $C(\Psi)$ be the family of all non-empty closed subsets of Ψ. Let $Y : \Psi \to N(\Psi)$ be a set-valued mapping, and $\varepsilon_0 \in \Psi$ be arbitrary and fixed. Define

$$\check{D}(Y, \varepsilon_0) = \{(\varepsilon_i)_{i\geq 0} : \varepsilon_i \in Y(\varepsilon_{i-1}), \text{ for all } i \in \mathbb{N}\}.$$

Any element of $\check{D}(Y, \varepsilon_0)$ is named as dynamic iterative-scheme of Y starting from point ε_0. The dynamic-iterative scheme $\left(\varepsilon_j\right)_{j\in\mathbb{N}\cup\{0\}}$ onward has the form $\left(\varepsilon_j\right)$ (see [17]).

Example 2.1. Let $\Psi = C([0, 1])$ be a Banach space with the norm $\|\varepsilon\| = \sup_{r\in[0,1]} |\varepsilon(r)|$ where $\varepsilon \in \Psi$.

Let $Y : \Psi \to 2^{\Psi}$ be so that for every $\varepsilon \in \Psi$, $Y(\varepsilon)$ is a collection of the function

$$r \longmapsto k \int_0^r \varepsilon(t)\, dt,\ k \in [0,1],$$

that is,

$$Y(\varepsilon)(r) = \left\{ k \int_0^r \varepsilon(t)\, dt : k \in [0,1] \right\},\ \varepsilon \in \Psi$$

and let $\varepsilon_0(u) = u$, $u \in [0,1]$, so $\left(\frac{1}{j!(j+1)!} r^{j+1}\right)$ is a dynamic process of Y with starting point ε_0. The mapping $Y : \Psi \to R$ is said to be $\check{D}(Y, \varepsilon_0)$ dynamic lower semi-continuous at $\varepsilon \in \Psi$, if for each dynamic iterative-scheme $\left(\varepsilon_j\right) \in \check{D}(Y, \varepsilon_0)$ and for each subsequence $\left(\varepsilon_{j(i)}\right)$ of $\left(\varepsilon_j\right)$ converges to ε, we write $Y(\varepsilon) \leq \liminf_{i\to\infty} Y\left(\varepsilon_{j(i)}\right)$. In this case, Y is $\check{D}(Y, \varepsilon_0)$-dynamic lower semi-continuous. If Y is $\check{D}(Y, \varepsilon_0)$ dynamic lower semi-continuous at each $\varepsilon \in \Psi$, then Y is known as lower semi-continuous. If for each sequence $\left(\varepsilon_j\right) \subset \Psi$ and $\varepsilon \in \Psi$ so that $\left(\varepsilon_j\right) \to \varepsilon$, we write $Y(\varepsilon) \leq \liminf_{i\to\infty} Y(\varepsilon(j))$ (more see [4, 17]).

Branciari [9] introduced the following concepts:

Definition 2.2. [9] Let (Ψ, δ) be a metric space and $Y : \Psi \to \Psi$ be so that

$$\int_0^{\delta(Y\varepsilon_1, Y\varepsilon_2)} \Phi(s)\, \delta_\varsigma s \leq \beta \int_0^{\delta(\varepsilon_1, \varepsilon_2)} \Phi(s)\, ds$$

for all $\varepsilon_1, \varepsilon_2 \in \Psi$, where $\beta \in (0,1)$, $\Phi : \kappa \to \kappa$ is a non-negative Lebesgue integrable mapping which is summable on each compact subset of κ $[\kappa = [0, +\infty)]$ and $\int_0^{\epsilon} \Phi(s)\, ds$ for all $\epsilon > 0$.Then, Y has fixed point.

Lemma 2.3. *[21] Let $(\varepsilon_i)_{i\in\mathbb{N}}$ be a sequence so that* $\lim_{i\to+\infty} \varepsilon_i = \varepsilon$. *Then*

$$\lim_{i\to+\infty} \int_0^{\varepsilon_i} \Phi(w)\, \delta w = \int_0^{\varepsilon} \Phi(w)\, dw.$$

Lemma 2.4. *[21] Let $(\varepsilon_i)_{i\in\mathbb{N}}$ be a sequence. Then*

$$\lim_{i\to+\infty} \int_0^{\varepsilon_i} \Phi(w)\, dw = 0 \Leftrightarrow \lim_{i\to+\infty} \varepsilon_i = 0.$$

For the last few years, contraction theorems have rapidly been evolving, not only in the metric frame, but also in many different extended spaces and the controlled metric space is one of them. In 2018, Mlaiki et al. [23] generalized the notion of an extended b-metric space to a controlled metric space. Alamgir et al. [1] introduced the idea of a Hausdorff controlled metric and proved some well-known results in control metric spaces.

Definition 2.5. [23] A controlled-metric space on a non-empty set Ψ is a function $\delta_\varsigma : \Psi \times \Psi \to R^+$ with $\varsigma : \Psi \times \Psi \to [1, \infty)$ so that $\forall\, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \Psi$
(C_1) : if $\delta_\varsigma(\varepsilon_1, \varepsilon_2) = 0$ if and only if $\varepsilon_1 = \varepsilon_2$;
(C_2) : $\delta_\varsigma(\varepsilon_1, \varepsilon_2) = \delta_\varsigma(\varepsilon_2, \varepsilon_1)$;
(C_3) : $\delta_\varsigma(\varepsilon_1, \varepsilon_3) \leq \varsigma(\varepsilon_1, \varepsilon_2)\, \delta_\varsigma(\varepsilon_1, \varepsilon_2) + \varsigma(\varepsilon_2, \varepsilon_3)\, \delta_\varsigma(\varepsilon_2, \varepsilon_3)$.

The pair (Ψ, δ_ς) is known as a controlled-metric space.

A sequence $\{\varepsilon_i\}$ in controlled m.s Ψ is convergent to some $\varepsilon \in \Psi$ if for every $\epsilon > 0$, there is $I = I(\epsilon) \in \mathbb{N}$ so that $\delta_\varsigma(\varepsilon_i, \varepsilon) < \epsilon$ for all $i \geq I$ and have write $\lim_{i\to\infty}(\varepsilon_i) = \varepsilon$, a sequence $\{\varepsilon_i\}$ in (Ψ, δ_ς) is Cauchy if for every $\epsilon > 0$, there is $I = I(\epsilon) \in \mathbb{N}$ so that $\delta_\varsigma(\varepsilon_i, \varepsilon_{i'}) < \epsilon$ for all $i, i' \geq I$ and a controlled-metric space (Ψ, δ_ς) is complete if every Cauchy sequence in Ψ converges.

Let $x \in \Psi$ and $\epsilon > 0$, the open ball $B(x, \epsilon)$ is

$$B(x, \epsilon) = \{y \in \Psi : \delta_\varsigma(x, y) < \epsilon\}.$$

The mapping $Y : \Psi \to \Psi$ is continuous at $x \in \Psi$ if for each $\epsilon > 0$, there is $\alpha > 0$ so that

$$Y(B(x, \alpha)) \subseteq B(Yx, \epsilon).$$

Owing to above proposition, we clearly say that if Y is continuous at $x \in \Psi$, then for $x_i \to x$, we have $Yx_i \to Yx$ as $i \to \infty$.

Further, Alamgir et al. [1] discussed some well-known results via the Hausdorff controlled metric. Define the Pompeiu-Hausdorff controlled metric $\hat{H}_\varsigma$ induced by δ_ς on $CB(\Psi)$ as follows:

$$\hat{H}_\varsigma(\theta_1, \theta_2) = \max\left\{\sup_{\varepsilon_1\in\theta_1} \check{D}_\varsigma(\varepsilon_1, \theta_2), \sup_{\varepsilon_2\in\theta_2} \check{D}_\varsigma(\varepsilon_2, \theta_1)\right\}$$

for each $\theta_1, \theta_2 \in CB(\Psi)$, where $\check{D}(\varepsilon_1, \theta_2) = \inf_{\varepsilon_2\in\theta_2} \delta_\varsigma(\varepsilon_1, \varepsilon_2)$.

Lemma 2.6. *[1] Let θ_1 be a nonempty subsets of a controlled-metric space (Ψ, δ), then*

$$\delta_\varsigma(\varepsilon, \theta_1) \leq \varsigma(\varepsilon_1, \varepsilon_2)\delta_\varsigma(\varepsilon_1, \varepsilon_2) + \varsigma(\varepsilon_2, \theta_1)\delta_\varsigma(\varepsilon_2, \theta_1)$$

for $\varepsilon_1, \varepsilon_2 \in \Psi$, where $\varsigma(\varepsilon_2, \theta_1) = \inf_{\varepsilon^*\in\theta_1} \varsigma(\varepsilon_2, \varepsilon^*)$ and $\delta_\varsigma(\varepsilon_2, \theta_1) = \inf_{\varepsilon^*\in\theta_1} \delta_\varsigma(\varepsilon_2, \varepsilon^*)$.

Lemma 2.7. *[1] Let $\theta_1, \theta_2 \in CB(\Psi)$, then for all $\beta > 0$ and $\varepsilon_2 \in \theta_2$, there is $\varepsilon_1 \in \theta_1$ so that*

$$\delta_\varsigma(\varepsilon_1, \varepsilon_2) \leq \hat{H}_\varsigma(\theta_1, \theta_2) + \beta.$$

Lemma 2.8. *[1] Let θ_1 and θ_2 be nonempty subsets of a controlled-metric space (Ψ, δ). If $\alpha \in \theta_1$, then*

$$\delta_\varsigma(\alpha, \theta_2) \leq \hat{H}_\varsigma(\theta_1, \theta_2).$$

In 2012, Wardowski [32] initiated F-contractions and a related fixed point result was presented.

Definition 2.9. [32] $Y : \Psi \to \Psi$ is called an F-contraction on a metric space (Ψ, δ), if there exist $F \in \nabla_{\mathcal{F}}$ and $\tau \in R^+/\{0\}$ so that $\delta(Y\varepsilon_1, Y\varepsilon_2) > 0$, implies

$$\tau + \mathcal{F}(\delta(Y\varepsilon_1, Y\varepsilon_2)) \leq \mathcal{F}(\delta(\varepsilon_1, \varepsilon_2)) \tag{2.1}$$

for each $\varepsilon_1, \varepsilon_2 \in \Psi$, where $\nabla_{\mathcal{F}}$ is the family of all functions $F : (0, +\infty) \longrightarrow (-\infty, +\infty)$, so that
$(\mathcal{F}_i)$ $\varepsilon_1 < \varepsilon_2$ implies $F(\varepsilon_1) < F(\varepsilon_2)$ for all $\varepsilon_1, \varepsilon_2 \in (0, +\infty)$;
$(\mathcal{F}_{ii})$ for each sequence $\{\varepsilon(j)\}$ of positive real numbers,

$$\lim_{j\longrightarrow\infty} \varepsilon(j) = 0 \text{ iff } \lim_{j\longrightarrow\infty} \mathcal{F}(\varepsilon(j)) = -\infty;$$

$(\mathcal{F}_{iii})$ there is $k \in (0, 1)$ such that $\lim_{c\to0}(c)^k F(c) = 0$. Then there is a unique fixed point of Y.

Example 2.10. As examples of F-contractions, one writes:
$(i) : F(\varepsilon) = \ln(\varepsilon)$;
$(ii) : F(\varepsilon) = \ln(\varepsilon) + \varepsilon$;
$(iii) : F(\varepsilon) = -\frac{1}{\sqrt{\varepsilon}}$;
$(iv) : F(\varepsilon) = \ln\left(\varepsilon^2 + \varepsilon\right)$.
Owing to $(\mathcal{F}_i)$ and (2.1), each F-contraction Y is a contractive mapping, and so each F-contraction mapping is continuous.

Our goal is to introduce a new concept of non-linear (F, F_H)-dynamic-iterative scheme for Branciari Ćirić type-contractions and establish some related multi-valued fixed point results on controlled-metric spaces. Finally, we give concrete examples, an application and some open questions.

3. A family of F-dynamic-iterative scheme: $\check{D}(Y, \varepsilon_0)$

First, we introduce the following definition.

Definition 3.1. Let (Ψ, δ_ς) be a controlled-metric space and $Y : \Psi \to CB(\Psi)$ be a set valued Branciari Ćirić type contraction based on F-dynamic-iterative scheme $\check{D}(Y, \varepsilon_0)$. If there are $F \in \nabla_{\mathcal{F}}$, $\tau : (0, +\infty) \to (0, +\infty)$ a non-constant function and $\Phi : \kappa \to \kappa$ a non-negative Lebesgue integrable mapping which is summable on each compact subset of κ so that

$$\hat{H}_\varsigma(Y\varepsilon_{i-1}, Y\varepsilon_i) > 0 \;\Rightarrow\; \tau(\Delta(\varepsilon_{i-1}, \varepsilon_i)) + \mathcal{F}\Big(\int_0^{\hat{H}_\varsigma(Y\varepsilon_i, Y\varepsilon_{i+1})} \Phi(s)\,\delta s \leq \mathcal{F}\Big(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)} \Phi(s)\,\delta s) \tag{3.1}$$

where

$$\Delta(\varepsilon_{i-1}, \varepsilon_i) = \max\left\{\delta_\varsigma(\varepsilon_{i-1}, \varepsilon_i), \check{D}_\varsigma(\varepsilon_{i-1}, Y\varepsilon_{i-1}), \check{D}_\varsigma(\varepsilon_i, Y\varepsilon_i), \frac{\check{D}_\varsigma(\varepsilon_{i-1}, Y\varepsilon_i) + \check{D}_\varsigma(\varepsilon_i, Y\varepsilon_{i-1})}{2}\right\}$$

for all $i \in \mathbb{N}$, $\varepsilon_i \in \check{D}(Y, \varepsilon_0)$ and for every given $\epsilon > 0$ so that $\int_0^\epsilon \Phi(s)\,\delta s > 0$.

Remark 3.2. In continuing way of our results, we consider only the dynamic iterative scheme $\varepsilon_i \in \check{D}(Y, \varepsilon_0)$ that verifies the following criteria:

$$\delta_\varsigma(\varepsilon_i, \varepsilon_{i+1}) > 0 \;\Rightarrow\; \delta_\varsigma(\varepsilon_{i-1}, \varepsilon_i) > 0 \text{ for each } i \in \mathbb{N}. \tag{3.2}$$

When the process does not verify (3.2), then there is $i_0 \in \mathbb{N}$ so that

$$\delta_\varsigma(\varepsilon_{i_0}, \varepsilon_{i_{0+1}}) > 0$$

and

$$\delta_\varsigma(\varepsilon_{i_{0-1}}, \varepsilon_{i_0}) = 0.$$

Then we get $\varepsilon_{i_{0-1}} = \varepsilon_{i_0} \in Y\varepsilon_{i_{0-1}}$ which ensures the existence of a fixed point. In view of this consideration of dynamic iterative scheme satisfying (3.2), it does not depreciate a generality of our analysis.

Remark 3.3. Upon setting, clearly Y is a contraction mapping with respect to F-dynamic iterative scheme $\check{D}_{\varsigma}(Y,\varepsilon_0)$ and in the light of $\Phi(s)\equiv 1$, we easily conclude that it is an F-contraction.

Theorem 3.4. *Let $(\Psi,\delta_{\varsigma})$ be a complete controlled-metric space and $Y:\Psi\to CB(\Psi)$ be a set valued Branciari Ćirić type contraction with respect to F-dynamic-iterative scheme $\check{D}_{\varsigma}(Y,\varepsilon_0)$. Assume that:*
(D1): There is a F-dynamic iterative scheme $\varepsilon_i\in\check{D}_{\varsigma}(Y,\varepsilon_0)$ such that

$$\lim_{k\to l^+}\inf\tau(k)>0\,\textit{for each}\; l\geq 0;$$

(D2): A mapping $\Psi\ni\varepsilon_i\longmapsto\delta_{\varsigma}(\varepsilon_i,Y\varepsilon_i)$ is $\check{D}_{\varsigma}(Y,\varepsilon_0)$-$F$-dynamic lower semi-continuous.
Then Y has a fixed point.

Proof. Choose $\varepsilon_0\in\Psi$ to be an arbitrary point. In view of $\varepsilon_i\in\check{D}_{\varsigma}(Y,\varepsilon_0)$, we design the F-dynamic iterative scheme by the following family:

$$\check{D}_{\varsigma}(Y,\varepsilon_0)=\left\{(\varepsilon_i)_{i\in N\cup\{0\}}:\varepsilon_{i+1}=\varepsilon_i\in Y\varepsilon_{i-1}\text{ for all } i\in\mathbb{N}\right\}.$$

In case, there is $i_0\in\mathbb{N}$ so that $\varepsilon_{i_0}\in Y\varepsilon_{i_0}$, then ε_{i_0} is a fixed point of Y is clear. Therefore, if we let $\varepsilon_i\notin Y\varepsilon_i$ then $\check{D}_{\varsigma}(Y,\varepsilon_0)>0$ for every $i\in\mathbb{N}$. Since $Y\varepsilon_i$ is compact, by (3.1) and Lemma 2.8, one writes

$$\begin{aligned}
\mathcal{F}(\int_0^{\check{D}(\varepsilon_i,Y\varepsilon_i)}\Phi(s)\,\delta s) &\leq \mathcal{F}(\int_0^{\hat{H}_{\varsigma}(Y\varepsilon_i,Y\varepsilon_{i+1})}\Phi(s)\,\delta s)\leq\mathcal{F}(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)}\Phi(s)\,\delta s)-\tau(\Delta(\varepsilon_{i-1},\varepsilon_i)) \quad (3.3)\\
&= \mathcal{F}(\int_0^{\max\left\{\begin{array}{c}\delta_{\varsigma}(\varepsilon_{i-1},\varepsilon_i),\check{D}(\varepsilon_{i-1},Y\varepsilon_{i-1}),\check{D}(\varepsilon_i,Y\varepsilon_i),\\ \frac{\check{D}(\varepsilon_{i-1},Y\varepsilon_i)+\check{D}(\varepsilon_i,Y\varepsilon_{i-1})}{2}\end{array}\right\}}\Phi(s)\,\delta s)\\
&\quad -\tau\left(\max\left\{\begin{array}{c}\delta_{\varsigma}(\varepsilon_{i-1},\varepsilon_i),\check{D}(\varepsilon_{i-1},Y\varepsilon_{i-1}),\check{D}(\varepsilon_i,Y\varepsilon_i),\\ \frac{\check{D}(\varepsilon_{i-1},Y\varepsilon_i)+\check{D}(\varepsilon_i,Y\varepsilon_{i-1})}{2}\end{array}\right\}\right)\\
&\leq \mathcal{F}(\int_0^{\max\left\{\delta_{\varsigma}(\varepsilon_{i-1},\varepsilon_i),\frac{\check{D}(\varepsilon_{i-1},Y\varepsilon_i)}{2}\right\}}\Phi(s)\,\delta s)\\
&\quad -\tau\left(\max\left\{\delta_{\varsigma}(\varepsilon_{i-1},\varepsilon_i),\frac{\check{D}(\varepsilon_{i-1},Y\varepsilon_i)}{2}\right\}\right)\\
&\leq \mathcal{F}(\int_0^{\max\left\{\delta_{\varsigma}(\varepsilon_{i-1},\varepsilon_i),\frac{\check{D}(\varepsilon_i,Y\varepsilon_i)}{2}\right\}}\Phi(s)\,\delta s)\\
&\quad -\tau\left(\max\left\{\delta_{\varsigma}(\varepsilon_{i-1},\varepsilon_i),\frac{\check{D}(\varepsilon_i,Y\varepsilon_i)}{2}\right\}\right)\\
&\leq \mathcal{F}(\int_0^{\delta_{\varsigma}(\varepsilon_{i-1},\varepsilon_i)}\Phi(s)\,\delta s)-\tau\left(\delta_{\varsigma}(\varepsilon_{i-1},\varepsilon_i)\right).
\end{aligned}$$

Moreover, since $Y\varepsilon_i$ is compact, we obtain $\varepsilon_{i+1}\in Y\varepsilon_i$ so that $\delta_{\varsigma}(\varepsilon_i,\varepsilon_{i+1})=\check{D}_{\varsigma}(\varepsilon_i,Y\varepsilon_i)$. Using (3.3), we have

$$\mathcal{F}(\int_0^{\delta_{\varsigma}(\varepsilon_i,\varepsilon_{i+1})}\Phi(s)\,\delta s)\;\leq\;\mathcal{F}(\int_0^{\hat{H}_{\varsigma}(Y\varepsilon_i,Y\varepsilon_{i+1})}\Phi(s)\,\delta s)$$

$$\leq \mathcal{F}(\int_0^{\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)} \Phi(s)\,\delta s) - \tau(\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i))$$
$$< \mathcal{F}(\int_0^{\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)} \Phi(s)\,\delta s).$$

Thus, the sequence $\{\delta_\varsigma(\varepsilon_i,\varepsilon_{i+1})\}$ is decreasing and hence it is convergent. Now, we show that $\lim_{i\to\infty}\delta_\varsigma(\varepsilon_i,\varepsilon_{i+1}) = 0$. From, (D1) there is $\sigma > 0$ and $i_0 \in \mathbb{N}$ so that $\tau(\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)) > \sigma$ for all $i > i_0$. Hence, we see that

$$\begin{aligned}
\mathcal{F}(\int_0^{\delta_\varsigma(\varepsilon_i,\varepsilon_{i+1})} \Phi(s)\,\delta s) &\leq \mathcal{F}(\int_0^{\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)} \Phi(s)\,\delta s) - \tau(\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)) \\
&\leq \mathcal{F}(\int_0^{\delta_\varsigma(\varepsilon_{i-2},\varepsilon_{i-1})} \Phi(s)\,\delta s) - \tau(\delta_\varsigma(\varepsilon_{i-2},\varepsilon_{i-1})) - \tau(\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)) \\
&\vdots \\
&\leq \mathcal{F}(\int_0^{\delta_\varsigma(\varepsilon_0,\varepsilon_1)} \Phi(s)\,\delta s) - \tau(\delta_\varsigma(\varepsilon_0,\varepsilon_1)) - \cdots - \tau(\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)) \\
&= \mathcal{F}(\int_0^{\delta_\varsigma(\varepsilon_0,\varepsilon_1)} \Phi(s)\,\delta s) - \tau((\delta_\varsigma(\varepsilon_0,\varepsilon_1)) + \cdots + \tau(\varepsilon_{i_{0-1}},\varepsilon_{i_0})) \\
&\quad -\tau(\delta_\varsigma(\varepsilon_{i_0},\varepsilon_{i_{0+1}})) + \cdots + \tau(\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)) \\
&\leq \mathcal{F}(\int_0^{\delta_\varsigma(\varepsilon_0,\varepsilon_1)} \Phi(s)\,\delta s) - (i - i_0)\,\sigma.
\end{aligned} \tag{3.4}$$

Let us set $\varpi_i = \int_0^{\delta_\varsigma(\varepsilon_i,\varepsilon_{i+1})} \Phi(s)\,\delta s > 0$ for $i = 0, 1, 2, \cdots$, and from (3.4), we obtain $\lim_{i\to\infty} F(\varpi_i) = -\infty$. Using (F_{ii}), we get

$$\lim_{i\to\infty} (\varpi_i) = 0. \tag{3.5}$$

In view of (F_{iii}), there is $\alpha \in (0, 1)$ so that

$$\lim_{i\to\infty} [\varpi_i]^k \mathcal{F}[\varpi_i] = 0. \tag{3.6}$$

By (3.4), the following holds for all $i > i_0$,

$$\begin{aligned}
[\varpi_i]^\alpha \mathcal{F}[\varpi_i] - [\varpi_i]^\alpha \mathcal{F}[\varpi_0] &\leq [\varpi_i]^\alpha (\mathcal{F}(\lambda_0) - (i - i_0)\,\sigma) - [\varpi_i]^\alpha \mathcal{F}[\varpi_0] \\
&= -[\varpi_i]^\alpha (i - i_0)\,\sigma \leq 0.
\end{aligned} \tag{3.7}$$

Taking limit as $i \to \infty$ in (3.7) and using (3.6), we have

$$\lim_{i\to\infty} i\,[\varpi_i]^\alpha = 0. \tag{3.8}$$

Due to (3.8), there is $i_1 \in \mathbb{N}$ so that $i\,[\varpi_i]^\alpha \leq 1$ for all $i \geq i_1$, we have

$$\varpi_i \leq \frac{1}{i^{\frac{1}{\alpha}}}. \tag{3.9}$$

Now, in order to show that $\{\varepsilon_i\}$ is a Cauchy sequence, we consider $j_1,\ j_2 \in \mathbb{N}$ so that $j_1 > j_2 \geq i_1$. From (3.9) and by the metric condition, we have

$$
\begin{aligned}
&\int_0^{\delta_\varsigma(\varepsilon_{j_1},\varepsilon_{j_2})} \Phi(s)\,\delta s \qquad (3.10)\\
\leq\ & \int_0^{\varsigma(\varepsilon_{j_1},\varepsilon_{j_{1+1}})\delta_\varsigma(\varepsilon_{j_1},\varepsilon_{j_{1+1}})+\varsigma(\varepsilon_{j_{1+1}},\varepsilon_{j_2})\delta_\varsigma(\varepsilon_{j_{1+1}},\varepsilon_{j_2})} \Phi(s)\,\delta s\\
\leq\ & \int_0^{\left\{\begin{array}{l}\varsigma\left(\varepsilon_{j_1},\varepsilon_{j_{1+1}}\right)\delta_\varsigma\left(\varepsilon_{j_1},\varepsilon_{j_{1+1}}\right)+\varsigma\left(\varepsilon_{j_{1+1}},\varepsilon_{j_2}\right)\varsigma\left(\varepsilon_{j_{1+1}},\varepsilon_{j_{1+2}}\right)\\ \delta_\varsigma\left(\varepsilon_{j_{1+1}},\varepsilon_{j_{1+2}}\right)+\varsigma\left(\varepsilon_{j_{1+1}},\varepsilon_{j_2}\right)\varsigma\left(\varepsilon_{j_{1+2}},\varepsilon_{j_2}\right)\delta_\varsigma\left(\varepsilon_{j_{1+2}},\varepsilon_{j_2}\right)\end{array}\right.} \Phi(s)\,\delta s\\
\leq\ & \int_0^{\left\{\begin{array}{l}\varsigma\left(\varepsilon_{j_1},\varepsilon_{j_{1+1}}\right)\delta_\varsigma\left(\varepsilon_{j_1},\varepsilon_{j_{1+1}}\right)+\varsigma\left(\varepsilon_{j_{1+1}},\varepsilon_{j_2}\right)\varsigma\left(\varepsilon_{j_{1+1}},\varepsilon_{j_{1+2}}\right)\\ \delta_\varsigma\left(\varepsilon_{j_{1+1}},\varepsilon_{j_{1+2}}\right)+\varsigma\left(\varepsilon_{j_{1+1}},\varepsilon_{j_2}\right)\varsigma\left(\varepsilon_{j_{1+2}},\varepsilon_{j_2}\right)\varsigma\left(\varepsilon_{j_{1+2}},\varepsilon_{j_{1+3}}\right)\\ \delta_\varsigma\left(\varepsilon_{j_{1+2}},\varepsilon_{j_{1+3}}\right)+\varsigma\left(\varepsilon_{j_{1+1}},\varepsilon_{j_2}\right)\varsigma\left(\varepsilon_{j_{1+2}},\varepsilon_{j_2}\right)\varsigma\left(\varepsilon_{j_{1+3}},\varepsilon_{j_2}\right)\\ \delta_\varsigma\left(\varepsilon_{j_{1+3}},\varepsilon_{j_2}\right)\end{array}\right.} \Phi(s)\,\delta s\\
\leq\ & \cdots\\
\leq\ & \int_0^{\left\{\begin{array}{l}\varsigma\left(\varepsilon_{j_1},\varepsilon_{j_{1+1}}\right)\delta_\varsigma\left(\varepsilon_{j_1},\varepsilon_{j_{1+1}}\right)+\sum_{i=j_{1+1}}^{j_2-2}\left(\prod_{r=j_{1+1}}^{i}\varsigma\left(\varepsilon_r,\varepsilon_{j_2}\right)\right)\\ \varsigma\left(\varepsilon_i,\varepsilon_{i+1}\right)\delta_\varsigma\left(\varepsilon_i,\varepsilon_{i+1}\right)+\prod_{l=j_{1+1}}^{j_2-1}\varsigma\left(\varepsilon_l,\varepsilon_{j_2}\right)\delta_\varsigma\left(\varepsilon_{j_{2-1}},\varepsilon_{j_2}\right)\end{array}\right.} \Phi(s)\,\delta s\\
\leq\ & \int_0^{\left\{\begin{array}{l}\varsigma\left(\varpi_{j_1},\varpi_{j_{1+1}}\right)\delta_\varsigma\left(\varpi_{j_1},\varpi_{j_{1+1}}\right)+\sum_{i=1}^{j_2-1}\left(\prod_{r=j_{1+1}}^{i}\varsigma\left(\varepsilon_r,\varepsilon_{j_2}\right)\right)\\ \varsigma\left(\varepsilon_i,\varepsilon_{i+1}\right)\delta_\varsigma\left(\varepsilon_i,\varepsilon_{i+1}\right)\end{array}\right.} \Phi(s)\,\delta_\varsigma s\\
\leq\ & \int_0^{\left\{\begin{array}{l}\varsigma\left(\varepsilon_{j_1},\varepsilon_{j_{1+1}}\right)\delta_\varsigma\left(\varepsilon_{j_1},\varepsilon_{j_{1+1}}\right)+\sum_{i=1}^{j_2-1}\left(\prod_{r=j_{1+1}}^{i}\varsigma\left(\varepsilon_r,\varepsilon_{j_2}\right)\right)\\ \varsigma\left(\varepsilon_i,\varepsilon_{i+1}\right)\delta_\varsigma\left(\varepsilon_i,\varepsilon_{i+1}\right)\end{array}\right.} \Phi(s)\,\delta_\varsigma s\\
=\ & \int_0^{\varsigma(\varepsilon_{j_1},\varepsilon_{j_{1+1}})\varpi_{j_1}+\sum_{i=1}^{j_2-1}\left(\prod_{r=j_{1+1}}^{i}\varsigma(\varepsilon_r,\varepsilon_{j_2})\right)\varsigma(\varepsilon_i,\varepsilon_{i+1})\varpi_i} \Phi(s)\,\delta s\\
\leq\ & \int_0^{\varsigma(\varepsilon_{j_1},\varepsilon_{j_{1+1}})\frac{1}{(j_1)^{\frac{1}{\alpha}}}+\sum_{i=1}^{j_2-1}\left(\prod_{r=j_{1+1}}^{i}\varsigma(\varepsilon_r,\varepsilon_{j_2})\right)\varsigma(\varepsilon_i,\varepsilon_{i+1})\frac{1}{(i)^{\frac{1}{\alpha}}}} \Phi(s)\,\delta s.
\end{aligned}
$$

Owing to (3.10) and in view of the convergence of series $\sum_{l=j_1}^{\infty}\frac{1}{l^{\frac{1}{\alpha}}}$, we get $\int_0^{\delta_\varsigma(\varepsilon_{j_1},\varepsilon_{j_2})}\Phi(s)\,\delta s \to 0$. Hence, $\{\varepsilon_i\}$ is Cauchy in (Ψ,δ_ς). Further, for the completeness of Ψ there is $\varepsilon^* \in \Psi$ so that $\lim_{i\to\infty}\varepsilon_i = \varepsilon^*$. Since Y is compact, we have $Y\varepsilon_i \to Y\varepsilon^*$ and by Lemma 2.8, one writes

$$\check{D}_\varsigma(\varepsilon_i, Y\varepsilon^*) \leq \hat{H}_\varsigma(Y\varepsilon_{i-1}, Y\varepsilon^*). \qquad (3.11)$$

So, $\check{D}_\varsigma(\varepsilon^*, Y\varepsilon^*) = 0$ and $\varepsilon^* \in Y\varepsilon^*$. Now, by right continuity of F we examine $\varepsilon^* \in Y\varepsilon^*$. Suppose on the contrary, $\varepsilon^* \notin Y\varepsilon^*$ then there are $i_0 \in \mathbb{N}$ and a subsequence $\{\varepsilon_{i_k}\}$ of $\{\varepsilon_i\}$ so that $\check{D}_\varsigma(\varepsilon_{i_k+1}, Y\varepsilon^*) > 0$ for each $i_k \geq i_0$ [Otherwise, there is $i_1 \in \mathbb{N}$ so that $\varepsilon_i \in Y\varepsilon^*$ for every $i \geq i_1$, which yields $\varepsilon^* \in Y\varepsilon^*$, it is a contradiction]. Since $\check{D}_\varsigma(\varepsilon_{i_k+1}, Y\varepsilon^*) > 0$ for each $i_k \geq i_0$, one writes

$$\begin{aligned}\mathcal{F}(\int_0^{\check{D}_\varsigma(\varepsilon_{i_k+1},Y\varepsilon^*)} \Phi(s)\,\delta s) &\leq \mathcal{F}(\int_0^{\hat{H}_\varsigma(Y\varepsilon_{i_k},Y\varepsilon^*)} \Phi(s)\,\delta s) \\ &\leq \mathcal{F}(\int_0^{\Delta(\varepsilon_{i_k},\varepsilon^*)} \Phi(s)\,\delta s) - \tau(\Delta(\varepsilon_{i_k}, \varepsilon^*)).\end{aligned} \tag{3.12}$$

Taking a limit as $k \to \infty$ in (3.12),

$$\begin{aligned}\mathcal{F}(\int_0^{\check{D}_\varsigma(\varepsilon^*,Y\varepsilon^*)} \Phi(s)\,\delta s) &\leq \mathcal{F}(\int_0^{\check{D}_\varsigma(\varepsilon^*,Y\varepsilon^*)} \Phi(s)\,\delta s) - \tau(\Delta(\varepsilon^*, \varepsilon^*)) \\ &< \mathcal{F}(\int_0^{\check{D}_\varsigma(\varepsilon^*,Y\varepsilon^*)} \Phi(s)\,\delta s),\end{aligned}$$

which is a contradiction. Thus, since $\Psi \ni \varepsilon_i \longmapsto \delta_\varsigma(\varepsilon_i, Y\varepsilon_i)$ is $\check{D}_\varsigma(Y, \varepsilon_0)$-$F$-dynamic lower semi-continuous, we have

$$\begin{aligned}\int_0^{\check{D}_\varsigma(\varepsilon^*,Y\varepsilon^*)} \Phi(s)\,\delta s &\leq \liminf_{n\to\infty} \int_0^{\check{D}_\varsigma(\varepsilon_{i_k},Y\varepsilon_{i_k})} \Phi(s)\,\delta s \\ &\leq \liminf_{n\to\infty} \int_0^{\check{D}_\varsigma(\varepsilon_i,Y\varepsilon_i)} \Phi(s)\,\delta s \\ &= 0.\end{aligned} \tag{3.13}$$

The closedness of $Y\varepsilon^*$ implies that $\varepsilon^* \in Y\varepsilon^*$ which means that ε^* has a fixed point of Y. □

Some direct consequences of Theorem 3.4 are as follows:

Remark 3.5. In light of Theorem 3.4, we derive the following contractive condition:

$$\tau(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)} \Phi(s)\,\delta s) + \mathcal{F}(\int_0^{\hat{H}_\varsigma(Y\varepsilon_i,Y\varepsilon_{i+1})} \Phi(s)\,\delta s) \leq \mathcal{F}(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)} \Phi(s)\,\delta s),$$

where

$$\Delta(\varepsilon_{i-1}, \varepsilon_i) = \max\left\{\delta_\varsigma(\varepsilon_{i-1}, \varepsilon_i), \check{D}_\varsigma(\varepsilon_{i-1}, Y\varepsilon_{i-1}), \check{D}_\varsigma(\varepsilon_i, Y\varepsilon_i), \frac{\check{D}_\varsigma(\varepsilon_{i-1}, Y\varepsilon_i) + \check{D}_\varsigma(\varepsilon_i, Y\varepsilon_{i-1})}{2}\right\}$$

for all $i \in \mathbb{N}$, $\varepsilon_i \in \check{D}_\varsigma(Y, \varepsilon_0)$ and $\hat{H}_\varsigma(Y\varepsilon_i, Y\varepsilon_{i+1}) > 0$. Then, Y has a fixed point.

Corollary 3.6. *Let* (Ψ, δ_ς) *be a complete controlled-metric space and* $Y : \Psi \to CB(\Psi)$ *be a set-valued Branciari Ćirić type contraction based on F-dynamic-iterative scheme* $\check{D}(Y, \varepsilon_0)$. *Suppose for some* $F \in \nabla_{\mathcal{F}}$, $\tau : (0, +\infty) \to (0, +\infty)$ *a non-constant function and* $\Phi : \kappa \to \kappa$ *a non-negative Lebesgue integrable mapping which is summable on each compact subset of* κ *so that*

$$2\tau(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)} \Phi(s)\,\delta s) + \mathcal{F}(\int_0^{\hat{H}_\varsigma(Y\varepsilon_i,Y\varepsilon_{i+1})} \Phi(s)\,\delta s) \leq \mathcal{F}(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)} \Phi(s)\,\delta s) \tag{3.14}$$

where

$$\Delta(\varepsilon_{i-1},\varepsilon_i)=\max\left\{\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i),\check{D}_\varsigma(\varepsilon_{i-1},Y\varepsilon_{i-1}),\check{D}_\varsigma(\varepsilon_i,Y\varepsilon_i),\frac{\check{D}_\varsigma(\varepsilon_{i-1},Y\varepsilon_i)+\check{D}_\varsigma(\varepsilon_i,Y\varepsilon_{i-1})}{2}\right\}$$

for all $i\in\mathbb{N}$, $\varepsilon_i\in\check{D}_\varsigma(Y,\varepsilon_0)$, $\delta_\varsigma(Y\varepsilon_i,Y\varepsilon_{i+1})>0$ *and for each given* $\epsilon>0$ *such that* $\int_0^\epsilon\Phi(s)\delta s>0$. *Assume that (3.4) and (3.4) are satisfied. Then Y has a fixed point.*

Remark 3.7. In view of Corollary 3.6, we state the following contractive condition:

$$\hat{H}_\varsigma(Y\varepsilon_i,Y\varepsilon_{i+1})>0\Rightarrow 2\tau(\Delta(\varepsilon_{i-1},\varepsilon_i))+\mathcal{F}(\int_0^{\hat{H}_\varsigma(Y\varepsilon_i,Y\varepsilon_{i+1})}\Phi(s)\delta s)\leq\mathcal{F}(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)}\Phi(s)\delta s),$$

where

(*i*) $\Delta_1(\varepsilon_{i-1},\varepsilon_i)=\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)$;
(*ii*) $\Delta_2(\varepsilon_{i-1},\varepsilon_i)=\max\left\{\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i),\check{D}_\varsigma(\varepsilon_{i-1},Y\varepsilon_{i-1}),\check{D}_\varsigma(\varepsilon_i,Y\varepsilon_i)\right\}$;
(*iii*) $\Delta_3(\varepsilon_{i-1},\varepsilon_i)=\max\left\{\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i),\frac{\check{D}_\varsigma(\varepsilon_{i-1},Y\varepsilon_{i-1}),\check{D}_\varsigma(\varepsilon_i,Y\varepsilon_i)}{2},\frac{\check{D}_\varsigma(\varepsilon_{i-1},Y\varepsilon_i)+\check{D}_\varsigma(\varepsilon_i,Y\varepsilon_{i-1})}{2}\right\}$

for all $i\in\mathbb{N}$, $\varepsilon_i\in\check{D}_\varsigma(Y,\varepsilon_0)$. Then Y has a fixed point.

Corollary 3.8. *Let* (Ψ,δ_ς) *be a complete controlled-metric space and* $Y:\Psi\to CB(\Psi)$ *be a set-valued Branciari Ćirić type contraction based on F-dynamic-iterative scheme* $\check{D}(Y,\varepsilon_0)$. *If for some* $F\in\nabla_{\mathcal{F}}$, $\tau_j:(0,+\infty)\to(0,+\infty)$, $j=1,2$ *a non-constant function and* $\Phi:\kappa\to\kappa$ *is a non-negative Lebesgue integrable mapping which is summable on each compact subset of* κ *so that one of the following holds:*
(G_1) $\delta_\varsigma(Y\varepsilon_i,Y\varepsilon_{i+1})>0\Rightarrow\tau_{j=1}(\Delta(\varepsilon_{i-1},\varepsilon_i))-\frac{1}{\int_0^{\hat{H}_\varsigma(Y\varepsilon_i,Y\varepsilon_{i+1})}\Phi(s)\delta s}\leq-\frac{1}{\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)}\Phi(s)\delta s}$;
(G_2) $\delta_\varsigma(Y\varepsilon_i,Y\varepsilon_{i+1})>0\Rightarrow\tau_{j=2}(\Delta(\varepsilon_{i-1},\varepsilon_i))+\frac{1}{1-\exp\int_0^{\hat{H}_\varsigma(Y\varepsilon_i,Y\varepsilon_{i+1})}\Phi(s)\delta s}\leq\frac{1}{1-\exp\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)}\Phi(s)\delta s}$,
where

$$\Delta(\varepsilon_{i-1},\varepsilon_i)=\max\left\{\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i),\delta_\varsigma(\varepsilon_{i-1},Y\varepsilon_{i-1}),\delta_\varsigma(\varepsilon_i,Y\varepsilon_i),\frac{\delta_\varsigma(\varepsilon_{i-1},Y\varepsilon_i)+\delta_\varsigma(\varepsilon_i,Y\varepsilon_{i-1})}{2}\right\}$$

for all $i\in\mathbb{N}$, $\varepsilon_i\in\check{D}_\varsigma(Y,\varepsilon_0)$ *and for each given* $\epsilon>0$ *so that* $\int_0^\epsilon\Phi(s)\delta s>0$. *Assume that (D1) and (D2) are satisfied. Then Y has a fixed point.*

Proof. The proof directly proceed from Corollary 3.8 based on the functions $F(\varepsilon)=-\frac{1}{\varepsilon}$ and $F(\varepsilon)=\frac{1}{1-\exp(\varepsilon)}$, which is also fulfilled for the family $\nabla_{\mathcal{F}}$, then the result follows. □

Example 3.9. Let $\Psi=R^+\cup\{0\}$. Define the complete controlled-metric spaces (Ψ,δ_ς) by

$$\delta_\varsigma(\varepsilon_1,\varepsilon_2)=\begin{cases}0,\ \varepsilon_1=\varepsilon_2;\\ \frac{1}{\varepsilon_1},\ \varepsilon_1\geq1\ \&\ \varepsilon_2\in[0,1);\\ \frac{1}{\varepsilon_2}\ \varepsilon_2\geq1\ \&\ \varepsilon_1\in[0,1);\\ 1,\ \text{otherwise,}\end{cases}$$

and $\varsigma:\Psi\times\Psi\to[1,\infty)$ as

$$\varsigma(\varepsilon_1,\varepsilon_2)=\begin{cases}1,\ \varepsilon_1,\varepsilon_2\in[0,1);\\ \max\{\varepsilon_1,\varepsilon_2\},\ \text{otherwise.}\end{cases}$$

Consider a mapping $Y : \Psi \to CB(\Psi)$ defined by $Y\varepsilon = \left[0, \frac{\varepsilon}{2}\right], \varepsilon > 0$ and τ a non-constant function, that is, $\tau : R^+ \to R^+$ is of the form

$$\tau(\varepsilon) = \varepsilon.\ln(\frac{101}{100}), \qquad \text{for } \varepsilon \in (0, +\infty).$$

Consider tghe dynamic iterative process $\check{D}(Y, \varepsilon_0)$: A sequence $\{\varepsilon_i\}$ is defined by $\varepsilon_i = \varepsilon_0 g^{i-1}$ for each $i \in \mathbb{N}$ with starting point $\varepsilon_0 = 2$ and $g = \frac{1}{2}$ so that (see Table 1):

Table 1. F-dynamic iterative process; for $i \geq 2$.

$i \geq 2$	$\varepsilon_i = \varepsilon_0 g^{i-1}$	$\in$	$Y\varepsilon_{i-1} = \left[0, \frac{\varepsilon}{2}\right]$
$\varepsilon_{i=2}$	1	−	$Y\varepsilon_{i=1} = [0, 1]$
$\varepsilon_{i=3}$	$\frac{1}{2}$	−	$Y\varepsilon_{i=2} = [0, \frac{1}{2}]$
$\varepsilon_{i=4}$	$\frac{1}{4}$	−	$Y\varepsilon_{i=2} = [0, \frac{1}{4}]$
$\varepsilon_{i=5}$	$\frac{1}{8}$	−	$Y\varepsilon_{i=2} = [0, \frac{1}{8}]$

By continuing the above iterative process, one asserts that

$$\check{D}(Y, \varepsilon_0) = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\} \tag{3.15}$$

is a F-dynamic iterative process of Y starting from $\varepsilon_0 = 2$.

For $\varepsilon_i \in \check{D}_\varsigma(Y, \varepsilon_0)$ and Y as a Branciari Ćirić type contraction mapping with respect to F-dynamic-iterative scheme $\check{D}_\varsigma(Y, \varepsilon_0)$, we obtain $\hat{H}_\varsigma(Y\varepsilon_i, Y\varepsilon_{i+1}) = \frac{|\varepsilon_{i-1} - \varepsilon_i|}{2}$ and $\Delta(\varepsilon_{i-1}, \varepsilon_i) = |\varepsilon_{i-1} - \varepsilon_i|$. Now, by contractive condition (3.1) upon setting of $F(\varepsilon) = \ln(\varepsilon)$ and $\Phi(s) = 1$ for all $s \in R$, we see that $\tau(h) \leq \Omega(\imath)$, where

$$\Omega(i) = \mathcal{F}(\int_0^{|\varepsilon_{i-1} - \varepsilon_i|} \delta s) - \mathcal{F}(\int_0^{\frac{|\varepsilon_{i-1} - \varepsilon_i|}{2}} \delta s).$$

Hence, all the required hypotheses of Theorem 3.4 are satisfied and consequently in view of Tables 1 and 2, and Figures 1 and 2, the required hypotheses of Theorem 4.4 regarding to $\tau(h) \leq \Omega(\imath)$, are satisfied for all possible values. Here, $0 \in Y(0)$ is a fixed point of Y for a Branciari Ćirić type contraction mapping with respect to F-dynamic-iterative scheme $\check{D}_\varsigma(Y, \varepsilon_0)$.

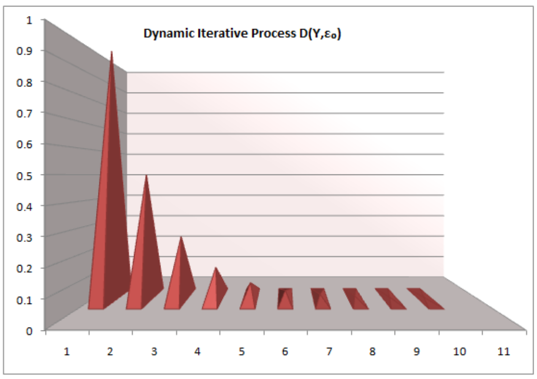

Figure 1. F-dynamic iterative process of Y starting from $\varepsilon_0 = 2$.

Table 2. Corresponding values of $\tau(h)$ & $\Omega(\iota)$.

ε_ι	$\varepsilon_{\iota-1}$	$\tau(h)$	$\Omega(\iota)$
1	0.25	0.00995033085	0.693147
.	0.0625	0.00497516542	.
.	0.0156	0.00248758271	.
.	0.003906	0.00124379136	.
.	0.000977	0.00062189568	.
.	0.000244	0.00031094784	.
.	0.0000152	0.00015547392	.
.	0.00006103515	0.00007773696	.
.	0.00001525878	0.00003886848	.
1	0.00000381469	0.00001943424	0.693147

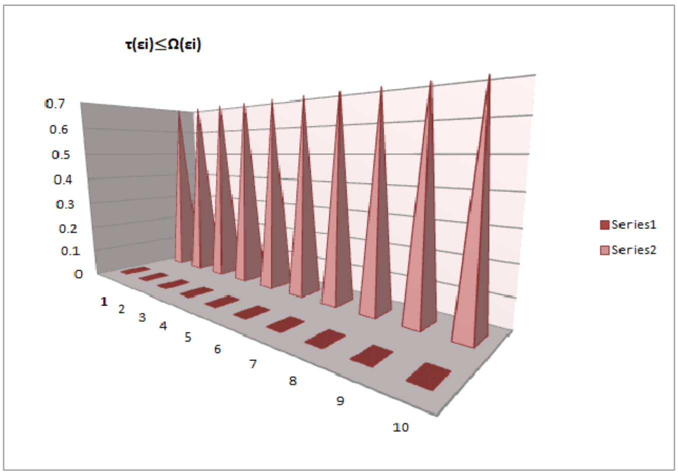

Figure 2. $\tau(h) \le \Omega(\iota)$.

4. A Family of F_H-dynamic-iterative scheme: $\check{D}(F, Y, \alpha_0)$

Here, we give our second general definition.

Definition 4.1. Let $F : \Psi \to \Psi$ and $Y : \Psi \to CB(\Psi)$ be so that

$$\check{D}(F, Y, \alpha_0) = \{(\alpha_j)_{j \in \mathbb{N} \cup \{0\}} : \alpha_{j+1} = F\alpha_j \in Y\alpha_{j-1}\} \tag{4.1}$$

for each integer $j \ge 1$. The set $\check{D}(F, Y, \alpha_0)$ is said to be a hybrid dynamic-iterative scheme of F and Y having the starting point α_0. The hybrid dynamic-iterative scheme $\check{D}(F, Y, \alpha_0)$ is shortly written as $F\left(\alpha_j\right)$.

Definition 4.2. Let $F : \Psi \to \Psi$ and $Y : \Psi \to CB(\Psi)$ be an hybrid Branciari Ćirić type contraction on the controlled-metric space (Ψ, δ_ς) with respect to F_H-dynamic-iterative scheme $\check{D}(F, Y, \varepsilon_0)$. Suppose there are $F_H \in \nabla_{\mathcal{F}}$, $\tau : (0,+\infty) \to (0,+\infty)$ a non-constant function and $\Phi : \kappa \to \kappa$ a non-negative Lebesgue integrable mapping which is summable on each compact subset of κ so that

$$\delta_\varsigma (F\varepsilon_i, F\varepsilon_{i+1}) > 0 \;\Rightarrow\; \tau (\Delta (\varepsilon_{i-1}, \varepsilon_i)) + \mathcal{F}_H(\int_0^{\delta_\varsigma (F\varepsilon_i, F\varepsilon_{i+1})} \Phi (s)\, \delta s) \leq \mathcal{F}_H(\int_0^{\Delta(\varepsilon_{i-1}, \varepsilon_i)} \Phi (s)\, \delta s) \tag{4.2}$$

where

$$\Delta (\varepsilon_{i-1}, \varepsilon_i) = \max \left\{ \delta_\varsigma (F\varepsilon_{i-1}, F\varepsilon_i), \delta_\varsigma (F\varepsilon_{i-1}, Y\varepsilon_{i-1}), \delta_\varsigma (F\varepsilon_i, Y\varepsilon_i), \frac{\delta_\varsigma (F\varepsilon_{i-1}, Y\varepsilon_i) + \delta_\varsigma (F\varepsilon_i, Y\varepsilon_{i-1})}{2} \right\}$$

for all $i \in \mathbb{N}$, $\varepsilon_i \in \check{D} (F, Y, \varepsilon_0)$ and for each given $\epsilon > 0$ so that $\int_0^\epsilon \Phi (s)\, \delta s > 0$.

Remark 4.3. Via Remark 3.2, we consider only the F_H-dynamic iterative scheme $\varepsilon_i \in \check{D} (F, Y, \varepsilon_0)$ that satisfying the following condition:

$$\delta_\varsigma (F\varepsilon_i, F\varepsilon_{i+1}) > 0 \;\Rightarrow\; \delta_\varsigma (F\varepsilon_{i-1}, F\varepsilon_i) > 0 \text{ for each } i \in \mathbb{N}. \tag{4.3}$$

If the investigated process that does not satisfy (4.3), then there is some $i_0 \in \mathbb{N}$ so that

$$\delta_\varsigma (F\varepsilon_{i_0}, F\varepsilon_{i_{0+1}}) > 0$$

and

$$\delta_\varsigma (F\varepsilon_{i_{0-1}}, F\varepsilon_{i_0}) = 0,$$

then we get $F\varepsilon_{i_{0-1}} = F\varepsilon_{i_0} \in Y\varepsilon_{i_{0-1}}$ which implies the existence of common fixed point. Due to this consideration of F_H-dynamic iterative scheme that satisfies (4.3), it does not depreciate a generality of our approach. Moreover, owing to Example 3.2, we easily conclude that the hybrid pair (F, Y) with respect to F_H-dynamic iterative scheme $\check{D} (F, Y, \varepsilon_0)$ is a contraction mapping.

Theorem 4.4. *Let $F : \Psi \to \Psi$ and $Y : \Psi \to CB (\Psi)$ be an hybrid Branciari Ćirić type contraction on the controlled-metric space (Ψ, δ_ς) with respect to F_H-dynamic-iterative scheme $\check{D} (F, Y, \varepsilon_0)$. Assume $F_H (F, Y) \neq \phi$, where $F_H (F, Y)$ provided that $F(\Psi)$ is complete and Y is a closed multivalued mapping such that*
(D3) there is an F_H-dynamic iterative scheme $\varepsilon_i \in \check{D} (F, Y, \varepsilon_0)$ such that such that

$$\liminf_{k \to l^+} \tau (k) > 0 \textit{ for each } l \geq 0;$$

(D4) for some $\varepsilon \in F_H (F, Y)$, F is Y-weakly commuting at ε so that $F^2\varepsilon = YF\varepsilon$.
Then the hybrid pair (F, Y) has a common fixed point.

Proof. Consider $\varepsilon_0 \in \Psi$ to be an arbitrary point. In view of (4.1), we have

$$\check{D}(F, Y, \varepsilon_0) = \{(\varepsilon_i)_{i \in \mathbb{N} \cup \{0\}} : \varepsilon_{i+1} = \varepsilon_i \in Y\varepsilon_{i-1}\}.$$

In case, if there is $i_0 \in \mathbb{N}$ so that $\varepsilon_{i_0} \in F\varepsilon_{i_0}$, then ε_{i_0} is a fixed point of F is clear. Therefore, if we let $\varepsilon_i \notin F\varepsilon_i$ then $\check{D}(F, Y, \varepsilon_0) > 0$ for every $i \in \mathbb{N}$. Using (4.2), one writes

$$\mathcal{F}_H(\int_0^{\delta_\varsigma (F\varepsilon_i, F\varepsilon_{i+1})} \Phi (s)\, \delta s) \leq \mathcal{F}_H(\int_0^{\Delta(\varepsilon_{i-1}, \varepsilon_i)} \Phi (s)\, \delta s) - \tau (\Delta (\varepsilon_{i-1}, \varepsilon_i))$$

$$= \mathcal{F}_H\Big(\int_0^{\max\left\{\begin{array}{c}\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_{i-1},Y\varepsilon_{i-1}),\delta_\varsigma(F\varepsilon_i,Y\varepsilon_i),\\ \frac{\delta_\varsigma(F\varepsilon_{i-1},Y\varepsilon_i)+\delta_\varsigma(F\varepsilon_i,Y\varepsilon_{i-1})}{2}\end{array}\right\}}\Phi(s)\,\delta s\Big)$$
$$-\tau\left(\max\left\{\begin{array}{c}\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_{i-1},Y\varepsilon_{i-1}),\delta_\varsigma(F\varepsilon_i,Y\varepsilon_i),\\ \frac{\delta_\varsigma(F\varepsilon_{i-1},Y\varepsilon_i)+\delta_\varsigma(F\varepsilon_i,Y\varepsilon_{i-1})}{2}\end{array}\right\}\right),$$

which implies

$$\begin{aligned}\mathcal{F}_H\Big(\int_0^{\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})}\Phi(s)\,\delta s\Big) &\leq \mathcal{F}_H\Big(\int_0^{\max\left\{\begin{array}{c}\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1}),\\ \frac{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_{i+1})+\delta_\varsigma(F\varepsilon_i,F\varepsilon_i)}{2}\end{array}\right\}}\Phi(s)\,\delta s\Big)\\ &\quad-\tau\left(\left\{\begin{array}{c}\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1}),\\ \frac{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_{i+1})+\delta_\varsigma(F\varepsilon_i,F\varepsilon_i)}{2}\end{array}\right\}\right)\\ &\leq \mathcal{F}_H\Big(\int_0^{\max\{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})\}}\Phi(s)\,\delta s\Big)\\ &\quad-\tau\left(\max\{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})\}\right).\end{aligned}\tag{4.4}$$

Based on (4.1) and (4.4), we have

$$\begin{aligned}\mathcal{F}_H\Big(\int_0^{\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})}\Phi(s)\,\delta s\Big) &\leq \mathcal{F}_H\Big(\int_0^{\max\{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})\}}\Phi(s)\,\delta s\Big)\\ &\quad-\tau\left(\max\{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})\}\right)\\ &< \mathcal{F}_H\Big(\int_0^{\max\{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})\}}\Phi(s)\,\delta s\Big),\end{aligned}$$

for all $i \in \mathbb{N}$. Due to (F_i), we obtain for some i,

$$\begin{aligned}\int_0^{\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})}\Phi(s)\,\delta s &< \int_0^{\max\{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i),\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})\}}\Phi(s)\,\delta s\\ &= \int_0^{\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})}\Phi(s)\,\delta s,\end{aligned}$$

which gives a contradiction. Thus, we get

$$\int_0^{\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})}\Phi(s)\,\delta s < \int_0^{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i)}\Phi(s)\,\delta s.$$

Consequently,

$$\mathcal{F}_H\Big(\int_0^{\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})}\Phi(s)\,\delta s\Big) \leq \mathcal{F}_H\Big(\int_0^{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i)}\Phi(s)\,\delta s\Big) - \tau\left(\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i)\right) \tag{4.5}$$

for all $i \in \mathbb{N}$. Thus, the sequence $\{\delta_\varsigma(\varepsilon_i,\varepsilon_{i+1})\}$ is decreasing and hence convergent. Now, we show that $\lim_{i\to\infty}\delta_\varsigma(\varepsilon_i,\varepsilon_{i+1}) = 0$. From (3.4) there is $\sigma > 0$ and $i_0 \in \mathbb{N}$, so that $\tau(\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)) > \sigma$ for all $i > i_0$. Thus, we have

$$\mathcal{F}_H\Big(\int_0^{\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})}\Phi(s)\,\delta s\Big) \leq \mathcal{F}_H\Big(\int_0^{\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i)}\Phi(s)\,\delta s\Big) - \tau\left(\delta_\varsigma(F\varepsilon_{i-1},F\varepsilon_i)\right) \tag{4.6}$$

$$
\begin{aligned}
&\leq \mathcal{F}_H(\int_0^{\delta_\varsigma(F\varepsilon_{i-2},F\varepsilon_{i-1})}\Phi(s)\,\delta s)-\tau\left(\delta_\varsigma\left(F\varepsilon_{i-2},F\varepsilon_{i-1}\right)\right)-\tau\left(\delta_\varsigma\left(F\varepsilon_{i-1},F\varepsilon_i\right)\right)\\
&\vdots\\
&\leq \mathcal{F}_H(\int_0^{\delta_\varsigma(F\varepsilon_0,F\varepsilon_1)}\Phi(s)\,\delta s)-\tau\left(\delta_\varsigma\left(F\varepsilon_0,F\varepsilon_1\right)\right)-\cdots-\tau\left(\delta_\varsigma\left(F\varepsilon_{i-1},F\varepsilon_i\right)\right)\\
&= \mathcal{F}_H(\int_0^{\delta_\varsigma(F\varepsilon_0,F\varepsilon_1)}\Phi(s)\,\delta s)-\tau\left(\left(\delta_\varsigma\left(F\varepsilon_0,F\varepsilon_1\right)\right)+\cdots+\tau\left(F\varepsilon_{i_{0-1}},F\varepsilon_{i_0}\right)\right)\\
&\quad -\tau\left(\delta_\varsigma\left(F\varepsilon_{i_0},F\varepsilon_{i_{0+1}}\right)\right)+\cdots+\tau\left(\delta_\varsigma\left(F\varepsilon_{i-1},F\varepsilon_i\right)\right)\\
&\leq \mathcal{F}_H(\int_0^{\delta_\varsigma(F\varepsilon_0,F\varepsilon_1)}\Phi(s)\,\delta s)-(i-i_0)\,\sigma.
\end{aligned}
$$

Setting $\varpi_i = \int_0^{\delta_\varsigma(F\varepsilon_i,F\varepsilon_{i+1})}\Phi(s)\,\delta s > 0$ for $i = 0, 1, 2, ...$ and from (4.6), we obtain $\lim_{i\to\infty} F(\varpi_i) = -\infty$. Using (F_{ii}) implies that

$$\lim_{i\to\infty}(\varpi_i) = 0. \tag{4.7}$$

From (F_{iii}), there is $\alpha \in (0, 1)$ so that

$$\lim_{i\to\infty}[\varpi_i]^k\,\mathcal{F}_H[\varpi_i] = 0. \tag{4.8}$$

By (4.6), the following holds for all $i > i_0$,

$$
\begin{aligned}
[\varpi_i]^\alpha\,\mathcal{F}_H[\varpi_i]-[\varpi_i]^\alpha\,\mathcal{F}_H[\lambda_0] &\leq [\varpi_i]^\alpha\left(\mathcal{F}_H(\varpi_0)-(i-i_0)\,\sigma\right)-[\varpi_i]^\alpha\,\mathcal{F}_H[\varpi_0] \\
&= -[\varpi_i]^\alpha\,(i-i_0)\,\sigma \leq 0.
\end{aligned} \tag{4.9}
$$

Taking limit as $i \to \infty$ in (4.9) and using (4.8), we have

$$\lim_{i\to\infty} i\,[\varpi_i]^\alpha = 0. \tag{4.10}$$

Let us perceive that from (4.10), there is $i_1 \in \mathbb{N}$ so that $i\,[\varpi_i]^\alpha \leq 1$ for all $i \geq i_1$. We have

$$\varpi_i \leq \frac{1}{i^{\frac{1}{\alpha}}}. \tag{4.11}$$

Now, we will show that $\{\varepsilon_i\}$ is a Cauchy sequence. For this mark, we consider $j_1,\ j_2 \in \mathbb{N}$ so that $j_1 > j_2 \geq i_1$. From (4.11),

$$
\begin{aligned}
&\int_0^{\delta_\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_2}\right)}\Phi(s)\,\delta s \qquad (4.12)\\
\leq\ &\int_0^{\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)\delta_\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)+\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_2}\right)\delta_\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_2}\right)}\Phi(s)\,\delta s\\
\leq\ &\int_0^{\left\{\begin{array}{l}\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)\delta_\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)+\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_2}\right)\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_{1+2}}\right)\\ \delta_\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_{1+2}}\right)+\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_2}\right)\varsigma\left(F\varepsilon_{j_{1+2}},F\varepsilon_{j_2}\right)\delta_\varsigma\left(F\varepsilon_{j_{1+2}},F\varepsilon_{j_2}\right)\end{array}\right.}\Phi(s)\,\delta s
\end{aligned}
$$

$$\leq \int_0^{\left\{\begin{array}{l}\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)\delta_\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)+\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_2}\right)\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_{1+2}}\right)\\ \delta_\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_{1+2}}\right)+\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_2}\right)\varsigma\left(F\varepsilon_{j_{1+2}},F\varepsilon_{j_2}\right)\varsigma\left(F\varepsilon_{j_{1+2}},F\varepsilon_{j_{1+3}}\right)\\ \delta_\varsigma\left(F\varepsilon_{j_{1+2}},F\varepsilon_{j_{1+3}}\right)+\varsigma\left(F\varepsilon_{j_{1+1}},F\varepsilon_{j_2}\right)\varsigma\left(F\varepsilon_{j_{1+2}},F\varepsilon_{j_2}\right)\varsigma\left(F\varepsilon_{j_{1+3}},F\varepsilon_{j_2}\right)\\ \delta_\varsigma\left(F\varepsilon_{j_{1+3}},F\varepsilon_{j_2}\right)\end{array}\right.}\Phi(s)\,\delta s$$

$$\leq \cdots$$

$$\leq \int_0^{\left\{\begin{array}{l}\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)\delta_\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)+\sum_{i=j_{1+1}}^{j_2-2}\left(\prod_{r=j_{1+1}}^{i}\varsigma\left(F\varepsilon_r,F\varepsilon_{j_2}\right)\right)\\ \varsigma\left(F\varepsilon_i,F\varepsilon_{i+1}\right)\delta_\varsigma\left(F\varepsilon_i,F\varepsilon_{i+1}\right)+\prod_{l=j_{1+1}}^{j_2-1}\varsigma\left(F\varepsilon_l,F\varepsilon_{j_2}\right)\delta_\varsigma\left(F\varepsilon_{j_{2-1}},F\varepsilon_{j_2}\right)\end{array}\right.}\Phi(s)\,\delta s,$$

which yields

$$\begin{aligned}\int_0^{\delta_\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_2}\right)}\Phi(s)\,\delta s &\leq \int_0^{\left\{\begin{array}{l}\varsigma\left(\varpi_{j_1},\varpi_{j_{1+1}}\right)\delta_\varsigma\left(\varpi_{j_1},\varpi_{j_{1+1}}\right)\\ +\sum_{i=1}^{j_2-1}\left(\prod_{r=j_{1+1}}^{i}\varsigma\left(\varepsilon_r,\varepsilon_{j_2}\right)\right)\\ \varsigma\left(\varepsilon_i,\varepsilon_{i+1}\right)\delta_\varsigma\left(\varepsilon_i,\varepsilon_{i+1}\right)\end{array}\right.}\Phi(s)\,\delta_\varsigma s\\ &\leq \int_0^{\left\{\begin{array}{l}\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)\delta_\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)\\ +\sum_{i=1}^{j_2-1}\left(\prod_{r=j_{1+1}}^{i}\varsigma\left(F\varepsilon_r,F\varepsilon_{j_2}\right)\right)\\ \varsigma\left(F\varepsilon_i,F\varepsilon_{i+1}\right)\delta_\varsigma\left(F\varepsilon_i,F\varepsilon_{i+1}\right)\end{array}\right.}\Phi(s)\,\delta_\varsigma s\\ &= \int_0^{\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)\varpi_{j_1}+\sum_{i=1}^{j_2-1}\left(\prod_{r=j_{1+1}}^{i}\varsigma\left(F\varepsilon_r,F\varepsilon_{j_2}\right)\right)\varsigma\left(F\varepsilon_i,F\varepsilon_{i+1}\right)\varpi_i}\Phi(s)\,\delta s\\ &\leq \int_0^{\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_{1+1}}\right)\frac{1}{(j_1)^{\frac{1}{\alpha}}}+\sum_{i=1}^{j_2-1}\left(\prod_{r=j_{1+1}}^{i}\varsigma\left(F\varepsilon_r,F\varepsilon_{j_2}\right)\right)\varsigma\left(F\varepsilon_i,F\varepsilon_{i+1}\right)\frac{1}{(i)^{\frac{1}{\alpha}}}}\Phi(s)\,\delta s.\end{aligned}$$

Owing to (4.12) and in view of convergence of series $\sum_{l=j_1}^{\infty}\frac{1}{l^{\frac{1}{\alpha}}}$, we get $\int_0^{\delta_\varsigma\left(F\varepsilon_{j_1},F\varepsilon_{j_2}\right)}\Phi(s)\,\delta s\to 0$. Hence, $\{F\varepsilon_i\}$ is Cauchy in $F(\Psi)$. Further, for the completeness of Ψ there is $\varepsilon^*\in F\Psi$ so that $\lim_{i\to\infty}F\varepsilon_i=\varepsilon^*$. Now, we claim that $F\varepsilon^*\in Y\varepsilon^*$. So, $\delta_\varsigma(F\varepsilon^*,Y\varepsilon^*)=0$ and $F\varepsilon^*\in Y\varepsilon^*$. In case, $F\varepsilon^*\notin Y\varepsilon^*$ then $\delta_\varsigma(F\varepsilon^*,Y\varepsilon^*)>0$ as F is compact. By (F_i) and Lemma 2.8, we see that

$$\delta_\varsigma(F\varepsilon_i,Y\varepsilon^*)\leq\hat{H}_\varsigma(Y\varepsilon_{i-1},F\varepsilon^*)<\Delta(\varepsilon_{i-1},\varepsilon^*).\tag{4.13}$$

Suppose on the contrary, $F\varepsilon^*\notin Y\varepsilon^*$ then there are an $i_0\in\mathbb{N}$ and a subsequence $\{\varepsilon_{i_k}\}$ of $\{\varepsilon_i\}$ so that $\delta_\varsigma(F\varepsilon_{i_k+1},Y\varepsilon^*)>0$ for each $i_k\geq i_0$ [Otherwise, there is $i_1\in\mathbb{N}$ so that $F\varepsilon_i\in Y\varepsilon^*$ for every $i\geq i_1$, which yields $F\varepsilon^*\in Y\varepsilon^*$, a contradiction]. Since $\delta_\varsigma(F\varepsilon_{i_k+1},Y\varepsilon^*)>0$ for each $i_k\geq i_0$, by the contractive condition, one writes

$$\mathcal{F}_H(\int_0^{\delta_\varsigma\left(F\varepsilon_{i_k+1},Y\varepsilon^*\right)}\Phi(s)\,\delta s)\leq\mathcal{F}_H(\int_0^{\Delta(\varepsilon_{i_k},\varepsilon^*)}\Phi(s)\,\delta s)-\tau(\Delta(\varepsilon_{i_k},\varepsilon^*)).\tag{4.14}$$

Letting $k \to \infty$ in (4.14),

$$\begin{aligned}\mathcal{F}_H\Big(\int_0^{\delta_\varsigma(F\varepsilon^*,Y\varepsilon^*)}\Phi(s)\,\delta s\Big) &\leq \mathcal{F}_H\Big(\int_0^{\delta_\varsigma(F\varepsilon^*,Y\varepsilon^*)}\Phi(s)\,\delta s\Big)-\tau(\delta_\varsigma(F\varepsilon^*,F\varepsilon^*))\\ &< \mathcal{F}_H\Big(\int_0^{\delta_\varsigma(F\varepsilon^*,Y\varepsilon^*)}\Phi(s)\,\delta s\Big),\end{aligned}$$

a contradiction. Thus, $F\varepsilon^* \in Y\varepsilon^*$, which means that ε^* has a common fixed point of the hybrid pair (F, Y). Further, for some $\varepsilon \in F_H(F, Y)$, F is Y-weakly commuting at ε so that $F\varepsilon^2 = F\varepsilon$. So we obtain $F^2\varepsilon \in YF\varepsilon$. In the light of given hypothesis, we see that $F\varepsilon = F^2\varepsilon$ and hence $F\varepsilon = F^2\varepsilon \in YF\varepsilon$, Consequently, $F\varepsilon \in F_H(F, Y)$. □

Some direct consequences of Theorem 4.4 are given.

Remark 4.5. In the light of Theorem 3.4, we obtain the following contractive conditions:
(i) $\tau(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)}\Phi(s)\,\delta s)+F_H(\int_0^{\delta_\varsigma(F\varepsilon_{i+1},F\varepsilon_i)}\Phi(s)\,\delta s)\leq F_H(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)}\Phi(s)\,\delta s)$;
(ii) $2\tau(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)}\Phi(s)\,\delta s)+F_H(\int_0^{\delta_\varsigma(F\varepsilon_{i+1},F\varepsilon_i)}\Phi(s)\,\delta s)\leq F_H(\int_0^{\Delta(\varepsilon_{i-1},\varepsilon_i)}\Phi(s)\,\delta s)$.
Due to above fashion, we easily see that the hybrid pair (F, Y) has a common fixed point.

Corollary 4.6. *Let $F : \Psi \to \Psi$ and $Y : \Psi \to CB(\Psi)$ be an hybrid Branciari Ćirić type contraction on the controlled-metric space (Ψ, δ_ς) with respect to F_H-dynamic-iterative scheme $\check{D}(F, Y, \varepsilon_0)$. Suppose there are $F_H \in \nabla_{\mathcal{F}}$, $\tau_j : (0,+\infty) \to (0,+\infty)$, $j = \overline{1,6}$ a non-constant function and $\Phi : \kappa \to \kappa$ a non-negative Lebesgue integrable mapping which is summable on each compact subset of κ so that $\delta(F\alpha_i, F\alpha_{i+1}) > 0$ and one of the following holds:*

$(G1) : \tau_{j=1}(\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\,\delta s)-\frac{1}{\int_0^{\delta(F\alpha_i,F\alpha_{i+1})}\Phi(s)\delta s}\leq-\frac{1}{\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\delta s}$;

$(G2) : \tau_{j=2}(\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\,\delta s)+\exp(\int_0^{\delta(F\alpha_i,F\alpha_{i+1})}\Phi(s)\,\delta s)\leq(\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\,\delta s)$;

$(G3) : \tau_{j=-3}(\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\,\delta s)+\frac{1}{1-\exp(\int_0^{\delta(F\alpha_i,F\alpha_{i+1})}\Phi(s)\delta s)}\leq\frac{1}{1-\exp(\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\delta s)}$;

$(G4) : \tau_{j=4}(\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\,\delta s)^c+(\int_0^{\delta(F\alpha_i,F\alpha_{i+1})}\Phi(s)\,\delta s)^c\leq(\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\,\delta s)^c$, $c > 0$;

$(G5) : \tau_{j=-5}(\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\,\delta s)-\frac{1}{\int_0^{\delta(F\alpha_i,F\alpha_{i+1})}\Phi(s)\delta s}+\int_0^{\delta(F\alpha_i,F\alpha_{i+1})}\Phi(s)\,\delta s\leq-\frac{1}{\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\delta s}$
$+\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\,\delta s$;

$(G6)\ \tau_{j=-6}(\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)\,\delta s)+\int_0^{\delta(F\alpha_i,F\alpha_{i+1})}\Phi(s)\,\delta s\exp\int_0^{\delta(F\alpha_i,F\alpha_{i+1})}\Phi(s)\,\delta s\leq\int_0^{\Delta(\alpha_{i-1},\alpha_i)}\Phi(s)$
$\delta s\exp\int_0^{\delta(F\alpha_i,F\alpha_{i+1})}\Phi(s)\,\delta s$,

where,
$\Delta(\alpha_{i-1},\alpha_i)=\max\left\{\delta(F\alpha_{i-1},F\alpha_i),\delta(F\alpha_{i-1},Y\alpha_{i-1}),\delta(F\alpha_i,Y\alpha_i),\frac{\delta(F\alpha_{i-1},Y\alpha_i)+\delta(F\alpha_i,Y\alpha_{i-1})}{2}\right\}$,
for all $i \in \mathbb{N}$, $\alpha_i \in \check{D}(F, Y, \alpha_0)$ and for each given $\epsilon > 0$ so that $\int_0^{\epsilon}\Phi(s)\,\delta s > 0$. Assume that (D3) and (D4) are satisfied. Then the hybrid pair (F, Y) has a common fixed point.

Proof. The proof follows directly from Corollary 4.6 based on the functions: $F(\alpha) = -\frac{1}{\alpha}$, $F(\alpha) = \exp(\alpha)$, $F(\alpha) = \frac{1}{1-\exp(\alpha)}$, $F(\alpha) = \alpha^{c>0}$, $F(\alpha) = -\frac{1}{\alpha} + \alpha$ and $F(\alpha) = \alpha.\exp(\alpha)$. Also, for the family $\nabla_{\mathcal{F}}$, the result follows. □

Example 4.7. Let $\Psi = R^+ \cup \{0\}$. Define the complete controlled-metric space (Ψ, δ_ς) by

$$\delta_\varsigma(\varepsilon_1, \varepsilon_2) = \begin{cases} 0, \ \varepsilon_1 = \varepsilon_2; \\ \frac{1}{\varepsilon_1}, \ \varepsilon_1 \geq 1 \ \& \ \varepsilon_2 \in [0,1); \\ \frac{1}{\varepsilon_2} \ \varepsilon_2 \geq 1 \ \& \ \varepsilon_1 \in [0,1); \\ 1, \ \text{otherwise}, \end{cases}$$

and $\varsigma : \Psi \times \Psi \to [1, \infty)$ as

$$\varsigma(\varepsilon_1, \varepsilon_2) = \begin{cases} 1, \ \varepsilon_1, \varepsilon_2 \in [0,1); \\ \max\{\varepsilon_1, \varepsilon_2\}, \ \text{otherwise}. \end{cases}$$

Let $F : \Psi \to \Psi$ and $Y : \Psi \to CB(\Psi)$ defined by $F\varepsilon = \frac{\varepsilon-1}{2}$ and

$$Y\varepsilon = \begin{cases} \left[\frac{1}{4}, \frac{\varepsilon}{2}\right], \varepsilon > 0 \\ \{0\}, \text{otherwise}. \end{cases}$$

Let τ be a non-constant function, that is, $\tau : R^+ \to R^+$ is of the form

$$\tau(\varepsilon) = \varepsilon.\ln(\frac{101}{100}), \qquad \text{for } \varepsilon \in (0, 70).$$

Design a sequence $\{\varepsilon_i\}$ by $\varepsilon_i = \varepsilon_{i-1} + 1$ with $\varepsilon_0 = 1$. Then the following estimates hold (see Table 3):

Table 3. F_H-dynamic iterative process; for $i \geq 1$.

$i \geq 1$	$\varepsilon_i = \varepsilon_{i-1} + 1$	$F\varepsilon_i$	$Y\varepsilon_{i-1} = \left[\frac{1}{4}, \frac{\varepsilon}{2}\right]$
$\varepsilon_{i=1}$	$\frac{1}{2}$	$F\varepsilon_{i=1} = \frac{1}{2}$	$Y\varepsilon_{i=0} = [\frac{1}{4}, \frac{1}{2}]$
$\varepsilon_{i=2}$	1	$F\varepsilon_{i=2} = 1$	$Y\varepsilon_{i=1} = [\frac{1}{4}, 1]$
$\varepsilon_{i=3}$	$\frac{3}{2}$	$F\varepsilon_{i=3} = \frac{3}{2}$	$Y\varepsilon_{i=2} = [\frac{1}{4}, \frac{3}{2}]$
$\varepsilon_{i=4}$	2	$F\varepsilon_{i=4} = 2$	$Y\varepsilon_{i=1} = [\frac{1}{4}, 2]$

Continuing in this way,

$$\check{D}(F, Y, \varepsilon_0) = \{\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\}$$

is an F_H-dynamic-iterative scheme of F and Y starting from the point $\varepsilon_0 = 1$.

For $\varepsilon_i \in \check{D}(F, Y, \varepsilon_0)$ and the hybrid pair (F, Y) for Branciari Ćirić type contraction mappings with respect to F_H-dynamic-iterative scheme $\check{D}(F, Y, \varepsilon_0)$, we see that $\hat{H}_\varsigma(F\varepsilon_i, F\varepsilon_{i+1}) = \frac{|\varepsilon_{i-1}-\varepsilon_i|}{2}$ and $\Delta(\varepsilon_{i-1}, \varepsilon_i) = |\varepsilon_{i-1} - \varepsilon_i|$. Now, in view of (4.2) with $F(\varepsilon) = \ln(\varepsilon)$ and $\Phi(s) = 1$ for $s \in R$, we have $\tau(h) \leq \Omega(\imath)$, where

$$\Omega(i) = \mathcal{F}_H(\int_0^{|\varepsilon_{i-1}-\varepsilon_i|} \delta s) - \mathcal{F}_H(\int_0^{\frac{|\varepsilon_{i-1}-\varepsilon_i|}{2}} \delta s).$$

Hence, all the required hypotheses of Theorem 4.4 are satisfied and consequently the hybrid pair (F, Y) for Branciari Ćirić type contraction mapping with respect to F_H-dynamic-iterative scheme $\check{D}(F, Y, \varepsilon_0)$ has a common fixed point. Hence, by Tables 3 and 4, and Figures 3 and 4, the required hypotheses of Theorem 4.4, regarding to $\tau(h) \leq \Omega(\iota)$, are satisfied for all possible values. Here, $0 = F(0) \in Y(0)$ is a common fixed point of F and Y. Next, observe that for $h \geq 70$ then $\tau(h) \nleq \Omega(\iota)$. So, Theorem 4.4 can not be satisfied.

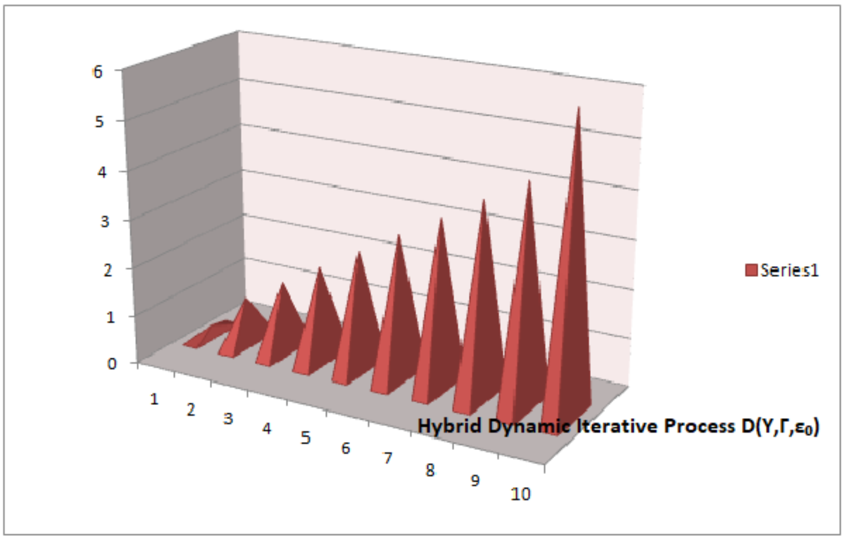

Figure 3. F_H-dynamic iterative process of F and Y starting from the point $\varepsilon_0 = 1$.

Table 4. Corresponding values of $\tau(h)$ & $\Omega(\iota)$.

ε_ι	$\varepsilon_{\iota-1}$	$\tau(h)$	$\Omega(\iota)$
0.5	0.5	0.00497516543	0.693147
.	1	0.00995033085	.
.	1.5	0.01492549628	.
.	2	0.01990066171	.
.	2.5	0.02487582713	.
.	3	0.02985099256	.
.	3.5	0.03482615799	.
.	4	0.03980132341	.
.	4.5	0.04477648843	.
0.5	5	0.04975165427	0.693147

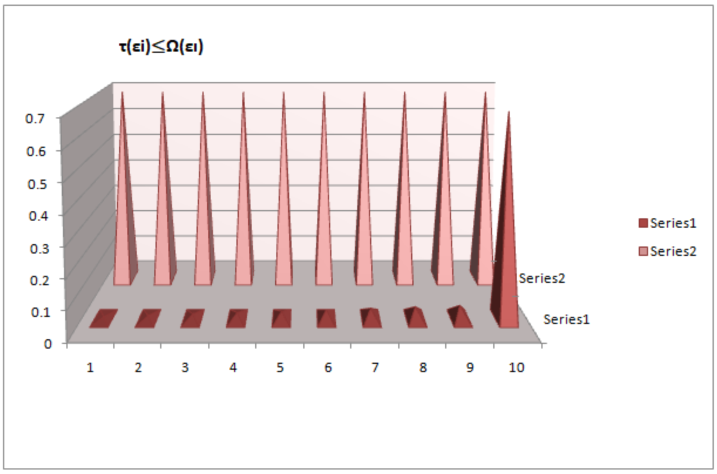

Figure 4. $\tau(h) \leq \Omega(\imath)$.

5. An application

Many recent developments on fractional calculus and fixed point theory are investigated in [16, 30], and also in the references therein.

Consider the Liouville-Caputo fractional differential equations viewed on order κ $(D_{(c,\kappa)})$ given as

$$D_{(c,\gamma)}(\omega(x)) = \frac{1}{\Gamma(i-\gamma)} \int_0^x (x-t)^{i-\gamma-1} \omega^{(i)}(t)\, dt \tag{5.1}$$

where $i-1 < \gamma < i$, $i = [\omega] + 1$, $\omega \in C^i([0, +\infty])$, the collection $[\gamma]$ corresponds to a positive real number and Γ is the Gamma function. Let the complete controlled-metric space $\delta_\varsigma : C(I) \times C(I) \to R^+$ be given as

$$\delta_\varsigma(g_{i-1}, g_i) = \left\| (g_1 - g_2)^2 \right\|_\infty = \sup_{a \in I} |g_1(a) - g_2(a)|^2 \tag{5.2}$$

with setting $\varsigma(g_1, g_2) = \varsigma(g_2, g_3) = 2$. Now, consider the following fashion of Liouville-Caputo fractional derivative

$$D_{(c,\gamma)}(\Xi(x)) = L_f(x, \Xi(x)), \tag{5.3}$$

where $x \in (0,1)$ and $\gamma \in (1,2]$ with

$$\begin{cases} \Xi(0) = 0, \\ \Xi(1) = \int_0^\vartheta \Xi(x)\, dx, \ \vartheta \in (0,1), \end{cases} \tag{5.4}$$

where $I = [0,1]$, $\Xi \in C(I, R)$ and $L : I \times R \to R$ is a continuous function. Take $P : \Psi \to \Psi$ as

$$Pv(r) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} L_f(t, v(t))\, dt \\ -\frac{2x}{(2-\vartheta^2)\Gamma(\gamma)} \int_0^1 (1-t)^{\gamma-1} L_f(t, v(t))\, dt \\ +\frac{2x}{(2-\vartheta^2)\Gamma(\gamma)} \int_0^\vartheta \left(\int_0^{x_1} (x_1 - t_1)^{\gamma-1} L_f(t_1, v(t_1))\, dt_1 \right) dt \end{cases} \tag{5.5}$$

for $v \in \Psi$ and $x \in [0,1]$. Now, we state the main result.

Theorem 5.1. *Suppose that L is non-decreasing on its second variable and there is* $\tau > 0$ *so that* $g_{i-1}, g_i \in D_{\varsigma}(\Upsilon, g_0)$ *and* $x \in [0,1]$ *implies*

$$|Pg_{i-1}(r) - Pg_i(r)| \leq \Omega \frac{\Delta(g_{i-1}, g_i)(r)}{\left(1 + \tau\sqrt{\max_{x\in I}\Delta(g_{i-1}, g_i)(r)}\right)^2}, \tag{5.6}$$

where $\Omega = \frac{(2\gamma-1)(\Gamma(\gamma+1))}{2(5\gamma+2)}$ *and*

$$\Delta(g_{i-1}, g_i)(r) = \max\left\{ \begin{array}{c} |g_{i-1}(r) - g_i(r)|^2, |g_{i-1}(r) - \Upsilon g_{i-1}(r)|^2, |g_i(r) - \Upsilon g_i(r)|^2, \\ \frac{|g_{i-1}(r) - \Upsilon g_i(r)|^2 + |g_i(r) - \Upsilon g_{i-1}(r)|^2}{2} \end{array} \right\}.$$

Then Eqs (5.3) and (5.4) have at least one solution, i.e., say $g^* \in \Psi$.

Proof. For each $x \in I$, consider

$$\begin{aligned}
& |Pg_{i-1}(r) - Pg_i(r)| \\
= & \Bigg|\Bigg(\frac{1}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1} L_f(t, g_{i-1}(t))\,dt \\
& - \frac{2x}{(2-\vartheta^2)\Gamma(\gamma)}\int_0^1 (1-t)^{\gamma-1} L_f(t, g_{i-1}(t))\,dt \\
& + \frac{2x}{(2-\vartheta^2)\Gamma(\gamma)}\int_0^{\vartheta}\left(\int_0^{x_1} (x_1-t_1)^{\gamma-1} L_f(t_1, g_{i-1}(t_1))\,dt_1\right)dt\Bigg) \\
& - \Bigg(\Bigg(\frac{1}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1} L_f(t, g_i(t))\,dt \\
& - \frac{2x}{(2-\vartheta^2)\Gamma(\gamma)}\int_0^1 (1-t)^{\gamma-1} L_f(t, g_i(t))\,dt \\
& + \frac{2x}{(2-\vartheta^2)\Gamma(\gamma)}\int_0^{\vartheta}\left(\int_0^{x_1} (x_1-t_1)^{\gamma-1} L_f(t_1, g_i(t_1))\,dt_1\right)dt\Bigg)\Bigg| \\
\leq & \frac{1}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}\left|L(t, g_{i-1}(t)) - L_f(t, g_i(t))\right|dt \\
& + \frac{2x}{(2-\vartheta^2)\Gamma(\gamma)}\int_0^1 (1-t)^{\gamma-1}\left|L_f(t, g_{i-1}(t)) - L_f(t, g_i(t))\right|dt \\
& + \frac{2x}{(2-\vartheta^2)\Gamma(\gamma)}\int_0^{\vartheta}\left|\int_0^{x_1} (x_1-t_1)^{\gamma-1}\left(L_f(t_1, g_{i-1}(t_1)) - L_f(t, g_i(t))\right)dt_1\right|dt.
\end{aligned}$$

Now, we have

$$\begin{aligned}
& |Pg_{i-1}(r) - Pg_i(r)| \\
\leq & \frac{1}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}\,\Omega\frac{\Delta(g_{i-1}, g_i)(r)}{\left(1 + \tau\sqrt{\max_{x\in I}\Delta(g_{i-1}, g_i)(r)}\right)^2}dt \\
& + \frac{2x}{(2-\vartheta^2)\Gamma(\gamma)}\int_0^1 (1-t)^{\gamma-1}\,\Omega\frac{\Delta(g_{i-1}, g_i)(r)}{\left(1 + \tau\sqrt{\max_{x\in I}\Delta(g_{i-1}, g_i)(r)}\right)^2}dt
\end{aligned}$$

$$+\frac{2x}{(2-\vartheta^2)\,\Gamma(\gamma)}\int_0^{\vartheta}\int_0^{x_1}(x_1-t_1)^{\gamma-1}\,\Omega\frac{\Delta(g_{i-1},g_i)\,(r)}{\left(1+\tau\sqrt{\max_{x\in I}\Delta(g_{i-1},g_i)\,(r)}\right)^2}dt_1dt$$

$$\leq \frac{\Omega\Delta(g_{i-1},g_i)\,(r)}{\Gamma(\gamma)\left(1+\tau\sqrt{\max_{x\in I}\Delta(g_{i-1},g_i)\,(r)}\right)^2}\left\{\begin{array}{c}\int_0^x(x-t)^{\gamma-1}\,dt\\ +\frac{2x}{(2-\vartheta^2)}\int_0^1(1-t)^{\gamma-1}\,dt\\ +\frac{2x}{(2-\vartheta^2)}\int_0^{\vartheta}\int_0^{x_1}(x_1-t_1)^{\gamma-1}\,dt_1dt\end{array}\right\}.$$

This yields that

$$\begin{aligned}|Pg_{i-1}(r)-Pg_i(r)| &\leq \frac{\Omega\Delta(g_{i-1},g_i)\,(r)}{\Gamma(\gamma)\left(1+\tau\sqrt{\max_{x\in I}\Delta(g_{i-1},g_i)\,(r)}\right)^2}\\ &\quad\left\{\frac{x^{\gamma}}{\gamma}+\frac{2x}{(2-\vartheta^2)}\frac{1}{\gamma}+\frac{2x}{(2-\vartheta^2)}\frac{\vartheta^{\gamma+1}}{\gamma(\gamma+1)}\right\}\\ &\leq \frac{\Omega\Delta(g_{i-1},g_i)\,(r)}{\left(1+\tau\sqrt{\max_{x\in I}\Delta(g_{i-1},g_i)\,(r)}\right)^2}\\ &\quad\sup_{x\in(0,1)}\left\{x^{\gamma}+\frac{2x}{(2-\vartheta^2)}+\frac{2x}{(2-\vartheta^2)}\frac{\vartheta^{\gamma+1}}{(\gamma+1)}\right\}\\ &= \frac{(2\gamma-1)}{2\,(5\gamma+2)}\frac{\Delta(g_{i-1},g_i)\,(r)}{\left(1+\tau\sqrt{\max_{x\in I}\Delta(g_{i-1},g_i)\,(r)}\right)^2}\\ &\quad\sup_{x\in(0,1)}\left\{x^{\gamma}+\frac{2x}{(2-\vartheta^2)}+\frac{2x}{(2-\vartheta^2)}\frac{\vartheta^{\gamma+1}}{(\gamma+1)}\right\}\\ &= \frac{(2\gamma-1)}{2\,(5\gamma+2)}\frac{\Delta(g_{i-1},g_i)\,(r)}{\left(1+\tau\sqrt{\max_{x\in I}\Delta(g_{i-1},g_i)\,(r)}\right)^2}.\end{aligned}$$

It implies that

$$|Pg_{i-1}(r)-Pg_i(r)|\leq\frac{\Delta(g_{i-1},g_i)\,(r)}{\left(1+\tau\sqrt{\max_{x\in I}\Delta(g_{i-1},g_i)\,(r)}\right)^2}. \tag{5.7}$$

Therefore,

$$\begin{aligned}\delta_{\varsigma}(Pg_{i-1}(r)-Pg_i(r)) &= \sup_{a\in I}|Pg_{i-1}(r)-Pg_i(r)|^2\\ &\leq \frac{\Delta(g_{i-1},g_i)\,(r)}{\left(1+\tau\sqrt{\max_{x\in I}\Delta(g_{i-1},g_i)\,(r)}\right)^2}.\end{aligned} \tag{5.8}$$

Now, by contractive condition (2.1) with $\Phi(s)=1$ for all $s\in R$ and $F(s)=-\frac{1}{\sqrt{s}}$, we have

$$\delta_{\varsigma}(\Upsilon g_i,\Upsilon g_{i+1})>0\ \Rightarrow\ \tau(\Delta(g_{i-1},g_i))+\mathcal{F}\left(\int_0^{\delta_{\varsigma}(\Upsilon g_i,\Upsilon g_{i+1})}\Phi(s)\,\delta s\right)\leq\mathcal{F}\left(\int_0^{\Delta(g_{i-1},g_i)}\Phi(s)\,\delta s\right),$$

for all $i\in\mathbb{N}$, $g_i\in\Psi$ and for each given $\epsilon>0$ so that $\int_0^{\epsilon}\Phi(s)\,\delta s>0$. Thus, all the required hypotheses of Theorem 3.4 are satisfied and we ensure that the Eqs (5.3) and (5.4) have at least one solution in P. □

Theorem 5.2. *Let $L : I \times R \to R$ be a continuous function, non-decreasing on second variable and there is $\tau > 0$ so that $g_{i-1}, g_i \in D(\Upsilon, Y, g_0)$ and $x \in [0,1]$ implies*

$$|Pg_{i-1}(r) - Pg_i(r)| \leq \Omega \frac{\Delta(g_{i-1}, g_i)(r)}{\left(1 + \tau\sqrt{\max_{x \in I} \Delta(g_{i-1}, g_i)(r)}\right)^2},$$

where $\Omega = \frac{(2\gamma-1)(\Gamma(\gamma+1))}{2(5\gamma+2)}$ and

$$\Delta(g_{i-1}, g_i)(r) = \max\left\{\begin{array}{c} |\Upsilon g_{i-1}(r) - \Upsilon g_i(r)|^2, |\Upsilon g_{i-1}(r) - Y g_{i-1}(r)|^2, |\Upsilon g_i(r) - Y g_i(r)|^2, \\ \frac{|\Upsilon g_{i-1}(r) - Y g_i(r)|^2 + |\Upsilon g_i(r) - Y g_{i-1}(r)|^2}{2} \end{array}\right\}.$$

In the light of Theorem 5.1 with Υ is Y-weakly commuting at g so that $\Upsilon^2 g = Y\Upsilon g$, we conclude that Eqs (5.3) and (5.4) have at least one solution.

Example 5.3. Consider the Liouville-Caputo fractional differential equations based on order $\gamma \left(D_{(c,\gamma)}\right)$

$$D_{\left(c,\frac{3}{2}\right)}(\Xi(x)) = \frac{1}{(x+3)^2} \frac{|\Xi(x)|}{1 + |\Xi(x)|}, \tag{5.9}$$

and its integral boundary valued problem:

$$\begin{cases} \Xi(0) = 0, \\ \Xi(1) = \int_0^{\frac{3}{4}} \Xi(x)\,dx, \ \vartheta \in (0,1), \end{cases} \tag{5.10}$$

where $\gamma = \frac{3}{2}$, $\vartheta = \frac{3}{4}$ and $L(x, v(x)) = \frac{1}{(x+3)^2} \frac{|\Xi(x)|}{1+|\Xi(x)|}$. So, the above setting is an example of Eqs (5.3) and (5.4). Hence, the Eqs (5.9) and (5.10) have at least one solution.

6. Open problems

In this section, we pose some challenging questions for the readers.

Problem 1: Can Theorems 3.4 and 4.4 be proved without the condition (F_{iii})?

Problem 2: Can Theorems 3.4 and 4.4 be proved by Semi-F-contraction and without the continuity of F-contraction?

7. Conclusions

In our present investigation, we have introduced and systematically studied an extension of the developments concerning F- contractions that were proposed, in the year 2012 by Wardowski. We have fruitfully developed and generalized the notion of the F- contractions to the case of non-linear F and F_H-dynamic-iterative scheme for Branciari Ćirić type-contractions and proved several multi-valued fixed point results on controlled-metric spaces. An approximations of the dynamic-iterative scheme instead of the conventional Picard sequence are also determined. The paper also includes a tangible example and a graphical interpretation that displays the motivation for such investigations. The work is completed by giving an application of the proposed non-linear F and F_H-dynamic-iterative scheme to the Liouville-Caputo fractional derivatives and fractional differential equations. In the future, these results can be furthered to acquire fixed point results for single and multi-valued mappings in the context of double controlled-metric space and triple controlled-metric spaces.

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Conflicts of interest

The authors declare that they have no competing interests.

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